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Many-Valueness and Modality

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Modal logic can be obtained either by extending classical (I omit non-classical cases, for instance intuitionistic system, and restrict further considerations to propositional calculus) logic or by using many-valued logic. The former strategy consists in adding modal operators, like necessity or possibility, to the stock of propositional constants, like negation, disjunction, conjunction, implication, etc. Łukasiewicz[1] proved that modalities cannot by defined inside two-valued logic. Hence, if one intends to define necessity or impossibility in the framework of non-modal propositional calculus, he or she must use many-valued logic. Łukasiewicz himself initially employed a three-valued system,[2] but he moved to a four-valued one in his later works.[3] My further remarks concern the three-valued case, that is the system $\mathbf{Ł}_3$, with some additional remarks about $\mathbf{Ł}_4$; more precisely I will point out some difficulties concerning the analysis of modalities in the former modal logic and Łukasiewicz's attempts to solve them in his later works. The comparison of modal logic

[1] Jan Łukasiewicz, "O pojęciu możliwości" (On the Concept of Possibility), *Ruch Filozoficzny* 6: 169–170.

[2] Idem, "O logice trójwartościowej" (On Three-Valued Logic), *Ruch Filozoficzny* 6: 170–171 (English translation in: Jan Łukasiewicz, *Selected Writings* (Warszawa–Amsterdam: PWN–Polish Scientific Publishers/North–Holland Publishing Company, 1970), 87–88.

[3] Idem, "A System of Modal Logic", *The Journal of Computing Systems* vol. 1, no. 3 (1953): 111–143 (reprinted in: Łukasiewicz, *Selected Writings*, 352–390); idem, *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic* (2nd ed. Oxford: At the Clarendon Press, 1957).

as extension of classical logic, best exemplified by so-called Lewis systems,[4] with Łukasiewicz modal logic (**Ł**–system for brevity)[5] is an interesting enterprise from formal as well as philosophical point view. One aspect is well-known and stressed, namely that although the **Ł**-system is extensional, Lewis' systems are not. The principle of extensionality (**PE** hereafter) says (it is a rough formulation) that if *A* is a compound sentence, its logical value is a function of logical values of its components. For instance, if $A = B \wedge C$, the value of *A* is uniquely determined by the values of *C* and *B* – more formally, if $A = B \wedge C$, $\mathbf{v}(A) = \mathbf{f}(\mathbf{v}(B), \mathbf{v}(C))$, where the function **f** is displayed by truth-tables, well-known in the classical propositional case.

Since **PE** was a dogma of the Warsaw School of Logic, modal logic must obey **PE** for being considered as logic. Hence, abandoning the principle of bivalence (**BI** hereafter) constituted the principal "Polish" step to build modal logic as based on many-valued semantic machinery. Yet both ways of obtaining modal logic share some important details, for instance, define $\Box A$ (it is necessary that *A*) as $\neg\Diamond\neg A$, where the diamond ($\Diamond$) means 'it is possible that'. Łukasiewicz, at least in his early works, interpreted the third value, usually denoted by ½, just as possibility (he changed later this way of speaking). Thus, the expression $\mathbf{v}(A) = ½$ can be read 'it is possible that *A*'. Łukasiewicz illustrated the case by so-called future contingents, that is, sentences about future contingent events. Consider the sentence (i) 'I will be in the place *P* at the time *t'* later than the present *t*'. Since (i) is neither true nor false at *t*, its value is ½. This example follows Aristotle's problem of the sea-battle tomorrow and raises several interpretative issues, for instance, the question whether true at present is equivalent to true. I will return to that later.

Lewis' systems are much more popular in present days than **Ł**-system. Thus, it is easier to outline formal aspects of modalities appealing to the former. I consider the generalized logical square (**PO** – the

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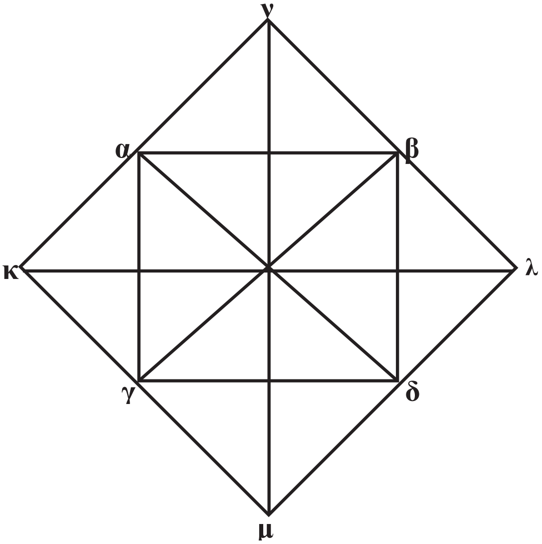
[4] For comprehensive surveys see: Brian F. Chellas, *Modal Logic: An Introduction* (Cambridge: Cambridge University Press, 1980); Urszula Żegleń, *Modalność w logice i filozofii. Podstawy ontyczne* (Modality in Logic and Philosophy. Ontic Foundations) (Warszawa: Polskie Towarzystwo Semiotyczne, 1990); Kazimierz Świrydowicz, *Podstawy logiki modalnej* (Foundations of Modal Logic) (Poznań: Wydawnictwo Naukowe UAM, 2004); James W. Garson, *Modal Logic for Philosophers* (Cambridge: Cambridge University Press, 2006).

[5] For some remarks see: Alexander A. Zinoview, *Philosophical Problems of Many-Valued Logic* (Dordrecht: D. Reidel Publishing Company, 1963); Nicolas Rescher, *Many-Valued Logic* (New York: McGraw Hill, 1969); Grzegorz Malinowski, *Many-Valued* Logic (Oxford: Clarendon Press, 1993) and Siegfried Gottwald, *A Treatise on Many-Valued Logic* (Baldock: Research Studies Press LTD, 2000).

logical octagon) for modalities as a convenient starting point for further analysis:[6]



The points *α*βγδ determine the traditional logical square. Interpret *α* as □*A*, β as □¬*A* (it is impossible that *A* = ¬◊*A*), γ as ◊*A* and δ as ◊¬*A*. We have the following dependencies (the symbol ├ refers to provability in logic):

(1)	$\vdash \neg(\alpha \wedge \beta)$	(α and β are contrary);
(2)	$\vdash (\alpha \Rightarrow \gamma)$	(α entails γ; γ is subordinated to α);
(3)	$\vdash (\beta \Rightarrow \delta)$	(β entails δ; δ is subordinated to β);
(4)	$\vdash (\alpha \Leftrightarrow \delta)$	(α and δ are contradictory);
(5)	$\vdash (\beta \Leftrightarrow \neg\gamma)$	(β and γ are contradictory);
(6)	$\vdash (\gamma \vee \delta)$	(γ and δ are complementary);
(7)	$\vdash (\Box A \Leftrightarrow \neg\Diamond\neg A)$	(□ is definable as ¬◊¬*A*);
(8)	$\vdash (\Diamond A \Leftrightarrow \neg\Box\neg A)$	(◊ is definable as ¬□¬);
(9)	$\vdash (\neg\Diamond A \Leftrightarrow \Box\neg A)$	(¬◊ is definable as □¬*A*).

6 For a more extensive analysis see: Jan Woleński, "Applications of Squares of Oppositions and Their Generalizations in Philosophical Analysis", *Logica Universalis* 2(1) (2008): 13–29 (reprinted in: Jan Woleński. *Essays on Logic and Its Applications in Philosophy* (Frankfurt am Main: Peter Lang, 2008), 255–269).

These theorems are analogical to dependencies between quantifiers – □ behaves as ∀ and ◊ as ∃. This assertion does not mean that modal operators are defined in non-modal logic, even first-order. We have just an analogy, which justifies (1)–(9). The strict expression of this fact points out that we have propositional calculus (**PC**) plus the logic of the logical square for modalities (**LS**). One could say that (1) – (9) express an amount of modal logic for operators with one argument. One problem related to (2) and (3) will be discussed below, because it leads to difficulties in analyzing modalities.

A more expressive content is displayed by the entire hexagon arising by adding the points νκλμ, interpreted as $\alpha \vee \beta$ ($\Box A \vee \Box\neg A$), A, $\neg A$, and $\Diamond A \wedge \Diamond\neg A$, respectively. We have new following principles

(10) $\vdash \alpha \Rightarrow \kappa$ ($\Box A \Rightarrow A$);
(11) $\vdash \beta \Rightarrow \lambda$ ($\Box\neg A \Rightarrow \neg A$);
(12) $\vdash \kappa \Rightarrow \gamma$ ($A \Rightarrow \Diamond A$);
(13) $\vdash \lambda \Rightarrow \delta$ ($\neg A \Rightarrow \Diamond\neg A$);
(14) $\vdash \neg(\kappa \wedge \lambda$ ($\neg(A \wedge \neg A)$);
(15) $\vdash \alpha \Rightarrow \nu$ ($\Box A \Rightarrow (\Box A \vee \Box\neg A$);
(16) $\vdash \beta \Rightarrow \nu$ ($\Box\neg A \Rightarrow (\Box\neg A \vee \Box A$);
(17) $\vdash \mu \Rightarrow$ (($\Diamond A \wedge \Diamond\neg A) \Rightarrow \Diamond A$);
(18) $\vdash \mu \Rightarrow \delta$ ($\Diamond A \wedge \Diamond\neg A) \Rightarrow \Diamond\neg A$);
(19) $\vdash \nu \Leftrightarrow \neg\mu$) ($\Box A \vee \Box\neg A \Leftrightarrow \neg(\Diamond A \wedge \Diamond\neg A)$);
(20) $\vdash \neg(\kappa \wedge \beta)$ ($\neg(A \wedge \Box\neg A$);
(21) $\vdash \neg(\lambda \wedge \Box A)$ ($\neg(\neg A \wedge \Box A$);
(22) $\vdash (\kappa \vee \gamma)$ ($A \vee \Diamond A$);
(23) $\vdash (\kappa \vee \delta)$ ($A \vee \Diamond A$);
(24) $\vdash (\lambda \vee \gamma)$ ($\neg A \vee \Diamond A$)
(26) $\vdash (\lambda \vee \delta)$ ($\neg A \vee \Diamond\neg A$);
(27) $\vdash (\alpha \vee \beta \vee \gamma \wedge \delta)$ ($\Box A \vee \Box\neg A \vee \Diamond A \wedge \Diamond\neg A$).

Denote μ by ♦A and read it as 'it is accidental (contingent) that A' (note, however, that contingency is also understood as non-necessity, that is δ in **(LO)**. Consequently, (27) can be rewritten as the formula $\Box A \vee \Box\neg A \vee$ ♦A asserting (in the stylization *de re* that is with modal words as predicates) that for any A, A is necessary, impossible or accidental (in a sense, the principle of the excluded four for modalities). Observe that the formula $\Box A \vee \Box\neg A$ is not equivalent to $\Box(A \vee \neg A)$ although the former implies the latter. Yet the logic of (**LO)** is classical due to (4) and (14) (there are other formulations).

Justification of (10) to (27) is similar as in the case of (1)–(9). Sometimes (see (10), (11), and (12)–13)) we have analogies of $\Box$ and $\Diamond$ with $\forall$ and $\exists$, respectively. In general, to repeat once again, the logic of (LO) follows from **PC** and special principles added for modalities. The typical way of doing modal logic (more precisely the Lewis systems) consists in extending **PC** by special axioms. To illustrate by some examples, the system **K** is the simplest (basic) one. It arises by adding the scheme $\Box(A \Rightarrow B) \Rightarrow (\Box A \Rightarrow \Box B)$ to **PC;** the system **T** – adding the scheme $\Box A \Rightarrow A$ (see (10) above) to **K**; the system **D** – by adding the scheme $\Box A \Rightarrow \Diamond A$ – to **K**; the system **S4** – by adding the scheme $\Box A \Rightarrow \Box\Box A$ to **T**, and the system **S5** – by adding the scheme $\neg\Box A \Rightarrow \Box\neg\Box A$ to **T**. However, this method clearly leads to the problem of the universality of modal logic. The possible world semantics is based on the accessibility relations between worlds (in fact, algebraic structures) as models. To simplify, let **W** serves as the distinguished possible world (sometimes called, the real world). Thus, the expression **W'RW** means 'the world **W'** is accessible from the world **W'** [usually the relation is the other way round] (or **W'** is an alternative for **W**). Now, we say that the formula $\Box A$ is true in **W**, provided that A is true in every world accessible from **W**, and the formula $\Diamond A$ is true in **W** – if A is true in some worlds accessible from **W**. The relation **R** acts dependently on its formal properties. No special condition is imposed on R^K (the accessibility relation associated with **K**), $\mathbf{R^T}$ is reflexive, $\mathbf{R^D}$ – serial (for any **W**, there is **W'** such that **W'RW**, R^{S4} – symmetric, and $\mathbf{R^{S5}}$ – equivalence relation (reflexive, symmetric and transitive).

Strictly speaking, only **K** is fully universal (valid in all possible worlds) modal logic, but other systems assume some additional semantic constraints with respect to **R**. Let us illustrate the problem by the system **D** associated with (2). The condition imposed on $\mathbf{R^D}$ states that every possible world has at least one alternative. This constraint implies that the set **PW** of possible worlds is non-empty. Assume that **PW** is just empty. By the truth-definition for $\Box A$, this formula is true in **PW**. On the other hand, the formula $\Diamond A$ must be false in **PW**. Hence, the implication (2) (and (3) as well) is also false in **PW**, contrary to the claim of its truth in the logic of (**LO**). We have here the next analogy holding between modal operators and quantifiers. Observe that the formula $\forall x A(x) \Rightarrow \exists x A(x)$ requires for its validity that the domain of discourse is not empty. The condition of seriality implies that **R** is not reflexive. This property has relevance for (10) and deontic logic. Assume that $\Box$ is read as 'it is obligatory that'. Clearly, $\Box A$ does not imply A, because if something is obligatory, it can be (and usually is) not the case. A similar situation occurs in epistemic logic, formalizing the formula 'it is known that A', unless a controversial definition of knowledge as true justified belief is adopted. Further considerations take into account alethic modal logic only, that is, formalizing necessity and possibility in the narrow (obligation is sometimes considered as deontic necessity) sense. Alethic modal logic (as well as deontic) obeys the principle of extensionality in a weaker form, namely that logical truths (and contradictions) are mutually substitutable *salva veritate* – the strong version says that formulas having the same logical value can be substituted with preserving the same logical value of the expression in which substitution is performed.

The hitherto presented considerations entirely ignored the rules of inference. In modal logic, they include usual rules of non-modal **PC**, for instance, the *modus ponens* or the rule of substitution (the latter if we work not with the schemes, like in the present paper, but with concrete formulas) plus the necessitation rule as specific for modalities. It has two different formulations:

(*) $A \vdash \Box A$;
(**) $\vdash A \vdash \Box A$.

The first version allows to infer $\Box A$ from A, whereas the second justifies inference of $\Box A$ from provable A. Roughly speaking, (*) admits "the usual" truth of its premise, but the second – "the qualified" truth of A. To illustrate, let the symbol $\vdash$ expresses 'A is provable in logic'. Thus, (**) says that the conclusion 'A is a logical necessity' follows from the assumption that A is logically provable from logical theorems. On the other hand, (*) legitimizes the conclusion that A is a necessary truth from the premises that A is true. A hot discussion[7] concerns whether modal logic should be based on (*) or (**). This issue is also very important for analysis of other kinds of modalities,[8] for instance, ontological, causal, etc. Another related aspect of the discussed problem concerns the reverse implication to (10), that is, the formula

(28) $A \Rightarrow \Box A$,

which together with (10) leads to

(29) $\Box A \Leftrightarrow A$.

However, (29) trivializes modal logic, because reduces it to non-modal propositional logic. In order to block this consequence, one has

[7] See: Jerzy Perzanowski, "Logiki modalne a filozofia" (Modal Logics and Philosophy), in: *Jak filozofować* (How to Philosophise?), ed. Jerzy Perzanowski (Warszawa: Państwowe Wydawnictwo Naukowe, 1989), 262–346.

[8] See: Urszula Żegleń, *Modalność w logice i filozofii. Podstawy ontyczne* (Modality in Logic and Philosophy. Ontic Foundations) (Warszawa: Polskie Towarzystwo Semiotyczne, 1990).

to restrict the deduction theorem, which is the formula (in the simplest shape)

(30) if $A \vdash B$, then $\vdash A \Rightarrow B$.

Anyway, even if we reject (28), the rule (*) explicitly suggests that every truth is necessary (Leibniz' view).

Łukasiewicz's in his analysis of modalities based on three-valued logic (in his later proposals he retained this intuition) wanted to retain the following traditional modal rules:[9]

(a) *Ab oportere ad esse valet consequential* (necessity entails actuality);
(b) *Ab esse ad posse valet consequentia* (actuality entails possibility);
(c) *Unumquodque, quando est, oportet est* (actuality entails necessity).

Now, (a) corresponds with (10), (b) – with (12), and (c) with (28). Ł-modal logic has the mutual of the operators □ and ◊ (see (7) and (8) as well as many other theorems occurring in Lewis' systems.

On the other hand, Łukasiewicz defined modal operators inside propositional calculus, but, due to his observation (see above), since it cannot be achieved in two-valued logic, $\mathbf{Ł}_3$ had to be taken into account. Tarski proposed the following definition of possibility

(31) $\lozenge A \Leftrightarrow (\neg A \Rightarrow A)$;

Correspondingly, we have

(32) $\Box A \Leftrightarrow \neg(A \Rightarrow \neg A)$.

The truth-equations for □ and ◊ are as follows: if $\mathbf{v}(A) = \mathbf{1}$, then $\mathbf{v}(\Box A) = 1$, in other cases $\mathbf{v}(\Box A) = 0$; if $\mathbf{v}(A) = \mathbf{1}$ or $\mathbf{v}(A) = ½$, then $\mathbf{v}(\lozenge A) = 1$; if $\mathbf{v}(A) = 0$, $\mathbf{v}(\lozenge A) = 0$. Using truth-tables for $\mathbf{Ł}_3$, one can verify (3) and (10). The corresponding equations for the value ½ are shaped by: $\mathbf{v}(A) = ½ = \mathbf{v}(\neg A)$; if $\mathbf{v}(A) = 1$, $\mathbf{v}(B) = ½$, then $\mathbf{v}(A \Rightarrow B) = ½$, if $\mathbf{v}(A) = ½ = \mathbf{v}(B) = ½$, then, $\mathbf{v}(A \Rightarrow B) = ½$; if $\mathbf{v}(A) = 0$, $\mathbf{v}(B) = ½$, then $\mathbf{v}(A \Rightarrow B) = ½$; if $\mathbf{v}(A) = ½$, $\mathbf{v}(B) = 1$, then $\mathbf{v}(A \Rightarrow B) = 1$; if $\mathbf{v}(A) = ½$, $\mathbf{v}(B) = 0$, then $\mathbf{v}(A \Rightarrow B) = 0$. A formula A is a tautology in $\mathbf{Ł}_3$ provided that $\mathbf{v}(A) = 1$ for every valuation. Assume that $\mathbf{v}(\neg(A \Rightarrow \neg A)) = 1$. Hence, $\mathbf{v}(A \Rightarrow \neg A) = 0$. This entails that $\mathbf{v}(\neg A) = 0$ and $\mathbf{v}(A) = 1$. Finally, $\mathbf{v}(\neg A \Rightarrow A) = 1$. Simi-

[9] See: Jan Łukasiewicz, "Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls", *Comptes renduz de la Société des Sciences et de Lettres de Varsovie*, cl. III, 23 (English translaton in: Łukasiewicz. *Selected Writings*), 154–155; page-reference to English translation.

larly, we prove that the formula $\neg(A \Rightarrow \neg A) \Rightarrow A$ (the counterpart of (1)) is a tautology. However, (28), that is $A \Rightarrow \neg(A \Rightarrow \neg A)$ is a critical point. Assume that $\mathbf{v}(A) = 1$. Hence, we obtain that $\mathbf{v}(\neg A) = 0$ and $\mathbf{v}(A \Rightarrow \neg A) = 0$. Since $\mathbf{v}((A \Rightarrow \neg A)) = 1$, (28) is tautological in $\mathbf{Ł}_3$. Since $A \Leftrightarrow B$ is defined in $\mathbf{Ł}_3$ by the conjunction (the same is in the classical **PC**) $(A \Rightarrow B) \wedge (B \Rightarrow A)$, we obtain (29) in the version of $A \Leftrightarrow \neg(A \Rightarrow \neg A)$.

Łukasiewicz returned to (29) in his article *A System of Modal Logic*[10] and his monograph on Aristotle's syllogistic.[11] Although we have no textual evidence, he probably saw inaccuracy of (b) as a theorem. However, $\mathbf{Ł}_4$ has neither (*) nor (**). In order to avoid (29), Łukasiewicz introduced the rejection rule, namely (***) $\dashv (A \vdash \Box A)$. Due to (***) no theorem of $\mathbf{Ł}_4$ begins with the symbol $\Box$. More importantly, $\mathbf{Ł}_4$ does not verify (28) and, *a fortiori*, (30). He also changed the definition of possibility (I skip details, in particular so-called twin-possibilities), although it is definable in $\mathbf{Ł}_4$. The principle of extensionality is valid with respect to this system – its general formulation requires an appeal to so-called variable propositional functors, a new formal machinery employed by Łukasiewicz in his post-war logical investigations (I omit details). $\mathbf{Ł}_4$ is still a part of propositional calculus, but extended by adding variable functors. These modifications make $\mathbf{Ł}_4$ much closer to Lewis' systems than it occurred in modal logic constructed on the base of $\mathbf{Ł}_3$, but there is still considerable difference (see below about (**)). In fact, propositional logic with variable functors is akin to Leśniewski's protothetic, an extended propositional logic with quantifiers binding propositional variables and variables of which propositional functors are values. Yet there is an important difference, namely protothetic respects the strong extensionality rule and the principle of bivalence. As far as I know, a modal extension of protothetic was not proposed. As a matter of fact, such an attempt would be incoherent with Leśniewski view against modal and many-valued logic.

Why Łukasiewicz rejected the rule of necessitation? His explanation is as follows:[12]

> This controversial rule [Łukasiewicz is speaking about (**)] [...] was the cause of many philosophical and theological speculations. [...]. After a long – but in my opinion unconvincing argumentation Von Wright says: "the proposition that a tautology is necessary and a contradiction impossible are truths of logic. This certainly agrees with our logical

[10] Idem, "A System of Modal Logic", *The Journal of Computing Systems* vol. 1, no. 3 (1953): 111–143 (reprinted in: Łukasiewicz. *Selected Writings*, 352–390).

[11] Idem, *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic* (2^{nd} ed; Oxford: At the Clarendon Press, 1957), chapters VI–VIII, especially 152–154.

[12] Idem, "A System of Modal Logic", 377.

> intuitions."[13] I am not certain that it does agree. I think, roughly speaking, that true propositions are simply true without being necessary, and false propositions are simply false without being impossible. This certainly does not hurt our logical intuitions, and may settle many controversies. [...]. It may be asked, however: Why should we introduce necessity and impossibility into logic if true apodeictic propositions are do not exist? I reply to this objection that we are primarily interested in problematic propositions of the form [in the symbolism of the present paper] $\lozenge A$ and $\lozenge\neg A$, which can be true and useful, although their arguments are rejected, and introducing problematic propositions we cannot omit their negations, i.e., apodeictic propositions, as both are inextricably connected with each other.

However, Łukasiewicz's view expressed in the last quotation essentially changes the entire traditional perspective for modalities displayed by (**LO**). The basic logical square is now reduced to the points κλγδ, regulated by $\mathbf{Ł}_4$ (or other chosen many-valued logics). We have (2), (3), (12) and (14), but the relation between κ and λ, and γ and δ must be modified, because inconsistency (κ and λ) is replaced by (generalized) contrariety (for instance, *A* and *A* cannot be together true, but it is possible that are valued by other not-zero identical values), but complementarity (γ and δ) remain as before (at least if $\mathbf{Ł}_3$ is assumed). If we add ν and μ, (19) can be not valid, relatively to a system of many-valued logic taken as basic for modal logic.

Łukasiewicz pointed the following problem (Łukasiewicz 1953, p. 377). Some authors observe that $\mathbf{Ł}_3$ violates the law of non-contradiction (as far as I know it was Ferdinand Gonseth who stated this objection first; hence I will speak about Gonseth's argument). Assume that $\mathbf{v}(A) = ½$. Hence (using truth-tables for $\mathbf{Ł}_3$, $\mathbf{v}(\neg A) = ½$, and $\mathbf{v}(A \wedge \neg A) = ½$. However, this result contradicts our expectations that every contradiction is false. In terms of possibility, we have that if $\mathbf{v}(A) = ½$, then $\mathbf{v}(\lozenge A) = 1$ as well as $\mathbf{v}(\lozenge\neg A) = \mathbf{1}$. Consequently, $\mathbf{v}(\lozenge A \wedge \lozenge\neg A) = 1$. To some extent, this result is fairly intuitive, because if it is true *A* is possible and it is true that $\neg A$ is possible, their conjunction is also true – it means that *A* is accidental. On the other hand, the conclusion that the sentence '*A* is accidental' is equivalent to the sentence '$A \wedge \neg A$ is possible' seems implausible, because it is radically at odds with our elementary intuitions. Łukasiewicz tried to defend his position by the following argument:[14]

> Let n be a positive integer. I contend that the following implication is true for all values of *n*: "If it is possible that *n* is even, and it is possible that *n*

[13] Georg H. Von Wright, *An Essay on Modal Logic* (Amsterdam: North–Holland Publishing Company, 1951), 14–15.

[14] Łukasiewicz, "A System of Modal Logic", 378; page-reference to the reprint.

> is not even, then it is possible that *n* is even, and it is possible that *n* is not even, that it is possible that than *n* is even and *n* is not even." If *n* = 4, it is true that *n* is possibly even, but it is not true that *n* is possibly not even; if *n* is 5, it is true that n is possibly not even, but it is not true that *n* is possibly even. The both premises are never true together, and the formula cannot be refuted.

This argument concerns the formula (A) $\Diamond A \wedge \Diamond B \Rightarrow \Diamond(A \wedge B)$ accepted by Łukasiewicz, but rejected in Lewis' systems.

The formula (B) $\Diamond A \wedge \Diamond\neg A \Rightarrow \Diamond(A \wedge \neg A)$ is a special case of (A), and the former is not legitimized by (**LO**). Łukasiewicz's argument for (B) is not convincing. In particular, it confuses possibility of satisfaction with possible (accidental, contingent) truth. Consider the formula (C) '*n* is even'. It is not a sentence, but an open formula, that is having a free variable. We can say that (C) is possibly satisfied by some numbers and possibly not satisfied by some numbers, but it does not imply that some numbers possibly satisfy and do not satisfy (C). A more precise formulation should note the difference between 'some numbers' and 'concrete numbers'. This observation leads to the assertion that the sentence '(C) is possibly satisfied by some number and possibly not satisfied by some numbers' does not entail that (C) is possibly satisfied and not satisfied by a concrete number, for instance 4 or 5. In fact, the expressions (D) '(C) is possibly satisfied by the number 4' and (E) '(C) is possibly satisfied by the number 5' are sentences that cannot illustrate the point μ in (**LO**), because the latter is not a denial of the former. Observe that the word 'possibly' is redundant in (D) and (E). In fact we have (F) '(C) is satisfied by 4' and (G) '(C) is satisfied by 5' – the former is true, but the latter is false, and thereby equivalent to (H) '(C) is not satisfied by 5'. (F) entails (D) by the logic of (**LO)**, and (I) '(C) is possibly not satisfied by 5' is a consequence of (H). Both sentences (D) and (I) are true, but they do not represent the conjunction $\Diamond(D) \wedge \Diamond\neg(D)$. Hence, the conjunction (D) $\wedge$ (I) does not illustrate the point μ in (**LO**). To conclude, Łukasiewicz's assertion that (A) and (B) cannot be refuted because their antecedents are always false is very problematic.

My refutation of Łukasiewicz's argument against Gonseth essentially employs (**LO**) and, tacitly, the metalogical principle of bivalence (the principle of non-contradiction is its part). Łukasiewicz himself mentioned the logical square for modalities as a historical peculiarity and never tried to modify it with respect to his logic, three-valued or other. However, he rejected[15] the definition of contingency as $\Diamond A \wedge \Diamond\neg A$. He argued that $\blacklozenge A$ should be understood either as $\Diamond A \wedge W\neg A$ or as $\Diamond\neg A \wedge WA$, where the letter *W* refers to a new concept of possibility de-

[15] Idem, *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*, 175–176.

fined within $Ł_4$. Since the full presentation of this concept requires rather a deep and extensive entering into formal details of Łukasiewicz's four-valued modal logic, I will give an informal account using Łukasiewicz's explanations:[16]

> Tossing the coin we may throw either a head or a tail; in other words, it is possible to throw a head and it is possible not to throw a head. We are inclined to regard both propositions as true. But they cannot be both true, if the first 'possible' is denoted by the same functor s the second. The first possibility is just the same as the second, but it just not follows that it should be denoted in the same word. The possibility of throwing a head is different from the possibility of not throwing a head. We may denote the one by [in the symbolism of the present paper] by ◊ and the second by *W*.
>
> (...).
>
> The true contingent refers to undecided events. Let us take the example with the coin which is the same sort as Aristotle's example with the sea-fight. Both examples are concerns events which are undecided at present, but will be decided in the future. Hence the premises 'It is possible to throw a head' and 'It is possible not to throw the head' may at present be both true, but the conclusion 'It is possible to throw a head and not to throw a had' is never true. We know, however, that contingency cannot be defined by the conjunction [in the symbolism of the present paper] $\Diamond A$ and $\Diamond\neg A$, but either $\Diamond A$ and $W\neg A$ or WA and $\Diamond\neg A$.

The above explanation avoids the confusion of truth and satisfaction, but still remains unclear.

First of all, I have difficulties with catching the reason why $\Diamond A$ and $\Diamond\neg A$ cannot be both true. If truth is explicated by model-theoretic means, the conjunction $\Diamond A \wedge \Diamond\neg A$ is consistent and, by the completeness theorem, has a model in which it is true. Łukasiewicz was probably confused by examples using the expression 'true at once'. However, truth in a model is not qualified by any temporal co-ordinates. This remark confirms an observation[17] that semantics for $Ł_3$ simultaneously employed two kinds of valuations, standard (model-theoretical) and temporal (it is indicated by the phrase "at once"). My argument is even stronger, namely that Łukasiewicz confused both valuations in his explanations concerning possibility and contingency. Moreover, to repeat earlier remarks, the shape of a logical square (if any) and its generalizations as displaying the basic logical behaviour of modalities remains unclear

[16] Ibidem, 178.

[17] See: Ludwik Borkowski. "On the Intuitive Interpretation of Three-Valued Logic", in: *Studies in Logic and the Theory of Knowledge 2*, ed. Ludwik Borkowski, Antoni Stepień (Lublin: Towarzystwo Naukowe KUL, 1991), 17–24.

in ◊–*W* logic of modalities. Although we have (17) and (18) the rest appears as quite problematic and requires further elaborations. Furthermore, Łukasiewicz's suggestion that we can eliminate necessities, looks as artificial, although it achieved popularity in Poland, for instance, in Tarski's[18] view that there is no sharp borderline between tautologies and empirical statements. I claim that many of Łukasiewicz's views on modality were dictated by the strong principle of extensionality and the claim that modalities should be defined inside propositional calculus – both assumptions pushed modal logic into many-valued one. On my part, (**LO**) and Lewis' systems offer more convincing means, in particular, more plausible from the intuitive point of view.

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[18] See: Alfred Tarski, "Über den Begriff der logischen Folgerung", *Actes du Congrés international de philosophie scientifique, Sorbonne, Paris 1935, Fasc.* 7, Paris: Hermann, 1936, 1–11 (English translation in: Alfred Tarski. 1956. *Logic, Semantics, Metamathematics. Papers since 1923 to 1939*. 409–420. Oxford: At Clarendon Press).

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Summary

Modal propositional logic can be obtained either by extending non-modal propositional logic (this is the case of Lewis' systems) or by using many-valued logic as the basic system. The second route was taken by Jan Łukasiewicz, who proved that modalities cannot be defined within two-valued logic. The principle of extensionality was a tacit Łukasiewicz's assumption. If we compare Lewis' modal systems with that of Łukasiewicz, we see that both solutions share most logical principles. Perhaps the most important difference concerns the formula $\Diamond(A \wedge \neg A)$, in words ($A$ and $\neg A$) is possible. Łukasiewicz argued that this formula has the value 1/2, if A and $\neg A$ have this value as well. I argue that Łukasiewicz's argument is not correct.

Keywords: logic, extensionality, two-valuedness, logical value

Streszczenie

Wielowartościowość i modalność

Modalną logikę zdań można otrzymać albo rozszerzając logikę zdań bez modalności (jak w przypadku systemów Lewisa), albo używając pewnej logiki wielowartościowej jako systemu podstawowego. To drugie podejście zostało obrane przez Jana Łukasiewicza, który udowodnił, że modalności nie można zdefiniować w ramach logiki dwuwartościowej. Łukasiewicz milcząco przyjął przy tym zasadę ekstensjonalności. Jeśli porównamy systemy modalne Lewisa z systemem Łukasiewicza, dostrzeżemy, że wiele zasad logicznych jest wspólnych dla tych rozwiązań. Z kolei prawdopodobnie najważniejsza różnica między nimi dotyczy formuły $\Diamond(A \wedge \neg A)$, słownie: (A i ¬A) jest możliwe. Łukasiewicz twierdził, że formuła ta przyjmuje wartość 1/2, jeśli wartość tę posiadają A i ¬A. Twierdzę, że argumentacja Łukasiewicza jest niepoprawna.

Słowa kluczowe: logika, ekstensjonalność, dwuwartościowość, wartość logiczna