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Hypothetical Structuralism

Introduction

Michael Resnik is considered to be one of the most well-known followers of structuralism in the philosophy of mathematics. In a series of works he proposed and developed an original structuralistic standpoint.[1] Many authors and he himself classify this version of his views as structuralism *sui generis*,[2] which assumes that the object studied by mathematics are the structures or positions in these structures. This stand-

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[1] Michael Resnik, "Mathematics as a Science of Patterns: Ontology and Reference", *Noûs* 15(4) (1981): 529–550, doi:10.2307/2214851; Michael Resnik, "Mathematics as a Science of Patterns: Epistemology", *Noûs* 16(1) (1982): 95–105, doi:10.2307/2215419; Michael Resnik, "Mathematics from the Structural Point of View", *Revue Internationale de Philosophie* 42(167) (1988): 400–424; Michael Resnik, "Structural Relativity", *Philosophia Mathematica* 4(2) (1996): 83–99, doi:10.1093/philmat/4.2.83.16; Michael Resnik, *Mathematics as a Science of Patterns* (Oxford: Clarendon Press, 1997), doi:10.1093/0198250142.001.0001.

[2] Geoffrey Hellman, "Three Varieties of Mathematical Structuralism", *Philosophia Mathematica* 9(2) (2001): 184–211, doi:10.1093/philmat/9.2.184; Michael Resnik, "Non-ontological Structuralism", *Philosophia Mathematica* 27(3) (2019): 303–315, doi.org/10.1093/philmat/nky002.

point is sometimes also classified as non-eliminative structuralism,[3] which is supposed to indicate the fact that the objects studied by mathematics are treated as existing in a certain definite way, most frequently as abstract entities.

In the subject literature are also structuralistic views included within the eliminative type. In particular, this is how modal structuralism is defined.[4] Their main idea is negation of the existence of mathematical structures as abstract objects.

If one were to seek analogy to the philosophical controversy about universals, then structuralisms of the *sui generis* type are similar to conceptual realism, whose different variants state that abstract universals have a reality that is independent from time-space world. They are in some way ontologically positive. On the other hand, the views of eliminative structuralists can be qualified as nominalist and thus considered as ontologically negative. So we have two opposing viewpoints, the first ontologically positive (structures as abstract mathematical objects exist), the second ontologically negative (structures as abstract objects do not exist). Together, both those standpoints are called ontological due to the fact that the views composing them include those which clearly refer to the nature and manner of the existence of mathematical structures.

1. From *sui generis* to non-ontological structuralism

The original position of Resnik's *sui generis* structuralism falls into the ontologically positive category, but in one of his more recent articles Resnik changed his views of mathematical structures, at the same time staying an advocate of mathematical structuralism.[5] He clearly distinguishes the above views from his new idea, which is non-ontological structuralism. Its name comes from the fact that it leaves "open questions concerning the existence and nature of mathematical objects".[6]

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[3] Erich Reck, Georg Schiemer, "Structuralism in the Philosophy of Mathematics", *The Stanford Encyclopedia of Philosophy*, access 11.07.2020. https://plato.stanford.edu/archives/spr2020/entries/structuralismmathematics.

[4] Geoffrey Hellman, *Mathematics without Numbers. Towards a Modal-Structural Interpretation* (Oxford: Clarendon Press, 1989); Geoffrey Hellman, "Structuralism without Structures", *Philosophia Mathematica* 4(2) (1996): 100–123, doi:10.1093/philmat/4.2.100.

[5] Resnik, "Non-ontological Structuralism".

[6] Ibidem, 303.

Originally, Resnik claimed under his ontological standpoint (*sui generis* structuralism) that the object of interest in mathematics are patterns, meaning the structures composed of certain objects taking the place of a given structure.

> The underlying philosophical idea here is that in mathematics the primary subject-matter is not the individual mathematical objects but rather the structures in which they are arranged. The objects of mathematics, that is, the entities which our mathematical constants and quantifiers denote, are themselves atoms, structureless points, or positions in structures. And as such they have no identity or distinguishing features outside a structure.[7]

The positions of the structure remain in various relations and they do not possess other properties except those determined by the structure.[8] In this sense, structures are treated as objects existing independently, in one way or the other. The mathematical theory can describe such a structure rightly or wrongly. Commenting on the cited fragment, Resnik states that there is too much ontology in these views.[9]

Non-ontological structuralism, proposed by Resnik now, preserves most of his views, only changing the point of view and restraining from an explicit ontological statement.

> So instead of expressing my view by putting the emphasis on objects I will put the emphasis on theories: Mathematics speaks of objects in order to describe or present structures; from the point of view of a mathematical theory, the denotations of its constants and quantifiers might as well be points or positions in a structure or structures; for the theory attributes to them no identifying features outside of the structure or structures in question.[10]

Hence, it is mathematical theories describing structures which become the object of his interest. As he admits, the change of views is inspired by Quine's ideas, especially the metaphysical concept of global structuralism.[11]

> Much of my old view survives. Now instead of talking about positions in patterns we talk about theories and singular terms and quantifiers. Instead of saying that there is no fact as to whether the positions of a natural-number sequence are identical to a certain sets, we say that there is no fact as to

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[7] Resnik, *Mathematics as a Science of Patterns*, 201.
[8] Ibidem, 202–203.
[9] Resnik, "Non-ontological Structuralism", 309.
[10] Ibidem.
[11] Ibidem, 306–308.

> which of the many interpretations of number theory in set theory is the correct one. This is just a consequence of ontological relativity without the explanation in terms of positions in patterns.[12]

According to Resnik, non-ontological structuralism should be clearly distinguished from the view which negates the existence of structures as abstract objects. Therefore, one thing is the statement that mathematical structures do not exist and mathematical theories describe some fictional (non-existing) objects, and another is suspending the judgment in this matter.

> On my view mathematics is like fiction in presenting incomplete descriptions of its objects, but that does not commit one to holding that mathematical objects are fictional.[13]

In this context, non-ontological means restraining from ontological judgments. A negation of the existence of structures is treated by Resnik as taking some ontological position; hence, it is classified as a kind of ontological structuralism.

In summary, we have divided the various structuralist concepts into two categories, the first ontologically positive, which includes, e.g., old Resnik's *sui generis* structuralism, and the second ontologically negative, an example of which is modal structuralism. The criterion for this division is to take a position on how mathematical structures exist, to acknowledge or deny their existence. This categorization is considered ontological because it refers to the way mathematical structures exist. Resnik believes that it is also possible to remain within mathematical structuralism while suspending judgment on how mathematical structures exist. He calls such a stance non-ontological strutkuralism. It is his new standpoint. In this article we will try to clarify this position.

In order to understand Resnik's position of non-ontological structuralism, we will briefly recall the Frege-Hilbert dispute and analyze Ajdukiewicz's concept of the methodology of deductive science. These two positions will allow us to present Resnik's positions in a new light.

[12] Ibidem, 310.

[13] Ibidem.

2. Hilbert style deductive theory

The followers of the ontologically negative vision of structuralism can include D. Hilbert. His concept of a deductive theory, where the meaning of primitive notions is completely abstracted from and hence no obviousness of axioms are expected, is interpreted by many structuralistically.[14] Hilbert wrote the following in a letter to Frege.

> […] it is surely obvious that every theory is only a scaffolding or schema of concepts together with their necessary relations to one another, and that the basic elements can be thought of in any way one likes. If in speaking of my points, I think of some system of things, e.g., the system love, law, chimney-sweep […] and then assume all my axioms as relations between these things, then my propositions, e.g., Pythagoras' theorem, are also valid for these things […] [A]ny theory can always be applied to infinitely many systems of basic elements. One only needs to apply a reversible one-one transformation and lay it down that the axioms shall be correspondingly the same for the transformed things.[15]

According to Hilbert, the meaning of primitive notions is established through their use in axioms which are treated as implicit definitions. As is known, this is how he reconstructed Euclidean geometry,[16] where such concepts as "point", "straight line" or "plane" receive precise meanings through being used in axioms. On the other hand, their intuitive meaning accepted by Euclid in *Elements* does not play any greater role in this theory, beside the heuristic role. In this way, those specific constants become a sort of variables prepared for any interpretation. Despite Hilbert's declarations, a purely formal axiomatic system, free from any errors and references to intuitive understanding of the terms used, was not managed to obtain until the seventh edition of the aforementioned work. The style of practicing mathematical theories along the lines of Hilbert's work is sometimes called the algebraic approach.[17]

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[14] Fiona T. Doherty, "Hilbertian Structuralism and the Frege-Hilbert Controversy", *Philosophia Mathematica* 27(3) (2019): 335–361, doi:10.1093/philmat/nkz016.

[15] Stewart Shapiro, "Categories, Structures, and the Frege-Hilbert Controversy: The Status of Meta-mathematics", *Philosophia Mathematica* (III) 13 (2005): 66, doi:10.1093/philmat/nki007.

[16] David Hilbert, *Grundlagen der Geometrie* (Leipzig: B.G. Teubner, 1903).

[17] Shapiro, "Categories, Structures, and the Frege-Hilbert Controversy: The Status of Meta-mathematics", 66; Patricia Blanchette, "The Frege-Hilbert Controversy", *The Stanford Encyclopedia of Philosophy*, access 20.07.2020, https://plato.stanford.edu/archives/fall2018/entries/frege-hilbert.

Frege took an opposing view as he thought that "arithmetic and geometry each have a specific subject matter, space in the one case and the realm of natural numbers in the other".[18] According to him, the meaning of primitive terms of deductive theories is established through the reference to concrete objects and this is how axioms become a certain kind of obvious theorems based on intuition and assumed to rightly describe the examined reality. This reference takes a place in geometry and arithmetic, and in any other deductive theory. With this approach, the truthfulness and thus non-contradiction of axioms does not require any special proof. The discussion between Hilbert and Frege is known today as the Frege-Hilbert Controversy about meaning of primitive notions.[19]

Frege's position is undoubtedly, in light of the above divergences, an ontological (but rather not structuralist) position. Resnik's[20] question of what kind of position Hilbert's views are is rightly asked. If we assume that Hilbert recognizes or denies the existence of mathematical objects that are described by axioms, then we can consider his position as some kind of ontological structuralism. Resnik leans toward the negative version of it. On the other hand, assuming that Hilbert's algebraic approach is merely an expression of some hypothetical attitude towards possibly existing or not mathematical structures, then we will classify him as a non-ontological structuralist. Thus, each of these assumptions refers only to the internal attitude that Hilbert takes towards the axioms of mathematical theories. This possible attitude toward the axioms of deductive theory will prove to be the key to understanding the structuralist positions discussed in this article.

3. Three stages of development of deductive sciences

In this context the concept of development of the methodology of deductive sciences as proposed by Ajdukiewicz is worth analyzing. It is well known that he was a student of Hilbert and Husserl in Goettingen, a student of Łukasiewicz in Lvov and he was considered to be one of the

[18] Shapiro, "Categories, Structures, and the Frege-Hilbert Controversy: The Status of Meta-mathematics", 66.

[19] Doherty, "Hilbertian Structuralism and the Frege-Hilbert Controversy".

[20] Resnik, "Non-ontological Structuralism".

most important representatives of the Polish School of Logic.[21] It can be supposed, and this is also confirmed by research[22] that Hilbert's concepts considerably influenced Ajdukiewicz's views.

His views on the structure of deductive theories became visible in his early works since 1920[23] and their study can also be found in a highly original textbook of logic.[24] A contemporary study of this concept can be found, for example, in Tkaczyk.[25] According to Ajdukiewicz, all deductive theories go through three stages of development:

(1) The first one is called pre-axiomatic intuitive stage. Its characteristic feature is that axioms and primitive terms used are accepted as obvious and intuitively understandable while their set is not ultimately established. The proofs of theorems are based on intuitions and their assessment of validity is connected with the obviousness of successive steps in the proving process. If only we do not encounter other researchers' objections, we can always make a new and obvious assumption to the theory which is found at this stage.

(2) Moving to the second stage, called axiomatic intuitive, takes place through the closing and ultimately establishing of the set of axioms and a dictionary of primitive terms. The basis to accept them is still obviousness and intuitive understandability but new axioms and terms which appear most obvious must not join the once established set. Therefore, the openness of the first stage is replaced by a closed and ultimately established set of expressions.

(3) The third and last stage, called axiomatic abstract, gives up obviousness and intuitive comprehensibility for the benefit of complete freedom in the choice of axioms. The freedom is limited solely by the set of expected theorems or unwanted theorems as well. The only acceptable way to justify theorems at this stage is deduction. In the previous stag-

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21 Jan Woleński, "Lvov-Warsaw School", *The Stanford Encyclopedia of Philosophy*, access 21.07.2020, https://plato.stanford.edu/archives/sum2020/entries/lvov-warsaw.

22 Roman Murawski, *The Philosophy of Mathematics and Logic in the 1920s and 1930s in Poland* (Birkhäuser Basel, 2014), 101–106, doi.org/10.1007/978-3-0348-0831-6.

23 Kazimierz Ajdukiewicz, Jerzy Giedymin, "From the Methodology of the Deductive Sciences", *Studia Logica: An International Journal for Symbolic Logic* 19(1966): 9–45.

24 Kazimierz Ajdukiewicz, *Pragmatic Logic*, transl. by Olgierd (Dordrecht–Boston: D. Reidel Publishing Company, 1974).

25 Marcin Tkaczyk, "Kazimierz Ajdukiewicz's Philosophy of Mathematics", *Stud East Eur Thought* 68 (2016): 21–38, doi.org/10.1007/s11212-016-9245-x.

es, it was required to provide some justification for the accepted axioms, whereas here this is not required or even more, it should not be done. This last stage is otherwise called axiomatic formalized. At this stage, the openness characteristic of the first stage, as well the obviousness and intuitiveness characteristic of the two earlier stages disappear. The sets of axioms and primitive terms are fixed and closed and there are no external requirements placed on the elements of those sets. Obviousness and intuitive comprehensibility do not play any role.

Coming back to the controversy between Frege and Hilbert, we can say that the third axiomatic abstract stage in the development of deductive sciences, spoken about by Ajdukiewicz, is a description of what we called Hilbert's algebraic approach. What is especially remarkable here is that the existing meanings of primitive terms and axioms are abstracted from and they are treated only as certain formulas prepared for later interpretation. In this sense, Frege's idea, saying that the meaning of primitive terms originates from the outside reality, can be considered close to the second axiomatic intuitive stage, where obviousness and intuitiveness are expected from primitive terms and axioms.

4. Hypothetical or assertive style

Keeping in mind the three-stage process of development of deductive sciences, let us note that elsewhere Ajdukiewicz proposes a slightly different division of axiomatic theories. This division is independent of the stage of development of a given theory. The criterion for the division is the attitude toward the axioms of the theory. Axioms can be regarded as certain hypotheses that are not asserted sentences, or they can be regarded as asserted sentences. The theories of the first type are called hypothetical-deductive, while the second type are assertive-deductive theories.

The hypothetical-deductive theories are characterized by Ajdukiewicz:

> A hypothetical deductive system is a science in which at the outset we list a number of statements without adopting any attitude toward them, i.e., without either accepting or rejecting them, and next we derive from them by deduction (but do not infer) other statements that follow from the former. The statements listed at the outset (but neither accepted nor rejected) also are called axioms, and the statements derived from them (but also neither accepted nor rejected on that account) also are called derived theorems.[26]

[26] Ajdukiewicz, *Pragmatic Logic*, 206–207.

On the other hand, the theories of the second assertive-deductive kind Ajdukiewicz describes:

> Those deductive systems in which the axioms are asserted, and hence accepted, and in which on the strength of the acceptance of the axioms we arrive, by deductive inference, at accepting derived theorems, are called assertive deductive systems.[27]

Thus, a researcher who accepts the axioms of, say, geometry, treats them as asserted sentences (stating something about a certain reality) is cultivating the theory in an assertive-deductive style. In contrast, the same researcher can propose a certain geometry as an axiomatic theory, but at the same time not treat its axioms as asserted sentences, in which case he cultivates the theory in a hypothetical-deductive style. The researcher's attitude toward the theory does not affect in any way the content of the theory, that is, the set of axioms and theorems derived from them. A deductive theory is itself a creation independent of the researcher's attitude (hypothetical or assertive) toward its axioms (theorems).

In light of the above differences, two cases are worth considering. How will the nature of mathematical objects be understood in a situation in which a mathematical theory is in the third axiomatic-abstract stage of development and treated in one of two ways, hypothetical or assertive. Since the third stage refers to the full formalization of the theory, in which the primary terms have no fixed meaning, it seems particularly important to point out the difference between these two points of view.

A researcher taking a deductive theory in the assertive-deductive manner assumes axioms and thus states something about a certain reality. These statements have a special character since they do not refer to any outside reality, so far used as the meaning of primitive terms. Building a deductive theory for primitive terms in this way they "establish (their) meanings anew by deciding that the said terms are to denote such objects (i.e., individuals, classes, relations) which satisfy the axioms of a given theory, i.e., satisfy the conditions formulated in those axioms".[28] The consequence is planned and desired ambiguity of primitive terms which is manifested in their denotation not established once and for all. Finding a model for a deductive theory built in such a way is like another step in its construction in the sense that an unequivocal denotation for primitive terms and thus axioms become true sentences concerning

[27] Ibidem, 207.
[28] Ibidem, 203.

the indicated domain, naturally if the established designates exist at all. Hence, the model in reality always has advantage over the model in another deductive theory. And from among two proofs of non-contradiction of a deductive theory, the more credible is the one which establishes the meaning of primate terms in such a way that they denote the objects whose existence is doubtless.

A deductive theory in the third stage practiced in the assertive-deductive style could be understood as stating something very general about a certain reality common to a lot of fields. This fact is emphasized by an ambiguity of primitive terms. If we decided that this is the realization of the one-over-many scheme, which was clearly posed in Plato's philosophy, then – under certain conditions – the statement that the theories understood in this way describe universal mathematical structures would be true. Let us notice that assertion of axioms or its lack seems to be a researcher's personal question. It can be said that this is a certain kind of a personal style of taking science. Somebody who builds deductive theories in the assertive-deductive style can additionally have certain philosophical convictions on the nature of the general object described by a given theory. If we consider this object to be a structure, then the views of this type will be classified as ontological structuralism. It will be classified as positive when this general object is considered to be existing in this or another way. On the other hand, with the same assumptions, this structuralism can be regarded as negative when the ontological standpoint referring to the nature of this structure is eliminative in some way. It can also happen that while practicing science in this manner (assertive style) we will suspend the judgment on the nature of this general object, at the same time acknowledging that only after establishing denotations for primitive notions in an unequivocal way it is possible to determine the nature of the objects under discussion. In this way, the establishment of different denotations for primitive terms can lead us to different ontologies. Posing the problem in such a way can bring us closer to the concept of Quine's global structuralism, which – according to Resnik – "is non-ontological and simply another formulation of ontological relativity".[29]

Therefore, a formalized deductive theory (in the third stage) taken in the assertive-deductive style, where no definite ontological attitude is adopted referring to the general object (structure) which this theory de-

[29] Resnik, "Non-ontological Structuralism", 307.

scribes, can be called non-ontological structuralism. We believe that this is what Resnik[30] meant by using this name to define his views.

Note in the margin that calling Frege's concept assertive-deductive can prove to be misleading. This is done by Shapiro,[31] but it is a different meaning of the 'assertive-deductive' concept. It happens especially when assertions of axioms are considered a difference from Hilbert's algebraic standpoint. Besides, we said that Resnik classifies Hilbert's views as negative ontological structuralism.[32] It seems that in the light of the above presented distinctions, this classification is not obvious and its justification requires finding the answer to the question about which of the attitudes, the assertive or the hypothetical one, is adopted by Hilbert in relation to the axioms of deductive theories. These questions, as a historical issue, will be left unsettled in the present article.

5. Hypothetical structuralism

The above discussion leads to the conclusion that Ajdukiewicz's concept sheds some new light on Resnik's proposal. We can consider the view of non-ontological mathematical structuralism as similar to the concept of doing mathematics as a deductive science located in the third axiomatic abstract stage, in which the researcher takes an assertive-deductive manner. On the other hand, it is still possible to take a hypothetical-deductive position, while assuming that mathematics is a deductive science located in the third axiomatic abstract stage.

It takes place when the primitive terms of the theory which occur in its axioms are treated as symbols of variables, without determining their meanings. They are not even connected with any quantifiers; only their semantic category is established.[33] In this way, both axioms and theorems derived from the former cease to be the statements which by their nature can be true or false and they become schemata of statements which are neither true or false.

An example of the schemata with established semantic category of variables is the formula $1 + 3x = 7$. It does not state anything which could be rejected or accepted. The x symbol is treated here as having no mean-

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[30] Ibidem.

[31] Shapiro, "Categories, Structures, and the Frege-Hilbert Controversy: The Status of Meta-mathematics", 66.

[32] Resnik, "Non-ontological Structuralism".

[33] Ajdukiewicz, *Pragmatic Logic*, 205.

ing. We only know its semantic category and as a consequence we know how to perform the proper substitution in order to obtain a theorem which can be true or false. Similarly, the formula without the free variables can be the other kind of schemata. The one of the axioms of elementary theory of inequality is an example:[34]

$$\forall x: \neg(x < x).$$

In it are few logical symbols and one primitive term “<”. The formula is closed, but it is in specific way a schemata. The meaning of the primitive term is not fixed. This statement may be interpreted as a description of a fact from the rational number structure. In that way, the meaning of the symbol “<” is fixed as “is less than”. But at the same time we may propose other interpretation, wherein the meaning the symbol is fixed as “lies to the left of” and the statement is about the points on a given straight line. Another possible interpretation of the symbol is “is earlier than”; then, the axiom is true in the range of time moments.

> Since […] the axioms and the derived theorems of an abstract deductive theory are not statements, but schemata of statements, hence they may be neither accepted nor rejected. Hence, in this approach, an abstract deductive theory does not consist of anything that could express the conviction of the researcher who is concerned with that theory. In pursuing his research he does not assert anything. His work is confined to deriving by deduction schemata of statements, called derived theorems, from schemata of statements, called axioms; the derived theorems, being not statements, but schemata of statements, also do not state anything.[35]

In this way, axioms as schema are linked with theorems which are also schema, thus creating schema of inference.[36] What is more, these schemas of inference prove to be deductively valid, which means that the conclusion is true whenever all premises are true. A deductive theory understood in such a way becomes a catalogue of schemas of inference ready to be used in any science.

> As he does not assert anything, a researcher who is concerned with an abstract deductive theory (in the case of the approach in which the specific primitive terms of that theory are treated as variables) does not contribute –

[34] Ibidem, 203.
[35] Ibidem, 206.
[36] Ibidem, 110.

> in contrast to researchers engaged in the pursuit of other disciplines – to our knowledge of the real world.[37]

According to Ajdukiewicz,[38] studying the correctness of deductive inference is known to consist of studying two of its aspects, namely material correctness and formal correctness. In other words, an argument is formally correct if it is deductively valid. Moreover, the inference that is at the same time formally and materially correct is sound.[39]

Material correctness consists in studying the truth of premises, which principally belongs to an expert in a given field. Therefore, material correctness of premises in mathematical reasoning can be examined by a mathematician, who finds statements true or false. Each science has the criteria of acceptability of statements, if mathematics has that criteria (e.g. obviousness), then mathematician as an expert may state a truth of premises. But this task is beyond a formalized deductive theory taken in the hypothetical style above all because the assumption is that the accepted statements are not expected to be true.

Therefore, a catalogue of schemas of inference are only required to fulfill the condition of formal correctness of reasoning, i.e., deductive validity. Formal correctness of inference consists the conclusion in such reasoning logically following from the premises. Hence, while building deductive theories in the axiomatic and abstract way, where axioms are not asserted statements, mathematicians increase the catalogue of formally correct inferences. It can be said that they sometimes do hard work which is but service to other sciences.

> For if a researcher who is studying real facts succeeds in finding out that the facts he is concerned with satisfy the axioms of a given abstract deductive theory (i.e., if the sphere of those facts is a model of that theory), then owing to the work done earlier by the scientist who studied that abstract theory by deducing derived theorems from axioms, the student of facts can learn, without any extra effort on his part, that the domain he is concerned with also satisfies the derived theorems of that theory; he thus signally broadens his knowledge of the sphere of facts he is studying.[40]

In this way, mathematicians, studying deductive theories, create an ever-growing catalog of correct inference schemes, which are, by defi-

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[37] Ibidem, 206.

[38] Ibidem, 107–110.

[39] Daniel Bonevac, *Deduction: Introductory Symbolic Logic* (Oxford: Blackwell Publishing, 2003), 17–18.

[40] Ajdukiewicz, *Pragmatic Logic*, 206.

nition, prepared for specially understood interpretation. Thus, each deductive theory (cultivated in the abstract axiomatic stage in the hypothetical style) separately and all together form structures that describe the relations that occur between certain objects. These relations are indicated in no other way than by their description contained in the axioms and derivative theorems of a given theory. This underdetermination or programmatic ambiguity of the primitive terms of a given theory is reinforced by the hypothetical approach. The researcher, practicing deductive theory in a hypothetical way, does not assume anything about the existence and nature of the object of mathematics (structure), she does not even suspend judgment about it. Mathematical structure is understood here in a different way. The structure is formed by a set of axioms, accepted as hypotheses by the researcher, and derived theorems, which can be derived from these axioms by established rules of proof.

Such a hypothetical approach to deductive theories as theories in the third axiomatic abstract stage, we will call hypothetical structuralism. This view, although inspired by Resnik's position of non-ontological structuralism, differs fundamentally in its attitude toward the axioms (and derivative theorems) of mathematical theories. A proponent of Resnik's non-ontological structuralism builds deductive theories in an assertive-deductive style, while a proponent of hypothetical structuralism cultivates theory in a hypothetical-deductive manner.

The style of practicing a dedicatory theory does not affect its content, so, we can say that as far as the mathematical content is concerned, it is neutral. To further illustrate this distinction, let us consider the example of the elementary theory of inequality mentioned above. This theory has seven axioms, one of which is the mentioned expression:

$$\forall \mathrm{x}: \neg(\mathrm{x} < x).$$

The only primitive term of this theory is $<$, the meaning of which is established precisely in these seven axioms, since it is a deductive theory in the third axiom abstract stage. The other terms used in the axioms, are logical terms with an established meaning in logic. The theorems of this theory derived from the axioms describe a certain reality in the context of this one primitive term.

A proponent of Resnik's non-ontological structuralism adopts an assertive-deductive perspective, that is, she accepts the axioms and the theorems derived from them. She accepts that the above axiom says something about a certain reality, in this case a certain mathematical structure, which we can call the structure of inequality relations. On the other hand, she does not make a statement about how this structure exists and what its nature is. Assertion of axioms and theorems, therefore, amounts to saying that the theory has a certain object of study, and this object is independent of the theory. This object is a structure of 'relation of inequality'. According to this view, judgment on the manner of existence and nature of this structure is suspended.

The same will be true of any other mathematical theory. A follower of Resnik's views within the theory of arithmetic of natural numbers will answer that there is a structure of natural numbers, which is described by G. Peano's axioms, while how this structure exists and what is its nature is not determined. The same can apply to Euclid's axiomatic theory of geometry and, say, Lobachevski's geometry. Nothing prevents one from claiming the existence of two different structures described by these two competing theories while suspending judgment on how they exist. How these structures exist, their relationship to each other, etc., is irrelevant when one takes the position of non-ontological structuralism.

It will be different for the proponent of hypothetical structuralism. Since its attitude is hypothetical-deductive, the axioms (and theorems) in this case are not accepted sentences. The aforementioned axiom of the elementary theory of inequality describes a certain hypothesis from which the theorems of this theory can be derived. Thus, the entire theory is a set of hypotheses that have been arbitrarily accepted (as axioms) or derived from the former. Within this view, the question of the existence of the object of mathematics or its nature is not posed in any way. It is not the case that the answer to this question is suspended (we abstain from judgment). Here, a question of this kind is not posed, because programmatically the practice of mathematics involves something other than the description of the objects of mathematics (a certain structure).

In hypothetical structuralism, the researcher focuses on building a certain theory, based on arbitrary axioms. This theory is built by means of proof, according to predetermined strict rules of proof procedure. From the fact that the axioms are only hypotheses, the entire theory, and therefore all theorems, are also hypothetical. The axioms, along with the theorems, describe certain general properties that, by definition, demand to be interpreted. The aforementioned elementary theory of inequality describes the properties of the relation $<$, which we call an inequality relation.

The interpretation referred to here, i.e., establishing that the symbol $<$ is understood in one way or another (e.g., the expression '$x < y$' reads 'x is prior to y') is the next step in the construction of a mathematical theory. This step, from the perspective of hypothetical structuralism, is not necessary, but it enriches the research being done. Many researchers will say that theories for which interpretations have been found in reality are better than those for which no such interpretation exists. Still others will say that finding an interpretation for a mathematical theory in another mathematical theory (e.g., reading the expression '$x < y$' as 'the natural number x is smaller than the natural number y') is also a momentous discovery.

Hypothetical structuralism is called structuralism not because it describes certain structures as objects studied by mathematics, but because it treats the axioms and theorems of deductive theories as elements arranged in a certain structure. This structure is built by means of relations of deriving new theorems from those already derived (or accepted as hypotheses, i.e., axioms). Theorems as formulas form a hierarchical structure linked by mutual relations of derivation with other formulas, i.e., other theorems. The mathematician's work consists in proving theorems on the basis of accepted axioms (the same is what the mathematician's work consists in within Resnik's position of non-ontological structuralism). The mathematician does not ask about the existence or nature of the object under study, since this question and the possible answer to it are of little importance in this approach.

Conclusion

The starting point of our considerations was the concept of non-ontological structuralism proposed by Resnik. This view arose through a modification of *sui generis* structuralism, which Resnik formerly advocated. Both of these views maintain that the objects described by mathematical theories are structures. The difference between the two, on the other hand, is that *sui generis* structuralism accepts the existence and nature of these structures in some articulated way, while non-ontological structuralism suspends judgment on that.

The purpose of the article was to discuss the concept of hypothetical structuralism, which also does not relate in any way to the existence and nature of mathematical objects, but at the same time cannot be considered to coincide with Resnik's view of non-ontological structuralism. The difference between these positions became apparent on the ground of Ajdukiewicz's conception of the development of deductive science. Inspired by Hilbert's considerations, Ajdukiewicz proposed three stages in the development of deductive sciences. The last stage is the abstract axiomatic stage, which is similar to Hilbert's algebraic approach. We recognized that both non-ontological structuralism and hypothetical structuralism treat mathematics as a deductive theory located at this stage. In contrast, what distinguishes them is the way they approach the axioms of deductive theory. A proponent of Resnik's non-ontological structuralism practices mathematics in an assertive-deductive style. On the other hand, a proponent of hypothetical structuralism practices mathematics in a hypothetical-deductive style. These positions are discussed in detail.

Finally, in order to conclude, let us try to answer a certain question related to the elementary minority concept presented as an example. Are the axioms of this theory true sentences? The non-ontological structuralist will answer: yes, I accept that these sentences describe a certain mathematical structure. The hypothetical structuralist will answer: the answer to this question does not affect the content of the theory, while if they are true, it is better for the theory. Are structures the object of mathematics under study? The hypothetical structuralist will answer: I do not exclude this, especially from this reason that the primitive term ‘<’ is a relational term, so the whole theory is a theory of a certain relation between any objects. So how do these structures exist and what is their nature? The non-ontological structuralist will answer: I suspend my judgment on this subject, and the hypothetical structuralist will do the same.

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Summary

Michael Resnik suggests a new version of structuralism which he calls non-ontological structuralism. In the present short article, I discuss this view-point in the context of the Frege-Hilbert controversy about meaning of primitive notions in deductive theory, with special regard to the original views of Ajdukiewicz, Hilbert's student. Following the proposed differentiations, I introduce a new type of structuralism which I call hypothetical structuralism, close to Resnik's non-ontological structuralism.

Keywords: structuralism, hypothetical structuralism, non-ontological structuralism, Resnik, Ajdukiewicz, philosophy of mathematics