THEORIA ET HISTORIA SCIENTIARUM, VOL. VIII, N° 1
Ed. Nicolaus Copernicus University 2008

Michael Leyton

Geometry as Transfer

1. Introduction

It is generally accepted that intelligent action involves considerable use of *transfer*. For example, Carbonell [1] has argued that learning proceeds by analogical reasoning; Rosch [12] has argued that categorization proceeds by seeing objects in terms of prototypes; and Leyton [9] has argued that the human perceptual system is organized as a hierarchy of transfer.

The role of *geometry* is also seen as fundamental to the representations produced by the cognitive system. For example, Gallistel [2] has elaborated the powerful role of geometry in animal learning and navigation; Lakoff [3] has emphasized the role of geometry in semantics; and Leyton [9] has proposed an extensive role for geometry in causal explanation.

We bring together the two above factors, *transfer* and *geometry,* in the book, Leyton [10], by developing a generative theory of shape in which transfer is a fundamental organizing principle. *In this approach, transfer is basic to the very meaning of geometry.* The purpose of the present paper is to give an introduction to this transfer-based theory of geometry.

2. Transfer

In this section, we will give a basic introduction to this theory of transfer, with examples from human perception, robot kinematics, and object-oriented programming. In the subsequent sections we will look at the structure of differential equations and scientific laws.

A generative theory of shape characterizes the structure of a shape by a sequence of actions needed to generate it. According to our theory, these actions must maximize transfer. That is:

MAXIMIZATION OF TRANSFER. *Make one part of the generative sequence a transfer of another part of the generative sequence, whenever possible.*

We will show that the appropriate formulation of this is as follows: A situation of transfer involves two levels: the *fiber group,* which is the group of actions to be transferred; and the *control group,* which is the group of actions that will transfer the fiber group. The justification for these structures algebraically being groups is given in Leyton [10], but the theory of transfer will work equally for semi-groups, which is the most general case we would need to consider for generativity.

Now, we can think of transfer as the control group moving the fiber group around some space. Furthermore, the action of transfer combines the fiber group and control group into a total group written thus:

Fiber Group ® Control Group

This total group contains all the information of the situation. It contains not only the fiber group and the control group, but the algebraic relation between them. We will argue that the 2-place operation ⓦ is what is group-theoretically called a *wreath product* to be fully described later. However, for now the reader needs to understand only that the operation encodes the fact that there is a transfer relationship between the control group and the fiber group; i.e., *the control group moves the fiber group around.* The purpose of the present section is to give the reader an intuitive description of transfer, together with several examples that will illustrate the power of transfer. The precise mathematical structure will be given in section 5, where we will elaborate our claim that transfer is best modeled by a wreath product. Although it will not be required in the present section, a rigorous definition of wreath product is given in the footnote below.[1]

[1] Consider two group actions: the actions of groups, $G(F)$ and $G(C)$, on sets, F and C, respectively. The wreath product $G(F) ⓦ G(C)$ is the semi-direct product $\{\Pi_{c\in C} G(F)_c\} ⓢ_\sigma G(C)$, where the product symbol Π means the (group) direct product, and the groups $G(F)_c$ are isomorphic copies of $G(F)$ indexed by the members c of the set C. The map $\sigma : G(C) \rightarrow Aut\ \{\Pi_{c\in C} G(F)_c\}$ is defined such that $\sigma(g)$ corresponds to the group action of $G(C)$ on C, now applied to the indexes c in $\Pi_{c\in G(C)} G(F)_c$. That is, $\sigma(g) : \Pi_{c\in G(C)} G(F)_c \rightarrow \Pi_{c\in G(C)} G(F)_{gc}$. Finally, we have a group action of $G(F) ⓦ G(C)$ on $F \times C$ defined thus: For $\phi \in G(F)$, $\kappa \in G(C)$, and (f, c) in $F \times C$, we have $[\phi, \kappa](f, c) = (\phi_c f, \kappa c)$, where $\phi_c \in G(F)_c$.

We shall often use another term for transfer: *nested control*. That is, the group **Fiber Group ⓦ Control Group** will be called a *structure of nested control*; the 2-place operation ⓦ will be referred to as the *control-nesting operation*; and the fiber group will be said to be *control-nested* in the control group.

The above discussion considered a 2-level structure of transfer; i.e., the movement of a fiber group by a control group. An n-level structure of transfer will be constructed by a recursive use of the 2-place operation ⓦ, and the resulting group will be written thus:

$$G_1 ⓦ G_2 ⓦ \dots ⓦ G_n.$$

In this group, each level G_i acts as a control group with respect to its left-subsequence $G_1 ⓦ G_2 ⓦ \dots ⓦ G_{i-1}$ as fiber. In other words, G_i transfers its left-subsequence around some environment; but this left-subsequence is itself a hierarchy of transfer, and so on, recursively downwards.

Shape generation is best described by an n-fold hierarchy of transfer; that is, by a group of the form $G_1 ⓦ G_2 ⓦ \dots ⓦ G_n$.

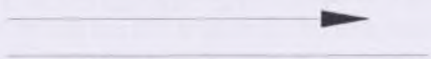

Figure 1. The generation of a side, using translations.

It is now necessary for us to work through several examples so that the reader can begin to become familiar with aspects of this approach, and see also the wide range of applications.

(1) Human Perception

In Leyton [4], [5], [6], [7], [8], [9], we put forward several hundred pages of psychological evidence showing that the human perceptual system is structured as a control-nested hierarchy of groups, $G_1 ⓦ G_2 ⓦ \dots ⓦ G_n$. We argued that only this view could explain a wealth of psychological results in the area of perceptual organization, shape representation, and motion perception. For example, all the Gestalt grouping phenomena can be explained very economically using this principle: the perceptual groupings correspond to symmetry groups G. and these are control-nested recursively in a manner described above. We showed that this applies not only to static shape perception but also to motion perception.

In order to illustrate this, let us begin with a very simple example. We will show the way in which the human visual system structures a square. In a sequence of psychological experiments, Leyton [6], [7], we showed that human vision represents a square *generatively*, in the following way. It begins with the top side. Perceptually the top side is generated by starting with a corner point, and applying translations to trace out the side, as shown in Fig. 1.

Next, this translational structure is *transferred* from one side to the next - rotationally around the square. In other words, we have a transfer of translation by rotation. This is illustrated in Fig. 2.

Therefore, the transfer structure is defined as:

$$\mathbf{Translations} \ \textcircled{w} \ \mathbf{Rotations}$$

where **Translations** is the fiber group and **Rotations** is the control group. Recall that, in any transfer situation, the control group moves the fiber group around.

We will represent the translations group simply as the additive group **R.** The rotations group is $\mathbf{Z_4}$**,** the cyclic group of order 4, represented as

$$\mathbf{Z}_4 = \{e, r_{90}, r_{180}, r_{270}\}.$$

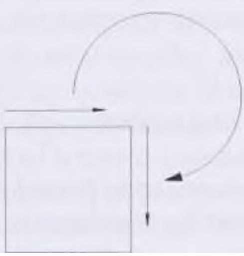

Figure 2. Transfer of translation by rotation.

where r_g means rotation by *d* degrees. The successive group elements are obviously rotations by successive 90° increments. Thus the transfer structure illustrated in Fig. 2, is this:

$$\mathbf{R} \ \textcircled{w} \ \mathbf{Z}_4. \tag{1}$$

At first, the reader might question our putting the *entire* group of translations in the fiber position, even though the group is “cut off” at the end points of a side. However, this is handled very easily in our system by placing what we call an **occupancy group, Z,** (a cyclic group of order 2), at each point along the infinite line. The group switches between two states, “occupied” and “non- occupied,” for the point at which it is located. Obviously, all the points along the finite side of the square are occupied, and all the points past its end-points are unoccupied. Algebraically, we place the occupancy group as an extra level, in the structure of nested control, below the **R** group, thus:

$$\mathbf{Z}_2 \textcircled{w} \mathbf{R} \textcircled{w} \mathbf{Z}_4.$$

In Leyton [10], we investigate occupancy structures in detail, and show that they elegantly represent many phenomena, e.g., in Gestalt perception, quantum physics, etc. In the present paper, however, we will omit the occupancy level, to keep the discussion focussed on the geometric (spatial) structure. Observe that the occupancy group is a color group, not a geometrical group; i.e., it has no spatial action.

Thus let us now return to the purely geometric structure, given in expression (I) above. The next thing we want to do is to show that this expression gives **generative coordinates** to the square. As before, we will assume, without loss of generality, that the top side is generated by translation from the left end; and that the set of sides is generated from the top side by clockwise rotations. In other words, we are using the standard scenario for drawing a square: simply trace out the sides successively in the clockwise direction. The group **R ® Z**$_+$ gives the structure of this trace, which we see is actually control-nested. The control-nested structure will give the *generative* coordinates of each point, as follows:

Because the structure is generative, we can consider the fiber group **R** as mapped onto each side. For example, consider the top side. As shown in Fig. 3, the zero translation, *e*, is mapped to the left corner on the side. Then, any other point on the side is uniquely described by the translation *t* that generated it from the initial point. Fig. 3a shows the actual translation that was applied, i.e., as an action, and Fig. 3b shows the action converted into the label for the point. The same structure occurs on any side.

Figure 3. Translation as a coordinate along side.

Similarly, consider the control group $\mathbf{Z}_4 = \{e, r_{90}, r_{180}, r_{270}\}$. Again because of generativity, this group is mapped onto the set of four sides, in the manner shown in Fig. 4. Because, the top side is the starting side it receives the identity element e from $\mathbf{Z}_4$. Any other side receives one of the rotations in Z_4, i.e., the generative operation that was used to create the side from the top side.

Any point on the square is therefore described by a pair of coordinates:

$$(t, r) \in \mathbf{R} \ \textcircled{w}\ \mathbf{Z}_4$$

where $t \in \mathbf{R}$ and $r \in \mathbf{Z}_4$. For example, as shown in Fig. 5, a point on the right side is labeled (t, r_{90}) where t is the translation that was used to generate it from the starting point of the side, and r_{90} was the rotation that was used to generate the right side from the top side.

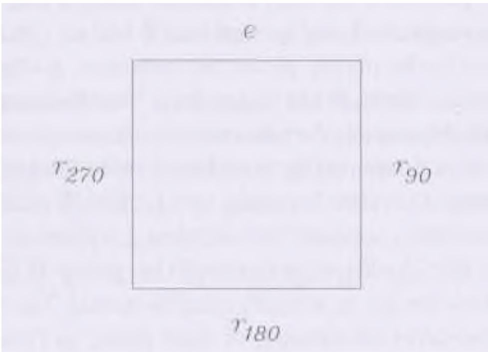

Figure 4. The mapping of $\mathbf{Z}_4$ onto a square.

In order to describe the phenomenon we are going to investigate, it is necessary to fill in some other coordinates on the square. Consider the top left- hand corner point, as shown in Fig 6. We have assumed that the entire history starts here. Therefore the amount of translation here is zero - i.e., the point is at the identity element $e_1$of the fiber group **R.** Furthermore, the amount of rotation that has been applied so far is also zero - i.e., the side on which the point sits is at the identity element e_2 of the control group $\mathbf{Z_4}$**.** Therefore, as shown in Fig. 6. the top left corner-point has the pair of coordinates (e_1, e_2) in **R (w)** $\mathbf{Z_4}$**.** Now consider the point at translational distance t along the top side. Its coordinates are clearly, (r, e_2) as shown in Fig. 6. Next consider the top right-hand corner point, and consider its description as the first point on the right-hand side. As such, it has undergone no translation along that side, and is therefore at the identity element e_1 of the fiber group **R.** However, the point must have coordinate r_{90} in the control group $\mathbf{Z_4}$ because the right side is achieved by a 90° rotation from the top side. Thus the point has the pair of coordinates (e , r). Finally, as we saw earlier, the lower labeled point on the right-hand side has coordinates (f, r_{90}).

The crucial thing we need to observe is the *transfer* structure involved in this. First observe that the relationship between the two points on the top side is the translation *t* given by the top straight arrow in Fig. 7. Similarly the relationship between the two points on the right side is the translation *t* given by the downward straight arrow. The *transfer* effect of rotation is to send the translation on the top side to the translation on the right side. This is shown by the circular arrow in Fig 7, which sends the straight arrow on the top side to the straight arrow on the right side. As we have said before, the control group **Z_4,** takes the fiber group **R** and transfers it from one side on to the next.

We now need to observe that the group we are studying, **R ® Z_4,** satisfies the following three conditions:

391,: *The group is decomposable into a control-nested structure* G, ® G_2@ ... ® G_n.

39? : Each level is "1-dimensional", i.e., either a cyclic group (in the discrete case) or a 2 1-parameter group (in the continuous case).

39? : Each level is represented as an isometry group.

3

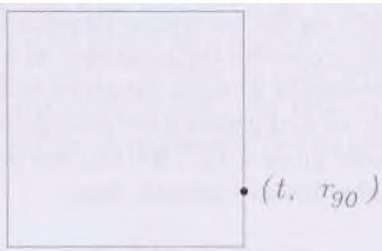

Figure 5. The coordinates of a point on a square.

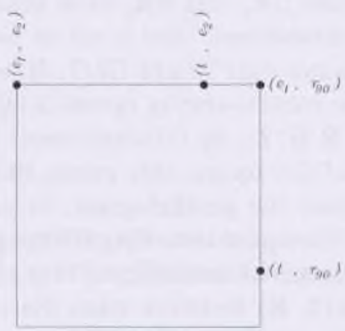

Figure 6. The coordinates of four points.

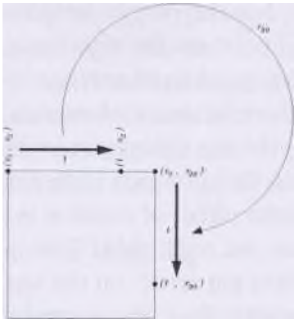

Figure 7. The control-nested structure of those coordinates.

We will call a group that obeys the above three conditions, an **iso-regular group.** Intuitively, we will summarize the three conditions by saying that the group is a *controlnested hierarchy of repetitive isometries.* Iso-regular groups will be fundamental to our theory of shape-generation. In our theory, such groups describe non-deformed objects. The theory states that any shape has an underlying iso-regular group. To generate the shape, one starts by generating its iso-regular group, and then adding actions that create deformation. These actions are imposed as further levels of transfer on the iso-regular structure.

As a simple illustration, consider the generation of a parallelogram. We shall see that its underlying iso-regular group is the group $\mathbf{R} \textcircled{w} \mathbf{Z}_4$ of a square. Thus to generate a parallelogram, we first generate the iso-regular group of a square, and then add the general linear group $GL(2, \mathbf{R})$, i.e., the group of invertible linear transformations, as a higher level of control, thus:

$$\mathbf{R} \textcircled{w} \mathbf{Z}_4 \textcircled{w} GL(2, \mathbf{R}). \tag{3}$$

Notice that, with this extra level, we no longer have an iso-regular group, because the iso-regularity conditions $\mathfrak{IR}_2$ and $\mathfrak{IR}_3$ have been broken; i.e., the added level $GL(2, \mathbf{R})$ is not “1-dimensional” and is not an isometry group.

Notice that the operation used to add $GL(2, \mathbf{R})$ on to the lower structure $\mathbf{R}$ © $\mathbf{Z}_4$ is, once again, the control-nesting operation $\textcircled{w}$ which means that $GL(2, \mathbf{R})$ acts by *transferring* $\mathbf{R} \textcircled{w} \mathbf{Z}_4$, as follows: Since the fiber group $\mathbf{R} \textcircled{w} \mathbf{Z}_4$ represents the structure of the square, this means that $GL(2, \mathbf{R})$ transfers the structure of the square onto the parallelogram. In particular, it transfers the generative coordinates of the square onto the parallelogram. For example, recall that Fig. 6 showed the generative coordinates of four of the points on the square. The control action of $GL(2, \mathbf{R})$ therefore takes the coordinates of these four points and transfers them onto the corresponding four points on the parallelogram, as shown in Fig 8.

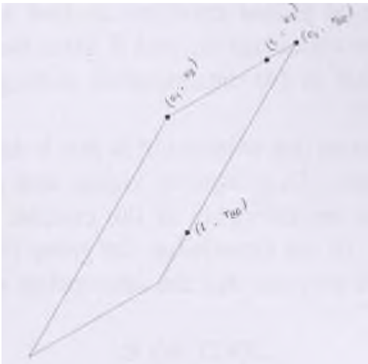

Figure 8: The transferred coordinates from a square.

More deeply still, the fiber group $\mathbf{R}$ ⓦ $\mathbf{Z}_4$ in expression (3) is itself a transfer structure, as we saw in Fig. 7, where rotation transferred the translation process from the top side onto the right side. This transfer structure is itself transferred, by $GL(2, \mathbf{R})$, onto the parallelogram, as shown in Fig. 9. That is, we have **transfer of transfer**. This recursive transfer is encoded by the successive ⓦ operations in expression (3). This illustrates what we said earlier, that, given a transfer hierarchy G_1 ⓦ G_2 ⓦ ... ⓦ G_n, each level G_i acts as a control group with respect to its left-subsequence G_1 ⓦ G_2 ⓦ ... ⓦ G_{i-1} as fiber. In other words, G_i transfers its left-subsequence around some environment; but this left-subsequence is itself a hierarchy of transfer, and so on recursively downwards.

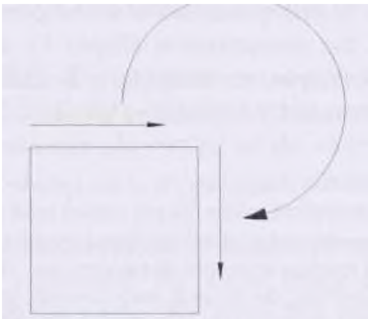

Figure 9. The transfer of transfer.

The theory we give is equally applicable to 3-dimensional shape. For example, consider the structure of a cylinder. The standard group-theoretic description of a cylinder is

$$SO(2) \times \mathbf{R} \tag{4}$$

where 5(9(2), the group of planar rotations around a fixed point, gives the rotational symmetry of the cross-section, and **R** gives the translational symmetry along the axis. Notice that in (4) the operation linking the two groups is the direct product operation x.

For us, the problem with this expression is that it does not give a generative description of the cylinder. In computer vision and graphics, cylinders are described generatively as the sweeping of the circular cross-section along the axis, as shown in Fig. 10. To our knowledge, the group of this sweeping structure has never been given. We propose that the appropriate group is:

$$SO(2) \textcircled{w} \mathbf{R}. \tag{5}$$

Notice that it uses the control-nesting operation ⓦ rather than the direct product $\times$, and therefore the group has a very different structure from that in expression (4). The operation ⓦ means that this new group has a *fiber-control* structure, in which $SO(2)$ is the fiber group and **R** is the control group. This is exactly what we see in the sweeping structure shown in Fig. 10. The cross-section is generated first as a fiber, and then its position is controled by translation.[2]

The reader should observe that the control-nested group in (5) is what we call an *iso-regular group*; i.e., it satisfies the conditions $\mathfrak{IR}_1$-$\mathfrak{IR}_3$ on page 229. This fact is critical: The cylinder is an example of a standard shape primitive in graphics. In Leyton [10], we argue that each of the standard primitives is characterized by an iso-regular group. In fact, we show that our algebraic methods lead to *a systematic classification of shape primitives.*

We also argue that, having generated a shape primitive via an iso-regular group, one then obtains the non-primitive shapes by applying additional fiber and control levels. For example, we show how Boolean operations and spline deformations can be algebraically formulated within this framework.

[2] Although the the direct product description (4) of the cylinder is used universally, we argue that that there is a strong mathematical reason why it cannot model the cylinder as a generative structure, and is therefore inappropriate for modeling crystal growth in physics, drilling and milling in manufacturing, assembly of revolute structures in robotics, etc. The reason is as follows: In the generative representation of a cylinder, the group **R** must move the group 50(2) along the cylinder. This movement must take place by the conjugation g - g^{-1} of 50(2) by the elements g of **R** (conjugation is the group-theorists tool for movement). However, in the direct product formulation (4), the rotation group 50(2) is a normal subgroup; which means that conjugation of 50(2) by **R** will leave 50(2) invariant. Therefore it will not be able to move 50(2) along the cylinder. This means that the direct product formulation cannot model generative structure (i.e., crystal growth, drilling and milling, robot assembly, etc.). In contrast, we shall see that, in the control-nested formulation (5), the rotation group 50(2) is not a normal subgroup. Therefore, in this latter formulation, it can move 50(2). Indeed the fibering that occurs in a wreath product operation will ensure that **R** moves 50(2) along the cylinder in the correct way.

Figure 10. The sweep structure of a cylinder.

Now let us turn to the deepest problem in human perception: the problem of *perceptual organization.* This problem is standardly formulated as the following question: What are the structural principles by which the perceptual system forms *groupings?* The problem of grouping is the longest unsolved problem in perception - having been investigated for the entire 20th century. It underlies all aspects of perception, from image segmentation to 3D shape representation. Yet literally no progress has been made in solving this problem. However, using the theory developed in Leyton [10], the solution naturally drops out of our algebraic theory of *transfer*, as follows: We have said that the human perceptual system organizes any stimulus set generatively into a recursive hierarchy of transfer, i.e., into a control-nested hierarchy of groups: $G_1 ⓦ G_2 ⓦ \ldots ⓦ G_n$. We show that the perceptual groupings come directly from this recursive transfer structure, as follows:

GROUPING PRINCIPLE. *The groupings formed by perception, correspond to the left-subsequences $G_1 ⓦ G_2 ⓦ \ldots ⓦ G_{i-1}$ of the control-nested hierarchy. In fact, they can be systematically elaborated as the conjugates $g - g^{-1}$ of the left-subsequences.*

Let us conclude this initial review of human perception by considering the basic visual problem of *projection.* The visual image is the projection of some *environmental* shape onto the retina. We argue that the appropriate approach to handle this is to describe the environmental shape generatively, and to add the projective process as an extra generative level, resulting in the shape on the image. Thus, the image shape is given a complete generative description in which the projective process is merely the last phase. The powerful thing is that the entire generativity is handled by a control-nested hierarchy of groups, $G_1 ⓦ G_2 \ldots ⓦ G_n$, in which the left-subsequence $G_1 ⓦ G_2 ⓦ \ldots ⓦ G_{n-1}$ represents the generation of the environmental shape, and the final control group G_n represents the projective group.

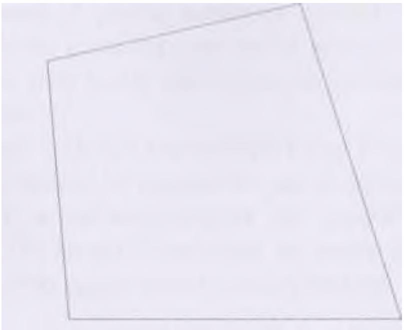

Figure 11. A square distorted by projection.

As a simple example, consider the projection of a square onto the retina, producing the projectively distorted square shown in Fig. 11. We have seen that the undistorted square is represented as the group $\mathbf{R} \textcircled{w} \mathbf{Z}_4$. This group gives the generative structure of the square *in the environment*. Now to add the effect of projecting the square onto the retina, we merely add the projective group $PGL(3, \mathbf{R})$ onto the generative sequence of the square thus:

$$\mathbf{R} \textcircled{w} \mathbf{Z}_4 \textcircled{w} PGL(3, \mathbf{R}). \tag{6}$$

Once again, notice that the operation used to add $PGL(3, \mathbf{R})$ onto the lower group $\mathbf{R} \textcircled{w} \mathbf{Z}_4$, is the control-nesting operation w; which means that $PGL(3, \mathbf{R})$ acts by *transferring* $\mathbf{R} \textcircled{w} \mathbf{Z}_4$ from the undistorted square in the environment onto the distorted square in the image[3].

This approach differs substantially from the standard one in projective geometry. Notice, for example, that the group sequence in expression (6) above, is built up from the Euclidean structure, which is given here by the iso-regular group $\mathbf{R} \textcircled{w} \mathbf{Z}_4$. This means that the undistorted square is a privileged figure in the space or quadrilaterals, the phenomenon ol privileged figures in a space ot distorted figures is a basic inviolable result in human perceptual psychology. This completely violates Klein's principle that geometric objects are the *invariants* of the specified transformation group - which is the most famous principle of 20th century geometry and physics. As we shall see, our generative theory of geometry is the direct opposite of Klein's approach. In our system, geometric objects are characterized by generative sequences. This means that they cannot be invariants, because invariance destroys recoverability of the applied operations,

[3] The algebraic action of $PGL(3, \mathbf{R})$ with respect to $\mathbf{R} \textcircled{w} \mathbf{Z}_4$ will be defined via the action of $PGL(3, \mathbf{R})$ on the projective plane represented intrinsically.

and recoverability is basic to the computation of the sequences. Quite simply: You cannot characterize geometric objects generatively, if you cannot recover their generative history. But you cannot recover their generative history if they are invariants under generative actions.

(2) Serial-Link Manipulators

A generative theory of shape encodes shape by a system of *actions.* Since we argue that the human perceptual system encodes shape generatively, this means that the perceptual system represents shapes in terms of actions. We argue that a major consequence of this is that the human perceptual system is structured by the same principles as the motor system, since the motor system is structured by action.

Now, we have said that actions are *intelligently* organized if they are organized by transfer; and we have given an initial set of illustrations of how the perceptual system is organized by transfer. We will now show that the motor system must also be organized by transfer. To do so, we consider the most common type of motor system, the serial-link manipulator.

Review of serial-link manipulators: The most famous example of a serial-link manipulator is the human arm: Such a structure is a series of rigid links going from the base to the hand. Each link corresponds biologically to a bone. Furthermore, each successive pair of links is connected by a joint. The base end is called the proximal (near) end of the manipulator, and the hand end is called the distal (far) end of the manipulator. Standardly, a serial-link manipulator is specified by embedding a coordinate frame in each successive link. Each frame is judged relative to the next frame in the proximal direction, e.g., the frame of the hand is judged relative to the frame of the forearm, and the frame of the forearm is judged relative to the frame of the upper arm, etc. The relationship between two successive frames is given by a matrix *A.* Thus the overall relationship between the hand coordinate frame and the base coordinate frame is given by the product of matrices

$$A_1\, A_2 \,\ldots\, A_n$$

corresponding to the succession of links. In robotics, each matrix A. is modeled as a rigid motion, and is therefore a member of the special Euclidean group 5E(3), the group generated by translations and rotations (but no reflections). Standardly, the order from left to right along the matrix sequence (7) corresponds to the order from base to hand (proximal to distal). However, without loss of generality, we will choose the left-to-right order as corresponding to the hand-to-base order (distal-to-proximal). This will maintain consistency with our other notation.

An Algebraic Theory of Serial-Link Manipulators. According to the theory in this book, the basic property of serial-link manipulators is *transfer*: The hand has a space of actions that is transferred through the environment by the fore-arm, which has a space of actions that is transferred through the environment by the upper-arm, and so on down to the base (e.g., the torso). Thus, we argue that the group of a serial-link manipulator has the following *control-nested* structure:

$$SE(3)_1 \circledw SE(3)_2 \circledw \ldots \circledw SE(3)_n \quad (8)$$

where each level $SE(3)_i$ is isomorphic to the special Euclidean group $SE(3)$, and the succession from left to right corresponds to the succession from hand to base (distal to proximal). Thus, each matrix A_i in expression (7) is taken from its corresponding Euclidean group $SE(3)_i$ in (8). Although ordinary matrix multiplication is used between any two successive matrices in (7), we now see that the group product in the corresponding position in (8) is actually the control-nesting operation, Ⓦ , which is the wreath product. Thus each group $SE(3)_i$ along the sequence (8) acts as a control group with respect to its left-subsequence $SE(3)_1$ Ⓦ $SE(3)_2$ Ⓦ ... Ⓦ $SE(3)_{i-1}$ and this corresponds to the fact that $SE(3)_i$ transfers the action structure of its left-subsequence around the environment. As usual, the successive use of the Ⓦ operation is interpreted recursively, and it is this that defines the hierarchical nature of the motion spaces.

The entire group we have given in (8) for the serial-link manipulator, is very different from the group that is normally given in robotics for serial link- manipulators. Standardly, it is assumed that, because one is multiplying the matrices in (7) together, and therefore producing an overall Euclidean motion T between hand and base, the group of such motions T is simply 5E(3). However, we argue that this is not the case. The group is the much more complicated group given in expression (8). This group encodes the complex link-configurations that can occur between the hand and base. If the group were simply 5E(3), then there would be a single configuration of links between hand and base and this would remain rigidly unaltered as the hand moves. However, there are infinitely many different configurations that the links can take between hand and base, and the group in (8) gives the relationships between all these configurations. To put it another way: It is conventionally assumed that, because the overall relation between hand and base is a Euclidean motion, the group of motions between the hand and base is the group of rigid motions. However, the structure between hand and base is *not rigid.* Therefore the group is not *SE(3).* It is the much more complicated group (8).

Most crucially, notice that we produced this group by considering the *transfer* relationships involved. It is this that allowed us to specify the algebraic structure.

(3) Object-Oriented Programming: Inheritance

The fact that each frame in a serial-link manipulator is judged relative to the next frame in the distal-to-proximal direction, means that serial-link manipulators are an example of what are called *parent-child* structures in object-oriented programming. Parent-child relationships express the fundamental structuring principle of object-oriented software called inheritance. Meyer [11] defines *inheritance* as a classification scheme in which one class is "an heir of another if it incorporates the other's features in addition to its own." Such object relationships are basic, for instance, to assembly-subassembly organization in mechanical CAD. For example, most major mechanical programs such as *Pro/ ENGINEER* provide menus which allow the designer to determine the parent-child relationships in an assembly hierarchy, and most part information windows in the program provide the user with the parent-child positioning of any selected part, because feasible modification of an individual part is impossible without knowing these relationships. Parent-child relationships are also a major explicit part of all animation software, such as *3D Studio Viz/Max,* where kinematic relationships between limbs are given exactly as defined in robotics. Again, all object-subobject relations in architectural CAD are parent-child relations; e.g., doors are placed relative to walls and move with them as the designer modifies the room.

The examples mentioned in the previous paragraph are all geometric parent- child relationships. A major part of Leyton [10] gives an algebraic theory of such relationships in object-oriented programming. We claim that the inheritence.

structure of parent-child hierarchies is given algebraically by control-nested groups $G_1 \textcircled{w} G_2 \textcircled{w} \ldots \textcircled{w} G_n$, where, as usual, the $\textcircled{w}$ operation is the wreath product. This means that geometric parent-child hierarchies follow from our generative theory of shape.

3. Transfer in Differential Equations

In the next section we will look at the nature of scientific laws, and see that they are structured by transfer. However, the topic of transfer in science has a more general setting within the theory of differential equations. Transfer is, in fact, fundamental to methods of solving differential equations. Most methods exploit the fact that the solutions of a differential equation can be *transferred* onto each other. This phenomenon is considerably more profound than it might at first seem. For example, it is basic to the structure of scientific laws.

Differential equations are by far the most frequently used modeling method throughout the world. It is no exaggeration to say that more than several trillion differential equations are solved per second across the world, e.g., in electrical power plants, factories, financial institutions, etc. Clearly all this depends on methods for *solving* differential equations, and a large variety of such methods have been developed. However, basic to these methods is symmetry. This is the modern approach that was created by Sophus Lie, and for which he formulated the machinery of Lie groups and Lie algebras. In fact, the use of symmetry to solve differential equations is very familiar to high-school students, as follows:

Consider the first-order differential equation:

$$\frac{dy}{dx} = F(x). \tag{9}$$

From high-school, we are all familiar with the fact its solution is the integral

$$y = \int F(x)dx + C. \tag{10}$$

where C is a constant of integration. Because of this constant, we know that there are a whole set of solution curves, each one obtained by substituting a particular number for

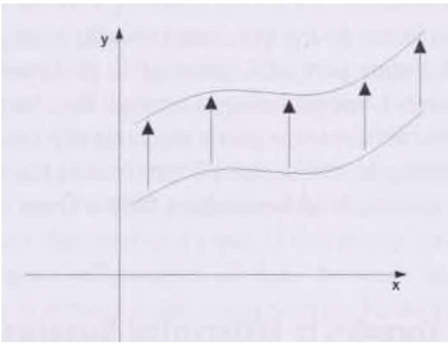

Figure 12. Transfer of solutions onto solutions, in a differential equation.

This is the first example of Lie theory that anyone encounters. The differential equation (11) admits a 1-parameter Lie group of translations in the *y* direction. The consequence is that you can map the solutions to each other using this translation group. The group is represented by the constant of integration C in (10).

Other types of differential equations can have other types of symmetry groups. For example, a first-order differential equation of this type:

A=/2) <>›› *ax* \xy'

admits a 1-parameter group of *scalings.* You can then map its solution curves onto each other via this group. Thus the constant *C* of integration will occur not as an *addition* onto a solution, as in (10), but as a multiplicative factor on the solution. This constant will actually represent the scaling group involved.

What we have seen in this section can be summarized as follows:

Solving differential equations depends on the *transfer* of solutions onto solutions.

In the next section, we will see that this is fundamental to the structure of scientific laws.

4. Scientific Structure

At the foundations of any branch of physics there is a dynamical equation, which is regarded as the fundamental dynamical *law* of that branch of physics. This law determines the evolution of a system state. For example, in Newtonian mechanics, the dynamic equation is Newton's second law, $F = ma$, which determines the trajectory of a system in classical mechanics; in quantum mechanics, the dynamical law is Schrodinger's equation which determines how a quantum-mechanical state will evolve over time; in Hamiltonian mechanics, we have Hamilton's equations which determine how a point will move in phase space.

The law, being a dynamical equation, is expressed as a differential equation. Very profoundly, the *lawful* nature of the equation is given by the symmetries of the equation, as follows:

Consider Fig. 13. The bottom flow-line in the figure shows an experiment being run in a laboratory in New York. The system is set up at time 0, in initial state s(0), which is the left-most point on that flow-line. The flow-line then represents the evolution of the system's state in the experiment. Suppose that the evolution is found to be governed by a particular dynamic equation.

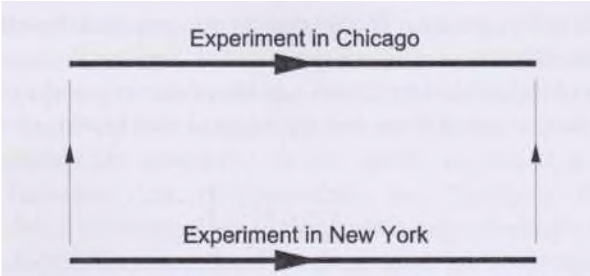

Figure 13. The transfer of a scientific experiment.

Now, the upper flow-line in Fig. 13 shows the same experiment being run in a laboratory in Chicago. By this we mean that the system in Chicago is started in a translated version T[s(0)] of the initial conditions s(0) in New York. That is, the lefthand point of the upper flow-line is T[s(0)]. The upper flow-line then represents the evolution of the system's state in the Chicago experiment. Let us assume that the upper flow-line turns out to be a translated version of the lower flow-line. This translation is shown by the vertical arrows in Fig. 13.

The important question is this: Is the upper flow-line described by the same dynamic equation that was discovered for the New York experiment? In other words, can one say that both flow-lines are solution-curves for the same dynamic equation? If one can, then the dynamic equation begins to appear *lawful*, i.e., to apply everywhere. This lawfulness is equivalent to discovering that the equation has translational symmetry.

What we mean by this is the following: A dynamical equation prescribes flow-lines; these are the solution-curves to the equation. We ask: Does the translation of one flow-line in the set of solution-curves produce another flow- line in that set? If it does, then we say that the dynamical equation has translational symmetry. This is equivalent to saying that it is a law; i.e., that it works anywhere.

We have illustrated the relation between symmetries and laws using translational symmetry as an example. However, the same argument applies to the choice of any other kind of symmetry, e.g., rotational symmetry. In physics, the basic program is to hunt for dynamical equations that have symmetries; i.e., are lawful. Conversely, one can start with a symmetry group and use it to help construct a lawful dynamical equation. For example, this was Einstein's technique in establishing the correct form of Maxwell's electromagnetic equations.

In any branch of physics, the appropriate symmetry group will be one that sends solution-curves to other solution-curves of the dynamic equation. The appropriate groups for the following branches of physics are:

Newtonian mechanics <-> Galilean group Special relativity <-> Lorentz group

Hamiltonian mechanics <-> Symplectic group
Quantum mechanics <-> Unitary group

It is clear that the phenomenon we have been describing above is one of *transfer*. That is, a dynamical equation permits transfer if the transferred version of any solution-curve (flow-line) is also a solution-curve. It is this that makes the equation lawful. Therefore the phenomenon of transfer is equivalent to the lawful property of the equation.

Lawfulness Transfer

Now, we have said that the lawful property is due to symmetries in the equation. However, we shall see that, to describe this structure in terms of *transfer* gives a deeper description - one that captures more fully the process of scientific discovery.

Let us therefore describe the situation in terms of transfer. Observe first that the flow itself is a symmetry across the state-space. This is because a dynamical equation (differential equation) prescribes a vector field and a vector field prescribes a 1-parameter group G_1 of actions along the flow-lines of the vector field[4].

In particular, let us now isolate any individual flow-line. The group G_1 can be considered as "confined" to that flow-line. For example, in Fig. 13, the group G_1 would be moving along any one of the horizontal lines.

Now, let us consider the symmetry discussed above: the symmetry G_2 of the differential equation. This maps flow-lines to flow-lines. This is illustrated by the vertical arrows in Fig. 13. Thus G_2 is acting across the flow-lines, and G_1 is acting along any flow-line. This means that we can consider the flow-lines as fibers, and G_2, as a control group transferring G_1 from one flow-line (fiber) to another. That is. we have this control-nested structure:

$$G_1 \circledw G_2. \tag{12}$$

This combined group fits the rigorous definition of wreath product. Notice that this a richer algebraic structure than is normally used to express symmetries in

---

[4] For ease of discussion we are assuming that the dynamical equation is *a first-order* differential equation. A first-order equation prescribes a flow like a "fluid" directly on the the space of independent variables. This is the situation for example in quantum mechanics and Hamiltonian mechanics. If, however, the dynamic equation is *second-order* - e.g., as in the case of Newtonian mechanics or Lagrangian mechanics - then we will consider the bundle of independent worldlines. This is handled using our concept of a *wreath covering* in Leyton [10]. In this way, the second-order case also has a flow structure, and we can describe it using a wreath product.

physics. First, there is an independent copy of G_1 on each of the fibers. We can think of this as representing experiments that were independently done before a process of induction discovered a relation between these experiments. Then after induction had established the control group G_2, experiments could be coordinated and one could, for example, establish a single "wave front" of points moving along the flow. This in fact, corresponds to the "diagonal" of the wreath product (see later). Thus all the stages of scientific discovery are contained in the wreath structure, as opposed to the conventional symmetry structure in physics. This is fully exaborated in Leyton [10] where we deal with wreath products as hierarchies of detection.

A related reason why one searches for symmetries of the dynamic equation comes from Noether's theorem, which states that, to each continuous symmetry of the dynamic equation, there is a conservation law; i.e., a conserved quantity such as energy, linear momentum, angular momentum, etc. It is clear therefore that the possible control groups G, in the wreath product (12) above, correspond to the possible conservation laws of the system. As illustrations, let us consider quantum mechanics and Hamiltonian mechanics:

Quantum Mechanics

In quantum mechanics, a state of the world is given by a wave function. The space of wave functions (world states) is called Hilbert space; i.e., any *point* in Hilbert space is a world state. The dynamic equation tells us how the world states evolve over time. This equation is called Schrodinger's equation. Schrodinger's equation specifies a *rigid rotation* of Hilbert space; i.e., Schrodinger's equation says that any point in Hilbert space will simply rotate around Hilbert space over time. Therefore the *flow-lines* generated by Schrodinger's equation correspond to a rotation group acting on Hilbert space. We shall denote this rotation group by G_1

Now, because one wants to identify conservation laws, one wants to find symmetry groups of the flow. These will send flow-lines onto flow-lines. Remarkably, any such symmetry group will also be a rotation group G_2 of Hilbert space. This will rigidly rotate the flow-lines of the Schrodinger equation onto each other.

Thus we have two groups G_1 the rotation group prescribed by Schrodinger's equation, and G_2, the rotation group of symmetries. According to our generative theory of shape, we should regard these two groups, respectively, as the fiber group and control group of the wreath product in expression (12).

What we have just said illustrates a basic point that we make in Leyton [10]: With respect to scientific structure, there is the following correspondence.

Conservation Laws <--> Wreath Products.

Mathematically we will construct this by setting up a correspondence between any pair of commuting observables V and W and the wreath product of their 1- parameter groups, G_V and G_W:

$$[V, W] = \leftrightarrow G_W \; \textcircled{w} \; G_V.$$

Hamiltonian Mechanics

Hamiltonian mechanics is generally regarded as the most powerful formulation of classical mechanics. In fact, it depends on exactly the type of structure defined above. In Hamiltonian mechanics, the space of states is called *phase space* - its independent dimensions are the position and momentum variables of the system. The *total energy* of the system is given by a smooth function H on this space, and is called the Hamiltonian. The "gradient" of this function generates a flow on phase space. These are the flow-lines that describe the system's evolution over time. The flow corresponds to a 1-parameter group G_H along its flow-lines.

Now, any other dynamical variable that one might wish to measure besides energy, e.g., angular momentum, is also given by a function F on phase space. The "gradient" of this function also generates a flow across phase space. This fiow corresponds to its own 1-parameter group G_F along its flow-lines.

Fig. 14 illustrates the two flows we have just considered: The flow of the energy function H. which corresponds to the time-evolution of the system; and the flow of the dynamical variable F, which corresponds to some other property we might wish to measure; e.g., angular momentum.

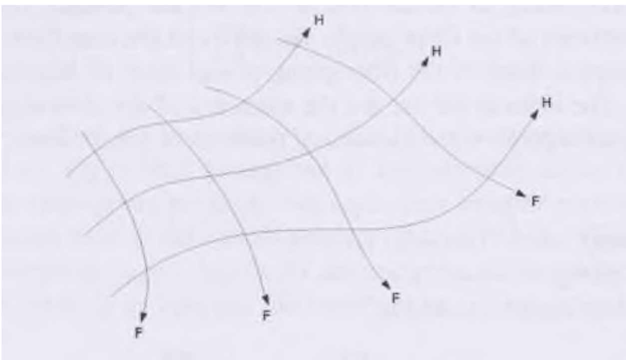

Figure 14. The flow of the Hamiltonian *H*, and the flow of another dynamical variable *F*.

The crucial issue is this: Do the flow-lines of *F* transfer the flow-lines of *H* onto each other? If they do, then Noether's theorem states that there will be a *conservation law* corresponding to F; that is, *F* will be conserved; e.g., angular momentum will be conserved.

Now the group generating the *H* flow is G_H and the group generating the *F* flow is G_F. According to our generative theory of shape, the appropriate way to describe this is as the wreath product:

$$G_H \text{ Ⓦ } G_F. \tag{13}$$

Notice that the *H* flow goes in the fiber position, and the *F* flow goes in the control position. Careful consideration reveals that this is an example of the diagram we gave earlier on p. 240. The *H* lines in the Hamiltonian system correspond to the horizontal lines in that diagram, because these lines are the evolution lines of the system. The *F* lines in the Hamiltonian system correspond to the vertical lines in that diagram, because these lines are the symmetries that transfer the evolution lines onto each other.

5. The Wreath Product Theory of Transfer

In this final section, we fully describe the structure of a wreath product. Fig. 15 gives an intuitive sense of the structure. A wreath product is a group in which there are two levels: The upper level corresponds to the control group. The lowei level is the set of transferred versions of the fiber group, including the origina non-transferred version. The versions are represented by the vertical columns it the diagram. As indicated by the long arrow, the control group transfers thesi versions onto each other.

The crucial thing to notice is that the wreath product contains *all* the transferred versions of the fiber group. We shall call the transferred versions, the fiber-group *copies*. Each of the fiber-group copies must be labelled individually by an index. The indexes we use are the elements of the control group. *There is one fiber-group copy for each element of the control group.* Thus, let us use this notation:

Fiber group	$= G(F)$
Control group	$= G(C)$
Fiber-group copy	$= G(F)g$

where the index g on a fiber-group copy is an element of the control group G(C).

Figure 15. A wreath product.

The transferring action of the control group on the fiber-group copies is then easy to model: Any element *h*, in the control group, translates the fiber- group copies onto each other as indicated by the long arrow. This is achieved by having *h* act on the *indexes* of the fiber-group copies. That is, *h* sends the fiber- group copy $C(F)_e$ to the fiber-group copy $G(F)_{hg}$.

The above action on indexes corresponds to a deep algebraic aspect of wreath products, as follows: There are two group products that are needed to fully define a wreath product: (1) The fiber-group copies are combined using the direct product operation. The reader should think of the entire bottom block in Fig. 15 as the direct product of the fiber-group copies. We will call this the *fiber-group product*, and denote it thus:

$$\prod_{g \in G(C)} G(F)_g .$$

(2) The fiber-group product (lower level) and the control group (upper level) are combined using a semi-direct product. Thus we have:

$$\left\{ \prod_{g \in G(C)} G(F)_g \right\} ⓢ G(C)$$

where ⓢ is the symbol for semi-direct product. In any semi-direct product the upper group (here the control group) has an *automorphic action* on the lower group (here the fiber-group product). We argue that *transfer* corresponds to the automorphic action used in the wreath-product case, as follows: Given a member *h* of the control group (upper level), its automorphic action on the fiber-group product (lower level) is to translate the latter's index structure by *h*. thus:

$$\prod_{g \in G(C)} G(F)_g \rightarrow \prod_{g \in G(C)} G(F)_{hg} .$$

This automorphic action corresponds algebraically to conjugation of the fiber- group product by *h*. Thus, the transfer of the fiber-group copy $G(F)_g$ to the fiber- group copy $G(F)_{hg}$ is given algebraically by conjugation of $G(F)_g$ by h. That is:

$$G(F)_{hg} = h^{-1}\left[G(F)_g\right]h.$$

We are therefore lead to the following crucial conclusion:

Theory of transfer. *Transfer corresponds to the automorphic action of the control group on the fiber-group product in a wreath product.*

The type of wreath product we have described above is called a regular wreath product. It is useful also to consider a generalization of this called a permulational wreath product. Here, the fiber group acts on a set we will call *the fiber set F.* Each fiber-group copy acts on its own copy of the fiber set. In addition, each copy of the fiber group is indexed not directly in the control group, but in a set we will call the *control set C.* The control group is then regarded as acting on the control set. In this way, we get the movement of the fiber-group copies. In fact, quite literally, they will be transferred from one copy of the fiber set to another.

References

[1] Carbonell, J.G (1986). Derivational analogy. In R.S. Michalski, J.G. Carbonell. & T.M. Mitchell (Ed.), *Machine Learning,* vol 2. Los Altos, CA: Morgan-Kaufman

[2] Gallistel, R.C. (1990). *The Organization of Learning.* Cambridge, Mass: MIT Press

[3] Lakoff, G. (1987). *Women, Fire, and Dangerous Things.* Chicago: University of Chicago Press.

[4] Leyton, M. (1984) Perceptual organization as nested control. *Biological Cybernetics* 51, 141-153.

[5] Leyton, M. (1986a) Principles of information structure common to six levels of the human cognitive system. *Information Sciences,* 38, 1-120. Entire journal issue.

[6] Leyton, M. (1986b) A theory of information structure I: General principles. *Journal of Mathematical Psychology,* 30, 103-160.

[7] Leyton, M. (1986c) A theory of information structure II: A theory of perceptual organization *Journal of Mathematical Psychology,* 30, 257-305.

[8] Leyton, M. (1987a) Nested structures of control: An intuitive view. *Computer Vision Graphics, and Image Processing,* 37, 20-53.

[9] Leyton, M. (1992). *Symmetry, Causality, Mind.* Cambridge, Mass: MIT Press.

[10] Leyton, M. (2001). *A Generative Theory of Shape.* Book. Submitted.

[11] Meyer, B. (1997). *Object-Oriented Software Construction.* New Jersey: Prentice Ha Hall.

[12] Rosch, E. (1978). Principles of categorization. In E. Rosch & B.B. Lloyd (Ed: *Cognition and Categorization.* Hillsdale, NJ: Lawrence Erlbaum.