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PERIODIC SOLUTIONS FOR SECOND ORDER SINGULAR DIFFERENTIAL SYSTEMS WITH PARAMETERS

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Abstract. In this paper we consider the existence of periodic solutions of one-parameter and two-parameter families of second order singular differential equations.

1. Introduction

We say that a vector valued function $f\colon \mathbb{R}\times D\to\mathbb{R}^N$, $D\subseteq\mathbb{R}^N$, is singular if for a non-empty subset $\Omega\subset\partial D$ and any $x_0\in\Omega$

$$\lim_{x\to x_0}\|f(t,x)\|=\infty,\quad \text{uniformly in } t\in\mathbb{R}.$$

Equivalently, we say that the differential equation

$$\ddot{x}+a(t)x=f(t,x)+e(t) \tag{1.1}$$

is singular if the nonlinear term f is singular, where $a,e\in C(\mathbb{R},\mathbb{R}^N)$ are T-periodic functions, $f\colon\mathbb{R}\times D\to\mathbb{R}^N$ is continuous for some $D\subseteq\mathbb{R}^N$ and periodic in t with period T.

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During the last two decades, the existence of nontrivial periodic solutions of (1.1) has been studied by many researchers in the nonsingular case as well as in the singular case. See [2], [11], [12], [15], [16], [18], [23] for the scalar case and [3], [4], [6], [7], [17] for the higher dimensional case. Usually, the proof is based on either the method of upper and lower solutions [1], [10], [15], fixed point theorems [6], [7], [16]–[18], alternative principle of Leray–Schauder [3], [11] or topological degree theory [23], [26].

On the other hand, second-order nonlinear differential equations or systems with parameters have also been studied by some researchers. See, for example [8], [13], [19]–[22], [25] and the references therein. Based on a fixed point theorem in cones, under different combinations of superlinearity and sublinearity of the function f, Graef, Kong and Wang in [8] studied the existence, multiplicity, and nonexistence results for positive solutions of the following scalar nonsingular periodic boundary value problem

$$\begin{cases} \ddot{y} - \rho^2 y + \lambda g(t) f(y) = 0, \\ y(0) = y(2\pi), \quad \dot{y}(0) = \dot{y}(2\pi), \end{cases} \tag{1.2}$$

for different parameter values $\lambda \in \mathbb{R}^+ = (0, \infty)$. Later, Wang in [21] extended the similar idea to the singular periodic systems. For systems with two parameters, one nice result was proved in [22] by Wu and Wang. They studied the existence of periodic solutions of the following system with two parameters

$$\begin{cases} \ddot{u} + a_1(t) u = \lambda b_1(t) f_1(u, v), \\ \ddot{v} + a_2(t) v = \mu b_2(t) f_2(u, v), \end{cases} \tag{1.3}$$

where $(\lambda, \mu) \in (\mathbb{R}^+)^2$ and a_i, b_i, f_i, $i = 1, 2$, satisfy some additional conditions (see [22, conditions (H_1)–(H_5)]). By employing fixed point index theory, they show that there exist three nonempty subsets of $(\mathbb{R}^+)^2$: Γ, Δ_1, Δ_2 such that $(\mathbb{R}^+)^2 = \Gamma \cup \Delta_1 \cup \Delta_2$ and (1.3) has at least two positive periodic solutions for $(\lambda, \mu) \in \Delta_1$, one positive periodic solution for $(\lambda, \mu) \in \Gamma$ and no positive periodic solutions for $(\lambda, \mu) \in \Delta_2$. Note that the results in [22] can only be applied to the nonsingular case. Yang in [24] established the existence results for $2m$-order differential systems with two parameters.

Motivated by these recent developments, in this paper, we investigate the existence and multiplicity of T-periodic solutions of the following special case of system (1.3),

$$\begin{cases} \ddot{x} + a_1(t) x = \lambda f_1(x, y), \\ \ddot{y} + a_2(t) y = \mu f_2(x, y), \end{cases} \tag{1.4}$$

where $\lambda, \mu \in \mathbb{R}^+$. However, we consider (1.4) in the singular case, which is the main difference between our results and those in the literature.