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Peirce's Triadic Logic and Its (Overlooked) Connexive Expansion

In memoriam Professor Vladimir V. Mironov (1953–2020)

Abstract. In this paper, we present two variants of Peirce's Triadic Logic within a language containing only conjunction, disjunction, and negation. The peculiarity of our systems is that conjunction and disjunction are interpreted by means of Peirce's mysterious binary operations Ψ and Φ from his 'Logical Notebook'. We show that semantic conditions that can be extracted from the definitions of Ψ and Φ agree (in some sense) with the traditional view on the semantic conditions of conjunction and disjunction. Thus, we support the conjecture that Peirce's special interest in these operations is due to the fact that he interpreted them as conjunction and disjunction, respectively. We also show that one of our systems may serve as a suitable base for an interesting implicative expansion, namely the connexive three-valued logic by Cooper. Sound and complete natural deduction calculi are presented for all systems examined in this paper.

Keywords: Peirce; Triadic Logic; conjunction; disjunction; connexive logic; natural deduction; generalized truth values

1. Introduction

In 1964 Fisch and Turquette published the paper [11] in which they suggested an analysis of a three-page manuscript by Peirce from his 'Logical Notebook'. Alongside their analysis the authors published the manuscript itself. This publication is important from a historical perspective, because it permits the claim that Peirce had been developing a

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three-valued logic long before Łukasiewicz's publication on the same issues [14, 15]. On one page of the manuscript (which is dated 23 February 1909) Peirce suggested a sketch of what he called a 'Triadic Logic':

> Triadic logic is that logic which, though not rejecting entirely the Principle of Excluded Middle, nevertheless recognizes that every proposition, S is P, is either true, or false, or else S has a lower mode of being such that it can neither be determinately P, nor determinately not-P, but is at the limit between P and not-P. [11]

Unfortunately, the manuscript gives no clue to what definition of consequence relation (or validity) Peirce had in mind, hence we cannot confidently claim that Peirce developed any completed logical theory. However, despite this fact, he explicitly postulated three truth values: $\mathbb{V}$ – «verum» (truth), $\mathbb{F}$ – «falsum» (falsity), $\mathbb{L}$ – «the limit». Moreover, he was concerned with different operations which are explicitly defined in terms of truth tables over the set of these truth values. Altogether Peirce mentioned six[1] binary operations: Φ, Ψ, Θ, Z, Υ and Ω. It is worth noting that four of these operations were (re)discovered much later, and became very widespread in the literature. Operations Θ and Z are, in fact, disjunction and conjunction in the strong Kleene logic $\mathbf{K_3}$, Lukasiewicz logic $\mathbf{Ł_3}$, Priest's logic of paradox **LP**, Post's logic $\mathbf{P_3}$, Dunn-McCall logic $\mathbf{RM_3}$, etc. In turn, Ω and Υ are nothing other than conjunction and disjunction in the week Kleene logics ($\mathbf{K_3^w}$ and **PWK**) and other related logics which can be grouped as infectious ones. However, the remaining Φ and Ψ operations did not receive comparable attention in the literature on many-valued logics. At the same time, exploring the manuscript, we can observe that Peirce paid a lot of attention to these very operations. Nevertheless, as rightfully claimed by Parks [24] in his reply to Fisch and Turquette, we can find at least three little known occurrences of operations, which coincide with Φ and Ψ, in the works of Sobocinski [27][2], Cooper [9] and Belnap [5]. It is worth noting that in these works the corresponding operations are used to interpret conjunction and disjunction. Moreover, note that Ebbinghaus's logic $\mathbf{E_3}$ of nonsense [10] contains Φ as an interpretation of disjunction. In [12] Finn and Grigolia studied some generalizations of the logics of significance,

[1] In this paper we are primarily interested in binary operations, but we note that Peirce also mentioned four unary operations.

[2] By the way, Sobocinski's paper was published long before [11].

which contain Ψ as an interpretation of conjunctions. Not one of these works referred to Peirce's manuscript.

Some attempts to explain the nature of Peirce's interest in Φ and Ψ have been made in the literature. For example, Turquette in [31] and [32] proposed that Peirce was motivated by issues regarding functional completeness. It was shown in [31] that combining either Φ or Ψ together with one of Peirce's unary operations $'$ or $`$ (whose matrix definitions coincide with Post's cyclic negation and its dual) forms a functionally complete set. Though we find this version interesting we do not find it fully justified. If we take a look at the manuscript, it is easy to observe that it contains no calculations showing that Peirce tried to express some operations by means of the others. The most part of the text consists of calculations by which Peirce probably tested Φ and Ψ for the properties of associativity and commutativity. The only thing that alludes to the link between Peirce's operations and functional completeness is the occurrence of $'$ and $`$, which, as we said above, coincide with Post's negation and its dual. Of course, the occurrence of the latter operations can hardly be explained only by claiming that Peirce treated them as candidates for negation because their definitions do not reflect any customary intuitions about this notion. But we find the shift from such a coincidence to the claim that Peirce was seeking for a functionally complete set of operations much harder to explain.[3]

In this paper, we try to answer the following question: is it possible to claim that Peirce considered Φ and Ψ as candidates for disjunction and conjunction, respectively? Answering this question should help us to understand why these operations occupied him so much. As a result of such inquiry, we will provide a positive answer. Technically, we present two versions of his Triadic Logic (one is paracomplete, the other is paraconsistent), which can be classified as systems belonging to the family of first-degree entailment logics; that is, those which are built over a language containing only disjunction, conjunction, and negation. We find this strategy quite reasonable, because, as is clear from Peirce's manuscript, he did not deal with any implication-like operations. In both logics conjunction and disjunction are interpreted by means of Peirce's operations Ψ and Φ, respectively. Besides matrix semantics, we present an equivalent one in terms of generalized truth values, which allows us to work with customary semantic conditions in terms of the traditional

[3] Some new and interesting research into Φ and Ψ was proposed recently in [18].

categories of truth and falsity. The resulting truth and falsity conditions for disjunction and conjunction, respectively, coincide with the traditional ones. In turn, the resulting truth condition for conjunction and the falsity condition for disjunction are shown to be consistent with the traditional ones. We provide a natural deduction formalization for both logics and prove their completeness and soundness. Finally, we show that Cooper's logic from [9] can be seen as a connexive extension of our paraconsistent variant of the Triadic Logic. We also equip it with a sound and complete natural deduction calculus and briefly discuss its relationships with other connexive logics.

2. Two Versions of Triadic Logic

2.1. Semantics

We fix a propositional language $\mathcal{L}$ containing conjunction, disjunction, and negation. The notion of a formula is standard. The set of all propositional variables of $\mathcal{L}$ is denoted as $\mathcal{P}$, the set of all its formulae as $\mathcal{F}$. By a *logic* we mean a pair $\langle \mathcal{L}, \vDash \rangle$, where $\mathcal{L}$ is the propositional language defined above and $\vDash$ is a semantic consequence relation induced by some semantic structure. A logic can be defined from the proof-theoretical point of view if we replace $\vDash$ with its syntactical counterpart $\vdash$. According to Definition 3 below, we define $\mathbf{TL_1} = \langle \mathcal{L}, \vDash_{\mathcal{M}_1} \rangle$ and $\mathbf{TL_2} = \langle \mathcal{L}, \vDash_{\mathcal{M}_2} \rangle$. The abbreviation **TL** stands for «**T**riadic **L**ogic».

Definition 1. A $\mathbf{TL_1}$-*matrix* for $\mathcal{L}$ is a tuple $\mathcal{M}_1 = \langle \mathcal{V}_{\mathbf{TL_1}}, \mathcal{D}_{\mathbf{TL_1}}, \mathcal{O}_{\mathbf{TL_1}} \rangle$, where: (a) $\mathcal{V}_{\mathbf{TL_1}} = \{\mathbb{V}, \mathbb{L}, \mathbb{F}\}$, (b) $\mathcal{D}_{\mathbf{TL_1}} = \{\mathbb{V}\}$, (c) for every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}_{\mathbf{TL_1}}$ contains a corresponding n-ary function $f_\diamond$: $\mathcal{V}^n_{\mathbf{TL_1}} \to \mathcal{V}_{\mathbf{TL_1}}$. The functions included in $\mathcal{O}_{\mathbf{TL_1}}$ are defined by means of the following tables:

$f_\neg$	φ
$\mathbb{F}$	$\mathbb{V}$
$\mathbb{L}$	$\mathbb{L}$
$\mathbb{V}$	$\mathbb{F}$

$f_\wedge$	$\mathbb{V}$	$\mathbb{L}$	$\mathbb{F}$
$\mathbb{V}$	$\mathbb{V}$	$\mathbb{V}$	$\mathbb{F}$
$\mathbb{L}$	$\mathbb{V}$	$\mathbb{L}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$

$f_\vee$	$\mathbb{V}$	$\mathbb{L}$	$\mathbb{F}$
$\mathbb{V}$	$\mathbb{V}$	$\mathbb{V}$	$\mathbb{V}$
$\mathbb{L}$	$\mathbb{V}$	$\mathbb{L}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{V}$	$\mathbb{F}$	$\mathbb{F}$

A $\mathbf{TL_1}$-*valuation* in a $\mathbf{TL_1}$-matrix $\mathcal{M}_1$ is a function $v\colon \mathcal{F} \to \mathcal{V}_{\mathbf{TL_1}}$ that satisfies the following condition for every n-ary connective $\diamond$ of $\mathcal{L}$ and $\varphi_1, \ldots, \varphi_n \in \mathcal{F}$:

$$v(\diamond(\varphi_1, \ldots, \varphi_n)) = f_\diamond(v(\varphi_1), \ldots, v(\varphi_n)).$$

Definition 2. A $\mathbf{TL_2}$-*matrix* for $\mathcal{L}$ is a tuple $\mathcal{M}_\mathbf{2} = \langle \mathcal{V}_{\mathbf{TL_2}}, \mathcal{A}_{\mathbf{TL_2}}, \mathcal{O}_{\mathbf{TL_2}} \rangle$, where: (a) $\mathcal{V}_{\mathbf{TL_2}} = \{\mathbb{V}, \mathbb{L}, \mathbb{F}\}$, (b) $\mathcal{A}_{\mathbf{TL_2}} = \{\mathbb{F}\}$, (c) for every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}_{\mathbf{TL_2}}$ contains a corresponding n-ary function $f_\diamond$: $\mathcal{V}^n_{\mathbf{TL_2}} \to \mathcal{V}_{\mathbf{TL_2}}$, and $\mathcal{O}_{\mathbf{TL_2}} = \mathcal{O}_{\mathbf{TL_1}}$. A $\mathbf{TL_2}$-*valuation* in a $\mathbf{TL_2}$-matrix $\mathcal{M}_\mathbf{2}$ is a function $v : \mathcal{F} \to \mathcal{V}_{\mathbf{TL_2}}$ that satisfies the following condition for every n-ary connective $\diamond$ of $\mathcal{L}$ and $\varphi_1, \ldots, \varphi_n \in \mathcal{F}$:

$$v(\diamond(\varphi_1, \ldots, \varphi_n)) = f_\diamond(v(\varphi_1), \ldots, v(\varphi_n)).$$

Remark 1. As is clear from the definitions above, $f_\wedge$, $f_\vee$, and $f_\neg$ are Peirce's Ψ, Φ and $\bar{.}$ operations, respectively.

Remark 2. $\mathcal{D}_{\mathbf{TL_1}}$ and $\mathcal{A}_{\mathbf{TL_2}}$ are the sets of *designated* and *anti-designated* values, respectively. We employ the notion of anti-designated values for the following reason. Based on the manuscript only, we cannot extract a definition of a consequence relation that is appropriate for Peirce. Thus, the introduction of anti-designated values on a par with the designated ones allows us to work with two standard (for the modern many-valued logic) ways of defining a consequence relation. Indeed, it is easy to observe that $\mathcal{M}_1$ contains the sole value $\mathbb{V}$ as designated, thereby leading to a truth-preserving definition of the consequence relation. In turn, $\mathcal{M}_\mathbf{2}$ contains $\mathbb{F}$ as an anti-designated value, thereby allowing us to define the consequence relation via non-falsity preservation. Note, however, that the latter strategy can be objected to. One may argue that we can replace this definition simply by taking both $\{\mathbb{V}, \mathbb{L}\}$ as designated. But, as far as we can see, this contradicts Peirce's idea that $\mathbb{L}$ represents something not determinately true. Hence, we feel a bit of tension in treating $\mathbb{L}$ as designated. Nevertheless, it is worth observing that Peirce had clear views on the meaning of the consequence relation in general elsewhere. For a detailed discussion see [16].

Definition 3. For any $\Gamma \cup \{\varphi\} \subseteq \mathcal{F}$:

(C1) $\Gamma \vDash_{\mathcal{M}_1} \varphi \iff$ for every $\mathbf{TL_1}$-valuation v: if $v(\gamma) \in \mathcal{D}_{\mathbf{TL_1}}$ for every $\gamma \in \Gamma$, then $v(\varphi) \in \mathcal{D}_{\mathbf{TL_1}}$.

(C2) $\Gamma \vDash_{\mathcal{M}_\mathbf{2}} \varphi \iff$ for every $\mathbf{TL_2}$-valuation v: if $v(\gamma) \notin \mathcal{A}_{\mathbf{TL_2}}$ for every $\gamma \in \Gamma$, then $v(\varphi) \notin \mathcal{A}_{\mathbf{TL_2}}$.

Remark 3. In fact, $\mathbf{TL_2}$ is nothing other than the first-degree fragment of Sobocinski's logic from [27]. Recall that Sobocinski used $f_\neg$ and $f^*_\to$ (presented below) as initial operations.

$f^*_{\rightarrow}$	$\mathbb{V}$	$\mathbb{L}$	$\mathbb{F}$
$\mathbb{V}$	$\mathbb{V}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{L}$	$\mathbb{V}$	$\mathbb{L}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{V}$	$\mathbb{V}$	$\mathbb{V}$

However, $f^*_{\rightarrow}$ can be defined as $f^*_{\rightarrow}(\varphi, \psi) = f_{\vee}(f_{\neg}(\varphi), \psi)$. The converse also holds, i.e., $f_{\vee}$ and $f_{\wedge}$ are both definable by the use of $f^*_{\rightarrow}$ and $f^{\neg}$. Indeed, $f_{\vee}(\varphi, \psi) = f^*_{\rightarrow}(f_{\neg}(\varphi), \psi)$ and $f_{\wedge}(\varphi, \psi) = f_{\neg}(f^*_{\rightarrow}(f_{\neg}(f_{\neg}(\varphi)), f_{\neg}(\psi)))$. The operation $f^*_{\rightarrow}$ is well-known as the implication of Dunn-McCall logic $\mathbf{RM_3}$ (see [1]).

First of all, it is clear that $\mathbf{TL_1}$ and $\mathbf{TL_2}$ are paracomplete and paraconsistent, respectively. It is not difficult to show, using Definition 3, that

- $\varphi \wedge {\sim}\varphi \vDash_{\mathcal{M}_1} \psi$,
- $\psi \nvDash_{\mathcal{M}_1} \varphi \vee {\sim}\varphi$,
- $\varphi \wedge {\sim}\varphi \nvDash_{\mathcal{M}_2} \psi$,
- $\psi \vDash_{\mathcal{M}_2} \varphi \vee {\sim}\varphi$.

Now we turn to the more interesting properties of $\mathbf{TL_1}$ and $\mathbf{TL_2}$. Notice that Simplification fails in $\mathbf{TL_1}$ and Addition fails in $\mathbf{TL_2}$. Again, this may be easily shown, using Definition 3:

- $\varphi \wedge \psi \nvDash_{\mathcal{M}_1} \varphi$,
- $\varphi \wedge \psi \nvDash_{\mathcal{M}_1} \psi$,
- $\varphi \nvDash_{\mathcal{M}_2} \varphi \vee \psi$,
- $\psi \nvDash_{\mathcal{M}_2} \varphi \vee \psi$.

This feature relates $\mathbf{TL_1}$ and $\mathbf{TL_2}$ with the family of the weak Kleene logics and the infectious ones in general [see, e.g., 4, 7, 22, 28]. We know that Addition fails in $\mathbf{K_3^w}$, whereas Simplification fails in **PWK**. Notice that $\mathbf{K_3^w}$ is paracomplete. Generally, for any paracomplete infectious logic (in some precise sense of the term) it holds that Addition fails. In turn, for any paraconsistent infectious logic, it holds that Simplification fails. In the case of $\mathbf{TL_1}$ and $\mathbf{TL_2}$ the situation is the mirror image: the paracomplete $\mathbf{TL_1}$ does not validate Simplification, whereas the paraconsistent $\mathbf{TL_2}$ does not validate Addition. Interestingly, the failure of these logical principles within infectious logics is motivated by philosophical intuitions about the nature of the consequence relation and the epistemic status of truth value gaps and gluts. Hence, the question about the existence of a rational connection between the absence of Simplification and Addition within suggested versions of 'Triadic

Logic' and the philosophical interpretation of Peirce's intermediate truth value $\mathbb{L}$ is of great interest.

The problems with Addition and Simplification, obviously, lead to the problems with distributivity between conjunction and disjunction. In this respect $\mathbf{TL_1}$ and $\mathbf{TL_2}$ are balanced. As a consequence, in both logics, the same set of distribution laws fails. It is not difficult to show, using Definition 3, that

- $(\varphi \wedge \psi) \vee (\varphi \wedge \chi) \nvDash_x \varphi \wedge (\psi \vee \chi)$,
- $\varphi \wedge (\psi \vee \chi) \vDash_x (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$,
- $(\varphi \vee \psi) \wedge (\varphi \vee \chi) \vDash_x \varphi \vee (\psi \wedge \chi)$,
- $\varphi \vee (\psi \wedge \chi) \nvDash_x (\varphi \vee \psi) \wedge (\varphi \vee \chi)$,
- $(\varphi \vee \psi) \wedge \chi \nvDash_x \varphi \vee (\psi \wedge \chi)$,

where $x \in \{\mathcal{M}_1, \mathcal{M}_\mathbf{2}\}$.

In order to address our main question, it is instructive to formulate the semantics of $\mathbf{TL_1}$ and $\mathbf{TL_2}$ in terms of generalized truth values.

Remark 4. Notice that Peirce's truth values can be rewritten as elements of $\{T, F, \varnothing\}$, thereby transforming truth tables of negation, conjunction, and disjunction into the following form.

$f_\neg$	φ
$\{F\}$	$\{T\}$
$\varnothing$	$\varnothing$
$\{T\}$	$\{F\}$

$f_\wedge$	$\{T\}$	$\varnothing$	$\{F\}$
$\{T\}$	$\{T\}$	$\{T\}$	$\{F\}$
$\varnothing$	$\{T\}$	$\varnothing$	$\{F\}$
$\{F\}$	$\{F\}$	$\{F\}$	$\{F\}$

$f_\vee$	$\{T\}$	$\varnothing$	$\{F\}$
$\{T\}$	$\{T\}$	$\{T\}$	$\{T\}$
$\varnothing$	$\{T\}$	$\varnothing$	$\{F\}$
$\{F\}$	$\{T\}$	$\{F\}$	$\{F\}$

On that understanding, it is easy to extract semantic conditions for $\wedge$, $\vee$ and $\neg$ in terms of truth and falsity.

Definition 4. Let $\mathfrak{M} = \langle\{T, F, \varnothing\}, \xi\rangle$ be a *generalized truth values model*, where $\{T, F, \varnothing\}$ is the set of generalized truth values[4], and ξ is a *valuation* mapping $\mathcal{P}$ into $\{T, F, \varnothing\}$. Thus, ξ can be extended to $\mathcal{F}$ by means of the following semantic conditions:

(S1) $T \in \xi({\sim}\varphi) \Leftrightarrow F \in \xi(\varphi)$,
(S2) $F \in \xi({\sim}\varphi) \Leftrightarrow T \in \xi(\varphi)$,

[4] This set of values can be seen as the set of all proper subsets of $\{T, F\}$.

(S3) $T \in \xi(\varphi \wedge \psi) \Leftrightarrow [T \in \xi(\varphi)$ and $F \notin \xi(\psi)]$ or $[T \in \xi(\psi)$ and $F \notin \xi(\varphi)]$,
(S4) $F \in \xi(\varphi \wedge \psi) \Leftrightarrow F \in \xi(\varphi)$ or $F \in \xi(\psi)$,
(S5) $F \in \xi(\varphi \vee \psi) \Leftrightarrow [F \in \xi(\varphi)$ and $T \notin \xi(\psi)]$ or $[F \in \xi(\psi)$ and $T \notin \xi(\varphi)]$,
(S6) $T \in \xi(\varphi \vee \psi) \Leftrightarrow T \in \xi(\varphi)$ or $T \in \xi(\psi)$.

Using standard induction on the complexity of a formula φ, we can prove the next lemma.

Lemma 1. *Let ξ be a valuation. Then for any $\varphi \in \mathcal{F}$ it holds that:*

(a) $F \notin \xi(\varphi) \Rightarrow T \in \xi(\varphi)$ *or* $T \notin \xi(\varphi)$,
(b) $T \notin \xi(\varphi) \Rightarrow F \in \xi(\varphi)$ *or* $F \notin \xi(\varphi)$.

Proof. Straightforward. As an example we consider only one case. Let $\varphi = \neg\pi$ and $F \notin \xi(\neg\pi)$. Then, by (S2), we obtain $T \notin \xi(\pi)$. Applying the inductive hypothesis we have $F \in \xi(\pi)$ or $F \notin \xi(\pi)$, from which, using (S1), we obtain the desired $T \in \xi(\neg\pi)$ or $T \notin \xi(\neg\pi)$. Suppose that $T \notin \xi(\neg\pi)$. Then, by (S1), we have $F \notin \xi(\pi)$. Applying the inductive hypothesis we have $T \in \xi(\pi)$ or $T \notin \xi(\pi)$, and, using (S2), we obtain the desired $F \in \xi(\neg\pi)$ or $F \notin \xi(\neg\pi)$. □

As to the consequence relations, they can be defined, as in the case of matrix semantics, in two ways: either through the preservation of truth or the preservation of non-falsity.

Definition 5. For any $\Gamma \cup \{\varphi\} \subseteq \mathcal{F}$:

(C3) $\Gamma \vDash_a \varphi \Leftrightarrow$ for every ξ: if $T \in \xi(\gamma)$ for every $\gamma \in \Gamma$, then $T \in \xi(\varphi)$.
(C4) $\Gamma \vDash_b \varphi \Leftrightarrow$ for every ξ: if $F \notin \xi(\gamma)$ for every $\gamma \in \Gamma$, then $F \notin \xi(\varphi)$.

In order to prove the equivalence between the matrix and generalized truth values semantics of $\mathbf{TL_1}$ and $\mathbf{TL_2}$, we first need to prove the next lemma.

Lemma 2. *Let v be a $\mathbf{TL_1}$-valuation ($\mathbf{TL_2}$-valuation), ξ be a valuation in the generalized truth values model. Then for any $\varphi \in \mathcal{F}$:*

$$(a)\ v(\varphi) = \mathbb{V} \ \Leftrightarrow\ T \in \xi(\varphi) \text{ and } F \notin \xi(\varphi),$$
$$(b)\ v(\varphi) = \mathbb{L} \ \Leftrightarrow\ T \notin \xi(\varphi) \text{ and } F \notin \xi(\varphi),$$
$$(c)\ v(\varphi) = \mathbb{F} \ \Leftrightarrow\ T \notin \xi(\varphi) \text{ and } F \in \xi(\varphi).$$

Proof. By induction on the complexity of φ, using Definition 4, Definition 1, and Definition 2. □

Then we obtain the following theorem.

Theorem 1. *For any* $\Gamma \cup \{\varphi\} \subseteq \mathcal{F}$*:*

$$\Gamma \models_{\mathcal{M}_1} \varphi \;\Leftrightarrow\; \Gamma \models_a \varphi \quad \text{and} \quad \Gamma \models_{\mathcal{M}_2} \varphi \;\Leftrightarrow\; \Gamma \models_b \varphi.$$

Proof. Using Lemma 2, Definition 3, and Definition 5. □

Thus, we have shown the equivalence between the two types of semantics. Henceforth, we will use $\Gamma \models_{\mathbf{TL_1}} \varphi$ to denote that φ is a logical consequence of the set of formulae Γ in $\mathbf{TL_1}$ with respect to Definition 5, and analogously in the case of $\Gamma \models_{\mathbf{TL_2}} \varphi$.

2.2. Conjunction? Disjunction?

As we can see from Definition 4, $\wedge$ is characterized by the falsity condition (S4), which coincides with the traditional interpretation of conjunction, while $\vee$ is characterized by the truth condition (S6), which is, in turn, standard for disjunction. Curiously, the truth of $\wedge$ (S3) and falsity of $\vee$ (S5) are defined in a rather unusual way. The meaning of $\wedge$ and $\vee$ is halfway between the the meaning of conjunction and disjunction, respectively. As to the traditional truth condition for conjunction and the falsity condition for disjunction, they have the following form:

(S3*) $T \in v(\varphi \wedge \psi) \;\Leftrightarrow\; T \in v(\varphi)$ and $T \in v(\psi)$,
(S5*) $F \in v(\varphi \vee \psi) \;\Leftrightarrow\; F \in v(\varphi)$ and $F \in v(\psi)$.

In the case of $\mathbf{TL_1}$ and $\mathbf{TL_2}$, these conditions are insufficient. Let us take a closer look at $\wedge$. The standard truth condition tells us that the truth of conjunction is determined by the simultaneous truth of its conjuncts. However, if we compare this interpretation with Peirce's tables and semantic conditions corresponding to them, it becomes clear that we are dealing with a more general condition. In other words, $\wedge$ is true, only if *one of its constituents is true, and the second is not false.* In principle, this interpretation agrees with the traditional view of the truth-condition for conjunction.

In order to present (S3) and (S5) in a more compact form, let us introduce a special notation. Let $\varphi, \psi \in \mathcal{F}$ and $X, Y \in \{\varphi, \psi\}$, where $X \neq Y$. Then (S3)* can be rewritten as:

$$T \in \xi(\varphi \wedge \psi) \;\Leftrightarrow\; T \in \xi(X) \text{ and } T \in \xi(Y).$$

Obviously, depending on the relationships between truth and falsity in a given semantics, this condition may be transformed. For instance, if we are working with a classical logic, then, in virtue of the equivalences $T \in \xi(\varphi) \Leftrightarrow F \notin \xi(\varphi)$ and $F \in \xi(\varphi) \Leftrightarrow T \notin \xi(\varphi)$, we obtain:

$$\begin{aligned} T \in \xi(\varphi \wedge \psi) \;\Leftrightarrow\; & T \in \xi(X) \text{ and } T \in \xi(Y) && \text{(s1)} \\ & \text{or} \\ & T \in \xi(X) \text{ and } F \notin \xi(Y) && \text{(s2)} \\ & \text{or} \\ & F \notin \xi(X) \text{ and } F \notin \xi(Y) && \text{(s3).} \end{aligned}$$

In the case of $\mathbf{TL_1}$ and $\mathbf{TL_2}$, (S3) can be rewritten as:

$$T \in \xi(\varphi \wedge \psi) \;\Leftrightarrow\; T \in \xi(X) \text{ and } F \notin \xi(Y).$$

We can see that this condition coincides with (s2). Hence, we can conclude that $\wedge$ is characterized by the classical truth condition for conjunction. Moreover, taking into account (S4), we can assert that $\wedge$ is conjunction in the traditional sense. Nevertheless, $\wedge$ is more than just a conjunction, because, despite the fact that (s2) allows us to derive (s1), it also allows us to derive something more:

$$\begin{aligned} T \in \xi(\varphi \wedge \psi) \Longleftrightarrow\; & T \in \xi(X) \text{ and } F \notin \xi(Y) \\ \Longrightarrow\; & T \in \xi(X) \text{ and } T \in \xi(Y) \\ & \text{or} \\ & T \in \xi(X) \text{ and } T \notin \xi(Y) \qquad \text{(Lemma 1).} \end{aligned}$$

The right disjunct of the last statement is, of course, unacceptable as a truth condition not only for classical conjunction but also for many non-classical ones, even including conjunctions from the infectious logics. However, two remarks should be made. Firstly, this condition is *one of many* possible options, which is supplied in addition to the classical one. Secondly, non-truth in the Triadic logic does not necessarily mean falsity; the corresponding sentence can be neither determinately truth nor determinately false, that is at the limit between truth and falsity, as Peirce would have said. Thus, depending on which truth values are treated as designated (or not treated as anti-designated) this condition can conform with the traditional view to a greater or lesser extent. Nevertheless, this condition challenges those who tend to say that Peirce perceived Ψ to have a conjunctive nature. It would be instructive to address this issue in the context of Peirce's philosophy, but we leave it for future work.

An analogous argument can be established with respect to the falsity condition for $\vee$. As a result, we obtain that $\vee$ is false, only if *one of its constituents is false, and the second if not true*. This interpretation also rather agrees with the traditional view on the falsity condition for disjunction.

(S5*) can be rewritten as

$$F \in \xi(\varphi \vee \psi) \;\Leftrightarrow\; F \in \xi(X) \text{ and } F \in \xi(Y).$$

Again, if we work within the classical logic, using $T \in \xi(\varphi) \Leftrightarrow F \notin \xi(\varphi)$ and $F \in \xi(\varphi) \Leftrightarrow T \notin \xi(\varphi)$, we obtain:

$$\begin{aligned} F \in \xi(\varphi \vee \psi) \Leftrightarrow\ & F \in \xi(X) \text{ and } F \in \xi(Y) && \text{(s1*)} \\ & \text{or} \\ & F \in \xi(X) \text{ and } T \notin \xi(Y) && \text{(s2*)} \\ & \text{or} \\ & T \notin \xi(X) \text{ and } T \notin \xi(Y) && \text{(s3*).} \end{aligned}$$

If we shift to $\mathbf{TL_1}$ and $\mathbf{TL_2}$, then (S5) is transformed into:

$$F \in \xi(\varphi \vee \psi) \Leftrightarrow\; F \in \xi(X) \text{ and } T \notin \xi(Y).$$

We can see that this condition coincides with (s2*). From this, we can conclude that $\vee$ is characterized by the classical falsity condition for disjunction. Taking into account (S6) we can say that $\vee$ can be rightfully treated as a disjunction. But again, $\vee$ is more than just a disjunction. Besides (s1*) and (s2*) it allows to derive something more:

$$\begin{aligned} F \in \xi(\varphi \vee \psi) &\Leftrightarrow F \in \xi(X) \text{ and } T \notin \xi(Y) \\ &\Rightarrow F \in \xi(X) \text{ and } F \in \xi(Y) \\ &\quad\ \text{or} \\ &\quad\ F \in \xi(X) \text{ and } F \notin \xi(Y) && \text{(Lemma 1).} \end{aligned}$$

Analogously to the case of $\wedge$, here we can see that besides the standard falsity condition we obtain a non-standard one: $F \in \xi(X)$ and $F \notin \xi(Y)$. But in contrast to the case of the non-standard truth condition for $\wedge$, this condition seems to be not so extraordinary. Indeed, in virtue of the fact that in Triadic logic non-falsity is not equivalent to truth, we can rightfully (from the traditional point of view) assert the falsity of a disjunction when one of its disjuncts is false, and another is not false. This echoes the 'infectious' interpretation of truth value gaps and their role in the epistemic status of disjunctive sentences (see [3, 22, 29, 30] for the discussion on that topic).

3. Formalization

3.1. Natural deduction systems

In this section, we present natural deduction systems for $\mathbf{TL_1}$ and $\mathbf{TL_2}$. We denote them as $\mathcal{N}_{\mathbf{TL_1}}$ and $\mathcal{N}_{\mathbf{TL_2}}$, respectively.

Below is the list of the inference rules of $\mathcal{N}_{\mathbf{TL_1}}$. The double line in (R8)–(R10) says that the rules work in both directions.

$$\text{(R1)}\ \frac{\varphi, \psi}{\varphi \wedge \psi} \qquad \text{(R2)}\ \frac{\varphi \wedge \psi}{\varphi \vee \psi} \qquad \text{(R3)}\ \frac{\varphi \wedge \psi}{\psi \wedge \varphi} \qquad \text{(R4)}\ \frac{\varphi}{\varphi \wedge (\psi \vee {\sim}\psi)}$$

$$\text{(R5)}\ \frac{(\varphi \vee \psi) \wedge \chi}{\varphi \vee (\psi \wedge \chi)} \qquad \text{(R6)}\ \frac{\varphi}{\varphi \vee \psi} \qquad \text{(R7)}\ \frac{\psi}{\varphi \vee \psi} \qquad \text{(R8)}\ \frac{\neg\neg\varphi}{\varphi}$$

$$\text{(R9)}\ \frac{\neg(\varphi \wedge \psi)}{\neg\varphi \vee \neg\psi} \qquad \text{(R10)}\ \frac{\neg(\varphi \vee \psi)}{\neg\varphi \wedge \neg\psi} \qquad \text{(R11)}\ \frac{\varphi, \neg\varphi}{\psi}$$

$$\text{(R12)}\ \frac{\neg(\varphi \vee \psi)}{\neg\varphi \vee \neg\psi} \qquad \text{(R13)}\ \frac{\varphi \vee \psi, \quad \begin{matrix}[\varphi] \\ \chi\end{matrix}, \quad \begin{matrix}[\psi] \\ \chi\end{matrix}}{\chi}$$

In order to obtain $\mathcal{N}_{\mathbf{TL_2}}$, it is sufficient to replace (R3), (R4), (R6), (R7), (R11) and (R12) with:

$$\text{(R14)}\ \frac{\varphi \vee \psi}{\psi \vee \varphi} \qquad \text{(R15)}\ \frac{\varphi \wedge \psi}{\varphi} \qquad \text{(R16)}\ \frac{\varphi \wedge \psi}{\psi}$$

$$\text{(R17)}\ \frac{\varphi \vee (\psi \wedge \neg\psi)}{\varphi} \qquad \text{(R18)}\ \frac{\neg\varphi \wedge \neg\psi}{\neg(\varphi \wedge \psi)} \qquad \text{(R19)}\ \frac{}{\varphi \vee \neg\varphi}$$

The notion of a proof (tree-shaped) is defined as usual for both systems. Given a formula π, $[\pi]$ denotes an assumption that should be discharged after te application of the corresponding rule. It is worth noting that

(R17) is derivable in $\mathcal{N}_{\mathbf{TL_1}}$, while its dual counterpart (R4) is derivable in $\mathcal{N}_{\mathbf{TL_2}}$, as the following proof schemata show:

$$\dfrac{\varphi \vee (\psi \wedge \neg\psi) \qquad [\varphi] \qquad \dfrac{[\psi \wedge \neg\psi]}{\varphi}\,(\text{R11})}{\varphi}\,(\text{R13})$$

$$\dfrac{\varphi \qquad \dfrac{}{\psi \vee \neg\psi}\,(\text{R19})}{\varphi \wedge (\psi \vee \neg\psi)}\,(\text{R1})$$

It is worth observing that $\mathcal{N}_{\mathbf{TL_1}}$ and $\mathcal{N}_{\mathbf{TL_2}}$ have something in common with the natural deduction systems of Ebbinghaus and Hałkowska logics $\mathbf{E_3}$ and $\mathbf{Z}$ [see 25]. For example, the rule (R4) seems to be a variation of the rules $(\wedge I')$ and $(\wedge I'')$ from [25].

3.2. Completeness and soundness

We use Henkin-style technique to prove the completeness of $\mathcal{N}_{\mathbf{TL_1}}$ and $\mathcal{N}_{\mathbf{TL_2}}$. Let us lay down some auxiliary notions.

DEFINITION 6. Let $\Gamma \subseteq \mathcal{F}$, and L be a logic. Then the set of formulae Γ is a *theory*, only if it satisfies the following condition: if $\Gamma \vdash_L \varphi$, then $\varphi \in \Gamma$. Obviously, if $\vdash_L$ is a Tarskian-type consequence relation, then, in virtue of its reflexivity, the converse also holds: if $\varphi \in \Gamma$, then $\Gamma \vdash_L \varphi$. A theory Γ is *prime*, only if for any $\varphi, \psi \in \mathcal{F}$ it holds that if $\varphi \vee \psi \in \Gamma$, then $\varphi \in \Gamma$ or $\psi \in \Gamma$. A theory Γ is *consistent*, only if for any $\varphi, \psi \in \mathcal{F}$ it holds that $\varphi \notin \Gamma$ or $\neg\varphi \notin \Gamma$. A theory Γ is *decisive*, only if for any $\varphi, \psi \in \mathcal{F}$ it holds that $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

We define the notion of canonical valuation for $\mathbf{TL_1}$. For any consistent prime theory $\mathcal{T}$, let $\xi_{\mathcal{T}}$ be a $\mathbf{TL_1}$-*canonical valuation*, which is defined by means of the following clauses (for any $p \in \mathcal{P}$):

$$T \in \xi_{\mathcal{T}}(p) \;\Leftrightarrow\; p \in \mathcal{T}, \qquad\qquad F \in \xi_{\mathcal{T}}(p) \;\Leftrightarrow\; \neg p \in \mathcal{T}.$$

Now we prove the following lemma.

LEMMA 3. $\mathbf{TL_1}$-*canonical valuation* $\xi_{\mathcal{T}}$ *can be extended to* $\mathcal{F}$.

PROOF. By induction on the complexity of a formula φ. The case where $\varphi = \neg\pi$ is proven in a standard way, using (R9). We consider the case where $\varphi = \pi_1 \wedge \pi_{\mathbf{2}}$. Suppose that $T \in \xi_{\mathcal{T}}(\pi_1 \wedge \pi_{\mathbf{2}})$. By Definition 4, we obtain two sub-cases (a) $T \in \xi_{\mathcal{T}}(\pi_1)$ and $F \notin \xi_{\mathcal{T}}(\pi_{\mathbf{2}})$, or (b) $T \in \xi_{\mathcal{T}}(\pi_{\mathbf{2}})$ and $F \notin \xi_{\mathcal{T}}(\pi_1)$. The proofs of both sub-cases are analogous, so we consider only (a). By the inductive hypothesis, we have $\pi_1 \in \mathcal{T}$ and $\neg\pi_{\mathbf{2}} \notin \mathcal{T}$. Suppose that $\pi_{\mathbf{2}} \wedge \pi_1 \notin \mathcal{T}$. Then, using the primeness of $\mathcal{T}$, we have $\neg\pi_{\mathbf{2}} \vee (\pi_{\mathbf{2}} \wedge \pi_1) \notin \mathcal{T}$. Using the deductive closure of $\mathcal{T}$ and subsequently applying (R5), (R3), and (R4), we obtain $\pi_1 \notin \mathcal{T}$, thereby getting a contradiction. Thus, $\pi_{\mathbf{2}} \wedge \pi_1 \in \mathcal{T}$. Again, using the deductive closure of $\mathcal{T}$ and (R3), we obtain the desired result $\pi_1 \wedge \pi_{\mathbf{2}} \in \mathcal{T}$.

Now suppose that $\pi_1 \wedge \pi_{\mathbf{2}} \in \mathcal{T}$. Since $\mathcal{T}$ is consistent, we have $\sim(\pi_1 \wedge \pi_{\mathbf{2}}) \notin \mathcal{T}$. Using the deductive closure of $\mathcal{T}$ and subsequently applying (R9), (R6), and (R7), we obtain $\neg\pi_1 \notin \mathcal{T}$ and $\neg\pi_{\mathbf{2}} \notin \mathcal{T}$. From this, by the inductive hypothesis, we have $F \notin \xi_{\mathcal{T}}(\pi_1)$ and $F \notin \xi_{\mathcal{T}}(\pi_{\mathbf{2}})$. It remains to show that either $T \in \xi_{\mathcal{T}}(\pi_1)$, or $T \in \xi_{\mathcal{T}}(\pi_{\mathbf{2}})$. In this case, using Definition 4, we'd obtain $T \in \xi_{\mathcal{T}}(\pi_1 \wedge \pi_{\mathbf{2}})$. Suppose that $\pi_1 \notin \mathcal{T}$ and $\pi_{\mathbf{2}} \notin \mathcal{T}$. Then, using the primeness of $\mathcal{T}$, we have $\pi_1 \vee \pi_{\mathbf{2}} \notin \mathcal{T}$. But this, in virtue of the deductive closure of $\mathcal{T}$ and (R2), leads to $\pi_1 \wedge \pi_{\mathbf{2}} \notin \mathcal{T}$, thereby producing a contradiction with the initial assumption. Then either $\pi_1 \in \mathcal{T}$ or $\pi_{\mathbf{2}} \in \mathcal{T}$. By the inductive hypothesis, we get $T \in \xi_{\mathcal{T}}(\pi_1)$ or $T \in \xi_{\mathcal{T}}(\pi_{\mathbf{2}})$, which leads us to the desired result. Thus, we have shown that $T \in \xi_{\mathcal{T}}(\pi_1 \wedge \pi_{\mathbf{2}}) \Leftrightarrow \pi_1 \wedge \pi_{\mathbf{2}} \in \mathcal{T}$.

The statement $F \in \xi_{\mathcal{T}}(\pi_1 \wedge \pi_{\mathbf{2}}) \Leftrightarrow \neg(\pi_1 \wedge \pi_{\mathbf{2}}) \in \mathcal{T}$ is proven in a standard way, using (R9), (R6), (R7), and primeness of $\mathcal{T}$.

Let us consider the case $\varphi = \pi_1 \vee \pi_{\mathbf{2}}$. Analogously to the previous case, the statement $T \in \xi_{\mathcal{T}}(\pi_1 \vee \pi_{\mathbf{2}}) \Leftrightarrow \pi_1 \vee \pi_{\mathbf{2}} \in \mathcal{T}$ can be proven in a standard way, using the primeness of $\mathcal{T}$, (R6), and (R7). In fact, this is possible because of the usual truth condition for $\vee$ and falsity condition for $\wedge$.

Suppose that $F \in \xi_{\mathcal{T}}(\pi_1 \vee \pi_{\mathbf{2}})$. By Definition 4 we have two sub-cases: (a) $F \in \xi_{\mathcal{T}}(\pi_1)$ and $T \notin \xi_{\mathcal{T}}(\pi_{\mathbf{2}})$, or (b) $F \in \xi_{\mathcal{T}}(\pi_{\mathbf{2}})$ and $T \notin \xi_{\mathcal{T}}(\pi_1)$. The proofs of these cases are analogous, so we, again, consider only one of them. Let (b) $F \in \xi_{\mathcal{T}}(\pi_{\mathbf{2}})$ and $T \notin \xi_{\mathcal{T}}(\pi_1)$. Then, by the inductive hypothesis, we obtain $\neg\pi_{\mathbf{2}} \in \mathcal{T}$ and $\pi_1 \notin \mathcal{T}$. Suppose that $\neg(\pi_1 \vee \pi_{\mathbf{2}}) \notin \mathcal{T}$. Then, using the deductive closure of $\mathcal{T}$ and (R10), we have $\neg\pi_1 \wedge \neg\pi_{\mathbf{2}} \notin \mathcal{T}$. Due to the primeness of $\mathcal{T}$ it holds that $\pi_1 \vee (\neg\pi_1 \wedge \neg\pi_{\mathbf{2}}) \notin \mathcal{T}$. Using the deductive closure of $\mathcal{T}$ and subsequently applying (R5), (R3), and (R4), we obtain $\neg\pi_{\mathbf{2}} \notin \mathcal{T}$, thereby getting a contradiction. Therefore, $\neg(\pi_1 \vee \pi_{\mathbf{2}}) \in \mathcal{T}$.

Suppose that $\neg(\pi_1 \vee \pi_{\mathbf{2}}) \in \mathcal{T}$. Due to the consistency of $\mathcal{T}$ we have $\pi_1 \vee \pi_{\mathbf{2}} \notin \mathcal{T}$. Using the deductive closure of $\mathcal{T}$ and subsequently applying (R6) and (R7), we obtain $\pi_1 \notin \mathcal{T}$ and $\pi_\mathbf{2} \notin \mathcal{T}$. By the inductive hypothesis, we have $T \notin \xi_\mathcal{T}(\pi_1)$ and $T \notin \xi_\mathcal{T}(\pi_\mathbf{2})$. It remains to show that $F \in \xi_\mathcal{T}(\pi_1)$ or $F \in \xi_\mathcal{T}(\pi_\mathbf{2})$, because in this case, using Definition 4, we'd obtain $F \in \xi_\mathcal{T}(\pi_1 \vee \pi_\mathbf{2})$. Suppose that $\neg\pi_1 \notin \mathcal{T}$ and $\neg\pi_\mathbf{2} \notin \mathcal{T}$. Using the primeness of $\mathcal{T}$ and (R12), we obtain $\neg(\pi_1 \vee \pi_\mathbf{2}) \notin \mathcal{T}$. Contradiction. Thus, $\neg\pi_1 \in \mathcal{T}$ or $\neg\pi_\mathbf{2} \in \mathcal{T}$, which, by the inductive hypothesis, gives $F \in \xi_\mathcal{T}(\pi_1)$ or $F \in \xi_\mathcal{T}(\pi_\mathbf{2})$. Therefore, we have proven the statement $F \in \xi_\mathcal{T}(\pi_1 \vee \pi_\mathbf{2}) \Leftrightarrow \neg(\pi_1 \vee \pi_\mathbf{2}) \in \mathcal{T}$. □

An analogous lemma is needed for $\mathbf{TL_2}$. We define a canonical valuation $\theta_\mathcal{T}$ over decisive theories instead of consistent ones. For any decisive prime theory $\mathcal{T}$, let $\theta_\mathcal{T}$ be a $\mathbf{TL_2}$-*canonical valuation*, which is defined by means of the following clauses (for any $p \in \mathcal{P}$):

$$T \notin \theta_\mathcal{T}(p) \;\Leftrightarrow\; \neg p \in \mathcal{T}, \qquad\qquad F \notin \theta_\mathcal{T}(p) \;\;\Leftrightarrow\;\; p \in \mathcal{T}.$$

Now we extend this definition.

Lemma 4. *$\mathbf{TL_2}$-canonical valuation $\theta_\mathcal{T}$ can be extended to $\mathcal{F}$.*

Proof. Analogous to the proof of Lemma 3. The peculiarity consists in that this particular proof requires the usage of inference rules which are characteristic of $\mathcal{N}_{\mathbf{TL_2}}$. The case where $\varphi = \neg\pi$ is proven as usual, using (R9). Consider the case $\varphi = \pi_1 \wedge \pi_\mathbf{2}$. Suppose that $T \notin \theta(\pi_1 \wedge \pi_\mathbf{2})$. By Definition 4 we have ($T \notin \theta_\mathcal{T}(\pi_1)$ or $F \in \theta_\mathcal{T}(\pi_\mathbf{2})$) and ($T \notin \theta_\mathcal{T}(\pi_\mathbf{2})$ or $F \in \theta_\mathcal{T}(\pi_1)$). We have four sub-cases. We consider only three of them because the remaining one can be proven analogously. Let (a) $T \notin \theta_\mathcal{T}(\pi_1)$ and $T \notin \theta_\mathcal{T}(\pi_\mathbf{2})$. By the inductive hypothesis, we have $\neg\pi_1 \in \mathcal{T}$ and $\neg\pi_\mathbf{2} \in \mathcal{T}$. From this, using the deductive closure of $\mathcal{T}$, (R1), and (R18), we obtain $\neg(\pi_1 \wedge \pi_\mathbf{2}) \in \mathcal{T}$. Let (b) $T \notin \theta_\mathcal{T}(\pi_1)$ and $F \in \theta_\mathcal{T}(\pi_1)$. By the inductive hypothesis, we obtain $\neg\pi_1 \in \mathcal{T}$ and $\pi_1 \notin \mathcal{T}$. Suppose that $\neg(\pi_1 \wedge \pi_\mathbf{2}) \notin \mathcal{T}$. From this, since $\mathcal{T}$ is decisive, we have $\pi_1 \wedge \pi_\mathbf{2} \in \mathcal{T}$. Further, using the deductive closure of $\mathcal{T}$ and (R15), we obtain $\pi_1 \in \mathcal{T}$, thereby getting a contradiction. Thus, $\neg(\pi_1 \wedge \pi_\mathbf{2}) \in \mathcal{T}$. Let (c) $F \in \theta_\mathcal{T}(\pi_1)$ and $F \in \theta_\mathcal{T}(\pi_\mathbf{2})$. By the inductive hypothesis, we have $\pi_1 \notin \mathcal{T}$ and $\pi_\mathbf{2} \notin \mathcal{T}$. Since $\mathcal{T}$ is prime we have $\pi_1 \vee \pi_\mathbf{2} \notin \mathcal{T}$. From this, using the deductive closure, of $\mathcal{T}$ and (R2), we obtain $\pi_1 \wedge \pi_\mathbf{2} \notin \mathcal{T}$. But then, by the decisiveness of $\mathcal{T}$, we obtain $\neg(\pi_1 \wedge \pi_\mathbf{2}) \in \mathcal{T}$. Thus, we have proven the statement $T \notin \theta_\mathcal{T}(\pi_1 \wedge \pi_\mathbf{2}) \Rightarrow \neg(\pi_1 \wedge \pi_\mathbf{2}) \in \mathcal{T}$.

Now we prove $\neg(\pi_1 \wedge \pi_\mathbf{2}) \in \mathcal{T} \Rightarrow T \notin \theta_\mathcal{T}(\pi_1 \wedge \pi_\mathbf{2})$. We argue by contraposition; that is, it is our goal to prove the statement $T \in \theta_\mathcal{T}(\pi_1 \wedge$

$\pi_2) \Rightarrow \neg(\pi_1 \wedge \pi_2) \notin \mathcal{T}$. Suppose $T \in \theta_{\mathcal{T}}(\pi_1 \wedge \pi_2)$. By Definition 4 we obtain (a) $T \in \theta_{\mathcal{T}}(\pi_1)$ and $F \notin \theta_{\mathcal{T}}(\pi_2)$ or (b) $T \in \theta_{\mathcal{T}}(\pi_2)$ and $F \notin \theta_{\mathcal{T}}(\pi_1)$. The proofs of both cases are identical, so we consider only (a). Let $T \in \theta_{\mathcal{T}}(\pi_1)$ and $F \notin \theta_{\mathcal{T}}(\pi_2)$. By the inductive hypothesis, we have $\neg\pi_1 \notin \mathcal{T}$ and $\pi_2 \in \mathcal{T}$. Suppose that $\neg(\pi_1 \wedge \pi_2) \in \mathcal{T}$. Then, using $\pi_2 \in \mathcal{T}$, the deductive closure of $\mathcal{T}$ and subsequently applying (R9), (R14), (R1), (R15), (R16), (R1), (R5), and (R17), we have $\neg\pi_1 \in \mathcal{T}$. Contradiction. Thus, $\neg(\pi_1 \wedge \pi_2) \notin \mathcal{T}$.

The statement $F \notin \theta_{\mathcal{T}}(\pi_1 \wedge \pi_2) \Leftrightarrow \pi_1 \wedge \pi_2 \in \mathcal{T}$ is proven in a standard way, using (R1), (R15) and (R16).

Consider the case $\varphi = \pi_1 \vee \pi_2$. The statement $T \notin \theta_{\mathcal{T}}(\pi_1 \vee \pi_2) \Leftrightarrow \neg(\pi_1 \vee \pi_2) \in \mathcal{T}$ is also proven in the standard way, suing (R10), (R15), and (R16).

Let us prove the statement $F \notin \theta_{\mathcal{T}}(\pi_1 \vee \pi_2) \Rightarrow \pi_1 \vee \pi_2 \in \mathcal{T}$. Let $F \notin \theta_{\mathcal{T}}(\pi_1 \vee \pi_2)$. By Definition 4 we obtain ($F \notin \theta_{\mathcal{T}}(\pi_1)$ or $T \in \theta_{\mathcal{T}}(\pi_2)$) and ($F \notin \theta_{\mathcal{T}}(\pi_2)$ or $T \in \theta_{\mathcal{T}}(\pi_1)$). We have four sub-cases. Let (a) $F \notin \theta_{\mathcal{T}}(\pi_1)$ and $F \notin \theta_{\mathcal{T}}(\pi_2)$. By the inductive hypothesis, we have $\pi_1 \in \mathcal{T}$ and $\pi_2 \in \mathcal{T}$. Using the deductive closure of $\mathcal{T}$, (R1) and (R2), we have $\pi_1 \vee \pi_2 \in \mathcal{T}$. Let (b) $F \notin \theta_{\mathcal{T}}(\pi_1)$ and $T \in \theta_{\mathcal{T}}(\pi_1)$. By the inductive hypothesis, we have $\pi_1 \in \mathcal{T}$ and $\neg\pi_1 \notin \mathcal{T}$. Suppose that $\pi_1 \vee \pi_2 \notin \mathcal{T}$. Then, using the fact that $\mathcal{T}$ is decisive, we obtain $\neg(\pi_1 \vee \pi_2) \in \mathcal{T}$. From this, using the deductive closure of $\mathcal{T}$, (R10) and (R15), we obtain $\neg\pi_1 \in \mathcal{T}$. Contradiction. Thus, $\pi_1 \vee \pi_2 \in \mathcal{T}$. Let (c) $T \in \theta_{\mathcal{T}}(\pi_1)$ and $T \in \theta_{\mathcal{T}}(\pi_2)$. By the inductive hypothesis, we obtain $\neg\pi_1 \notin \mathcal{T}$ and $\neg\pi_2 \notin \mathcal{T}$. Since $\mathcal{T}$ is prime we have $\neg\pi_1 \vee \neg\pi_2 \notin \mathcal{T}$. Since $\mathcal{T}$ is decisive we have $\neg(\neg\pi_1 \vee \neg\pi_2) \in \mathcal{T}$. From this, using the deductive closure of $\mathcal{T}$ and subsequently applying (R10), (R15), (R16), (R8), (R1), and (R2), we obtain $\pi_1 \vee \pi_2 \in \mathcal{T}$. The case where $T \in \theta_{\mathcal{T}}(\pi_2)$ and $F \notin \theta_{\mathcal{T}}(\pi_2)$ can be proven analogously to (b).

Now we prove the statement $\pi_1 \vee \pi_2 \in \mathcal{T} \Rightarrow F \notin \theta_{\mathcal{T}}(\pi_1 \vee \pi_2)$. Again, we argue by contraposition; that is, our goal is to prove $F \in \theta_{\mathcal{T}}(\pi_1 \vee \pi_2) \Rightarrow \pi_1 \vee \pi_2 \notin \mathcal{T}$. Let $F \in \theta_{\mathcal{T}}(\pi_1 \vee \pi_2)$. By Definition 4 we have (a) $F \in \theta_{\mathcal{T}}(\pi_1)$ and $T \notin \theta_{\mathcal{T}}(\pi_2)$ or (b) $F \in \theta_{\mathcal{T}}(\pi_2)$ and $T \notin \theta_{\mathcal{T}}(\pi_1)$. Both cases are proven in analogous manner, so as an example we consider only (b). Let $F \in \theta_{\mathcal{T}}(\pi_2)$ and $T \notin \theta_{\mathcal{T}}(\pi_1)$. By the inductive hypothesis, we obtain $\pi_2 \notin \mathcal{T}$ and $\neg\pi_1 \in \mathcal{T}$. Suppose that $\pi_1 \vee \pi_2 \in \mathcal{T}$. Then, using $\neg\pi_1 \in \mathcal{T}$, the deductive closure of $\mathcal{T}$ and subsequently applying (R14), (R1), (R5), and (R17), we obtain $\pi_2 \in \mathcal{T}$. Contradiction. Therefore, $\pi_1 \vee \pi_2 \notin \mathcal{T}$. □

The next step is to prove Lindenbaum's lemma.

Lemma 5. *For any* $\Gamma \cup \{\varphi\} \subseteq \mathcal{F}$:

1. *if* $\Gamma \nvdash_{\mathcal{N}_{\mathbf{TL_1}}} \varphi$, *then there exists a consistent prime theory* $\Gamma^{'}$, *such that* $\Gamma \subseteq \Gamma^{'}$ and $\Gamma^{'} \nvdash_{\mathcal{N}_{\mathbf{TL_1}}} \varphi$.
2. *if* $\Gamma \nvdash_{\mathcal{N}_{\mathbf{TL_2}}} \varphi$, *then there exists a decisive prime theory* $\Gamma^{'}$, *such that* $\Gamma \subseteq \Gamma^{'}$ and $\Gamma^{'} \nvdash_{\mathcal{N}_{\mathbf{TL_2}}} \varphi$.

Proof. The proofs of both (1) and (2) are standard and, in fact, coincide with the ones for $\mathbf{K_3}$ and **LP**, respectively. The reader may consult, for example, [26] for details. □

Using Lemma 3, Lemma 4 and Lemma 5 we can prove the following completeness theorem.

Theorem 2. *Let* $x \in \{\mathbf{TL_1}, \mathbf{TL_2}\}$. *For any* $\Gamma \cup \{\varphi\} \subseteq \mathcal{F}$:

$$\Gamma \vDash_x \varphi \ \Rightarrow\ \Gamma \vdash_{\mathcal{N}_x} \varphi.$$

Proof. As an example we consider the case of $x = \mathbf{TL_2}$. Let $\Gamma \nvdash_{\mathcal{N}_{\mathbf{TL_2}}} \varphi$. By Lemma 5 there exists a decisive prime theory Γ', such that $\Gamma \subseteq \Gamma'$ and $\Gamma' \nvdash_{\mathcal{N}_{\mathbf{TL_2}}} \varphi$. Due to the reflexivity of $\vdash_{\mathcal{N}_{\mathbf{TL_2}}}$ we get $\varphi \notin \Gamma'$. By Lemma 4 we obtain that there exists a $\mathbf{TL_2}$-canonical valuation $\theta_{\Gamma'}$ such that $F \notin \theta_{\Gamma'}(\psi)$ (for every $\psi \in \Gamma'$) and $F \in \theta_{\Gamma'}(\varphi)$. Then, by Definition 5, we obtain $\Gamma' \nvDash_{\mathbf{TL_2}} \varphi$. Since $\Gamma \subseteq \Gamma'$ we have $\Gamma \nvDash_{\mathbf{TL_2}} \varphi$, by the monotonicity of $\vDash_{\mathbf{TL_2}}$. □

As to the soundness theorem, it can be proven in a standard manner by checking that all inference rules are sound with respect to Definition 5.

Theorem 3. *Let* $x \in \{\mathbf{TL_1}, \mathbf{TL_2}\}$. *For any* $\Gamma \cup \{\varphi\} \subseteq \mathcal{F}$:

$$\Gamma \vdash_{\mathcal{N}_x} \varphi \Rightarrow \Gamma \vDash_x \varphi.$$

In virtue of Theorem 1 we obtain the following corollary.

Corollary 1. *For any* $\Gamma \cup \{\varphi\} \subseteq \mathcal{F}$:

$$\Gamma \vDash_{\mathcal{M}_1} \varphi \ \Leftrightarrow\ \Gamma \vdash_{\mathcal{N}_{\mathbf{TL_1}}} \varphi \quad \text{and} \quad \Gamma \vDash_{\mathcal{M}_2} \varphi \Leftrightarrow \Gamma \vdash_{\mathcal{N}_{\mathbf{TL_2}}} \varphi.$$

4. Cooper's Logic of Ordinary Discourse as a connexive extension of $\mathbf{TL_2}$

In 1968 Cooper presented the 'Logic of Ordinary Discourse' (henceforth, **OL**) [9]. Surprisingly, this logic can be seen as an implicative expansion of $\mathbf{TL_2}$. Consider the propositional language $\mathcal{L}_{\rightarrow}$, which is obtained through the expansion of $\mathcal{L}$ by $\rightarrow$. Throughout this section, we will use $\mathcal{F}$ to denote the set of all formulae of $\mathcal{L}_{\rightarrow}$.

Definition 7. An **OL**-*matrix* $\mathcal{M}_{\mathbf{3}}$ for $\mathcal{L}_{\rightarrow}$ is obtained from $\mathbf{TL_2}$-matrix $\mathcal{M}_{\mathbf{2}}$ by adding the function $f_{\rightarrow}$.

$f_{\rightarrow}$	$\mathbb{V}$	$\mathbb{L}$	$\mathbb{F}$
$\mathbb{V}$	$\mathbb{V}$	$\mathbb{L}$	$\mathbb{F}$
$\mathbb{L}$	$\mathbb{V}$	$\mathbb{L}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{L}$	$\mathbb{L}$	$\mathbb{L}$

An **OL**-*valuation* in an **OL**-matrix $\mathcal{M}_{\mathbf{3}}$ is a function $v\colon \mathcal{F} \rightarrow \mathcal{V}_{\mathbf{OL}}$ that satisfies the following condition for every n-ary connective $\diamond$ of $\mathcal{L}_{\rightarrow}$ and $\varphi_1, \ldots, \varphi_n \in \mathcal{F}$:

$$v(\diamond(\varphi_1, \ldots, \varphi_n)) = f_{\diamond}(v(\varphi_1), \ldots, v(\varphi_n)).$$

Remark 5. We use $\mathbb{V}$, $\mathbb{L}$ and $\mathbb{F}$ so as to maintain a unified notation. Of course, Cooper himself used different truth values. Nevertheless, they rather conforming Peirce's interpretation: **T** ('true'), **F** ('false') and **G** ('gap') instead of $\mathbb{V}$, $\mathbb{F}$ and $\mathbb{L}$, respectively.

The consequence relation in $\mathcal{M}_{\mathbf{3}}$ is defined according to (C2) from Definition 3. The notion of validity is defined in a standard way: for any $\varphi \in \mathcal{F}$, φ is called valid in $\mathcal{M}_{\mathbf{3}}$, only if there is no such **OL**-valuation v in $\mathcal{M}_{\mathbf{3}}$, that $v(\varphi) = \mathbb{F}$.

In order to obtain a generalized truth values semantics for **OL**, it is sufficient to enrich Definition 4 with the following semantic conditions for implication.

(S7) $T \in v(\varphi \rightarrow \psi) \;\Leftrightarrow\;$ if $T \in v(\varphi)$ then $T \in v(\psi)$,
(S8) $F \in v(\varphi \rightarrow \psi) \;\Leftrightarrow\;$ if $T \in v(\varphi)$ then $F \in v(\psi)$.

(S7) represents the traditional truth-condition for implication, whereas (S8) represents the falsity condition for connexive implication in the style of Wansing [34]. Let us denote the resulting model by $\mathfrak{M}_{\mathbf{OL}}$. The definition of the consequence relation remains unchanged (see Definition 5), we only add the notion of validity with respect to the generalized truth values models. A formula φ is valid in a generalized truth values model $\mathfrak{M}_{\mathbf{OL}}$, only if $F \notin \xi(\varphi)$ holds for every valuation ξ in $\mathfrak{M}_{\mathbf{OL}}$. Let us use $\vDash_{\mathbf{OL}}$ to denote the corresponding consequence relation. It can be easily proved, using an argument analogous to Lemma 2 and Theorem 1, that the two semantics for **OL** just defined are equivalent; we leave the details to the interested reader.

Cooper's logic is remarkable since from one side it is linked with Peirce's Φ and Ψ operations, and from the other with connexive logics, a family of systems which is enjoying a resurgence of popularity in the recent literature. As far as we know, Cooper himself was not aware of any of these connections.

The reader familiar with the state of art in the field of connexive logics may have noticed that the definition of $f_{\rightarrow}$ coincides with the implication of Cantwell's logic **CN** from [6]. It also occurred in Olkhovikov's logic **LImp** from [19, 20] and Omori's dialetheic version of the logic of paradox **dLP** from [21]. This definition, as is well known, (in the presence of standardly defined negation) guarantees the validity of characteristic principles of connexive implication:

$$(\varphi \rightarrow \psi) \rightarrow \neg(\varphi \rightarrow \neg\psi), \qquad \text{(Boethius Thesis I)}$$

$$(\varphi \rightarrow \neg\psi) \rightarrow \neg(\varphi \rightarrow \psi), \qquad \text{(Boethius Thesis II)}$$

$$\neg(\varphi \rightarrow \neg\varphi), \qquad \text{(Aristotle Thesis I)}$$

$$\neg(\neg\varphi \rightarrow \varphi). \qquad \text{(Aristotle Thesis II)}$$

Despite the fact that Olkhovikov was concerned with $f_{\rightarrow}$ much earlier than Cantwell, the discovery of this operation is usually associated with the latter. Notice, however, that Cooper's paper was published forty years earlier than Cantwell's one. It is also worth mentioning that Cooper's paper, though written independently, was published around the same time when the issues concerning connexive logics had been under active discussion through the works of McCall [17] and Angell [2]. Moreover, in [9] Cooper cited his Ph.D. dissertation [8], written in 1964, which is two years earlier than McCall's [17]. Cooper was, perhaps, the first to introduce explicitly $f_{\rightarrow}$, and this fact seems to be little known since we cannot find any references to it either in the recently revised entry on connexive logic [33] at Stanford Encyclopedia of Philosophy or in recently published overviews from the special issues on connexive logic [35] and [23]. At the same time, however, the link between Cooper's logic and Peirce's Φ and Ψ was well discussed for example in [13].

A natural deduction system for **OL** (we call it $\mathcal{N}_{\mathbf{OL}}$) can be obtained from $\mathcal{N}_{\mathbf{TL_2}}$ by adding the following rules:

$$\text{(R20)}\ \frac{\varphi\to\psi,\quad \varphi}{\psi}\qquad\qquad \text{(R21)}\ \frac{\neg(\varphi\to\psi),\quad \varphi}{\neg\psi}$$

$$\text{(R22)}\ \frac{\begin{array}{c}[\varphi]\\ \vdots\\ \psi\end{array}}{\varphi\to\psi}\qquad\qquad \text{(R23)}\ \frac{\begin{array}{c}[\varphi]\\ \vdots\\ \neg\psi\end{array}}{\neg(\varphi\to\psi)}$$

BOETHIUS THESES andARISTOTLE THESES can be easily proven in $\mathcal{N}_{\mathbf{OL}}$. We show some examples below:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\varphi\to\psi]\quad[\varphi]}{\psi}\ \text{(R20)}}{\neg\neg\psi}\ \text{(R8)}}{\neg(\varphi\to\neg\psi)}\ \text{(R23)}}{(\varphi\to\psi)\to\neg(\varphi\to\neg\psi)}\ \text{(R22)}$$

$$\dfrac{\dfrac{\dfrac{[\neg\varphi]}{\neg\neg\neg\varphi}\ \text{(R8)}}{\neg\varphi}\ \text{(R8)}}{\neg(\neg\varphi\to\varphi)}\ \text{(R23)}\qquad\qquad \dfrac{\dfrac{[\varphi]}{\neg\neg\varphi}\ \text{(R8)}}{\neg(\varphi\to\neg\varphi)}\ \text{(R23)}$$

Remark 6. A simple natural deduction system $\mathcal{N}_{\mathbf{CN}}$, formalizing Cantwell's **CN**, can be obtained from $\mathcal{N}_{\mathbf{OL}}$ by replacing (R2), (R5), (R14), (R17), (R18) with (R6) and (R7). Correspondingly, a natural deduction system $\mathcal{N}_{\mathbf{MC}}$, formalizing the 'material connexive logic' **MC** [33], can be obtained from $\mathcal{N}_{\mathbf{CN}}$ by dropping (R19).

In order to prove the adequacy of $\mathcal{N}_{\mathbf{OL}}$ with respect to the generalized truth values semantics, we, first, run through the completeness theorem. It is sufficient to take care of an additional case which arises in the context of Lemma 4. In turn, Lemma 5 remains unchanged.

Additional Case of Lemma 4. Recall that the proof of Lemma 4 is by induction on the complexity of a formula φ. So, if we are working within $\mathcal{L}_{\to}$, then we need to consider the remaining case, namely $\varphi = \pi_1 \to \pi_{\mathbf{2}}$. Let us prove the statement $\neg(\pi_1 \to \pi_{\mathbf{2}}) \notin \mathcal{T} \Rightarrow T \in \theta_{\mathcal{T}}(\pi_1 \to \pi_{\mathbf{2}})$. Suppose that $\neg(\pi_1 \to \pi_{\mathbf{2}}) \notin \mathcal{T}$ and $T \notin \theta_{\mathcal{T}}(\pi_1 \to \pi_{\mathbf{2}})$. Since $\mathcal{T}$ is decisive we obtain $\pi_1 \to \pi_{\mathbf{2}} \in \mathcal{T}$. By (S7) we have $T \in \theta_{\mathcal{T}}(\pi_1)$ and

$T \notin \theta_{\mathcal{T}}(\pi_{\mathbf{2}})$. Using the inductive hypothesis, we obtain $\neg\pi_1 \notin \mathcal{T}$ and $\neg\pi_{\mathbf{2}} \in \mathcal{T}$. From $\neg\pi_1 \notin \mathcal{T}$, by the decisiveness of $\mathcal{T}$, follows $\pi_1 \in \mathcal{T}$. Moreover, using the deductive closure of $\mathcal{T}$, $\neg\pi_1 \notin \mathcal{T}$ and (R21), we get $\pi_1 \notin \mathcal{T}$ or $\neg(\pi_1 \to \neg\pi_1) \notin \mathcal{T}$. The first disjunct leads to contradiction, hence $\neg(\pi_1 \to \neg\pi_1) \notin \mathcal{T}$. From this, due to (R23), we have $\pi_1 \nvdash \neg\neg\pi_1$, thereby getting a contradiction, because $\mathcal{T}$ is closed under (R8). Thus, we have proven the statement $\neg(\pi_1 \to \pi_{\mathbf{2}}) \notin \mathcal{T} \Rightarrow T \in \theta_{\mathcal{T}}(\pi_1 \to \pi_{\mathbf{2}})$, which, by contraposition, gives us the desired result. Suppose that $\neg(\pi_1 \to \pi_{\mathbf{2}}) \in \mathcal{T}$. Our goal is to show $T \notin \theta_{\mathcal{T}}(\pi_1 \to \pi_{\mathbf{2}})$. Suppose that $T \in \theta_{\mathcal{T}}(\pi_1 \to \pi_{\mathbf{2}})$. Obviously due to the impossibility of gluts we obtain also $F \notin \theta_{\mathcal{T}}(\pi_1 \to \pi_{\mathbf{2}})$. From $T \in \theta_{\mathcal{T}}(\pi_1 \to \pi_{\mathbf{2}})$, using (S7), we have $T \notin \theta_{\mathcal{T}}(\pi_1)$ or $T \in \theta_{\mathcal{T}}(\pi_{\mathbf{2}})$. From $F \notin \theta_{\mathcal{T}}(\pi_1 \to \pi_{\mathbf{2}})$, using (S8), we have $T \in \theta_{\mathcal{T}}(\pi_1)$ and $F \notin \theta_{\mathcal{T}}(\pi_{\mathbf{2}})$. Since $T \notin \theta_{\mathcal{T}}(\pi_1)$ leads to contradiction we are working with $T \in \theta_{\mathcal{T}}(\pi_{\mathbf{2}})$. From $T \in \theta_{\mathcal{T}}(\pi_1)$, by the inductive hypothesis, we have $\neg\pi_1 \notin \mathcal{T}$. From $T \in \theta_{\mathcal{T}}(\pi_{\mathbf{2}})$, by the inductive hypothesis, we have $\neg\pi_{\mathbf{2}} \notin \mathcal{T}$. From the latter, using the deductive closure of $\mathcal{T}$ and (R21), we obtain $\neg(\pi_1 \to \pi_{\mathbf{2}}) \notin \mathcal{T}$ or $\pi_1 \notin \mathcal{T}$. Both cases lead to contradiction: the first contradicts the initial assumption, the second contradicts the fact that $\mathcal{T}$ is decisive. Therefore, $T \notin \theta_{\mathcal{T}}(\pi_1 \to \pi_{\mathbf{2}})$.

Suppose that $F \notin \theta_{\mathcal{T}}(\pi_1 \to \pi_{\mathbf{2}})$. From this, using (S8), we have $T \in \theta_{\mathcal{T}}(\pi_1)$ and $F \notin \theta_{\mathcal{T}}(\pi_{\mathbf{2}})$. By the inductive hypothesis we have $\neg\pi_1 \notin \mathcal{T}$ and $\pi_{\mathbf{2}} \in \mathcal{T}$. Since $\mathcal{T}$ is decisive we obtain $\pi_1 \in \mathcal{T}$. Hence, using the deductive closure of $\mathcal{T}$ and (R22), we obtain $\pi_1 \to \pi_{\mathbf{2}} \in \mathcal{T}$. Let us prove the converse. Suppose that $\pi_1 \to \pi_{\mathbf{2}} \in \mathcal{T}$ and $F \in \theta_{\mathcal{T}}(\pi_1 \to \pi_{\mathbf{2}})$. Due to the impossibility of gluts, we have $T \notin \theta_{\mathcal{T}}(\pi_1 \to \pi_{\mathbf{2}})$, from which, by (S7), we obtain $T \in \theta_{\mathcal{T}}(\pi_1)$ and $T \notin \theta_{\mathcal{T}}(\pi_{\mathbf{2}})$. Using the inductive hypothesis, we have $\neg\pi_1 \notin \mathcal{T}$ and $\neg\pi_{\mathbf{2}} \in \mathcal{T}$. In turn, from $F \in \theta_{\mathcal{T}}(\pi_1 \to \pi_{\mathbf{2}})$, by (S8), we have $T \notin \theta_{\mathcal{T}}(\pi_1)$ or $F \in \theta_{\mathcal{T}}(\pi_{\mathbf{2}})$. Obviously, the first sub-case leads to contradiction, hence we have $F \in \theta_{\mathcal{T}}(\pi_{\mathbf{2}})$ and, by the inductive hypothesis, $\pi_{\mathbf{2}} \notin \mathcal{T}$. From this, using the deductive closure of $\mathcal{T}$ and (R20), we obtain $\pi_1 \to \pi_{\mathbf{2}} \notin \mathcal{T}$ or $\pi_1 \notin \mathcal{T}$. Both cases lead to contradiction: the first contradicts the initial assumption, whereas the second contradicts fact that $\mathcal{T}$ is decisive.

Thus, holding in mind this case and using Lemma 5, the completeness result can be easily proven.

Theorem 4. *For any $\Gamma \cup \{\varphi\} \subseteq \mathcal{F}$: $\Gamma \vDash_{\mathbf{OL}} \varphi \Rightarrow \Gamma \vdash_{\mathcal{N}_{\mathbf{OL}}} \varphi$.*

The soundness part is straightforward as usual.

Theorem 5. *For any* $\Gamma \cup \{\varphi\} \subseteq \mathcal{F}$: $\Gamma \vdash_{\mathcal{N}_{\mathbf{OL}}} \varphi \Rightarrow \Gamma \vDash_{\mathbf{OL}} \varphi$.

From Theorem 4 and Theorem 5 we obtain the corollary.

Corollary 2. *For any* $\Gamma \cup \{\varphi\} \subseteq \mathcal{F}$: $\Gamma \vDash_{\mathbf{OL}} \varphi \Leftrightarrow \Gamma \vdash_{\mathcal{N}_{\mathbf{OL}}} \varphi$.

5. Concluding remarks

In this work, we were concerned with two ways of reconstructing Peirce's Triadic Logic. We obtained two systems, $\mathbf{TL_1}$ and $\mathbf{TL_2}$, which can be classified as members of the first-degree entailment family. Both logics are equipped with two kinds of semantics: matrix and generalized truth values ones. The characteristic feature of the former is that it contains the mysterious binary operations Φ and Ψ from Peirce's manuscript, which we used to interpret disjunction and conjunction, respectively. The point of introducing a semantics of the latter kind is in that it allows us to give a positive answer to the main question of our paper, namely: is it possible to claim that Peirce considered Φ and Ψ as candidates for disjunction and conjunction, respectively? As it was shown in Section 2.2, the semantic conditions for $\wedge$ and $\vee$ within both $\mathbf{TL_1}$ and $\mathbf{TL_2}$ allow the derivation of conditions which are consistent with the traditional view of the truth and falsity conditions for conjunction and disjunction.

Besides the main question, we obtained some interesting technical results. We equipped $\mathbf{TL_1}$ and $\mathbf{TL_2}$ with sound and complete natural deduction calculi. It turned out that $\mathbf{TL_2}$ may serve as a convenient base for obtaining a connexive extension, coinciding with the little-known three valued logic **OL** by Cooper. The resulting system can be seen as a weakening of Cantwell's logic **CN**. We presented a sound and complete natural deduction calculus for **OL**, from which it is also possible to obtain simple natural deduction calculi for **CN** and **MC** (see Remark 6).

As to the future work, we find it promising to investigate $\mathbf{TL}_1$, $\mathbf{TL}_2$ and their possible generalizations in relation to the family of infectious logics. As we remarked in Section 2.1, some distinctive features of $\mathbf{TL_1}$ and $\mathbf{TL_2}$ are related to the failure of Addition and Simplification, which, in turn, is also essential for many infectious logics. Thus, the issues concerning the characterization of logical consequence, epistemic interpretations of truth value gaps, informativeness of disjunction and conjunction are of great interest here.

Last but not least, it would be interesting to explore an application of Peirce's diagrammatic (graphical) technique to **OL**. It is known that Peirce's thoughts on existential graphs were very much interspersed with those pages that are devoted to the triadic logic in his 'Logical Notebook' and written in the very same weeks and days. Thus, this would be an interesting problem for a separate and detailed study.

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