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Barbershop Paradox and Connexive Implication

Barbershop paradox. In *A logical paradox* Lewis Carroll[1] presented a paradox concerning properties of conditionals. It is based on a story which could be summarized as follows: Allen, Brown and Carr run a barbershop. Somebody must be in to take care of the customers. Moreover, Allen and Brown are always together. Carrol formulated the issue in the following two hypotheticals:

First. If Carr is out, it follows that if Allen is out Brown must be in.

Second. Allen always takes Brown with him.

Then Lewis Carrol writes: "If Carr is out, these two Hypotheticals are true together. And we know that they cannot be true together. Which is absurd. Therefore Carr cannot be out. There's a nice Reductio ad Absurdum for you!". This way we proved in a logical way that Carr must be always in. It is obviously paradoxical because one can easily see that Allen and Brown might be in and then Carr might be out. So, if Allen and Brown are in and Carr is out then both hypotheticals are true.

Let us take a closer look at this. Let propositions *A*, *B*, *C* denote respectively 'Allen is out', 'Brown is out', 'Car is out'. This way Carrol's hypotheticals take the form:

(1) $C \rightarrow (A \rightarrow \sim B)$;

(2) $A \rightarrow B$.

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[1] Lewis Carroll, "A logical paradox", *Mind* 3, no. 11 (Jul., 1894): 436–438.

Lewis Carrol claims that:

(3) $A \rightarrow \sim B$ and $A \rightarrow B$ are incompatible.

If (1), (2) and (3) are together true then we get paradoxical result that C is false.

A solution — material implication. Suppose that we use the classical logic and $\rightarrow$ means material implication. Then (3) does not work as $A \rightarrow \sim B$ and $A \rightarrow B$ are both true provided A is false. Paradox disappears. Moreover, since A is false which means that Allen is in, then C, (1) and (2) are both true only when Allen is in. So, this apparent paradox is based on a mistake. This is not a real paradox like for example liar paradox but just a paralogism — unexpected result based on a mistake in reasoning.

However it could be a bad idea to blame Lewis Carrol. He did not write that he interpreted the conditional as material implication. 40 years earlier, in 1854 George Boole published his famous book *The Laws of Thoughts*. Probably we might recognize this book as a first publication where classical propositional logic was precisely defined. We can take for granted that when Lewis Carrol wrote his paper all major concepts of classical logic with the notion of material implication had already been well established. Nevertheless classical propositional logic in general and material implication in particular were not considered as a default logical system. The big part of the history of contemporary logic is a history of criticism of material implication. There are many natural examples of conditionals with false assumptions which considered as material implication bring us to a counterintuitive true proposition. This is why we should consider the conditional in the barbershop paradox in a non-classical way.

Let us note that within the classical logic each of the following conditions is equivalent to (1):

(1.1) $\sim C \vee \sim B \vee \sim A$;
(1.2) $(C \wedge A) \rightarrow \sim B$;
(1.3) $A \rightarrow (C \rightarrow \sim B)$.

For this reason within classical logic we can replace (1) by any condition from among (1.1), (1.2), (1.3) without any change in the reasoning about barbershop. It will be very different if we interpret the conditional in a non-classical way. We will come back to it later.

Connexivity. In the philosophical society of ancient Greece many kinds of conditionals were considered. S. McCall, in his review paper 'A history of connexivity' [6], expressed this fact in picturesque words: 'that time the very crows on the rooftops were croaking about what conditionals were true'.

Sextus Empiricus distinguished among others three types of implication: material (Philonian) implication, strict (Diodorean) implication and most important for us connexive implication. William Kneale and Martha Kneale quote Sextus Empiricus:

> And those who introduce the notion of connection say that a conditional is sound when the contradictory of its consequent is incompatible with its antecedent.[2]

We should mean that two sentences are incompatible if and only if they cannot be true at the same time. Hence it follows from this definition that no conditional of the form 'If p then not-p' can be true, since the contradictory of not-p, i.e. p, is never incompatible with p.

In *Prior Analytics*, Aristotle seems to be saying that it is impossible for a proposition to be implied by its own negation. It is impossible that if A then $\sim A$:

> It is impossible that the same thing should be necessitated by the being and by the not-being of the same thing. I mean, for example, that it is impossible that B should necessarily be great if A is white, and that B should necessarily be great if A is not white. For if B is not great A cannot be white. But if, when A is not white, it is necessary that B should be great, it necessarily results that if B is not great, B itself is great. But this is impossible".[3]

It seems that Aristotle here, like Sextus Empiricus in the previous case, is trying to show that two implications of the form 'If p then q' and 'If not-p then q' cannot both be true.

The main idea of connexivity is that proposition A has nothing in common with proposition $\sim A$. Similarly, if A has something in common with B, it cannot have anything in common with $\sim B$, and vice versa, if A has something in common with $\sim B$, it cannot have anything in common with B. A necessary truth condition of implication is that the implication antecedent and consequent relate in some sense. Such intuitions form the motivation for connexive logics.

The roots of connexive logic date back to the ancient times.

The concept of connexivity is expressed by the following theses:

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[2] William Kneale, Martha Kneale, *The Development of Logic* (London: Duckworth, 1962), 5, 129.

[3] Aristotle, *Prior Analytics*, 57b3–14, see: Storrs McCall, "A History of Connexivity", in: *Handbook of the History of Logic*, ed. Dov M. Gabbay et al., *Logic: A History of its Central Concepts*, vol. 11 (Elsevier: Amsterdam: 2012), 415–449.

(A1) $\sim(A \rightarrow \sim A)$

(A2) $\sim(\sim A \rightarrow A)$

(B1) $(A \rightarrow B) \rightarrow \sim(A \rightarrow \sim B)$

(B2) $(A \rightarrow \sim B) \rightarrow \sim(A \rightarrow B).$

(A1), (A2) are referred to as Aristotle's Theses, while (B1), (B2) — Boethius' Theses.

None of the propositions (A1), (A2), (B1), (B2) is a thesis of the classical logic. At the same time it is well known that if we add to classical logic any new proposition and consider it in the same way as a classical tautology, then any proposition could be proved in such a system. This means that we would produce an inconsistent logic. As a consequence, if we interpret negation and implication in a classical manner, we will produce an inconsistent logic where everything is true accepted. The only way to avoid it is to interpret either the implication or the negation or both of them in a non-classical way. On the other hand, if we do not assume the classical logic as a background propositions (A1), (A2), (B1), (B2) can be independent.

If we take as a basis the sole Aristotelian and Boethian theses (A1), (A2), (B1), (B2) we receive a very weak logical system. It is as weak that it allows very strange and counterintuitive connectives interpretations. Let us note two examples. Propositions (A1), (A2), (B1), (B2) are valid in binary matrix $\{1,0\}$ with distinguished value of 1, with classical material implication and negation defined as $\sim 1 = \sim 0 = 1$. Similarly, these propositions are true in a binary matrix with classical negation and implication defined as $x \rightarrow y = 1$ iff $x = y$. Obviously, the above examples do not mirror any desired properties of connexive implication. We set them forth just to show that a right connexive logic should be stronger than that defined by the sole Aristotelian and Boethian axioms.

Obviously to investigate connexive logics we need to interpret negation or implication in a non-classical way. However, at the same time we would like to keep as close as possible to classical logic while interpreting Aristotelian and Boethian laws. This very idea guided us in our research published in the paper.[4] We interpreted there negation, conjunction and disjunction in the classical Boolean way leaving a broad spectrum of possible interpretations of implication. This way we do not decide which of possible connexive logics is the right one. In a sense we consider all of them at the same time, investigating properties of any connexive logic satisfying Aristotelian and Boethian laws. Let us apply the same idea for

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[4] Tomasz Jarmużek, Jacek Malinowski, "Boolean Connexive Logics, Semantics and Tableau Approach", *Logic and Logical Philosophy* 28(2) (2019): 1–22.

barbershop paradox. In *Boolean Connexive Logics, Semantics and Tableau Approach* we constructed 32 logics related to connexivity applying a related semantics.

Relating semantics. This kind of semantics is based on a binary relation R on formulas. The fact that *ARB* may be interpret on various ways. Two formulas could be related analytically, causally, thematically, temporally, etc. Relating connectives have intensional character, since the Boolean conditions are not sufficient. Conjunction is a good example of such a connective. Although it is symmetrical in classical logic it behaves much worse in everyday use. It might have a causal or temporal component like for example in propositions: "He fell down and broke his leg". Such a component could be expressed by relating semantics only.

Here we use relating semantics to interpret connexive implication. The truth conditions for the implication consist not only of classical requirement that a predecessor is false or a successor is true. There is an additional requirement that predecessor and successor are related to each other in a connection determined by some binary relation R.

The idea underlying relating semantics based on the binary relation defined on a set of formulas probably has its origin in the works by Douglas N. Walton[5] and Richard L. Epstein.[6] A more general approach — without assumptions imposed on the relating relation — was proposed in an article by Tomasz Jarmużek and Bartosz J. Kaczkowski.[7] They also suggested some philosophical interpretations of relating relations, as for example casual, temporal or analytical interpretations. It should be stressed that the idea of casual interpretation of the relating relation, as one of the many philosophical interpretations of such relation, was also indicated by Walton in *Philosophical Basis of Relatedness Logic*.[8]

In *Boolean Connexive Logics, Semantics and Tableau Approach* we proved that provided R is *closed under negation*, i.e. it satisfies the condition (c1) if *ARB* then $\sim AR\sim B$ then Aristotelian and Boethian axioms are satisfied by the relating semantics if and only if its relating relations R satisfy the following conditions:

(4) $AR_e\sim A$, $\sim AR_e A$,
   if ARB then $AR_e\sim B$,
   $(A \rightarrow B)R\sim(A \rightarrow\sim B)$, $(A \rightarrow\sim B)R\sim(A \rightarrow B)$.

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[5] Douglas N. Walton, "Philosophical basis of relatedness logic", *Philosophical Studies* 36/2 (1979): 115–136.

[6] Richard L. Epstein, R. L., "Relatedness and implication", *Philosophical Studies* 36 (1979): 137–173.

[7] Tomasz Jarmużek, Bartosz J. Kaczkowski, "On Some Logic with a Relation Imposed on Formulae: Tableau System F", *Bulletin of the Section of Logic* 43(1/2) (2014): 53–72.

[8] Walton, "Philosophical basis of relatedness logic".

To make the above formulas more clear AR_eB means here that it is not true that ARB. A relation satisfying condition (4) will be called a *connexive relation*. The above result gives us a tool to construct, in an easy way, various connexive logics. Any relation R, and even any class of relations satisfying the above conditions determine a connexive logic.

Barbershop paradox in connexive interpretation. If we interpret barbershop conditional as connexive implication, then from (B1) and (B2) it is easy to deduce that Lewis Carrol's claim (3) is true. However conditions (1) and (2) are more problematic than in the classical case. For (2) we need that ARB A relates to B with respect to R. It sounds quite reasonable as according to the story Allen and Brown are always together. For (1) we need that C relates to connexive implication $A \rightarrow{\sim}B$. One could hardly understand what such a relation means. It is hard to justify such a connection. As a consequence, under natural connexive interpretations one could incline to accept (2) and reject (1).

Propositions (1), (1.1), (1.2), (1.3) express the same condition — somebody must be in to run a shop. They are mutually equivalent with respect to classical logic. However, for any two of those propositions it is easy to construct a connexive logic such that only one of the propositions is true. To accomplish this goal it is sufficient to define relation R satisfying appropriate conditions. Let us note however that some of such relations could not fit to the scenario of barbershop paradox. One can argue that (1) is a very sophisticated form of expressing a simple idea. It seems (1.1) expresses it in the simplest way. (1.2) looks also natural. However (1.3) is as sophisticated as (1). One could easily prove that with respect to connexive implication each of them expresses a different thought.

Let us consider a simple example of connexive logic defined by relation R on some language containing sentences A, B, C. Let us take our intuition about the relation R as 'both out'. Suppose that each two of A, B, C relates, i.e.

ARB, ARC, BRC. Let us close this relation under reflexivity, symmetry, (c1) and (4). We get ARA, BRB, CRC, BRA, CRA, CRB, ${\sim}AR{\sim}B$, ${\sim}AR{\sim}C$, ${\sim}$ CR ${\sim}A$, ${\sim}AR$ ${\sim}A$, etc. It is easy to find that R satisfies (4) so it defines a logic satisfying Aristotelian and Boethian axioms.

This way barbershop paradox disappears. In the connexive logic defined by R (2) and (3) are satisfied. Although (1) is false, its classical equivalent (1.1) is true. This way we have a right description of the barbershop scenario without falling into contradiction. Conditions (1.2), (1.3) are false. In general, propositions with implication connective have more severe truth conditions than other ones. It seems to be possible to define such a relation that also (1.2) is satisfied, however such a relation would unlikely be as natural as relation above. Let us add $(C \wedge B)R{\sim}B$ to

R and close it under (c1) and (4), then we get such the relation R_0 that (2), (3), (1.1) (1.2) are satisfied.

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Summary

In this paper we remind Barbershop paradox noted by Lewis Carroll in 1894. We apply relating semantics to investigate this paradox in detail and to show its relation with connexive logic.

Keywords: Lewis Carroll, barbershop paradox, connexive logics, connexive implication, relating semantics

Streszczenie

Paradoks golibrody i implikacja koneksywna

Praca dotyczy paradoksu golibrody rozważanego przez Lewisa Carrolla w artykule opublikowanym w „Mind" w 1894 roku. Prezentowana jest tutaj semantyczna analiza tego paradoksu za pomocą semantyki wiążącej oraz pokazane są związki paradoksu z implikacją koneksywną.

Słowa kluczowe: Lewis Carroll, paradoks golibrody, logika koneksywna, implikacja koneksywna, sematyka wiążąca.