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Expressive Power of the Positional Operator *R*: a Case Study in Modal Logic and Modal Philosophy

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§ 1. From Logical Modalities to Realization Operator

Theories of modalities deal with fundamental philosophical problems and were even called by medieval thinkers *crux logicorum*, i.e. a Good Friday like passion of logicians, due to the dimensions of difficulties attached to those theories. Complex questions of metaphysics, epistemology, linguistics, semiotics and logic are related here, creating a true Gordian knot. In her comprehensive monograph[1] Urszula Żegleń has been dealing with that bunch of problems with special emphasis on ontological basis of modality theories.

[1] Urszula Żegleń, *Modalność w logice i filozofii. Podstawy ontyczne* (Modality in logic and philosophy. Ontic foundations) (Warszawa: Polskie Towarzystwo Semiotyczne, 1990).

The slightly exotic term "modality" is derived from the Latin noun "modus" which means approximately the same as English "mode" (with Latin plural "modi"). The concept of modality refers a mode objects, properties, states of affairs, matters of facts, propositions, etc. exist, happen, are realized etc. Actually anything we speak of may be looked at under different angles, creating different *modes*. That is exactly why it is so difficult to dispense oneself from taking modalities into account when debating philosophical matters, and hence, why theories of modalities belong to basics of philosophy.

As other philosophical questions are so difficult to be dealt with profoundly without invoking modalities, the problem of modality can be in a sense regarded as a philosophical meta-problem. Diversity of philosophical investigations and their complex involvement in modalities creates large amount of modal concepts, categories of modality, and modal aspects of things. Consequently there are not many successful comprehensive accounts of the whole field of theories of modalities. Therefore it is no exaggeration to say that after nearly thirty years Urszula Żegleń's monograph (1990) lost nothing of its topicality and remains a true rare book.

Here our aim is to deliver a contribution and supplement to the wide account of modalities. We consider an interesting peculiar modality which has some qualities of the meta-level. Having noted some metatheoretical connections of the problem of modalities within philosophy, we find it interesting to ask whether there are analogical connections specifically in the area of logic. As the ideas to be presented require further research we rather ask questions than deliver firm claims.

§ 2. Between Modality and Modal Logic

We focus exclusively on modalities (modes) which refer to propositions or sentences. They are usually called *de dicto* modalities or propositional modalities. Standard way of indicating such modalities is by connectives to express modalities and to be attached to the sentence in question. The connectives usually are unary (i.e. one-place) and take the form of prefixes. Select any admissible sentence A. In chart (1) we present some typical kinds of *de dicto* modalities with some examples.

alethic modalities indicate the mode of *A*'s being true (or false) and may also be considered modes of existence of the state of affairs the sentence *A* refers to, e.g.: it is necessary that *A*, it is possible that *A*, it is contingent that *A*, it is impossible that *A*, (1a)

deontic modalities indicate the mode of *A*'s being morally or legally binding (or permitted), e.g.: it is obligatory that *A*, it is mandatory that *A*, it is permitted that *A*, it is prohibited that *A*, it is impossible that *A*, (1b)

axiological modalities deal with values as modes of *A*, e.g.: it is good that *A*, It is bad that *A*, it is right that *A*, it is wrong that *A*, it is indifferent that *A*, (1c)

tense modalities indicate the tense of the sentence *A* which is considered a mode, e.g.: it has once been that *A*, it has always been that *A*, it is going to once be that *A*, it is always going to be that *A*, it is and always will be that *A*, (1d)

epistemic modalities indicate the knowledge of *A* by a certain agent, group, institution etc., e.g.: it is known that *A*, (1e)

doxastic modalities are similar to epistemic, but usually involve the agent's knowledge-like attitudes toward *A*, e.g.: it is believed that *A*, it is doubted that *A*, it is certain that *A*, it is doubtful that *A*, it is probable that *A*, it is agreed that *A*, it is said by ancients that *A*, it is granted that *A*, (1f)

apodeictic modalities are similar to epistemic and doxastic, but allow only the strongest epistemic attitude, namely provability, e.g.: it provable in Peano's arithmentic that *A*, it is disprovable in Peano's arithmentic that *A*, (1g)

civil modalities dealing with social or cultural behavioural patterns, e.g.: it is lawful that *A*, it is unlawful that *A*, it is convenient that *A*, it is inconvenient that *A*, (1h)

The list of kinds and variants of modal expressions may by extended with practically no limits. Theory of alethic modalities was originated by Aristotle and is considered the most classical one, hence, alethic modalities are considered standard and by modality in a very narrow sense is often meant alethic modality and other kinds of modal notions may by considered extensions of the notion of modality.[2]

In formal logic standard modal propositional language is an absolutely free algebra with sentence letters as free generators. More specifically the alphabet of modal propositional logic consists of a denumerable set *At* of atomic formulas which are simply sentence letters, classical connectives: $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, of negation, conjunction, disjunction, con-

[2] Arthur Norman Prior, *Formal Logic* (Oxford: Claredndon Press, 1962); Kazimierz Świrydowicz, *Podstawy logiki modalnej* (Basics in Modal Logic, Poznań: Wydawnictwo Naukowe UAM, 2014).

ditional and biconditional (equivalence) in succession, and two unary modal connectives: M of possibility and L of necessity, of course with addition of parentheses. The set *Form* of formulas of modal propositional logic is the smallest collection containing

all members of the set *At*,		(2a)
$(\neg A)$,	for every $A \in Form$,	(2b)
$(A \wedge B)$,	for every $A, B \in Form$,	(2c)
$(A \vee B)$,	for every $A, B \in Form$,	(2d)
$(A \Rightarrow B)$,	for every $A, B \in Form$,	(2e)
$(A \Leftrightarrow B)$,	for every $A, B \in Form$,	(2f)
(MA)	for every $A \in Form$,	(2g)
(LA)	for every $A \in Form$.	(2h)

Of course, formula (2b) is to be read: in is not the case that A; formula (2c) is to be read: A and B; formula (2d) is to be read: A or B; formula (2e) is to be read: if A, then B; formula (2f) is to be read: A if and only if B; formula (2g) is to be read: it is possible that A; and finally formula (2h) is to be read: it is necessary that A. It is convenient to agree that in the absence of parentheses latter connectives in the sequence: L, M, $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$ have longer scopes then former ones. Of course, should the clauses (2g) and (2h) be omitted, *Form* would be the set of formulas of classical propositional calculus.

We have clearly here to do with the default alethic interpretation of modal connectives. However, the interpretations listed in chart (1) and even other interpretations are possible, with the same symbols or their graphical variants. This is how formulas of modal (alethic), deontic, tense, epistemic or other kinds of modal logics are constructed. Although formally those calculi are analogical, philosophically we have to do with different kinds of modalities.

The most popular formal description of modal logics is the relational semantics, i.e. the possible worlds semantics. It is based on three basic ideas or assumptions. Firstly, there is no absolute truth-values, rather formulas are true or false in relation to points of some kind, typically to possible states of affairs or possible worlds. We normally say formulas be true or false in a possible world. Secondly, for every conceivable state of affairs there exists a prearranged set of conceivable sates of affairs which are possible relative to the very state of affairs in question. Thirdly, truth values of formulas (2b)-(2f), built by means of classical connectives are functions of truth values of their components in the same point (state of affairs), whereas truth values of formulas (2g) and (2h), built by means of modal connectives are functions of truth values of their components within the set of points (states of affairs) possible relative to the point in question. A standard account of that semantics we rely on may be found in the famous work by George Edward Hughes and Maxwell John Cresswell.[3] Formally, relational model is an ordered triple

$$\mathbf{M} = \langle W,Q,v\rangle, \tag{3}$$

where W is a nonempty set, identified as the set of all conceivable states of affairs (possible worlds), Q is a binary relation in W and v is a function from the set of all ordered pairs $\langle A,w\rangle$, with $A\in At$ and $w\in W$, to the set $\{1,0\}$ of two classical truth values: truth and falsehood in succession. It is convenient to agree that $Q(w)$ is the set of exactly such $u\in W$ that wQu. It is easily to observe that members of the set W are points or contexts formulas are evaluated in as true or false and the relation Q makes some contests possible relative to others. Formulas are considered true or false in states of affairs according to the following recursive definition. Take any formulas A, $B\in Form$ and any relational model $\mathbf{M}$. Let $V_{\mathbf{M}}(A) \subseteq W$ be the set of exactly those states of affairs formula A is true in. Then, for any $w\in W$ (with "iff" standing for "if and only if")

$$w\in V_{\mathbf{M}}(A) \text{ iff } v(A,w)=1, \text{ provided } A\in At, \tag{4a}$$

$$w\in V_{\mathbf{M}}(\neg A) \text{ iff } w\notin V_{\mathbf{M}}(A), \tag{4b}$$

$$w\in V_{\mathbf{M}}(A\wedge B) \text{ iff both } w\in V_{\mathbf{M}}(A) \text{ and } w\in V_{\mathbf{M}}(B), \tag{4c}$$

$$w\in V_{\mathbf{M}}(A\vee B) \text{ iff either } w\in V_{\mathbf{M}}(A) \text{ or } w\in V_{\mathbf{M}}(B), \tag{4d}$$

$$w\in V_{\mathbf{M}}(A\Rightarrow B) \text{ iff either } w\notin V_{\mathbf{M}}(A) \text{ or } w\in V_{\mathbf{M}}(B), \tag{4e}$$

$$w\in V_{\mathbf{M}}(A\Leftrightarrow B) \text{ iff either } w\in V_{\mathbf{M}}(A) \text{ and } w\in V_{\mathbf{M}}(B), \text{ or } w\notin V_{\mathbf{M}}(B) \text{ and } w\notin V_{\mathbf{M}}(B), \tag{4f}$$

$$w\in V_{\mathbf{M}}(MA) \text{ iff } u\in V_{\mathbf{M}}(A) \text{ for some } u\in Q(w), \tag{4g}$$

$$w\in V_{\mathbf{M}}(LA) \text{ iff } u\in V_{\mathbf{M}}(A) \text{ for all } u\in Q(w). \tag{4h}$$

A formula is considered true in a model $\mathbf{M}$ if and only if it is true in every point (state of affairs) in this model. Valid formulas of different calculi are formulas true in different classes of models.[4]

It seems quite clear that due to conditions (4a)-(4f) being analogical to classical propositional calculus, formulas of modal logic being

[3] George Edward Hughes, John Cresswell Maxwell, *A New Introduction to Modal Logic* (London and New York: Routledge 1996).

[4] George Edward Hughes, John Maxwell Cresswell, *A New Introduction to Modal Logic* (London and New York: Routledge 1996) 36–39.

uniform substitution of classical tautologies, i.e. theorems of classical propositional calculus, are valid in every modal logic under present consideration. Hence modal propositional logics are extensions of classical propositional calculus unlike so called deviant logics, e.g. many-valued calculi.

Modal logics (more precisely: normal modal logics) are semantically defined by imposing special formal conditions on relation Q. The weakest (normal) modal logic is the system K. The set of theorems of K is identical with the set of formulas true in every relational model **M**, defined as above. Formulas:

$$LA \Leftrightarrow \neg M \neg A, \tag{5a}$$

$$MA \Leftrightarrow \neg L \neg A, \tag{5b}$$

$$L(A \Rightarrow B) \Rightarrow (LA \Rightarrow LB), \tag{5c}$$

$$L(A \wedge B) \Leftrightarrow (LA \wedge LB) \tag{5d}$$

are well known examples of theorems of the system K. Imposing conditions on the relation Q we construct smaller sets of models and stronger calculi with new theorems, for example:

Seriality	$(\forall x)\ (\exists y)\ xQy,$	(6a)
Reflexivity	$(\forall x)\ xQx,$	(6b)
Transivity	$(\forall x,y,z)\ (xQy \wedge yQz \Rightarrow xQz),$	(6c)
Symmetry	$(\forall x,y,z)\ (xQy \Rightarrow yQx),$	(6d)
Euclideness	$(\forall x,y,z)\ (xQy \wedge xQz \Rightarrow yQz).$	(6e)

The set of formulas true in every model meeting condition (6a) are identical with the set of theorems of the system D. An example of theorem of D is the formula:

$$LA \Rightarrow MA. \tag{6}$$

The set of formulas true in every model meeting condition (6b) are identical with the set of theorems of the system T. An example of theorem of T is the formula:

$$LA \Rightarrow A. \tag{7}$$

The set of formulas true in every model meeting conditions (6b) and (6c) are identical with the set of theorems of the system S4. An example of theorem of S4 is the formula:

$$LA \Rightarrow LLA. \tag{8}$$

The set of formulas true in every model meeting conditions (6b) and (6d) are identical with the set of theorems of the system B. An example of theorem of B is the formula:

$$A \Rightarrow LMA. \tag{9}$$

The set of formulas true in every model meeting conditions (6b) and (6e) are identical with the set of theorems of the system S5. An example of theorem of S5 is the formula:

$$MA \Rightarrow LMA. \tag{10}$$

The last set is actually also identical with the set of formulas true in every model meeting all the conditions: (6b), (6c) and (6d). Other systems may be described and axiomatized in an analogical way.[5]

In this paper, however, we do not focus on modal propositional calculi themselves. We rather intend to show that such connectives as *M* and *L*, regardless their intended interpretation, can be described – at least to some extent – by means of another modal connective of a different formal kind. It seems therefore that the modality we focus on should be considered more basic.

§ 3. Modal Connectives Reduced to the Operator R

By the more basic modality we mean the operator *R*, referred to as realization operator, of the following syntax. The alphabet is to be extended with the sign *R* itself as well as a denumerable set *Ind* of terms, serving as indicators. Recursive definition (2) of the set *Form* is to be extended to the effect that the set *Form* contains

$$R_aA, \qquad \text{for every } A \in Form,\ a \in Ind. \tag{7}$$

[5] Cf. Hughes & Cresswell, *A New Introduction to Modal Logic*; Świrydowicz, *Podstawy logiki modalnej*.

Formula (7) means that the formula *A* is related to the reference of the term *a* in a certain way. The relation depends on the interpretation of the operator *R*. For example, formula (7) could mean that *A* is true at the time *a*, is true in the possible world *a*, is known as true by the agent *a*, etc. No restriction exists here pertaining to interpretation of *R* and *Ind*.

In many cases modal connectives *M* and *L* can be reduced (defined) to the operator *R*, provided in the alphabet there are quantifiers and a binary predicate. So far we have no proof that it is generally true and we restrict ourselves to propose a hypothesis that the operator *R* allows definability of other modal connectives. To support the hypothesis some interesting examples are delivered in the final part of this paper.

Regardless the scope of our hypothesis (and it is certainly non-zero) the operator *R* shows its expressive power. It is itself a flexible modality and, once interpreted, allows definitions of different kinds of modalities.

The origin of the operator *R* is directly connected to search for a logic of time. Calculi involving the operator *R* are called positional logics. The first positional logic, being simultaneously the first logic of time, was constructed by Jerzy Łoś in his master's thesis *Podstawy analizy metodologicznej kanonów Milla*,[6] written in Lublin and supervised by Jerzy Słupecki.[7] Łoś employed the letter *U* rather than *R* and it is Nicolas Rescher who established the shape of the operator. It seems likely that *R* stands here for realization, although in some works by Rescher the letter *P* is being used instead, especially when the system is called positional or topological.[8] However, whenever Rescher speaks of some logic of time, he always uses the letter *R* as the operator of realization.[9] The custom to use the letter *R* with no indication to its interpretation comes from Tomasz Jarmużek and Andrzej Pietruszczak[10] and was established in later works in the field.[11] Regardless its graphical shapes the operator *R* and key ideas of its appli-

[6] Jerzy Łoś, "Podstawy analizy metodologicznej kanonów Milla" (Foundations of methodological analysis of Mill's canons, *Annales Universitatis Mariae Curie-Skłodowska* 1947).

[7] The whole history of Łoś' discovery is told in Marcin Tkaczyk and Tomasz Jarmużek, "Jerzy Łoś Positional Calculus and the Origin of Temporal Logic". *Logic and Logical Philosophy* 2018, DOI: 10.12775/LLP.2018.013. Published online: September 23, 2018.

[8] Nicolas Rescher, "Topological Logic", in: Nicolas Rescher, *Topics in Philosophical Logic* (Dordrecht-Holland: D. Reidel Publishing Company 1968); idem, Alasdair Urquhart, *Temporal Logix* (Wien, New York: Springer Verlag 1971).

[9] Nicolas Rescher, "Chronological Logic", in: Nicolas Rescher, *Topics in Philosophical Logic* (Dordrecht-Holland: D. Reidel Publishing Company 1968); idem, Urquhart, *Temporal Logic*.

[10] Tomasz Jarmużek, Andrzej Pietruszczak, "Completeness of Minimal Positional Calculus", *Logic and Logical Philosophy* 2004, 13, 146–162.

[11] Marcin Tkaczyk, *Logika czasu empirycznego* (The Logic of Physical Time) (Lublin: Wydawnictwo KUL, 2009); idem, "Negation in weak positional calculi", *Logic*

cation come from Łoś. The seminal work[12] was published in Polish, but its crucial ideas were presented shortly after thank to the review Henryk Hiż published in *The Journal of Symbolic Logic*.[13] The operator R was of an interest also in another seminal work from Łoś, concerning epistemic modalities.[14] The latter work was reviewed by Roman Suszko[15] to allow the idea to circulate beyond the Iron Curtain. However, logics with the operator R have been first and foremost taken into account with relation to analyses of concepts and inferences dealing with time.[16]

The most simple positional logic is the system MR, extracted from the system of Łoś by Jarmużek and Pietruszczak.[17] The alphabet of the system MR consists of the sets *At* and *Ind*, the operator R, classical connectives: $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, and parentheses. The set *For** of quasi-formulas is defined recursively by the conditions (2a)-(2f) and is identical with the set of formulas of classical propositinal calculus. Then the recursive definition of the set *Form* comes as follows: the set *Form* of formulas of the system MR is the smallest collection containing

$R_a A$, for every $A \in Form^*$, $a \in Ind$. (8a)

$(\neg A)$, for every $A \in Form$, (8b)

$(A \wedge B)$, for every $A, B \in Form$, (8c)

$(A \vee B)$, for every $A, B \in Form$, (8d)

$(A \Rightarrow B)$, for every $A, B \in Form$, (8e)

$(A \Leftrightarrow B)$, for every $A, B \in Form$, (8f)

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and Logical Philosophy 2013, 22: 3–19; idem, “Distribution Laws in Weak Positional Logics”, *Roczniki Filozoficzne* 2018, 66, 3: 163–179.

12 Jerzy Łoś, “Podstawy analizy metodologicznej kanonów Milla” (Foundations of methodological analysis of Mill’s canons), *Annales Universitatis Mariae Curie-Skłodowska* 1947, 2.5.F: 269–301.

13 Henryk Hiż, Review: Jerzy Łoś, “Foundations of the Methodological Analysis of Mill’s Canons”. *Journal of Symbolic Logic* 1951, 16: 58–59.

14 Jerzy Łoś, “Logiki wielowartościowe a formalizacja funkcji intensjonalnych” (Multivalued logics and formalization of intensional functions), *Kwartalnik Filozoficzny* 1948, 17: 1–2.

15 Roman Suszko, Review: Jerzy Łoś, „Many-Valued Logics and Formalization of Intensional Functions”, *Journal of Symbolic Logic* 1949, 14: 64–65.

16 E.g. Tomasz Jarmużek, “Minimal Logical Systems with R-operator: Their Metalogical Properties and Ways of Extensions”, in: *Perspectives on Universal Logic*, ed. J. Bézieau, A. Costa-Leite, Rome: Polimetrica Publisher 2007; Tomasz Jarmużek, *Jutrzejsza bitwa morska. Rozumowanie Diodora Kronosa* (Tomorrow Sea-Fight: Diodorus Cronus’ Argument, Toruń: Wydawnictwo Naukowe UMK, 2013); Tomasz Jarmużek, *On the Sea-Battle Tomorrow That May Not Happen* (Berlin: Peter Lang, 2018).

17 Jarmużek, Pietruszczak, “Completeness of Minimal Positional Calculus”, *Logic and Logical Philosophy* 2004, 13: 146–162.

Definition (8) excludes nested tokens of the operator R, like R_aR_aA, $R_a(A \Rightarrow R_aB)$ or $R_a(R_aA \vee R_aB)$. Only R-formulas of the shape (8a) and their combinations built by means of classical connectives are allowed. Jarmużek and Pietruszczak[18] presented the first axiomatics of the system MR. The set of axioms of the system MR is the smallest set containing

$$A, \quad \text{for every } A \in Form \text{ being a substitution of a classical tautology} \tag{9a}$$

$$R_aA, \quad \text{for every } A \in Form^* \text{ being a classical tautology} \tag{9b}$$

$$\neg R_aA \Leftrightarrow R_a\neg A, \quad \text{for every } A \in Form^*,\ a \in Ind \tag{9c}$$

$$(R_aA \wedge R_aB) \Rightarrow R_a(A \wedge B), \quad \text{for every } A, B \in Form^*,\ a \in Ind. \tag{9d}$$

The set of theorems of the system MR is the smallest collection containing all the axioms from (9a) to (9d) and closed under the rule of *modus ponens*:

$$\text{if } A \text{ and } (A \Rightarrow B) \text{ are theorems then so is } B, \text{ for every } A, B \in Form. \tag{9e}$$

Several other axiomatizations have been offered of the same system MR by Tkaczyk[19], which are free of the necessitation-like axiom (9b). Typical theorems of the system MR are laws of distribution of the operator R over classical connectives. Beside formula (9c) formula

$$(R_aA \otimes R_aB) \Leftrightarrow R_a(A \otimes B) \tag{10}$$

is a theorem of the system MR for every $A \in Form^*$, $a \in Ind$, and for any of the classical connectives: $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, being uniformly substituted for the sign $\otimes$.

A number of semantic approaches to the system MR have been proposed. In the original version from Jarmużek and Pietruszczak[20] a model is an ordered triple

$$\mathbf{M} = \langle W, d, v \rangle, \tag{11}$$

[18] Ibidem.

[19] Marcin Tkaczyk, *Logika czasu empirycznego* (The Logic of Physical Time, Lublin: Wydawnictwo KUL, 2009); idem, "Negation in weak positional calculi", *Logic and Logical Philosophy* 2013, 22: 3–19; idem, "Distribution Laws in Weak Positional Logics", Roczniki Filozoficzne 2018, 66, 3: 163–179.

[20] Jarmużek, Pietruszczak, "Completeness of Minimal Positional Calculus".

where W is a nonempty set, d is a function from the set Ind into the set W, and v is a function assigning classical truth values 1 or 0 to ordered pairs $\langle A,w\rangle$, where $A \in Form^*$ and $w \in W$. The function v meets classical conditions, hence,

$$v(w,\neg A) = 1 \text{ iff } v(w,A) = 0, \tag{12a}$$

$$v(w,A \wedge B) = 1 \text{ iff } v(w,A) = 1 \text{ and } v(w,B) = 1, \tag{12b}$$

and analogically for other classical connectives.[21] It has been observed that the original models can be upgraded and simplified[22], and other models have been constructed like extended value functions[23] or set-theoretical models.[24]

For the sake of systematicity a concept of *normal* positional calculus has been introduced. A positional logic is considered normal if and only if it meets three following conditions: it includes classical propositional calculus in the sense of axioms (9a) and (9b), laws of distribution for all classical connectives are theorems, i.e. (9c) and (10), and finally the rule of *modus ponens* (9e).[25] The system MR is the weakest normal positional logic. Interestingly enough it is also in a sense a maximal system, namely it is maximal with respect to theorems and rules involving a single term $a \in Ind$.[26]

There exist systems weaker than MR which are called non-normal. They may be described by means of algebraic semantics[27], set-theoretical semantics[28], or some special modifications put on understanding of the operator R itself.[29]

An extremely important enhancement of the system MR is the system MRQ, which is a first-order version of the system MR and involves quantifiers $\forall$ and $\exists$, a denumerable set *Var* of variables, and some set of predicates and function symbols.[30] The system MR is a cross between predicate logic and positional logic. Quantification over points and re-

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[21] Ibidem.

[22] Tomasz Jarmużek, Marcin Tkaczyk, *Normalne logiki pozycyjne* (Normal positional logics) (Lublin: Towarzystwo Naukowe KUL, 2015).

[23] Ibidem.

[24] Tkaczyk, "Distribution Laws in Weak Positional Logics".

[25] Jarmużek, Tkaczyk, *Normalne logiki pozycyjne*.

[26] Anna Maria Karczewska, "Maximality of the Minimal R-logic", *Logic and Logical Philosophy* 2018, 27: 193–203.

[27] Tkaczyk, *Logika czasu empirycznego*; idem, "Negation in weak positional calculi".

[28] Idem, "Distribution Laws in Weak Positional Logics".

[29] Jarmużek, "Minimal Logical Systems with R-operator: Their Metalogical Properties and Ways of Extensions".

[30] Idem, Tkaczyk, *Normalne logiki pozycyjne*.

lations between points may be expressed in the language of MRQ. Of course, the system MRQ shares undecidability with first-order logic, unless its alphabet is restricted to monadic predicates.[31] We extend the set *At* by addition of – some or all – atomic formulas of first order logic. However, we do not change conditions (2a)-(2f) when defining the set *Form** of quasi-formulas. Therefore quantifiers do not appear in quasi-formulas. On the other hand, we define the set *Form* of formulas of the system MRQ as the smallest collection containing all formulas (8) as well as

$$\exists x\, A, \qquad \text{for every } A \in Form,\ x \in Var, \tag{13a}$$

$$\forall x\, A, \qquad \text{for every } A \in Form,\ x \in Var, \tag{13b}$$

which allows full quantificational theory with respect to members of the set *Form*. Of course, we read the formula $\exists x\, A$: A for some x, and the formula $\forall x\, A$: A for every x. Axiomatizations and especially semantics of the system MRQ are rather complicated and we dispense ourselves from any extended introduction to it. It has been already delivered elsewhere in detail.[32] For the sake of this paper it is sufficient to understand that quantifiers and other classical connectives mirror standard logic, whereas the connective *R* mirrors the system MR.

It is fair to say that the system MRQ is a regimented and refined counterpart of the original system of Łoś. The language of the latter system was slightly more complicated, especially because it allowed quantification over propositional variables. Those problems, debates and inspirations originated by Łoś are accounted for by Jarmużek and Tkaczyk.[33]

Jarmużek and Tkaczyk[34] presented formal way to reduce modal connectives *M* and *L* to the operator *R*. The environment for that reduction is the system MRQ with a single binary predicate *Q*. Hence, we assume that $aQb \in Form^*$ is a quasi-formula for all terms (including variables) $a, b \in (Ind \cup Var)$. The basic idea for that reduction may be expressed by means of two following definitions (14) within the language of the system MRQ:

$$R_a MA = \exists x\, (aQx \wedge R_x A), \tag{14a}$$

$$R_a LA = \forall x\, (aQx \Rightarrow R_x A). \tag{14b}$$

31 Ibidem.
32 Ibidem.
33 Ibidem.
34 Ibidem.

It seems clear that definitions (14a) and (14b) are analogical to the semantic conditions (4g) and (4h) respectively. Due to normality of the system MRQ connectives work in the classical way, which delivers a base for a more formalized version of relational models (3). It has been shown that under definitions (14) in the plain system MRQ the connectives *M* and *L* are algebraically identical to their counterpart connectives of the system K, the weakest normal modal logic. The system K turns out to be a proper part of the system MRQ. Which shows, among others, how deeply grounded is the concept of normality in positional logic.

Furthermore, imposing conditions (6a)-(6e) on the predicate *Q*, we get some theories based on the logic MRQ. We demonstrated that modal connectives introduced in such theories by means of definitions (14) are algebraically identical with connectives of respective modal calculi. Normal modal logics turn out to be proper parts of respective theories of one binary predicate, based on the logic MRQ.[35]

It has also been demonstrated that not only modal connectives can be defined by means of the operator *R*, but also proofs of modal theorems may be reconstructed in the system MRQ. And what is even more interesting, there is a theory of the predicate *Q*, based on the system MRQ that allows to formally reconstruct metalogical proofs of soundness and completeness of systems of modal logic within the confines of the system MRQ. This shows the expressive power of this system is really high, and makes describing the connective *R* as meta-modality even more intriguing.[36]

There is a philosophical conclusion to be drawn from our research report: typical logical modalities and their theories may be expressed – at least to some extent we have covered so far – by means of one peculiar modality *R*. However, the possible borders of such reductions remain an unanswered question.

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[35] Ibidem.

[36] Ibidem.

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Summary

Theories of modal notions belong to subjects of permanent study for logicians as well as all philosophers. We describe some fundamental ideas concerning kinds of modal expressions. Then we deliver a concise introduction to basic concepts of modal logic with connectives *M* of possibility and *L* of necessity. We describe typical normal modal logic as sets of theorems as well as by means of relational semantics. In the focal points we present some important outcomes in the field of positional logic containing the operator *R*, with special emphasis on systems MR and MRQ. We show how relational semantics of typical modal logic may be reconstructed within positional logic MRQ with one binary predicate, and therefore normal modal logics turn out to be algebraically proper parts of positional logic or theories based on it. Some philosophical questions and insights are being raised on the ground of those formal research.

Keywords: expressive power, modal logic, positional logic, possible worlds, realization operator, reduction.

Streszczenie

Siła wyrazu pozycyjnego operatora *R*: studium przypadku z logiki modalnej i modalnej filozofii

Praca została poświęcona zagadnieniu operatora realizacji stosowanemu w logice pozycyjnej. Po krótkim przeglądzie podstawowych systemów logiki modalnej, jej semantyki oraz filozoficznych interpretacji operatorów modalnych prezentujemy podstawowe fakty na temat logiki pozycyjnej i operatora realizacji. Następnie pokazujemy, w jaki sposób można zredukować modalności zdefiniowane przez wybrane systemy logiki modalnej do odpowiednich teorii logiki pozycyjnej MRQ. Za możliwość tej redukcji odpowiadają operator realizacji R oraz siła wyrazu, która za nim stoi. Przykłady takiej redukcji skłaniają do przyjęcia filozoficznego wniosku, iż operator realizacji może być traktowany jako pewna meta-modalność.

Słowa kluczowe: siła wyrazu, logika modalna, logika pozycyjna, możliwe światy, operator realizacji, redukcja