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EXISTENCE OF SOLUTIONS IN THE SENSE OF DISTRIBUTIONS OF ANISOTROPIC NONLINEAR ELLIPTIC EQUATIONS WITH VARIABLE EXPONENT

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Abstract. The aim of this paper is to study the existence of solutions in the sense of distributions for a strongly nonlinear elliptic problem where the second term of the equation f is in $W^{-1,\overrightarrow{p}'(\,\cdot\,)}(\Omega)$ which is the dual space of the anisotropic Sobolev $W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$ and later f will be in $L^1(\Omega)$.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$. For the variable vectorial exponent $\overrightarrow{p}(\,\cdot\,) = (p_0(\,\cdot\,), \ldots, p_N(\,\cdot\,))$, we assume that for $i = 0, \ldots, N$, the functions $p_i(x) \in \mathcal{C}_+(\overline{\Omega})$ (defined in Section 2), where

$$p_0(x) \geq \max\{p_i(x),\ i = 1, \ldots, N\}, \quad \text{for any } x \in \overline{\Omega}. \tag{1.1}$$

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Our aim is to prove the existence of solutions in the sense of distributions to the anisotropic nonlinear elliptic problem:

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i} a_i(x,u,\nabla u) + g(x,u,\nabla u) + d(x)|u|^{p_0(x)-2}u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where the right-hand side f is in $W^{-1,\overrightarrow{p}'(\,\cdot\,)}(\Omega)$ which is the dual space of the anisotropic Sobolev space $W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$ and later f will be in $L^1(\Omega)$. The positive function $d(x)$ belong to $L^\infty(\Omega)$, and there exists a constant $d_0 > 0$ such that $d(x) \geq d_0$ almost everywhere in Ω.

We assume that for $i = 1, \ldots, N$ the function $a_i \colon \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ is Carathéodory function (i.e. measurable with respect to x in Ω for every (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$ and continuous with respect to (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω) which satisfies the following conditions:

$$|a_i(x,s,\xi)| \leq \beta\,(K_i(x) + |s|^{p_i(x)-1} + |\xi_i|^{p_i(x)-1}) \quad \text{for } i = 1,\ldots,N, \tag{1.3}$$

$$a_i(x,s,\xi)\xi_i \geq \alpha|\xi_i|^{p_i(x)} \quad \text{for } i = 1,\ldots,N, \tag{1.4}$$

$a_i(\,\cdot\,,\,\cdot\,,\,\cdot\,)$ is strictly monotone, i.e. for all $\xi = (\xi_1,\ldots,\xi_N)$ and $\xi' = (\xi'_1,\ldots,\xi'_N)$ in $\mathbb{R}^N$, we have

$$(a_i(x,s,\xi) - a_i(x,s,\xi'))(\xi_i - \xi'_i) > 0, \quad \text{for } \xi_i \neq \xi'_i, \tag{1.5}$$

for almost every $x \in \Omega$ and all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, where $K_i(\,\cdot\,)$ is a non-negative function lying in $L^{p'_i(\,\cdot\,)}(\Omega)$ where $1/p_i(x) + 1/p'_i(x) = 1$ and $\alpha, \beta > 0$ are two positive constants.

Note that, Gwiazda et al. in [17] studied a steady and in [18] a dynamic model for non-Newtonian fluids under an additional strict monotonicity assumption on the operator. The authors used Young measure techniques in place of a monotonicity method. Moreover, a version of the Minty–Browder trick adapted to the setting of generalized Orlicz spaces was introduced in [27] (see also [19]) in framework of non-Newtonian fluids.

The nonlinear term $g(x,s,\xi)$ is a Carathéodory function which satisfies

$$g(x,s,\xi)s \geq 0, \tag{1.6}$$

$$|g(x,s,\xi)| \leq b(|s|)(c(x) + \sum_{i=1}^{N} |\xi_i|^{p_i(x)}), \tag{1.7}$$

where $b(\,\cdot\,) \colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous non-decreasing function, and $c(\,\cdot\,) \colon \Omega \to \mathbb{R}^+$ with $c(\,\cdot\,) \in L^1(\Omega)$.

In view of (1.7), the Carathéodory function $g(x,u,\nabla u)$ does not define a mapping from $W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$ into its dual, but from $W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega) \cap L^\infty(\Omega)$ into $L^1(\Omega)$ (see also [9]).