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Modal Multilattice Logics with Tarski, Kuratowski, and Halmos Operators

Abstract. In this paper, we consider modal multilattices with Tarski, Kuratowski, and Halmos closure and interior operators as well as the corresponding logics which are multilattice versions of the modal logics **MNT4**, **S4**, and **S5**, respectively. The former modal multilattice logic is a new one. The latter two modal multilattice logics have been already mentioned in the literature, but algebraic completeness results have not been established for them before. We present a multilattice version of **MNT4** in a form of a sequent calculus and prove the algebraic and neighbourhood completeness theorems for it. We extend the algebraic completeness result for the multilattice versions of **S4** and **S5** as well.

Keywords: multilattice logic; modal logic; sequent calculus; algebraic completeness; neighbourhood semantics; embedding theorem

1. Introduction

Shramko [24] introduced the multilattice logic $\mathbf{ML}_n$ in order to generalize frameworks of Arieli and Avron's bilattice logic [1], Shramko and Wansing's trilattice logic [25], and Zaitsev's tetralattice logic [26]. Later on Kamide and Shramko extended $\mathbf{ML}_n$ by adding quantifiers [13], Kamide, Shramko, and Wansing [15] explored bi-intuitionistic and connexive modifications of $\mathbf{ML}_n$, Kamide [11, 12] combined $\mathbf{ML}_n$ with linear logic, Kamide and Shramko [14] presented the modal multilattice logic $\mathbf{MML}_n$. This logic was supposed to be a multilattice version of **S4**. However, as argued in [6, 7], this point is a bit problematic. Namely, Kamide and Shramko exploited an incomplete **S4** sequent calculus for

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the embedding purposes. In particular, this calculus does not allow for the construction of a proof for formulas representing the interdefinability of necessity and possibility operators. As a result, their sequent calculus for $\mathbf{MML}_n$ lacks the multilattice analogues of interdefinability theorems (this fact can be easily checked due to the backward embedding from **S4** into $\mathbf{MML}_n$ proved in [14]). It has motivated us to present a logic $\mathbf{MML}_n^{\mathbf{S4}}$ (see [6]) which provides the proofs of interdefinability formulas. Nevertheless, the primary aim of [6] was to introduce a multilattice version of **S5** (which we call $\mathbf{MML}_n^{\mathbf{S5}}$ following the established tradition). However, we have not set up the algebraic completeness results for $\mathbf{MML}_n^{\mathbf{S4}}$ and $\mathbf{MML}_n^{\mathbf{S5}}$. Thus, one of the aims of the current paper is to introduce modal multilattices with Kuratowski and Halmos operators and show that they determine $\mathbf{MML}_n^{\mathbf{S4}}$ and $\mathbf{MML}_n^{\mathbf{S5}}$, respectively.

Let us notice that although Kamide and Shramko introduced the notion of a modal multilattice, they have not proved an algebraic completeness theorem for $\mathbf{MML}_n$. It was proven in [7], but with respect to the structure which we call a De Morgan modal multilattice. It appears that Kamide and Shramko's original algebraic structure is too weak for $\mathbf{MML}_n$. First, it does not have some postulates for the inversions of closure and interior operators. Second, the closure and interior operators introduced by Kamide and Shramko are multilattice versions of Tarski operators (which are suitable for a weaker logic **MNT4**), but *not* Kuratowski operators (which are actually needed for **S4**). It has inspired us to take a new look at the multilattices equipped with Tarski operators and explore a multilattice version of **MNT4**. Thus, yet another aim of this paper is to introduce such a logic (we denote it as $\mathbf{MML}_n^{\mathbf{MNT4}}$). We prove an algebraic completeness theorem for $\mathbf{MML}_n^{\mathbf{MNT4}}$, present a sequent calculus for it, prove syntactical and semantic embedding theorems from $\mathbf{MML}_n^{\mathbf{MNT4}}$ into **MNT4**, and develop a neighbourhood semantics for $\mathbf{MML}_n^{\mathbf{MNT4}}$.

The paper is structured in the following way. In Section 2, we make some preliminary remarks regarding multilattices and their logics as well as introduce the notions of modal multilattices with Tarski, Kuratowski, and Halmos operators. Section 3 contains preliminaries regarding sequent and hypersequent calculi for **MNT4**, **S4**, and **S5**. In Section 4, we introduce a sequent calculus for $\mathbf{MML}_n^{\mathbf{MNT4}}$, recall a sequent calculus for $\mathbf{MML}_n^{\mathbf{S4}}$ and a hypersequent calculus for $\mathbf{MML}_n^{\mathbf{S5}}$ from [6]. In Section 5, we prove the algebraic completeness theorems for $\mathbf{MML}_n^{\mathbf{MNT4}}$, $\mathbf{MML}_n^{\mathbf{S4}}$, and $\mathbf{MML}_n^{\mathbf{S5}}$. In Section 6, we present a neighbourhood semantics for

$\mathbf{MML}_n^{\mathbf{MNT4}}$, prove syntactic and semantic embeddings from $\mathbf{MML}_n^{\mathbf{MNT4}}$ into **MNT4** and as a consequence obtain the neighbourhood completeness and cut elimination theorems for $\mathbf{MML}_n^{\mathbf{MNT4}}$. Section 7 contains concluding remarks.

2. Preliminaries I: multilattices

Definition 2.1 (Languages). Let $n > 1$ and $1 \leqslant j \leqslant n$. $\mathcal{P} = \{p_i, q_i, r_i \mid i \in \mathbb{N}\}$ is a set of propositional variables; $\mathcal{P}^j = \{p^j \mid p \in \mathcal{P}\}$ is a set of indexed propositional variables; $\mathcal{P}^* = \bigcup_{i=1}^{i=n} \mathcal{P}^i$; $\mathcal{C} = \{\neg, \Box, \Diamond, \wedge, \vee, \rightarrow, \leftarrow\}$; $\mathcal{C}^* = \bigcup_{i=1}^{i=n}\{\neg_i, \wedge_i, \vee_i, \rightarrow_i, \leftarrow_i\}$; $\mathcal{C}^\star = \bigcup_{i=1}^{i=n}\{\Box_i, \Diamond_i\}$. Let us fix three languages which we are going to use in what follows.

- $\mathscr{L}$ is the language of the modal logics **MNT4**, **S4**, and **S5**. It has the alphabet $\langle \mathcal{P}, \mathcal{P}^*, \mathcal{C}, (,)\rangle$.
- $\mathscr{L}_\mathrm{N}$ is the language of multilattice logic $\mathbf{ML}_n$. It has the alphabet $\langle \mathcal{P}, \mathcal{C}^*, (,)\rangle$.
- $\mathscr{L}_\mathrm{M}$ is the language of the modal multilattice logics $\mathbf{MML}_n^{\mathbf{MNT4}}$, $\mathbf{MML}_n^{\mathbf{S4}}$, and $\mathbf{MML}_n^{\mathbf{S5}}$. It has the alphabet $\langle \mathcal{P}, \mathcal{C}^*, \mathcal{C}^\star, (,)\rangle$.

The sets $\mathscr{F}$, $\mathscr{F}_\mathrm{N}$, and $\mathscr{F}_\mathrm{M}$, respectively, of all $\mathscr{L}$-, $\mathscr{L}_\mathrm{N}$-, and $\mathscr{L}_\mathrm{M}$-formulas are defined in a standard inductive way.

Now we introduce the notion of a multilattice as it has been given in a number of papers, thus respecting an established tradition, although this concept may be found a bit confusing since a set of unary *j-inversion* operations is incorporated into the structure of a multilattice.

Definition 2.2 (Multilattice; 14, p. 319, Definitions 2.1 and 2.2). A *multilattice* (or n-dimensional multilattice or n-lattice) is a structure $\mathcal{M}_n = \langle \mathcal{S}, \leq_1, \ldots, \leq_n\rangle$, where $n > 1$, $\mathcal{S} \neq \emptyset$, $\leq_1$, ..., $\leq_n$ are partial orders such that $\langle \mathcal{S}, \leq_1\rangle, \ldots, \langle \mathcal{S}, \leq_n\rangle$ are lattices with the corresponding pairs of meet and join operations $\langle \cap_1, \cup_1\rangle, \ldots, \langle \cap_n, \cup_n\rangle$ as well as the corresponding j-inversion operations $-_1$, ..., $-_n$ which satisfy the following conditions, for each $j \leqslant n$, $k \leqslant n$, $j \neq k$ and $x, y \in \mathcal{S}$:

$$x \leq_j y \text{ implies } -_j\, y \leq_j -_j x; \tag{anti}$$

$$x \leq_k y \text{ implies } -_j\, x \leq_k -_j y; \tag{iso}$$

$$-_j -_j\, x = x. \tag{per2}$$

Definition 2.3 (Distributive multilattice; 14, p. 319, Definition 2.1). A multilattice $\mathcal{M}_n = \langle \mathcal{S}, \leq_1, \ldots, \leq_n \rangle$ is called *distributive* iff all $2(2n^2 - n)$ distributive laws hold, i.e. $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$, where $x, y, z \in \mathcal{S}$, $\otimes, \oplus \in \{\cup_1, \cap_1, \ldots, \cup_n, \cap_n\}$, and $\otimes \neq \oplus$.

Definition 2.4. A multilattice $\mathcal{M}_n = \langle \mathcal{S}, \leq_1, \ldots, \leq_n \rangle$ is *bounded* if it has greatest and zero elements (denoted as 1_j and 0_j correspondingly) for each order.

Definition 2.5 (Classical multilattice). A multilattice $\mathcal{M}_n = \langle \mathcal{S}, \leq_1, \ldots, \leq_n \rangle$ is called *j-classical* iff $-_k -_j$ is a complementation with respect to the j-th order, i.e. $0_j = x \cap_j -_k -_j x$ and $1_j = x \cup_j -_k -_j x$, for each $x \in \mathcal{S}$ and $j, k \leqslant n$ such that $j \neq k$. A multilatice is called *classical* if it is j-classical for all $j \leqslant n$.

Remark 2.1. In what follows, referring to multilattices, we tacitly suppose them to be distributive and classical. ⊣

Definition 2.6 (Ultralogical multilattice; 14, p. 319, Definitions 2.3 and 2.4). A pair $\langle \mathcal{M}_n, \mathcal{U}_n \rangle$ is called an *ultralogical multilattice* iff $\mathcal{M}_n = \langle \mathcal{S}, \leq_1, \ldots, \leq_n \rangle$ is a multilattice and $\mathcal{U}_n \subsetneq \mathcal{S}$ satisfies the following conditions, for each $j, k \leqslant n$, $j \neq k$, and $x, y \in \mathcal{S}$:

- $x \cap_j y \in \mathcal{U}_n$ iff $x \in \mathcal{U}_n$ and $y \in \mathcal{U}_n$ ($\mathcal{U}_n$ is a multifilter (n-filter) on $\mathcal{M}_n$);
- $x \cup_j y \in \mathcal{U}_n$ iff $x \in \mathcal{U}_n$ or $y \in \mathcal{U}_n$ ($\mathcal{U}_n$ is a prime multifilter on $\mathcal{M}_n$);
- $x \in \mathcal{U}_n$ iff $-_j -_k x \notin \mathcal{U}_n$ ($\mathcal{U}_n$ is an ultramultifilter (n-ultrafilter) on $\mathcal{M}_n$).

Remark 2.2 (See 6, Definition 2.4; 7, Observation 1). Let $\langle \mathcal{M}_n, \mathcal{U}_n \rangle$ be an ultralogical multilattice. Then we can introduce for the lattices $\langle \mathcal{S}, \leq_1 \rangle, \ldots, \langle \mathcal{S}, \leq_n \rangle$ the corresponding *pseudo-complement* and *pseudo-difference* operations $\supset_1, \ldots, \supset_n$ and $\subset_1, \ldots, \subset_n$, respectively, as follows, for any $x, y \in \mathcal{S}$, $j \leqslant n$, and some *fixed* $k \leqslant n$ such that $j \neq k$:

$$\begin{aligned} x \supset_j y &= -_k -_j x \cup_j y, \\ x \subset_j y &= x \cap_j -_k -_j y. \end{aligned} \qquad ⊣$$

Definition 2.7 (Standard valuation; 24, Definition 4.5; 6, Definition 2.5). Let $\mathcal{M}_n = \langle \mathcal{S}, \leq_1, \ldots, \leq_n \rangle$ be a multilattice. A function v from $\mathcal{P}$ to $\mathcal{S}$ is called a *standard valuation* and is extended for any $\varphi, \psi \in \mathscr{F}_{\mathrm{N}}$ as follows:

(1) $v(\neg_j\varphi) = -_j v(\varphi)$;
(2) $v(\varphi \wedge_j \psi) = v(\varphi) \cap_j v(\psi)$;
(3) $v(\varphi \vee_j \psi) = v(\varphi) \cup_j v(\psi)$;
(4) $v(\varphi \to_j \psi) = v(\varphi) \supset_j v(\psi)$;
(5) $v(\varphi \leftarrow_j \psi) = v(\varphi) \subset_j v(\psi)$.

Remark 2.3. In [7, Definition 2.10], the notion of a *paraconsistent* valuation was also introduced (defined as a mapping from $\mathcal{P} \cup \{\neg_j p \mid p \in \mathcal{P}$, for all $j \leqslant n\}$ to $\mathcal{S}$). It was shown [7, Theorem 3.6] by the algebraic embedding theorem that $\mathbf{ML}_n$ is sound and complete with respect to multilattices with paraconsistent valuations. However, for the purposes of this paper it should be enough to consider the standard valuations only. In what follows, when we deal with multilattices, we mean by a valuation a standard one. ⊣

DEFINITION 2.8 (Entailment for $\mathbf{ML}_n$; cf. 24, Definitions 4.7 and 5.3). The entailment relation in $\mathbf{ML}_n$ is defined as follows, for all $\varGamma, \varDelta \subseteq \mathscr{F}_{\mathrm{N}}$ and $\varphi, \psi \in \mathscr{F}_{\mathrm{N}}$:

(1) $\varphi \models_j \psi$ iff for each multilattice $\mathcal{M}_n$ and each valuation v, it holds that $v(\varphi) \leq_j v(\psi)$.
(2) $\varGamma \models_{\mathbf{ML}_n} \varDelta$ iff for each ultralogical multilattice $\langle \mathcal{M}_n, \mathcal{U}_n \rangle$ and each valuation v, it holds that if $v(\gamma) \in \mathcal{U}_n$ for each $\gamma \in \varGamma$, then $v(\delta) \in \mathcal{U}_n$ for some $\delta \in \varDelta$.

LEMMA 2.1. *$\varGamma \models_{\mathbf{ML}_n} \varDelta$ iff for all v and $j \leqslant n$, $v(\bigwedge_j \varGamma) \models_j v(\bigvee_j \varDelta)$.*

PROOF. 1. Assume that $\varGamma \models_{\mathbf{ML}_n} \varDelta$ and for some valuation v, $v(\gamma) \in \mathcal{U}_n$ for each $\gamma \in \varGamma$. This means that $v(\bigwedge_j \varGamma) \in \mathcal{U}_n$. On the other hand, $v(\delta) \in \varDelta$ for some $\delta \in \varDelta$, so $v(\bigvee_j \varDelta) \in \mathcal{U}_n$. Next suppose that $v(\bigwedge_j \varGamma) \not\models_j v(\bigvee_j \varDelta)$ for some j. By the definition of $\models_j$, $v(\bigwedge_j \varGamma) \nleq_j v(\bigvee_j \varDelta)$. But then $v(\bigwedge_j \varGamma) \nleq_j -_k -_j v(\bigvee_j \varDelta)$, otherwise $-_k -_j v(\bigvee_j \varDelta) \in \mathcal{U}_n$, since $\mathcal{U}_n$ is closed under each of the relations. Clearly, $v(\bigwedge_j \varGamma) \nleq_j v(\bigvee_j \varDelta) \cup_j -_k -_j v(\bigvee_j \varDelta) = 1_j$ which is impossible (since $1_j \in \mathcal{U}_n$ and for each $x \in \mathcal{U}_n$, $x \leq_j 1_j$).

2. For the other direction suppose that $v(\bigwedge_j \varGamma) \in \mathcal{U}_n$. Hence, for any j, $x \in \mathcal{S}$, if $v(\bigwedge_j \varGamma) \leq_j x$, then $x \in \mathcal{U}_n$. In particular, take $x = v(\bigvee_j \varDelta)$. It follows from the definition of $\models_j$ that $v(\bigwedge_j \varGamma) \leq_j v(\bigvee_j \varDelta)$ for all j, so $v(\bigvee_j \varDelta) \in \mathcal{U}_n$, thus $v(\delta) \in \mathcal{U}_n$ for some $\delta \in \varDelta$ by the primness of $\mathcal{U}_n$. It remains to notice that $v(\gamma) \in \mathcal{U}_n$ for each $\gamma \in \varGamma$. Consequently, $\varGamma \models_{\mathbf{ML}_n} \varDelta$. ⊣

To embed a modal toolkit into the structure of a multilattice we make use of lattice closure and interior operations. Their additional properties influence the type of modal multilattice we obtain.

Definition 2.9 (14, p. 320, Definition 2.5). A multilattice $\mathcal{M}_n = \langle \mathcal{S}, \leq_1, \ldots, \leq_n \rangle$ is said to be *modal* if it is equipped with unary operations of *interior* I_j and *closure* C_j for each $j \leqslant n$, that is operations satisfying the following conditions (for $x, y \in \mathcal{S}$):

$$
\begin{aligned}
I_j(x) &\leq_j x; && \text{(decreasing)}\\
I_j(x) &= I_j I_j(x); && \text{(I-idempotent)}\\
I_j(x \cap_j y) &\leq_j I_j(x) \cap_j I_j(y); && \text{(sub-multiplicative)}\\
x &\leq_j C_j(x); && \text{(increasing)}\\
C_j(x) &= C_j C_j(x); && \text{(C-idempotent)}\\
C_j(x) \cup_j C_j(y) &\leq_j C_j(x \cup_j y). && \text{(sub-additive)}
\end{aligned}
$$

Remark 2.4. As Kamide and Shramko note [14], they supply multilattices with Tarski interior and closure operators, following the lines of Cattaneo and Ciucci's work [4]. In the next definition, we add to a modal multilattice the postulates regarding the inversions of the interior and closure operators as well as the postulates ($\mathsf{1}_j$ is open) and ($\mathsf{0}_j$ is closed) which should be present in the definition of the operators in question, according to [4, p. 44–46]. ⊣

Definition 2.10 (Tarski multilattice). A modal multilattice $\mathcal{M}_n = \langle \mathcal{S}, \leq_1, \ldots, \leq_n \rangle$ is said to be *Tarski* multilattice (or a modal multilattice with Tarski operators) iff for each $j \leqslant n$ operations I_j and C_j satisfy the following condition:

$$
\begin{aligned}
I_j(\mathsf{1}_j) &= \mathsf{1}_j; && \text{($\mathsf{1}_j$ is open)}\\
C_j(\mathsf{0}_j) &= \mathsf{0}_j; && \text{($\mathsf{0}_j$ is closed)}\\
-_j I_j(x) &= C_j(-_j x); && \text{($-_j I_j$-definition)}\\
-_j C_j(x) &= I_j(-_j x); && \text{($-_j C_j$-definition)}\\
-_k I_j(x) &= I_j(-_k x); && \text{($-_k I_j$-definition)}\\
-_k C_j(x) &= C_j(-_k x); && \text{($-_k C_j$-definition)}\\
I_j(x) &= -_j -_k C_j(-_j -_k x); && \text{(I-definition)}\\
C_j(x) &= -_j -_k I_j(-_j -_k x). && \text{(C-definition)}
\end{aligned}
$$

In the subsequent definitions, we introduce the required conditions for Kuratowski and Halmos operators, following [4, pp. 48 and 57].

DEFINITION 2.11 (Kuratowski multilattice). A Tarski multilattice $\mathcal{M}_n = \langle \mathcal{S}, \leq_1, \ldots, \leq_n \rangle$ is said to be *Kuratowski* (or a modal multilattice with Kuratowski operators) iff for each $j \leqslant n$ operations I_j and C_j satisfy the following condition:

$$I_j(x \cap_j y) = I_j(x) \cap_j I_j(y); \qquad \text{(multiplicative)}$$
$$C_j(x) \cup_j C_j(y) = C_j(x \cup_j y). \qquad \text{(additive)}$$

DEFINITION 2.12 (Halmos multilattice). A Kuratowski multilattice $\mathcal{M}_n = \langle \mathcal{S}, \leq_1, \ldots, \leq_n \rangle$ is said to be *Halmos* (or a modal multilattice with Halmos operators) iff for each $j \leqslant n$ operations I_j and C_j satisfy the following condition:

$$I_j(-_k -_j I_j(x)) = -_k -_j I_j(x); \qquad \text{(interior interconnection)}$$
$$C_j(-_k -_j C_j(x)) = -_k -_j C_j(x). \qquad \text{(closure interconnection)}$$

FACT 2.1. *Each Tarski* (*and hence Kuratowski and Halmos*) *multilattice has the following properties*:

$$x \leq_j y \textit{ implies } I_j(x) \leq_j I_j(y); \qquad (I\text{-monotonicity})$$
$$x \leq_j y \textit{ implies } C_j(x) \leq_j C_j(y). \qquad (C\text{-monotonicity})$$

PROOF. If $x \leq_j y$, then $I_j(x) = I_j(x \cap_j y) \leq_j I_j(x) \cap_j I_j(y) \leq_j I_j(y)$.
If $x \leq_j y$, then $C_j(x) \leq_j C_j(x) \cup_j C_j(y) \leq_j C_j(x \cup_j y) = C_j(y)$. $\dashv$

DEFINITION 2.13. Let $\mathcal{M}_n = \langle S, \leq_1, \ldots, \leq_n \rangle$ be a Tarski (resp. Kuratowski, Halmos) multilattice and v be a valuation introduced in Definition 2.7. Then we extend it for the modal formulas as follows:

(1) $v(\Box_j \varphi) = I_j v(\varphi)$,
(2) $v(\Diamond_j \varphi) = C_j v(\varphi)$.

The definition of an entailment relation for the case of a modal multilattice logic $\boldsymbol{L} \in \{\mathbf{MML}_n^{\mathbf{S4}}, \mathbf{MML}_n^{\mathbf{MNT4}}, \mathbf{MML}_n^{\mathbf{S5}}\}$ is almost the same as for the $\mathbf{ML}_n$ case. The only difference concerns with a type (Tarski, Kuratowski or Halmos) of a corresponding ultralogical multilattice. Thus we presume that $\mathit{\Gamma} \models_{\mathbf{MML}_n^{\mathbf{MNT4}}} \mathit{\Delta}$ ($\mathit{\Gamma} \models_{\mathbf{MML}_n^{\mathbf{S4}}} \mathit{\Delta}$, $\mathit{\Gamma} \models_{\mathbf{MML}_n^{\mathbf{S5}}} \mathit{\Delta}$) iff for each Tarski (Kuratowski, Halmos) ultralogical multilattice $\langle \mathcal{M}_n, \mathcal{U}_n \rangle$ and each valuation v, it holds that if $v(\gamma) \in \mathcal{U}_n$ for each $\gamma \in \mathit{\Gamma}$, then $v(\delta) \in \mathcal{U}_n$ for some $\delta \in \mathit{\Delta}$.

3. Preliminaries II: (hyper)sequent calculi for MNT4, S4, and S5

Definition 3.1 (Sequent). A sequent is an ordered pair written as follows: $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of $\mathfrak{L}$-formulas ($\mathfrak{L} \in \{\mathscr{L}, \mathscr{L}_{\mathrm{N}}, \mathscr{L}_{\mathrm{M}}\}$). A sequent is called valid for $\boldsymbol{L} \in \{\mathbf{ML}_n, \mathbf{MML}_n^{\mathbf{S4}}, \mathbf{MML}_n^{\mathbf{S5}}, \mathbf{MML}_n^{\mathbf{MNT4}}\}$ iff $\Gamma \models_{\boldsymbol{L}} \Delta$ holds (the case when both Γ and Δ are not empty). When $\Gamma \Rightarrow \Delta$ is valid for $\boldsymbol{L}$, we write $\boldsymbol{L} \models \Gamma \Rightarrow \Delta$. When $\Gamma = \emptyset$, $\boldsymbol{L} \models \; \Rightarrow \Delta$ iff $v(\delta) = \mathsf{1}_j$ for some $\delta \in \Delta$ and $j \leqslant n$; when $\Delta = \emptyset$, $\boldsymbol{L} \models \Gamma \Rightarrow$ iff $v(\gamma) = \mathsf{0}_j$ for all $\gamma \in \Gamma$ and some $j \leqslant n$.

Let us introduce Indrzejczak's [8, 9] cut-free sequent calculus for **MNT4**. The only axiom is as follows (for any $p \in \mathcal{P} \cup \mathcal{P}^*$):

$$(\mathrm{Ax}) \quad p \Rightarrow p$$

The structural rules are as follows:

$$(\mathrm{Cut})\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Theta \Rightarrow \Lambda}{\Gamma, \Theta \Rightarrow \Delta, \Lambda} \qquad (\mathrm{W}{\Rightarrow})\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$$

$$({\Rightarrow}\mathrm{W})\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$$

The non-modal logical rules are as follows:

$$(\wedge{\Rightarrow})\ \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \qquad ({\Rightarrow}\wedge)\ \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$(\vee{\Rightarrow})\ \frac{\varphi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta} \qquad ({\Rightarrow}\vee)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$(\to{\Rightarrow})\ \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Theta \Rightarrow \Lambda}{\varphi \to \psi, \Gamma, \Theta \Rightarrow \Delta, \Lambda} \qquad ({\Rightarrow}\to)\ \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}$$

$$(\leftarrow{\Rightarrow})\ \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\varphi \leftarrow \psi, \Gamma \Rightarrow \Delta} \qquad ({\Rightarrow}\leftarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Theta \Rightarrow \Lambda}{\Gamma, \Theta \Rightarrow \Delta, \Lambda, \varphi \leftarrow \psi}$$

$$(\neg{\Rightarrow})\ \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta} \qquad ({\Rightarrow}\neg)\ \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$$

The modal logical rules are as follows:

$$(\mathrm{M}_\Box)\ \frac{\varphi \Rightarrow \psi}{\Box\varphi \Rightarrow \Box\psi} \qquad (\mathrm{N}_\Box)\ \frac{\Rightarrow \varphi}{\Rightarrow \Box\varphi}$$

$$(4)\ \frac{\Box\varphi \Rightarrow \psi}{\Box\varphi \Rightarrow \Box\psi} \qquad (\Box{\Rightarrow})\ \frac{\varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta}$$

The rule $(\mathrm{M}_\Box)$ is derivable due to (4) and $(\Box{\Rightarrow})$.

Now we present yet another sequent calculus for **MNT4** in the language with both $\Box$ and $\Diamond$. For the modal rules below we use the following convention: the letter γ stands for the empty set or a one element set $\{\Box\psi\}$, the letter δ stands for the empty set or a one element set $\{\Diamond\psi\}$. This sequent calculus is obtained from Indrzejczak's one by a replacement of the rules ($\mathrm{M}_\Box$), ($\mathrm{N}_\Box$), and (4) with the following ones:

$$(\Rightarrow\Box)\ \frac{\gamma \Rightarrow \Diamond\Lambda, \varphi}{\gamma \Rightarrow \Diamond\Lambda, \Box\varphi} \qquad (\Diamond\Rightarrow)\ \frac{\varphi, \Box\Lambda \Rightarrow \delta}{\Diamond\varphi, \Box\Lambda \Rightarrow \delta} \qquad (\Rightarrow\Diamond)\ \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \Diamond\varphi}$$

The rules (4), ($\mathrm{M}_\Box$), ($\mathrm{M}_\Diamond$), ($\mathrm{N}_\Box$), ($\mathrm{N}_\Diamond$) are derivable in this calculus, where ($\mathrm{M}_\Diamond$) and ($\mathrm{N}_\Diamond$) are presented below:

$$(\mathrm{M}_\Diamond)\ \frac{\varphi \Rightarrow \psi}{\Diamond\varphi \Rightarrow \Diamond\psi} \qquad (\mathrm{N}_\Diamond)\ \frac{\varphi \Rightarrow}{\Diamond\varphi \Rightarrow}$$

If one replaces γ and δ in the rules ($\Rightarrow\Box$) and ($\Diamond\Rightarrow$) with the sets of formulas (possibly, empty, but not necessarily one element ones) $\Box\Gamma$ and $\Diamond\Delta$, respectively, then one obtains a cut-free sequent calculus for **S4** [16]. If in the rule ($\Rightarrow\Box$) one replaces γ with $\Box\Gamma$ and puts $\delta = \emptyset$ as well as in the rule ($\Diamond\Rightarrow$) puts $\gamma = \emptyset$ and replaces δ with $\Diamond\Delta$, then one gets an *in*complete version of **S4** [16] (although its restrictions for the $\Box$-free and $\Diamond$-free languages are complete [19]), since the sequents $\Box\varphi \Rightarrow \neg\Diamond\neg\varphi$, $\neg\Diamond\neg\varphi \Rightarrow \Box\varphi$, $\neg\Box\neg\varphi \Rightarrow \Diamond\varphi$ and $\Diamond\varphi \Rightarrow \neg\Box\neg\varphi$ are unprovable in it [16]. We mention the latter fact, because Kamide and Shramko used this incomplete version of **S4** sequent calculus as a basis of their logic $\mathbf{MML}_n$. As a result (which can be checked by the use of Kamide and Shramko's embedding theorems [14]), in the system $\mathbf{MML}_n$ the sequents $\Box_j\varphi \Rightarrow \neg_k\neg_j\Diamond_j\neg_k\neg_j\varphi$, $\neg_k\neg_j\Diamond_j\neg_k\neg_j\varphi \Rightarrow \Box_j\varphi$, $\neg_k\neg_j\Box_j\neg_k\neg_j\varphi \Rightarrow \Diamond_j\varphi$ and $\Diamond_j\varphi \Rightarrow \neg_k\neg_j\Box_j\neg_k\neg_j\varphi$ are not provable. In [6], the authors presented the logic $\mathbf{MML}_n^{\mathbf{S4}}$ which extends $\mathbf{MML}_n$ by these sequents. Similarly to the case of **S4**, in **MNT4** we need $\Box\Lambda$ and $\Diamond\Lambda$ in the rules ($\Rightarrow\Box$) and ($\Diamond\Rightarrow$), respectively, to make the sequents $\Box\varphi \Rightarrow \neg\Diamond\neg\varphi$, $\neg\Diamond\neg\varphi \Rightarrow \Box\varphi$, $\neg\Box\neg\varphi \Rightarrow \Diamond\varphi$ and $\Diamond\varphi \Rightarrow \neg\Box\neg\varphi$ provable.

Notice that although Indrzejczak's sequent calculus for **MNT4** in the $\Box$-language is cut-free, our calculus for **MNT4** in the language with both $\Box$ and $\Diamond$, unfortunately, seems to be non-cut-free. Consider the following application of cut ($\mathcal{D}_1$ and $\mathcal{D}_2$ stand for the derivations; see fig. 1).

Providing Gentzen-style constructive cut elimination proof, we could try to apply cut at earlier stage reducing the height of the derivation (see fig. 2).

$$
\dfrac{(\Rightarrow\Box)\ \dfrac{\begin{matrix}\mathcal{D}_1\\ \Box\psi \Rightarrow \Diamond\chi, \varphi\end{matrix}}{\Box\psi \Rightarrow \Diamond\chi, \Box\varphi} \qquad \dfrac{\begin{matrix}\mathcal{D}_2\\ \chi, \Box\omega \Rightarrow \Diamond\pi\end{matrix}}{\Diamond\chi, \Box\omega \Rightarrow \Diamond\pi}\ (\Diamond\Rightarrow)}{\Box\psi, \Box\omega \Rightarrow \Box\varphi, \Diamond\pi}
$$

Figure 1.

$$
\dfrac{\begin{matrix}\mathcal{D}_1\\ \Box\psi \Rightarrow \Diamond\chi, \varphi\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{D}_2\\ \chi, \Box\omega \Rightarrow \Diamond\pi\end{matrix}}{\Diamond\chi, \Box\omega \Rightarrow \Diamond\pi}\ (\Diamond\Rightarrow)}{\Box\psi, \Box\omega \Rightarrow \varphi, \Diamond\pi}
$$

Figure 2.

But, unfortunately, we cannot apply the rule $(\Rightarrow\Box)$ at this step to obtain $\Box\psi, \Box\omega \Rightarrow \Box\varphi, \Diamond\pi$, since we have two formulas with $\Box$ in the antecedent of the sequent (it is not a problem for **S4**, but here we have a restriction for the rule $(\Rightarrow\Box)$). We leave as a task for future research to present a cut-free sequent calculus for **MNT4** in the language with both $\Box$ and $\Diamond$.

A standard **S5** sequent calculus [19, 20] is known to be not-cut-free. There are various attempts to present cut-free non-standard versions of sequent calculi (in particular, hypersequent calculi). A survey of these attempts (focusing mainly on hypersequent calculi) may be found in Bednarska and Indrzejczak's paper [3]. We mention only the pioneering works by Mints [17, 18], Pottinger [22], and Avron [2] as well as the latest one by Indrzejczak [10] who introduced a *bi*sequent calculus for **S5**. In what follows, we will use Restall's hypersequent calculus for **S5** [23], since a $\mathbf{MML}_n^{\mathbf{S5}}$ hypersequent calculus was presented on its basis in [6].

Definition 3.2 (Hypersequent). A hypersequent is a finite multiset of sequents written as follows: $H := \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_m \Rightarrow \Delta_m$. A hypersequent is called valid iff at least one of its sequents is valid.

In Restall's calculus, the (internal) structural and logical rules of classical logic are in the hypersequent form. For example, the rules for conjunction are as follows (here and below H and G are hypersequents):

$$
(\wedge\Rightarrow)\ \frac{\varphi, \psi, \Gamma \Rightarrow \Delta \mid H}{\varphi\wedge\psi, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow\wedge)\ \frac{\Gamma \Rightarrow \Delta, \varphi \mid H \qquad \Gamma \Rightarrow \Delta, \psi \mid G}{\Gamma \Rightarrow \Delta, \varphi\wedge\psi \mid H \mid G}
$$

Restall's calculus has several external structural rules:

$$(\mathrm{EW}{\Rightarrow})\ \frac{H}{\varphi \Rightarrow \mid H} \qquad ({\Rightarrow}\mathrm{EW})\ \frac{H}{\Rightarrow \varphi \mid H}$$

$$(\mathrm{Merge})\ \frac{\Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H}{\Gamma, \Theta \Rightarrow \Delta, \Lambda \mid H}$$

Finally, it has modal logical rules:

$$(\Box{\Rightarrow})\ \frac{\varphi, \Gamma \Rightarrow \Delta \mid H}{\Box\varphi \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H} \qquad ({\Rightarrow}\Box)\ \frac{\Rightarrow \varphi \mid H}{\Rightarrow \Box\varphi \mid H}$$

$$(\Diamond{\Rightarrow})\ \frac{\varphi \Rightarrow \mid H}{\Diamond\varphi \Rightarrow \mid H} \qquad ({\Rightarrow}\Diamond)\ \frac{\Gamma \Rightarrow \Delta, \varphi \mid H}{\Gamma \Rightarrow \Delta \mid \Rightarrow \Diamond\varphi \mid H}$$

The rules for $\Diamond$ were added to Restall's system in [6]. Fortunately, the sequents $\Box\varphi \Rightarrow \neg\Diamond\neg\varphi$, $\neg\Diamond\neg\varphi \Rightarrow \Box\varphi$, $\Diamond\varphi \Leftrightarrow \neg\Box\neg\varphi$, and $\neg\Box\neg\varphi \Rightarrow \Diamond\varphi$ are provable in this calculus without any changes of the shape of the modal rules.

4. (Hyper)sequent calculi for modal multilattice logics

The sequent calculus for $\mathbf{MML}_n^{\mathbf{MNT4}}$ is as follows. Consider the axioms ($p \in \mathcal{P}$):

$$(\mathrm{Ax})\ p \Rightarrow p \qquad (\mathrm{Ax}_{\neg})\ \neg_j p \Rightarrow \neg_j p$$

The structural rules are as follows: (Cut), (W⇒), and (⇒W). The non-negated logical rules are presented below:

$$(\wedge_j{\Rightarrow})\ \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge_j \psi, \Gamma \Rightarrow \Delta} \qquad ({\Rightarrow}\wedge_j)\ \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge_j \psi}$$

$$(\vee_j{\Rightarrow})\ \frac{\varphi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee_j \psi, \Gamma \Rightarrow \Delta} \qquad ({\Rightarrow}\vee_j)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee_j \psi}$$

$$(\to_j{\Rightarrow})\ \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Theta \Rightarrow \Lambda}{\varphi \to_j \psi, \Gamma, \Theta \Rightarrow \Delta, \Lambda} \qquad ({\Rightarrow}\to_j)\ \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to_j \psi}$$

$$(\leftarrow_j{\Rightarrow})\ \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\varphi \leftarrow_j \psi, \Gamma \Rightarrow \Delta} \qquad ({\Rightarrow}\leftarrow_j)\ \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Theta \Rightarrow \Lambda}{\Gamma, \Theta \Rightarrow \Delta, \Lambda, \varphi \leftarrow_j \psi}$$

The jj-negated logical rules are as follows:

$$(\neg_j\wedge_j\Rightarrow)\ \frac{\neg_j\varphi, \Gamma \Rightarrow \Delta \qquad \neg_j\psi, \Gamma \Rightarrow \Delta}{\neg_j(\varphi \wedge_j \psi), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\neg_j\wedge_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_j\varphi, \neg_j\psi}{\Gamma \Rightarrow \Delta, \neg_j(\varphi \wedge_j \psi)}$$

$$(\neg_j\vee_j\Rightarrow)\ \frac{\neg_j\varphi, \neg_j\psi, \Gamma \Rightarrow \Delta}{\neg_j(\varphi \vee_j \psi), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\neg_j\vee_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_j\varphi \qquad \Gamma \Rightarrow \Delta, \neg_j\psi}{\Gamma \Rightarrow \Delta, \neg_j(\varphi \vee_j \psi)}$$

$$(\neg_j\rightarrow_j\Rightarrow)\ \frac{\neg_j\psi, \Gamma \Rightarrow \Delta, \neg_j\varphi}{\neg_j(\varphi \rightarrow_j \psi), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\neg_j\rightarrow_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_j\psi \qquad \neg_j\varphi, \Theta \Rightarrow \Lambda}{\Gamma, \Theta \Rightarrow \Delta, \Lambda, \neg_j(\varphi \rightarrow_j \psi)}$$

$$(\neg_j\leftarrow_j\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \neg_j\psi \qquad \neg_j\varphi, \Theta \Rightarrow \Lambda}{\neg_j(\varphi \leftarrow_j \psi), \Gamma, \Theta \Rightarrow \Delta, \Lambda} \qquad (\Rightarrow\neg_j\leftarrow_j)\ \frac{\neg_j\psi, \Gamma \Rightarrow \Delta, \neg_j\varphi}{\Gamma \Rightarrow \Delta, \neg_j(\varphi \leftarrow_j \psi)}$$

$$(\neg_j\neg_j\Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta}{\neg_j\neg_j\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\neg_j\neg_j)\ \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \neg_j\neg_j\varphi}$$

The kj-negated logical rules as follows:

$$(\neg_k\wedge_j\Rightarrow)\ \frac{\neg_k\varphi, \neg_k\psi, \Gamma \Rightarrow \Delta}{\neg_k(\varphi \wedge_j \psi), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\neg_k\wedge_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_k\varphi \qquad \Gamma \Rightarrow \Delta, \neg_k\psi}{\Gamma \Rightarrow \Delta, \neg_k(\varphi \wedge_j \psi)}$$

$$(\neg_k\vee_j\Rightarrow)\ \frac{\neg_k\varphi, \Gamma \Rightarrow \Delta \qquad \neg_k\psi, \Gamma \Rightarrow \Delta}{\neg_k(\varphi \vee_j \psi), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\neg_k\vee_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_k\varphi, \neg_k\psi}{\Gamma \Rightarrow \Delta, \neg_k(\varphi \vee_j \psi)}$$

$$(\neg_k\rightarrow_j\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \neg_k\varphi \qquad \neg_k\psi, \Theta \Rightarrow \Lambda}{\neg_k(\varphi \rightarrow_j \psi), \Gamma, \Theta \Rightarrow \Delta, \Lambda} \qquad (\Rightarrow\neg_k\rightarrow_j)\ \frac{\neg_k\varphi, \Gamma \Rightarrow \Delta, \neg_k\psi}{\Gamma \Rightarrow \Delta, \neg_k(\varphi \rightarrow_j \psi)}$$

$$(\neg_k\leftarrow_j\Rightarrow)\ \frac{\neg_k\varphi, \Gamma \Rightarrow \Delta, \neg_k\psi}{\neg_k(\varphi \leftarrow_j \psi), \Gamma \Rightarrow \Delta} \qquad (\neg_k\leftarrow_j\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \neg_k\varphi \qquad \neg_k\psi, \Theta \Rightarrow \Lambda}{\Gamma, \Theta \Rightarrow \Delta, \Lambda, \neg_k(\varphi \leftarrow_j \psi)}$$

$$(\neg_k\neg_j\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg_k\neg_j\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\neg_k\neg_j)\ \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg_k\neg_j\varphi}$$

For the modal logical rules below we adopt the following convention: the letter π denotes a set which is empty or consists of exactly one formula from the list of formulas $\Box_j\psi, \neg_j\Diamond_j\psi, \neg_k\Box_j\psi$, where $k \neq j$; the letter δ denotes a set which is empty or consists of exactly one formula from the list $\Diamond_j\psi, \neg_j\Box_j\psi, \neg_k\Diamond_j\psi$, where $k \neq j$; the letter $\Lambda^\sharp$ stands for the set (possibly, empty) $\{\Box_j\Lambda_1, \neg_j\Diamond_j\Lambda_2, \neg_k\Box_j\Lambda_3\}$, where $k \neq j$; and the letter $\Lambda^\flat$ stands for the set (possibly, empty) $\{\Diamond_j\Lambda_1, \neg_j\Box_j\Lambda_2, \neg_k\Diamond_j\Lambda_3\}$, where again $k \neq j$.

The non-negated modal rules:

$$(\Box_j\Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta}{\Box_j\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\Box_j)\ \frac{\pi \Rightarrow \Lambda^\flat, \varphi}{\pi \Rightarrow \Lambda^\flat, \Box_j\varphi}$$

$$(\Diamond_j\Rightarrow)\ \frac{\varphi, \Lambda^\sharp \Rightarrow \delta}{\Diamond_j\varphi, \Lambda^\sharp \Rightarrow \delta} \qquad (\Rightarrow\Diamond_j)\ \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \Diamond_j\varphi}$$

The jj-negated modal logical rules:

$$(\neg_j\Box_j\Rightarrow)\ \frac{\neg_j\varphi, \Lambda^\sharp \Rightarrow \delta}{\neg_j\Box_j\varphi, \Lambda^\sharp \Rightarrow \delta} \qquad (\Rightarrow\neg_j\Box_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_j\varphi}{\Gamma \Rightarrow \Delta, \neg_j\Box_j\varphi}$$

$$(\neg_j\Diamond_j \Rightarrow)\ \frac{\neg_j\varphi, \Gamma \Rightarrow \Delta}{\neg_j\Diamond_j\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\neg_j\Diamond_j)\ \frac{\pi \Rightarrow \Lambda^\flat, \neg_j\varphi}{\pi \Rightarrow \Lambda^\flat, \neg_j\Diamond_j\varphi}$$

The kj-negated modal logical rules:

$$(\neg_k\Box_j\Rightarrow)\ \frac{\neg_k\varphi, \Gamma \Rightarrow \Delta}{\neg_k\Box_j\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\neg_k\Box_j)\ \frac{\pi \Rightarrow \Lambda^\flat, \neg_k\varphi}{\pi \Rightarrow \Lambda^\flat, \neg_k\Box_j\varphi}$$

$$(\neg_k\Diamond_j\Rightarrow)\ \frac{\neg_k\varphi, \Lambda^\sharp \Rightarrow \delta}{\neg_k\Diamond_j\varphi, \Lambda^\sharp \Rightarrow \delta} \qquad (\Rightarrow\neg_k\Diamond_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_k\varphi}{\Gamma \Rightarrow \Delta, \neg_k\Diamond_j\varphi}$$

Remark 4.1. One can obtain a sequent calculus for $\mathbf{MML}_n^{\mathbf{S4}}$ [6] from the set of above rules replacing in each of modal rule the letters π and δ, respectively, with the sets $\{\Box_j\Gamma_1, \neg_j\Diamond_j\Gamma_2, \neg_k\Box_j\Gamma_3\}$ and $\{\Diamond_j\Delta_1, \neg_j\Box_j\Delta_2, \neg_k\Diamond_j\Delta_3\}$, where $k \neq j$. ⊣

One can obtain a hypersequent calculus for $\mathbf{MML}_n^{\mathbf{S5}}$ [6] from the sequent calculus for $\mathbf{MML}_n^{\mathbf{MNT4}}$ or $\mathbf{MML}_n^{\mathbf{S4}}$ as follows: (1) all the (internal) structural and logical non-modal rules should be presented into the hypersequent form; (2) one should add the rules (EW⇒), (⇒EW), and (Merge); (3) one should replace all the modal rules with the following ones. The non-negated modal rules:

$$(\Box_j\Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta \mid H}{\Box_j\varphi \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow\Box_j)\ \frac{\Rightarrow \varphi \mid H}{\Rightarrow \Box_j\varphi \mid H}$$

$$(\Diamond_j\Rightarrow)\ \frac{\varphi \Rightarrow \mid H}{\Diamond_j\varphi \Rightarrow \mid H} \qquad (\Rightarrow\Diamond_j)\ \frac{\Gamma \Rightarrow \Delta, \varphi \mid H}{\Gamma \Rightarrow \Delta \mid \Rightarrow \Diamond_j\varphi \mid H}$$

The jj-negated modal rules:

$$(\neg_j\Box_j\Rightarrow)\ \frac{\neg_j\varphi \Rightarrow \mid H}{\neg_j\Box_j\varphi \Rightarrow \mid H} \qquad (\Rightarrow\neg_j\Box_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_j\varphi \mid H}{\Gamma \Rightarrow \Delta \mid \Rightarrow \neg_j\Box_j\varphi \mid H}$$

$$(\neg_j\Diamond_j\Rightarrow)\ \frac{\neg_j\varphi, \Gamma \Rightarrow \Delta \mid H}{\neg_j\Diamond_j\varphi \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow\neg_j\Diamond_j)\ \frac{\Rightarrow \neg_j\varphi \mid H}{\Rightarrow \neg_j\Diamond_j\varphi \mid H}$$

The kj-negated modal rules:

$$(\neg_k\Box_j\Rightarrow)\ \frac{\neg_k\varphi, \Gamma \Rightarrow \Delta \mid H}{\neg_k\Box_j\varphi \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow\neg_k\Box_j)\ \frac{\Rightarrow \neg_k\varphi \mid H}{\Rightarrow \neg_k\Box_j\varphi \mid H}$$

$$(\neg_k\Diamond_j\Rightarrow)\ \frac{\neg_k\varphi \Rightarrow \mid H}{\neg_k\Diamond_j\varphi \Rightarrow \mid H} \qquad (\Rightarrow\neg_k\Diamond_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_k\varphi \mid H}{\Gamma \Rightarrow \Delta \mid \Rightarrow \neg_k\Diamond_j\varphi \mid H}$$

5. Algebraic completeness

5.1. Soundness

Lemma 5.1. *All the rules of the sequent calculus* $\mathbf{MML}_n^{\mathbf{MNT4}}$ *are sound with respect to modal multilattices with Tarskian operators.*

Proof. Propositional cases do not present difficulties and are proved in the usual way [see, e.g., 7]. We consider some representative modal instances.

Ad $(\Rightarrow\Box_j)$. Suppose that $\mathbf{MML}_n^{\mathbf{MNT4}} \models \pi \Rightarrow \Lambda^\flat, \varphi$. For concreteness assume that π is of the form $\neg_j\Diamond_j\psi$, $\Lambda^\flat$ is non-empty and consists of the formula $\lambda^\flat$. So, $v(\neg_j\Diamond_j\psi) \in \mathcal{U}_n$ implies $v(\lambda^\flat) \in \mathcal{U}_n$ or $v(\varphi) \in \mathcal{U}_n$. Let $v(\neg_j\Diamond_j\psi) \in \mathcal{U}_n$, hence $v(\lambda^\flat) \in \mathcal{U}_n$ or $v(\varphi) \in \mathcal{U}_n$, and we have to show that $v(\lambda^\flat) \in \mathcal{U}_n$ or $v(\Box_j\varphi) \in \mathcal{U}_n$.

First of all, let us prove that $v(\neg_j\Diamond_j\psi) \leq_j v(\varphi)$ implies $v(\neg_j\Diamond_j\psi) \leq_j v(\Box_j\varphi)$. Let $v(\neg_j\Diamond_j\psi) \leq_j v(\varphi)$. By Definition 2.13, $-_jC_j(v(\psi)) \leq_j v(\varphi)$. By ($-_jC_j$-definition), $I_j(-_jv(\psi)) \leq_j v(\varphi)$. By ($I$-monotonicity), $I_j(I_j(-_jv(\psi))) \leq_j I_j(v(\varphi))$. Therefore, by ($I$-idempotent), $I_j(-_jv(\psi)) \leq_j I_j(v(\varphi))$. Hence, by ($-_jC_j$-definition) and Definition 2.13, $v(\neg_j\Diamond_j\psi) \leq_j v(\Box_j\varphi)$.

Recall that $v(\lambda^\flat) \in \mathcal{U}_n$ or $v(\varphi) \in \mathcal{U}_n$. Assume that $v(\lambda^\flat) \in \mathcal{U}_n$. Then $v(\lambda^\flat) \in \mathcal{U}_n$ or $v(\Box_j\varphi) \in \mathcal{U}_n$. Hence, $v(\neg_j\Diamond_j\psi) \in \mathcal{U}_n$ implies $v(\lambda^\flat) \in \mathcal{U}_n$ or $v(\Box_j\varphi) \in \mathcal{U}_n$, i.e. $\mathbf{MML}_n^{\mathbf{MNT4}} \models \pi \Rightarrow \Lambda^\flat, \Box_j\varphi$. Suppose that $v(\varphi) \in \mathcal{U}_n$. Then $v(\neg_j\Diamond_j\psi) \in \mathcal{U}_n$ implies $v(\varphi) \in \mathcal{U}_n$. Therefore, $v(\neg_j\Diamond_j\psi) \leq_j v(\varphi)$ which, as we already know, implies $v(\neg_j\Diamond_j\psi) \leq_j v(\Box_j\varphi)$. Hence, $v(\neg_j\Diamond_j\psi) \in \mathcal{U}_n$ implies $v(\Box_j\varphi) \in \mathcal{U}_n$. Since $v(\neg_j\Diamond_j\psi) \in \mathcal{U}_n$, we have $v(\Box_j\varphi) \in \mathcal{U}_n$, and so $v(\lambda^\flat) \in \mathcal{U}_n$ or $v(\Box_j\varphi) \in \mathcal{U}_n$. Therefore, $\mathbf{MML}_n^{\mathbf{MNT4}} \models \pi \Rightarrow \Lambda^\flat, \Box_j\varphi$.

Other instances of π are treated similarly.

Ad $(\Rightarrow\neg_j\Box_j)$. Assume that $\mathbf{MML}_n^{\mathbf{MNT4}} \models \Gamma \Rightarrow \Delta, \neg_j\varphi$. Then for any valuation v, $v(\psi) \in \mathcal{U}_n$ for all $\psi \in \Gamma$ implies $v(\xi) \in \mathcal{U}_n$ for some $\xi \in \Delta$ or $v(\neg_j\varphi) \in \mathcal{U}_n$. The first of the disjuncts directly yields $\mathbf{MML}_n^{\mathbf{MNT4}} \models \Gamma \Rightarrow \Delta, \neg_j\Box_j\varphi$, while the second gives $v(\neg_j\Box_j\varphi) \in \mathcal{U}_n$. Indeed, $v(\neg_j\varphi) \leq_j C_j(v(\neg_j\varphi))$ by increasing. But $C_j(v(\neg_j\varphi)) = v(\Diamond_j\neg_j\varphi) \in \mathcal{U}_n$ by Definiton 2.13 and since $\mathcal{U}_n$ is a filter. Thus, $v(\neg_j\Box_j\varphi) \in \mathcal{U}_n$.

Ad ($\Rightarrow\neg_k\Box_j$). Using the similar conventions and general scheme of reasoning as in case ($\Rightarrow\Box_j$), we need to show $\mathbf{MML}_n^{\mathbf{MNT4}} \models \pi \Rightarrow \lambda^\flat, \neg_k\Box_j\varphi$ under the assumption $\mathbf{MML}_n^{\mathbf{MNT4}} \models \pi \Rightarrow \lambda^\flat, \neg_k\varphi$. So, suppose that $\lambda^\flat \in \mathcal{U}_n$ or $\neg_k\varphi \in \mathcal{U}_n$. Let us consider only the crucial sub-case $\neg_k\varphi \in \mathcal{U}_n$. This time we need to show that $v(\neg_j\Diamond_j\psi) \leq_j v(\neg_k\varphi)$ implies $v(\neg_j\Diamond_j\psi) \leq_j v(\neg_k\Box_j\varphi)$. It is done similarly to the case of ($\Rightarrow\Box_j$), but with the use of ($-_k I_j$-definition). Then, applying (I-monotonicity) and (I-idempotent), we get $I_j(v(\neg_j\psi)) \leq_j I_j(v(\neg_k\varphi))$. Thus, $I_j(v(\neg_k\varphi)) = -_k I_j(v(\varphi)) \in \mathcal{U}_n$, by Definition 2.10 and, finally, $v(\neg_k\Box_j\varphi) \in \mathcal{U}_n$, by Definition 2.13. Hence, $\mathbf{MML}_n^{\mathbf{MNT4}} \models \pi \Rightarrow \lambda^\flat, \neg_k\Box_j\varphi$.

Ad ($\Diamond_j\Rightarrow$). Suppose that $\mathbf{MML}_n^{\mathbf{MNT4}} \models \varphi, \Lambda^\sharp \Rightarrow \delta$. For concreteness assume that δ is of the form $\neg_j\Box_j\psi$, $\Lambda^\sharp$ is non-empty and consists of the formula $\lambda^\sharp$. Thus, if $v(\varphi) \in \mathcal{U}_n$ and $v(\lambda^\sharp) \in \mathcal{U}_n$, then $v(\neg_j\Box_j\psi) \in \mathcal{U}_n$.

Let us show that $v(\varphi) \leq_j v(\neg_j\Box_j\psi)$ implies $v(\Diamond_j\varphi) \leq_j v(\neg_j\Box_j\psi)$. Assume that $v(\varphi) \leq_j v(\neg_j\Box_j\psi)$, i.e. $v(\varphi) \leq_j -_j I_j(v(\psi))$. Then, by ($-_j I_j$-definition), $v(\varphi) \leq_j C_j(-_j v(\psi))$. By ($C$-monotonicity), $C_j(v(\varphi)) \leq_j C_j(C_j(-_j v(\psi)))$. By ($C$-idempotent), $C_j(v(\varphi)) \leq_j C_j(-_j v(\psi))$. Then $C_j(v(\varphi)) \leq_j -_j I_j(v(\psi))$, i.e. $v(\Diamond_j\varphi) \leq_j v(\neg_j\Box_j\psi)$.

Suppose $v(\Diamond_j\varphi) \in \mathcal{U}_n$ and $v(\lambda^\sharp) \in \mathcal{U}_n$. Assume that $v(\varphi) \in \mathcal{U}_n$. Then $v(\varphi) \in \mathcal{U}_n$ and $v(\lambda^\sharp) \in \mathcal{U}_n$. Hence, $v(\neg_j\Box_j\psi) \in \mathcal{U}_n$. Therefore, $v(\varphi) \in \mathcal{U}_n$ implies $v(\neg_j\Box_j\psi) \in \mathcal{U}_n$. Thus, $v(\varphi) \leq_j v(\neg_j\Box_j\psi)$ which gives us $v(\Diamond_j\varphi) \leq_j v(\neg_j\Box_j\psi)$. Consequently, $v(\Diamond_j\varphi) \in \mathcal{U}_n$ implies $v(\neg_j\Box_j\psi) \in \mathcal{U}_n$. Since $v(\Diamond_j\varphi) \in \mathcal{U}_n$, we have $v(\neg_j\Box_j\psi) \in \mathcal{U}_n$. Finally, we get if $v(\Diamond_j\varphi) \in \mathcal{U}_n$ and $v(\lambda^\sharp) \in \mathcal{U}_n$, then $v(\neg_j\Box_j\psi) \in \mathcal{U}_n$, i.e. $\mathbf{MML}_n^{\mathbf{MNT4}} \models \Diamond_j\varphi, \Lambda^\sharp \Rightarrow \delta$.

The other instances of δ are treated similarly.

Ad ($\Rightarrow\Diamond_j$). Suppose that $\mathbf{MML}_n^{\mathbf{MNT4}} \models \Gamma \Rightarrow \Delta, \varphi$. Then if $v(\chi) \in \mathcal{U}_n$ for each $\chi \in \Gamma$, then $v(\omega) \in \mathcal{U}_n$ for some $\omega \in \Delta$ or $v(\varphi) \in \mathcal{U}_n$. Assume that $v(\chi) \in \mathcal{U}_n$ for each $\chi \in \Gamma$. Then $v(\omega) \in \mathcal{U}_n$ for some $\omega \in \Delta$ or $v(\varphi) \in \mathcal{U}_n$. The former disjunct straightforwardly implies $\mathbf{MML}_n^{\mathbf{MNT4}} \models \Gamma \Rightarrow \Delta, \Diamond_j\varphi$. So $v(\varphi) \in \mathcal{U}_n$. By (increasing), $v(\varphi) \leq_j C_j(v(\varphi))$. Thus, $v(\Diamond_j\varphi) \in \mathcal{U}_n$ which gives us $\mathbf{MML}_n^{\mathbf{MNT4}} \models \Gamma \Rightarrow \Delta, \Diamond_j\varphi$. ⊣

LEMMA 5.2. *All the rules of the sequent calculus* $\mathbf{MML}_n^{\mathbf{S4}}$ *are sound with respect to modal multilattices with Kuratowski operators.*

PROOF. Recall that each Kuratowski multilattice is a Tarski multilattice and that the sequent calculus for $\mathbf{MML}_n^{\mathbf{S4}}$ differs from the one for $\mathbf{MML}_n^{\mathbf{MNT4}}$ by the formulation of the rules ($\Rightarrow\Box_j$), ($\Diamond_j\Rightarrow$), ($\neg_j\Box_j\Rightarrow$),

$(\Rightarrow\neg_j\Diamond_j)$, $(\Rightarrow\neg_k\Box_j)$, and $(\neg_k\Diamond_j\Rightarrow)$. Thus, we need to consider only these rules.

Ad $(\Rightarrow\Box_j)$. This proof is a generalization of the proof of soundness of the right rule for $\Box_j$ of $\mathbf{MML}_n$ with respect to De Morgan modal multilattices presented in [7, Lemma 5.4]. Suppose that

$$\mathbf{MML}_n^{\mathbf{S4}} \models \Box_j\Gamma_1, \neg_j\Diamond_j\Gamma_2, \neg_k\Box_j\Gamma_3 \Rightarrow \Diamond_j\Lambda_1, \neg_j\Box_j\Lambda_2, \neg_k\Diamond_j\Lambda_3, \varphi.$$

We leave it to the reader to show that $\mathbf{MML}_n^{\mathbf{S4}} \models \Box_j\Gamma_1, \neg_j\Diamond_j\Gamma_2, \neg_k\Box_j\Gamma_3, \neg_j\neg_k\Diamond_j\Lambda_1, \neg_j\neg_k\neg_j\Box_j\Lambda_2, \neg_k\neg_j\neg_k\Diamond_j\Lambda_3 \Rightarrow \varphi$. For simplicity we assume that Γ_1, $\Gamma_2$and Γ_3 consist of the formulas γ_1, γ_2 and γ_3, respectively, and Λ_1, Λ_2 and Λ_3 consist of the formulas λ_1, λ_2 and λ_3, respectively. By Definitions 2.8 and 2.13, we have $\Phi_1 \cap_j \Phi_2 \leq_j v(\varphi)$, where $\Phi_1 = I_j(v(\gamma_1)) \cap_j -_jC_j(v(\gamma_2)) \cap_j -_kI_j(v(\gamma_3))$ and $\Phi_2 = -_j-_kC_j(v(\lambda_1)) \cap_j -_j-_k-_jI_j(v(\lambda_2)) \cap_j -_j-_k-_kC_j(v(\lambda_3))$.

Using Definition 2.11, we modify Φ_1 as follows:

$$\Phi_1 = I_j(v(\gamma_1)) \cap_j I_j(-_jv(\gamma_2)) \cap_j I_j(-_kv(\gamma_3)).$$

By Definition 2.11, $-_j-_k-_jI_j(v(\lambda_2)) = -_j-_kC_j(-_jv(\lambda_2))$. Using this Definition again, we have:

$$\begin{aligned}
-_j-_kC_j(v(\lambda_1)) &= -_j-_k-_j-_kI_j(-_j-_kv(\lambda_1)),\\
-_j-_k-_jI_j(v(\lambda_2)) &= -_j-_k-_j-_kI_j(-_j-_k-_jv(\lambda_2)),\\
-_j-_k-_kC_j(v(\lambda_3)) &= -_k-_j-_k-_j-_kI_j(-_j-_kv(\lambda_3)).
\end{aligned}$$

We leave it to the reader to prove that $-_j-_k-_j-_kv(\psi) = v(\psi)$ and $-_k-_j-_k-_jv(\psi) = v(\psi)$, for each $\psi \in \mathscr{F}_{\mathrm{M}}$. We modify Φ_2 as follows:

$$\Phi_2 = I_j(-_j-_kv(\lambda_1)) \cap_j I_j(-_j-_k-_jv(\lambda_2)) \cap_j -_kI_j(-_j-_kv(\lambda_3)).$$

Using Definition 2.11, we modify the last conjunct of Φ_2:

$$\Phi_2 = I_j(-_j-_kv(\lambda_1)) \cap_j I_j(-_j-_k-_jv(\lambda_2)) \cap_j I_j(-_k-_j-_kv(\lambda_3)).$$

Using Definition 2.11 (the property (multiplicative)), we have:

$$\begin{aligned}
\Phi_1 &= I_j\big(v(\gamma_1) \cap_j -_jv(\gamma_2) \cap_j -_kv(\gamma_3)\big),\\
\Phi_2 &= I_j\big(-_j-_kv(\lambda_1) \cap_j -_j-_k-_jv(\lambda_2) \cap_j -_k-_j-_kv(\lambda_3)\big).
\end{aligned}$$

Let us introduce two new abbreviations:

$$\begin{aligned}
\Psi_1 &:= v(\gamma_1) \cap_j -_jv(\gamma_2) \cap_j -_kv(\gamma_3),\\
\Psi_2 &:= -_j-_kv(\lambda_1) \cap_j -_j-_k-_jv(\lambda_2) \cap_j -_k-_j-_kv(\lambda_3).
\end{aligned}$$

We have $\Phi_1 = I_j(\Psi_1)$ and $\Phi_2 = I_j(\Psi_2)$. Then $\Phi_1 \cap_j \Phi_2 = I_j(\Psi_1) \cap_j I_j(\Psi_2)$. Using (multiplicative) once again, we have $I_j(\Psi_1) \cap_j I_j(\Psi_2) = I_j(\Psi_1 \cap_j \Psi_2)$. Therefore, $I_j(\Psi_1 \cap_j \Psi_2) \leq_j v(\varphi)$. By I-monotonicity, $I_j(I_j(\Psi_1 \cap_j \Psi_2)) \leq_j I_j(v(\varphi))$. By Definition 2.11 ($I$-idempotent), $I_j(\Psi_1 \cap_j \Psi_2) \leq_j I_j(v(\varphi))$. Then $I_j(\Psi_1) \cap_j I_j(\Psi_2) \leq_j I_j(v(\varphi))$, i.e. $\Phi_1 \cap_j \Phi_2 \leq_j I_j(v(\varphi))$. By Definition 2.13, $\mathbf{MML}_n^{\mathbf{S4}} \models \Box_j \Gamma_1, \neg_j \Diamond_j \Gamma_2, \neg_k \Box_j \Gamma_3, \neg_j \neg_k \Diamond_j \Lambda_1, \neg_j \neg_k \neg_j \Box_j \Lambda_2, \neg_k \neg_j \neg_k \Diamond_j \Lambda_3 \Rightarrow \Box_j \varphi$. The latter fact implies the required one:

$$\mathbf{MML}_n^{\mathbf{S4}} \models \Box_j \Gamma_1, \neg_j \Diamond_j \Gamma_2, \neg_k \Box_j \Gamma_3 \Rightarrow \Diamond_j \Lambda_1, \neg_j \Box_j \Lambda_2, \neg_k \Diamond_j \Lambda_3, \Box_j \varphi.$$

The other cases are proved similarly. ⊣

In order to simplify a soundness proof for $\mathbf{MML}_n^{\mathbf{S5}}$ we need the following definition [see 21 for the translations of hypersequents and tree-hypersequents to formulas].

Definition 5.1. A transtation τ of hypersequents to $\mathscr{L}_{\mathrm{M}}$-formulas is inductively defined as follows:

- $\tau(\Gamma \Rightarrow \Delta) = \bigwedge_j \Gamma \rightarrow_j \bigvee_j \Delta$,[1]
- $\tau(\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_m \Rightarrow \Delta_m) = \Box_j \tau(\Gamma_1 \Rightarrow \Delta_1) \vee_j \cdots \vee_j \Box_j \tau(\Gamma_m \Rightarrow \Delta_m)$.

Proposition 5.1. *A hypersequent H is valid in $\mathbf{MML}_n^{\mathbf{S5}}$ iff the formula $\tau(H)$ is valid in $\mathbf{MML}_n^{\mathbf{S5}}$.*

Proof. Suppose that a hypersequent $H = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_m \Rightarrow \Delta_m$ is valid in $\mathbf{MML}_n^{\mathbf{S5}}$. Then there is at least one i such that $1 \leqslant i \leqslant m$ and $\Gamma_i \Rightarrow \Delta_i$ is valid. Thus, if $v(\varphi) \in \mathcal{U}_n$ for all $\varphi \in \Gamma_i$, then $v(\psi) \in \mathcal{U}_n$ for some $\psi \in \Delta_i$. Since $\mathcal{U}_n$ is a prime multifilter, we have that if $v(\bigwedge_j \Gamma_i) \in \mathcal{U}_n$, then $v(\bigvee_j \Delta_i) \in \mathcal{U}_n$. Thus, $v(\bigwedge_j \Gamma_i \rightarrow_j \bigvee_j \Delta_i) \in \mathcal{U}_n$. Therefore, $1_j \leq_j v(\bigwedge_j \Gamma_i \rightarrow_j \bigvee_j \Delta_i)$. By ($I$-monotonicity), $I_j(1_j) \leq_j I_j(v(\bigwedge_j \Gamma_i \rightarrow_j \bigvee_j \Delta_i))$. By ($1_j$ is open), $1_j \leq_j I_j\big(v(\bigwedge_j \Gamma_i \rightarrow_j \bigvee_j \Delta_i)\big)$. So $I_j\big(v(\bigwedge_j \Gamma_i \rightarrow_j \bigvee_j \Delta_i)\big) \in \mathcal{U}_n$, i.e. $v\big(\Box_j(\bigwedge_j \Gamma_i \rightarrow_j \bigvee_j \Delta_i)\big) \in \mathcal{U}_n$. Since $\mathcal{U}_n$ is prime, $v\big(\Box_j(\bigwedge_j \Gamma_1 \rightarrow_j \bigvee_j \Delta_1)\big) \vee \cdots \vee v\big(\Box_j(\bigwedge_j \Gamma_m \rightarrow_j \bigvee_j \Delta_m)\big) \in \mathcal{U}_n$, i.e. $\tau(H) \in \mathcal{U}_n$. So $\tau(H)$ is valid in $\mathbf{MML}_n^{\mathbf{S5}}$.

Suppose that $\tau(H)$ is valid in $\mathbf{MML}_n^{\mathbf{S5}}$. Thus, $v\big(\Box_j(\bigwedge_j \Gamma_1 \rightarrow_j \bigvee_j \Delta_1)\big) \vee \cdots \vee v\big(\Box_j(\bigwedge_j \Gamma_m \rightarrow_j \bigvee_j \Delta_m)\big) \in \mathcal{U}_n$. Then there is an i such that $1 \leqslant$

[1] To be more precise, this definition holds, if both $\Gamma \neq \emptyset$ and $\Delta \neq \emptyset$. If $\Gamma = \emptyset$, then $\tau(\Gamma \Rightarrow \Delta) = \bigvee_j \Delta$; if $\Delta = \emptyset$, then $\tau(\Gamma \Rightarrow \Delta) = \neg_k \neg_j \bigwedge_j \Gamma$; if both $\Gamma = \emptyset$ and $\Delta = \emptyset$, then $\tau(\Gamma \Rightarrow \Delta) = p \wedge_j \neg_k \neg_j p$.

$i \leqslant m$ and $\Box_j\big(v(\bigwedge_j \Gamma_i \to_j \bigvee_j \Delta_i)\big) \in \mathcal{U}_n$. By (decreasing), $v(\bigwedge_j \Gamma_i \to_j \bigvee_j \Delta_i) \in \mathcal{U}_n$. Hence, $v(\bigwedge_j \Gamma_i) \in \mathcal{U}_n$ implies $v(\bigvee_j \Delta_i) \in \mathcal{U}_n$. So $v(\varphi) \in \mathcal{U}_n$ for any $\varphi \in \Gamma_i$ implies $v(\psi) \in \mathcal{U}_n$ for some $\psi \in \Delta_i$. Therefore, $\Gamma_i \Rightarrow \Delta_i$ is valid. Hence, H is valid in $\mathbf{MML}_n^{\mathbf{S5}}$. ⊣

Lemma 5.3. *All the rules of the sequent calculus* $\mathbf{MML}_n^{\mathbf{S5}}$ *are sound with respect to modal multilattices with Halmos operators.*

Proof. *Ad* ($\Box_j\Rightarrow$). Assume that $\mathbf{MML}_n^{\mathbf{S5}} \models \varphi, \Gamma \Rightarrow \Delta \mid H$. Then, by Proposition 5.1, $\mathbf{MML}_n^{\mathbf{S5}} \models \tau(\varphi, \Gamma \Rightarrow \Delta \mid H)$, i.e. $\mathbf{MML}_n^{\mathbf{S5}} \models \Box_j\big((\varphi \wedge_j \bigwedge_j \Gamma) \to_j \bigvee_j \Delta\big) \vee_j \Box_j\tau(H)$. Then $\mathbf{MML}_n^{\mathbf{S5}} \models \Box_j\big((\varphi \wedge_j \bigwedge_j \Gamma) \to_j \bigvee_j \Delta\big)$ or $\mathbf{MML}_n^{\mathbf{S5}} \models \Box_j\tau(H)$. The latter option immediately implies the required result. Let us consider the former one. We leave it to the reader to check that $\mathbf{MML}_n^{\mathbf{S5}} \models \Box_j(\chi \to_j \omega) \to_j (\Box_j\chi \to_j \Box_j\omega)$. Then $\mathbf{MML}_n^{\mathbf{S5}} \models \Box_j(\varphi \wedge_j \bigwedge_j \Gamma) \to_j \Box_j \bigvee_j \Delta$. By ($I$-idempotent), $\mathbf{MML}_n^{\mathbf{S5}} \models (\Box_j\varphi \wedge_j \Box_j \bigwedge_j \Gamma) \to_j \Box_j \bigvee_j \Delta$. Thus, $\mathbf{MML}_n^{\mathbf{S5}} \models \neg_k\neg_j(\Box_j\varphi \wedge_j \Box_j \bigwedge_j \Gamma) \vee_j \Box_j \bigvee_j \Delta$. Then $\mathbf{MML}_n^{\mathbf{S5}} \models \neg_k\neg_j\Box_j\varphi \vee_j \neg_k\neg_j\Box_j \bigwedge_j \Gamma \vee_j \Box_j \bigvee_j \Delta$ which implies $\mathbf{MML}_n^{\mathbf{S5}} \models \neg_k\neg_j\Box_j\varphi \vee_j (\Box_j \bigwedge_j \Gamma \to_j \Box_j \bigvee_j \Delta)$. By (interior interconnection), $\mathbf{MML}_n^{\mathbf{S5}} \models \neg_k\neg_j\Box_j\varphi$ iff $\mathbf{MML}_n^{\mathbf{S5}} \models \Box_j\neg_k\neg_j\Box_j\varphi$. We leave it to the reader to prove $\mathbf{MML}_n^{\mathbf{S5}} \models (\Box_j\chi \to_j \Box_j\omega) \to_j \Box_j(\chi \to_j \omega)$. Therefore, $\mathbf{MML}_n^{\mathbf{S5}} \models \Box_j\neg_k\neg_j\Box_j\varphi \vee_j \Box_j\big(\bigwedge_j \Gamma \to_j \bigvee_j \Delta\big)$. By the properties of $\vee_j$, we have $\mathbf{MML}_n^{\mathbf{S5}} \models \Box_j\neg_k\neg_j\Box_j\varphi \vee_j \Box_j\big(\bigwedge_j \Gamma \to_j \bigvee_j \Delta\big) \vee_j \Box_j\tau(H)$. Hence, by the definition of τ, $\mathbf{MML}_n^{\mathbf{S5}} \models \tau(\Box_j\varphi \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H)$. By Proposition 5.1, $\mathbf{MML}_n^{\mathbf{S5}} \models \Box_j\varphi \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H$.

Ad ($\Rightarrow\Diamond_j$). Assume that $\mathbf{MML}_n^{\mathbf{S5}} \models \Gamma \Rightarrow \Delta, \varphi \mid H$. Then $\mathbf{MML}_n^{\mathbf{S5}} \models \neg_k\neg_j\varphi, \Gamma \Rightarrow \Delta \mid H$. By Proposition 5.1, $\mathbf{MML}_n^{\mathbf{S5}} \models \tau(\neg_k\neg_j\varphi, \Gamma \Rightarrow \Delta \mid H)$, i.e. $\mathbf{MML}_n^{\mathbf{S5}} \models \Box_j\big((\neg_k\neg_j\varphi \wedge_j \bigwedge_j \Gamma) \to_j \bigvee_j \Delta\big) \vee_j \Box_j\tau(H)$. Then $\mathbf{MML}_n^{\mathbf{S5}} \models \Box_j\big((\neg_k\neg_j\varphi \wedge_j \bigwedge_j \Gamma) \to_j \bigvee_j \Delta\big)$ or $\mathbf{MML}_n^{\mathbf{S5}} \models \Box_j\tau(H)$. The latter option immediately implies the soundness of the rule in question. Let us consider the former one. Similarly to the previous case, we get $\mathbf{MML}_n^{\mathbf{S5}} \models (\Box_j\neg_k\neg_j\varphi \wedge_j \Box_j \bigwedge_j \Gamma) \to_j \Box_j \bigvee_j \Delta$. Then $\mathbf{MML}_n^{\mathbf{S5}} \models (\neg_k\neg_j\Diamond_j\varphi \wedge_j \Box_j \bigwedge_j \Gamma) \to_j \Box_j \bigvee_j \Delta$. By (closure interconnection), $\mathbf{MML}_n^{\mathbf{S5}} \models (\Diamond_j\neg_k\neg_j\Diamond_j\varphi \wedge_j \Box_j \bigwedge_j \Gamma) \to_j \Box_j \bigvee_j \Delta$. Then $\mathbf{MML}_n^{\mathbf{S5}} \models (\neg_k\neg_j\Box_j\Diamond_j\varphi \wedge_j \Box_j \bigwedge_j \Gamma) \to_j \Box_j \bigvee_j \Delta$. Therefore, $\mathbf{MML}_n^{\mathbf{S5}} \models \Box_j\Diamond_j\varphi \vee_j \neg_k\neg_j\Box_j \bigwedge_j \Gamma \vee_j \Box_j \bigvee_j \Delta$. Hence, $\mathbf{MML}_n^{\mathbf{S5}} \models \Box_j\Diamond_j\varphi \vee_j (\Box_j \bigwedge_j \Gamma \to_j \Box_j \bigvee_j \Delta)$. Thus, $\mathbf{MML}_n^{\mathbf{S5}} \models \Box_j\Diamond_j\varphi \vee_j \Box_j(\bigwedge_j \Gamma \to_j \bigvee_j \Delta)$ which implies $\mathbf{MML}_n^{\mathbf{S5}} \models \Gamma \Rightarrow \Delta \mid \Rightarrow \Diamond_j\varphi \mid H$.

Ad ($\Rightarrow\Box_j$). Assume that $\mathbf{MML}_n^{\mathbf{S5}} \models\, \Rightarrow \varphi \mid H$. Then $\mathbf{MML}_n^{\mathbf{S5}} \models\, \Rightarrow \varphi$ or $\mathbf{MML}_n^{\mathbf{S5}} \models H$. The latter case immediately implies $\mathbf{MML}_n^{\mathbf{S5}} \models\, \Rightarrow \Box_j\varphi \mid H$. Let us look at the former one. We have $v(\varphi) = \mathsf{1}_j$, for some $j \leqslant n$. Thus, $\mathsf{1}_j \leq_j v(\varphi)$ and $v(\varphi) \leq_j \mathsf{1}_j$. By ($I$-monotonicity), $I_j(\mathsf{1}_j) \leq_j I_j(v(\varphi))$ and $I_j(v(\varphi)) \leq_j I_j(\mathsf{1}_j)$, i.e. $I_j(v(\varphi)) = I_j(\mathsf{1}_j)$. By ($\mathsf{1}_j$ is open), $I_j(v(\varphi)) = \mathsf{1}_j$. Hence, $\mathbf{MML}_n^{\mathbf{S5}} \models\, \Rightarrow \Box_j\varphi \mid H$.

Ad ($\Diamond_j\Rightarrow$). Then $\mathbf{MML}_n^{\mathbf{S5}} \models \varphi \Rightarrow\, \mid H$. Then $\mathbf{MML}_n^{\mathbf{S5}} \models \varphi \Rightarrow$ or $\mathbf{MML}_n^{\mathbf{S5}} \models H$. The latter case immediately implies the required result. Let us consider the former one. We have $v(\varphi) = \mathsf{0}_j$, for some $j \leqslant n$. Thus, $v(\varphi) \leq_j \mathsf{0}_j$ and $\mathsf{0}_j \leq_j v(\varphi)$. By ($C$-monotonicity), $C_j(v(\varphi)) \leq_j C_j(\mathsf{0}_j)$ and $C_j(\mathsf{0}_j) \leq_j C_j(v(\varphi))$, i.e. $C_j(v(\varphi)) = C_j(\mathsf{0}_j)$. By ($\mathsf{0}_j$ is closed), $C_j(v(\varphi)) = \mathsf{0}_j$. Hence, $\mathbf{MML}_n^{\mathbf{S5}} \models \Diamond_j\varphi \Rightarrow\, \mid H$.

The other cases are proved similarly. ⊣

THEOREM 5.1. *Let $\boldsymbol{L} \in \{\mathbf{MML}_n^{\mathbf{MNT4}}, \mathbf{MML}_n^{\mathbf{S4}}, \mathbf{MML}_n^{\mathbf{S5}}\}$. For each pair of finite sets Γ and Δ of $\mathscr{L}_{\mathrm{M}}$-formulas, it holds that if $\boldsymbol{L} \vdash \Gamma \Rightarrow \Delta$, then $\boldsymbol{L} \models \Gamma \Rightarrow \Delta$.*

PROOF. By induction of the height of the derivation, using Lemmas 5.1, 5.2, and 5.3. ⊣

5.2. Completeness

DEFINITION 5.2. Let $\boldsymbol{L} \in \{\mathbf{MML}_n^{\mathbf{MNT4}}, \mathbf{MML}_n^{\mathbf{S4}}, \mathbf{MML}_n^{\mathbf{S5}}\}$ and let $[\varphi]$ be the class of equivalence of $\varphi \in \mathscr{F}_{\mathrm{M}}$, i.e. $\{\psi \in \mathscr{F}_{\mathrm{M}} \mid \boldsymbol{L} \vdash \varphi \Rightarrow \psi, \boldsymbol{L} \vdash \psi \Rightarrow \varphi\}$. Let $\Gamma \subseteq \mathscr{F}_{\mathrm{M}}$ and $[\Gamma]$ be $\{[\gamma] \mid \gamma \in \Gamma\}$. Thus, $[\mathscr{F}_{\mathrm{M}}]$ is the set of all the classes of equivalences, i.e. $\{[\varphi] \mid \varphi \in \mathscr{F}_{\mathrm{M}}\}$.

DEFINITION 5.3. Let $\varphi, \psi \in \mathscr{F}_{\mathrm{M}}$ and $\boldsymbol{L} \in \{\mathbf{MML}_n^{\mathbf{MNT4}}, \mathbf{MML}_n^{\mathbf{S4}}, \mathbf{MML}_n^{\mathbf{S5}}\}$. Then a *Lindenbaum-Tarski algebra* (*LT-algebra*) is a structure $\mathcal{M}_n^{\mathbf{L}} = \langle [\mathscr{F}_{\mathrm{M}}], \leq_1, \ldots, \leq_n \rangle$ which satisfies the following conditions:

$$
\begin{gathered}
[\varphi] \leq_j [\psi] \text{ iff } [\varphi] = [\varphi \wedge_j \psi];\\
-_j[\varphi] = [\neg_j\varphi];\\
[\varphi] \cap_j [\psi] = [\varphi \wedge_j \psi];\\
[\varphi] \cup_j [\psi] = [\varphi \vee_j \psi];\\
[\varphi] \supset_j [\psi] = [\varphi \rightarrow_j \psi];\\
[\varphi] \subset_j [\psi] = [\varphi \leftarrow_j \psi];\\
I_j[\varphi] = [\Box_j\varphi];\\
C_j[\varphi] = [\Diamond_j\varphi].
\end{gathered}
$$

Fact 5.1. *For each* $\varphi, \psi \in \mathscr{F}_{\mathrm{M}}$ *and* $\boldsymbol{L} \in \{\mathbf{MML}_n^{\mathbf{MNT4}}, \mathbf{MML}_n^{\mathbf{S4}}, \mathbf{MML}_n^{\mathbf{S5}}\}$, *it holds that* $\boldsymbol{L} \vdash \varphi \Rightarrow \psi$ *and* $\boldsymbol{L} \vdash \psi \Rightarrow \varphi$ *iff* $[\varphi] = [\psi]$.

Lemma 5.4. *The following sequents are provable in the sequent calculus for* $\mathbf{MML}_n^{\mathbf{MNT4}}$:

(1)	$\Box_j(p \vee_j \neg_k\neg_j p) \Rightarrow p \vee_j \neg_k\neg_j p$	(13)	$\neg_j\Box_j\varphi \Rightarrow \Diamond_j\neg_j\varphi$;
(2)	$p \vee_j \neg_k\neg_j p \Rightarrow \Box_j(p \vee_j \neg_k\neg_j p)$	(14)	$\Diamond_j\neg_j\varphi \Rightarrow \neg_j\Box_j\varphi$;
(3)	$\Diamond_j(p \wedge_j \neg_k\neg_j p) \Rightarrow p \wedge_j \neg_k\neg_j p$	(15)	$\neg_j\Diamond_j\varphi \Rightarrow \Box_j\neg_j\varphi$;
(4)	$p \wedge_j \neg_k\neg_j p \Rightarrow \Diamond_j(p \wedge_j \neg_k\neg_j p)$	(16)	$\Box_j\neg_j\varphi \Rightarrow \neg_j\Diamond_j\varphi$;
(5)	$\Box_j\varphi \Rightarrow \varphi$	(17)	$\neg_k\Box_j\varphi \Rightarrow \Box_j\neg_k\varphi$;
(6)	$\Box_j\varphi \Rightarrow \Box_j\Box_j\varphi$;	(18)	$\Box_j\neg_k\varphi \Rightarrow \neg_k\Box_j\varphi$;
(7)	$\Box_j\Box_j\varphi \Rightarrow \Box_j\varphi$;	(19)	$\neg_k\Diamond_j\varphi \Rightarrow \Diamond_j\neg_k\varphi$;
(8)	$\Box_j(\varphi \wedge_j \psi) \Rightarrow \Box_j\varphi \wedge_j \Box_j\psi$;	(20)	$\Diamond_j\neg_k\varphi \Rightarrow \neg_k\Diamond_j\varphi$;
(9)	$\varphi \Rightarrow \Diamond_j\varphi$	(21)	$\neg_k\neg_j\Box_j\neg_k\neg_j\varphi \Rightarrow \Diamond_j\varphi$;
(10)	$\Diamond_j\varphi \Rightarrow \Diamond_j\Diamond_j\varphi$;	(22)	$\Diamond_j\varphi \Rightarrow \neg_k\neg_j\Box_j\neg_k\neg_j\varphi$;
(11)	$\Diamond_j\Diamond_j\varphi \Rightarrow \Diamond_j\varphi$;	(23)	$\neg_k\neg_j\Diamond_j\neg_k\neg_j\varphi \Rightarrow \Box_j\varphi$;
(12)	$\Diamond_j\varphi \vee_j \Diamond_j\psi \Rightarrow \Diamond_j(\varphi \vee_j \psi)$;	(24)	$\Box_j\varphi \Rightarrow \neg_k\neg_j\Diamond_j\neg_k\neg_j\varphi$.

Lemma 5.5. $\mathcal{M}_n^{\mathbf{MML}_n^{\mathbf{MNT4}}}$ *is a Tarski multilattice.*

Proof. Due to Lemma 5.4, operations I_j and C_j on $[\mathscr{F}_{\mathrm{M}}]$ satisfy the conditions which are listed in Definitions 2.9 and 2.10. Specifically, the correspondence between the properties required by these definitions and the provable sequents listed above is as follows:

- (decreasing) is justified by the provability of (5); (I-idempotent) by (6)–(7); (sub-multiplicative) by (8); (increasing) by (9); (C-idempotent) by (10)–(11); finally, (sub-additive) by (12).
- ($\mathbf{1}_j$ is open) and ($\mathbf{0}_j$ is closed) are justified by (1)–(4); ($-_jI_j$-definition) by (13)–(14); ($-_jC_j$-definition) by (15)–(16); ($-_kI_j$-definition) by (17)–(18); ($-_kC_j$-definition) by (19)–(20); (C-definition) by (21)–(22); (I-definition) by (23)–(24). ⊣

Lemma 5.6. *The sequents* (1)–(24) *from Lemma* 5.4 *as well as the following ones are provable in the sequent calculus for* $\mathbf{MML}_n^{\mathbf{S4}}$:

(25) $\Box_j\varphi \wedge_j \Box_j\psi \Rightarrow \Box_j(\varphi \wedge_j \psi)$; (26) $\Diamond_j(\varphi \vee_j \psi) \Rightarrow \Diamond_j\varphi \vee_j \Diamond_j\psi$.

Proof. In virtue of Remark 4.1, we observe that the rules of $\mathbf{MML}_n^{\mathbf{MNT4}}$ are restrictions of the rules of $\mathbf{MML}_n^{\mathbf{S4}}$. This fact implies that the sequents

(1)–(24) from Lemma 5.4 are provable in $\mathbf{MML}_n^{\mathbf{S4}}$ as well. As for the latter ones, their proofs are below.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi, \psi \Rightarrow \varphi}\,(\mathrm{W}{\Rightarrow}) \qquad \dfrac{\psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \psi}\,(\mathrm{W}{\Rightarrow})}{\varphi, \psi \Rightarrow \varphi \wedge_j \psi}\,({\Rightarrow}\wedge_j)}{\Box_j\varphi, \Box_j\psi \Rightarrow \varphi \wedge_j \psi}\,(\Box_j{\Rightarrow})2\mathrm{x}}{\Box_j\varphi, \Box_j\psi \Rightarrow \Box_j(\varphi \wedge_j \psi)}\,({\Rightarrow}\Box_j)}{\Box_j\varphi \wedge_j \Box_j\psi \Rightarrow \Box_j(\varphi \wedge_j \psi)}\,(\wedge_j{\Rightarrow})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \varphi, \psi}\,({\Rightarrow}\mathrm{W}) \qquad \dfrac{\psi \Rightarrow \psi}{\psi \Rightarrow \varphi, \psi}\,({\Rightarrow}\mathrm{W})}{\varphi \vee_j \psi \Rightarrow \varphi, \psi}\,(\vee_j{\Rightarrow})}{\varphi \vee_j \psi \Rightarrow \Diamond_j\varphi, \Diamond_j\psi}\,({\Rightarrow}\Diamond_j)2\mathrm{x}}{\Diamond_j(\varphi \vee_j \psi) \Rightarrow \Diamond_j\varphi, \Diamond_j\psi}\,(\Diamond_j{\Rightarrow})}{\Diamond_j(\varphi \vee_j \psi) \Rightarrow \Diamond_j\varphi \vee_j \Diamond_j\psi}\,({\Rightarrow}\vee_j) \quad \dashv$$

LEMMA 5.7. *$\mathcal{M}_n^{\mathbf{MML}_n^{\mathbf{S4}}}$ is a Kuratowski multilattice.*

PROOF. Due to Lemma 5.6, operations I_j and C_j satisfy the conditions which are listed in Definition 2.11. In particular, (multiplicative) is justified by the provability of (8) and (25), (additive) is justified by (12) and (26). $\dashv$

LEMMA 5.8. *The sequents (1)–(24) and (25)–(26) from Lemmas 5.4 and 5.6, respectively, as well as the following ones are provable in the sequent calculus for $\mathbf{MML}_n^{\mathbf{S5}}$:*

(27) $\Box_j\neg_k\neg_j\Box_j\varphi \Rightarrow \neg_k\neg_j\Box_j\varphi$; (29) $\Diamond_j\neg_k\neg_j\Diamond_j\varphi \Rightarrow \neg_k\neg_j\Diamond_j\varphi$;
(28) $\neg_k\neg_j\Box_j\varphi \Rightarrow \Box_j\neg_k\neg_j\Box_j\varphi$; (30) $\neg_k\neg_j\Diamond_j\varphi \Rightarrow \Diamond_j\neg_k\neg_j\Diamond_j\varphi$.

PROOF. As an example, we proof of the sequents (29) and (30).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \mid \Rightarrow \Diamond_j\varphi}\,({\Rightarrow}\Diamond_j)}{\varphi \Rightarrow \mid \neg_k\neg_j\Diamond_j\varphi \Rightarrow}\,(\neg_k\neg_j \Rightarrow)}{\varphi \Rightarrow \mid \Diamond_j\neg_k\neg_j\Diamond_j\varphi \Rightarrow}\,(\Diamond_j{\Rightarrow})}{\Diamond_j\varphi \Rightarrow \mid \Diamond_j\neg_k\neg_j\Diamond_j\varphi \Rightarrow}\,(\Diamond_j{\Rightarrow})}{\Rightarrow \neg_k\neg_j\Diamond_j\varphi \mid \Diamond_j\neg_k\neg_j\Diamond_j\varphi \Rightarrow}\,(\Rightarrow \neg_k\neg_j)}{\Diamond_j\neg_k\neg_j\Diamond_j\varphi \Rightarrow \neg_k\neg_j\Diamond_j\varphi}\,(\mathrm{Merge})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \mid \Rightarrow \Diamond_j \varphi}(\Rightarrow\Diamond_j)}{\varphi \Rightarrow \mid \neg_k\neg_j\Diamond_j\varphi \Rightarrow}(\neg_k\neg_j \Rightarrow)}{\Diamond_j\varphi \Rightarrow \mid \neg_k\neg_j\Diamond_j\varphi \Rightarrow}(\Diamond_j\Rightarrow)}{\Rightarrow \neg_k\neg_j\Diamond_j\varphi \mid \neg_k\neg_j\Diamond_j\varphi \Rightarrow}(\Rightarrow \neg_k\neg_j)}{\Rightarrow \mid \Rightarrow \Diamond_j\neg_k\neg_j\Diamond_j\varphi \mid \neg_k\neg_j\Diamond_j\varphi \Rightarrow}(\Rightarrow\Diamond_j)}{\neg_k\neg_j\Diamond_j\varphi \Rightarrow \Diamond_j\neg_k\neg_j\Diamond_j\varphi}(\text{Merge})2\text{x} \quad \dashv$$

Lemma 5.9. *$\mathcal{M}_n^{\mathbf{MML}_n^{\mathbf{S5}}}$ is a Halmos multilattice.*

Proof. Due to Lemma 5.8, operations I_j and C_j satisfy the conditions which are listed in Definition 2.12. In particular, the condition (interior interconnection) is justified by the provability of (27) and (28), (closure interconnection) is justified by (29) and (30). $\dashv$

By structural induction on a formula φ, using Definition 5.3 we get:

Lemma 5.10. *Let $\widetilde{v}$ be a valuation introduced in Definition 2.13 such that $\widetilde{v}(p) = [p]$, for all $p \in \mathcal{P}$ (such a valuation is said to be canonical). Then $\widetilde{v}(\varphi) = [\varphi]$, for all $\varphi \in \mathscr{F}_{\mathrm{M}}$.*

For the elaborated proof of the following lemma see [7, Lemma 4.12]

Lemma 5.11 (Lindenbaum). *$\boldsymbol{L} \in \{\mathbf{MML}_n^{\mathbf{MNT4}}, \mathbf{MML}_n^{\mathbf{S4}}, \mathbf{MML}_n^{\mathbf{S5}}\}$. For every pair of finite sets Γ and Δ of $\mathscr{L}_{\mathrm{M}}$-formulas, it holds that $\nvdash_{\boldsymbol{L}} \Gamma \Rightarrow \Delta$ implies that there is an ultramultifilter $\mathcal{U}_n$ on Lindenbaum-Tarski algebra $\mathcal{M}_n^{\boldsymbol{L}}$ and $[\chi] \in \mathcal{U}_n$ for each $\chi \in \Gamma$, while $[\omega] \notin \mathcal{U}_n$ for each $\omega \in \Delta$.*

Theorem 5.2. *Let $\boldsymbol{L} \in \{\mathbf{MML}_n^{\mathbf{MNT4}}, \mathbf{MML}_n^{\mathbf{S4}}, \mathbf{MML}_n^{\mathbf{S5}}\}$. For every pair of finite sets of $\mathscr{L}_{\mathrm{M}}$-formulas Γ and Δ, it holds that $\boldsymbol{L} \models \Gamma \Rightarrow \Delta$ iff $\boldsymbol{L} \vdash \Gamma \Rightarrow \Delta$.*

Proof. One direction follows from Theorem 5.1. For the other direction we use contraposition and assume that $\boldsymbol{L} \nvdash \Gamma \Rightarrow \Delta$. Then, by Lemma 5.11, $\mathcal{M}_n^{\boldsymbol{L}}$ has an ultramultifilter $\mathcal{U}_n$ such that $[\varphi] \in \mathcal{U}_n$ (for all $\varphi \in \Gamma$) and $[\psi] \notin \mathcal{U}_n$ (for all $\psi \in \Delta$). By Lemmas 5.5, 5.7 and 5.9, if $\boldsymbol{L} = \mathbf{MML}_n^{\mathbf{MNT4}}$ (resp. $\mathbf{MML}_n^{\mathbf{S4}}$, $\mathbf{MML}_n^{\mathbf{S5}}$), then $\mathcal{M}_n^{\boldsymbol{L}}$ is a Tarski (resp. Kuratowski, Halmos) multilattice. By Lemma 5.10, there is canonical valuation $\widetilde{v}$ such that $\widetilde{v}(\varphi) \in \mathcal{U}_n$ (for all $\varphi \in \Gamma$) and $\widetilde{v}(\psi) \notin \mathcal{U}_n$ (for all $\psi \in \Delta$), i.e. $\boldsymbol{L} \not\models \Gamma \Rightarrow \Delta$. $\dashv$

6. Metatheoretical properties via embedding

Now we present some metatheoretical results for the system $\mathbf{MML}_n^{\mathbf{MNT4}}$ obtained via embedding techniques following the lines of [6]. For this purpose we provide syntactic embedding using as a target the language of the **MNT4** sequent calculus discussed in Section 3.

6.1. Syntactic embedding

Let us define a syntactical embedding function f from the language $\mathscr{L}_{\mathrm{M}}$ to the language $\mathscr{L}$. For the purpose of brevity we put $\odot_j \in \{\wedge_j, \vee_j, \rightarrow_j, \leftarrow_j\}$ and $\circledast_j \in \{\Box_j, \Diamond_j\}$ as a shorthand for propositional connectives of the language $\mathscr{L}_{\mathrm{M}}$, while $\odot \in \{\wedge, \vee, \rightarrow, \leftarrow\}$ and $\circledast \in \{\Box, \Diamond\}$ for $\mathscr{L}$. We also write $\odot_j^d$ ($\circledast_j^d$) for be the dual of $\odot_j$ ($\circledast_j$).[2]

Definition 6.1. Let $n > 1$, $j, k \leqslant n$, and $j \neq k$. Then a mapping f from $\mathscr{L}_{\mathrm{M}}$ to $\mathscr{L}$ is defined inductively as follows:

(1) $f(p) = p$ and $f(\neg_j p) = p^j$ (where $p^j \in \mathcal{P}^j$), for each $p \in \mathcal{P}$,
(2) $f(\varphi \odot_j \psi) = f(\varphi) \odot f(\psi)$,
(3) $f\neg_j(\varphi \odot_j \psi) = f(\neg_j \varphi) \odot^d f(\neg_j \psi)$,
(4) $f\neg_k(\varphi \odot_j \psi) = f(\neg_k \varphi) \odot f(\neg_k \psi)$,
(5) $f(\neg_j \neg_j \varphi) = f(\varphi)$,
(6) $f(\neg_k \neg_j \varphi) = f(\neg \varphi)$,
(7) $f(\circledast_j(\varphi)) = \circledast f(\varphi)$,
(8) $f(\neg_j \circledast_j \varphi) = \circledast^d f(\neg_j \varphi)$,
(9) $f(\neg_k \circledast_j \varphi) = \circledast f(\neg_k \varphi)$.

We adopt the usual notation: where Λ is a set of formulas, $f(\Lambda) := \{f(\varphi) \mid \varphi \in \Lambda\}$.

Theorem 6.1 (Syntactical embedding from $\mathbf{MML}_n^{\mathbf{MNT4}}$ into **MNT4**). *Let f be a mapping introduced in Definition 6.1. Then for each sequent $\Gamma \Rightarrow \Delta$, such that Γ and Δ are sets of $\mathscr{L}_{\mathrm{M}}$-formulas: $\mathbf{MML}_n^{\mathbf{MNT4}} \vdash \Gamma \Rightarrow \Delta$ iff $\mathbf{MNT4} \vdash f(\Gamma) \Rightarrow f(\Delta)$.*

Proof. "From left to right". The proof is obtained by induction on the construction of the derivation $\mathcal{D}$ of $\Gamma \Rightarrow \Delta$ in the sequent calculus $\mathbf{MML}_n^{\mathbf{MNT4}}$. Suppose that a rule $\mathfrak{r}$ is the last one which was applied in a sub-derivation $\mathcal{D}'$ of $\mathcal{D}$ and the induction hypothesis (that is the

[2] The following pairs of connectives of $\mathscr{L}_{\mathrm{M}}$ are considered as dual: $\wedge_j$ and $\vee_j$; $\leftarrow_j$ and $\rightarrow_j$; $\Box_j$ and $\Diamond_j$. Similarly for the $\mathscr{L}$-connectives (just ignore the indices).

assertion of the lemma) holds for the whole part of $\mathcal{D}'$ taken without the conclusion of $\mathfrak{r}$. We have to show that the whole of $\mathcal{D}'$ converts into a **MNT4** sub-derivation. Let us consider some typical propositional and modal cases of $\mathfrak{r}$.

$\mathfrak{r} := (\neg_j{\to_j}{\Rightarrow})$. The premise of the last rule of application in $\mathcal{D}'$ is $\neg_j\psi, \Gamma \Rightarrow \Delta, \neg_j\varphi$. Using induction hypothesis we construct the following **MNT4**-derivation:

$$\dfrac{\dfrac{\vdots}{f(\neg_j\psi), f(\Gamma) \Rightarrow f(\Delta), f(\neg_j\varphi)}}{f(\neg_j\varphi) \leftarrow f(\neg_j\psi), f(\Gamma) \Rightarrow f(\Delta)}\,(\leftarrow\Rightarrow)$$

where the conclusion is the translation of the conclusion of $\mathfrak{r}$.

$\mathfrak{r} := (\neg_k{\to_j}{\Rightarrow})$. Now the premises of $\mathfrak{r}$ are $\Gamma \Rightarrow \Delta, \neg_k\varphi$ and $\neg_k\psi, \Theta \Rightarrow \Lambda$. Thus, applying the induction hypothesis, we obtain

$$\dfrac{\dfrac{\vdots}{f(\Gamma) \Rightarrow f(\Delta), f(\neg_k\varphi)} \qquad \dfrac{\vdots}{f(\neg_k\psi), f(\Theta) \Rightarrow f(\Lambda)}}{f(\neg_k\varphi) \to f(\neg_k\psi), f(\Gamma), f(\Theta) \Rightarrow f(\Delta), f(\Lambda)}\,(\to\Rightarrow)$$

$\mathfrak{r} := (\neg_j\Box_j{\Rightarrow})$. The premise of $\mathfrak{r}$ is of the form $\neg_j\varphi, \Lambda^\sharp \Rightarrow \delta$. Using the induction hypothesis we construct:

$$\dfrac{\dfrac{\vdots}{f(\neg_j\varphi), f(\Lambda^\sharp)[= \Box f(\Lambda)] \Rightarrow f(\delta)}}{\Diamond f(\neg_j\varphi), \Box f(\Lambda) \Rightarrow f(\delta)}\,(\Diamond\Rightarrow)$$

As one can see, the expression below the line is a translation of the result of the application of $\mathfrak{r}$ to the indicated premise.

$\mathfrak{r} := (\neg_k\Box_j{\Rightarrow})$. This case is justified by the following $\mathbf{MML}_n^{\mathbf{MNT4}}$-derivation:

$$\dfrac{\dfrac{\vdots}{f(\neg_k\varphi), f(\Gamma) \Rightarrow f(\delta)}}{\Box f(\neg_k\varphi), f(\Gamma) \Rightarrow f(\delta)}\,(\Box\Rightarrow)$$

Again, it is easy to see that the last sequent is a correct translation of $\neg_k\Box_j\varphi, \Gamma \Rightarrow \Delta$, the conclusion of $\mathfrak{r}$.

"From right to left". To prove the assertion we utilize induction on the construction of **MNT4**-proof $\mathcal{E}$. Suppose that induction hypothesis holds for a sub-derivation $\mathcal{E}'$ (dropping out the last sequent). Let us inspect some cases of application of the rule $\mathfrak{r}$ in the last step in the construction of the $\mathcal{E}$.

$\mathfrak{r} := (\Box\Rightarrow)$. Consider the final step of $\mathcal{E}$:

$$\dfrac{\dfrac{\vdots}{\chi, f(\Gamma) \Rightarrow f(\delta)}}{\Box\chi, f(\Gamma) \Rightarrow f(\delta)}\,(\Box\Rightarrow)$$

where $\Box\chi$ is a translation of some appropriate $\mathscr{L}_{\mathrm{M}}$-formula, that is $\Box\chi$ is one of $f(\Box_j\varphi)$, $f(\neg_k\Box_j\varphi)$ or $f(\neg_j\Diamond_j\varphi)$ for some $\mathscr{L}_{\mathrm{M}}$-formula φ, according to the Definition 6.1. Thus χ itself is of the form $f(\varphi)$, $f(\neg_k\varphi)$ or $f(\neg_j\varphi)$. By the induction hypothesis we already have a $\mathbf{MML}_n^{\mathbf{MNT4}}$-derivation for $f^{-1}(\chi), \Gamma \Rightarrow \delta$. What is left is to apply one of the rules $(\Box_j\Rightarrow)$, $(\neg_k\Box_j\Rightarrow)$ or $(\neg_j\Diamond_j \Rightarrow)$, corresponding to the above listed form of χ.

$\mathfrak{r} := (\Diamond\Rightarrow)$. The last inference of $\mathcal{E}$ has the following form:

$$\dfrac{\dfrac{\vdots}{\chi, f(\Xi) \Rightarrow f(\psi)}}{\Diamond\chi, f(\Xi) \Rightarrow f(\psi)}\,(\Diamond\Rightarrow)$$

Note that $f(\Xi)$ is of the form $\Box\Gamma$, while $f(\psi)$ is empty or is of the form $f(\delta)$ (in order to apply the rule $(\Diamond\Rightarrow)$ correctly). Thus $\Diamond\chi$ is an image of one of the following formulas: $f(\Diamond_j\varphi)$, $f(\neg_k\Diamond_j\varphi)$ or $f(\neg_j\Box_j\varphi)$. So χ coincides with one of the $f(\varphi)$, $f(\neg_k\varphi)$ or $f(\neg_j\varphi)$. Again, we already know a derivation of $f^{-1}(\chi), \Xi \Rightarrow \psi$ in $\mathbf{MML}_n^{\mathbf{MNT4}}$ and we complete it by applying $(\Diamond_j\Rightarrow)$, $(\neg_k\Diamond_j\Rightarrow)$ or $(\neg_j\Box_j\Rightarrow)$ depending on the structure of χ. $\dashv$

COROLLARY 6.1 (Decidability). *$\mathbf{MML}_n^{\mathbf{MNT4}}$ is decidable.*

PROOF. Follows from Theorem 6.1 and decidability of **MNT4**. $\dashv$

6.2. Neighbourhood semantics for $\mathbf{MML}_n^{\mathbf{MNT4}}$

DEFINITION 6.2. A *neighbourhood frame* is a structure $\mathcal{F} = (W, N)$, where W is an non-empty set, $N\colon W \to \mathscr{P}(\mathscr{P}(W))$ is a *neighbourhood function*. A *neighbourhood model* based on $\mathcal{F}$ is a pair $\mathcal{M} = (\mathcal{F}, \theta)$, where $\mathcal{F}$ is a neighbourhood frame, $\theta\colon \mathcal{P} \to \mathscr{P}(W)$ is a *valuation function*. The truth relation $\models$ is defined inductively in a standard way, we just spell out the modal condition (where $\|\varphi\|_{\mathcal{M}}$ is a truth set of formula φ in $\mathcal{M}$):

$$\mathcal{M}, x \models \Box\varphi \text{ iff } \|\varphi\|_{\mathcal{M}} \in N(x).$$

We introduce the very basic, *minimal* neighbourhood models. We need to add some extra conditions to obtain **MNT4**-neighbourhood models. An **MNT4**-neighbourhood model is a minimal neighbourhood model endowed with the following properties [5]:

supplemented model (sm) for all $X, Y \subseteq W$ such that $X \subseteq Y$, for all $x \in W$, if $X \in N(x)$ then $Y \in N(x)$;
N condition (nc) $W \in N(x)$ for all $x \in W$;
t condition (tc) for all $X \subseteq W$, if $X \in N(x)$ then $x \in X$, for all $x \in X$;
4 condition (4c) for all $X \subseteq W$, for all $x \in W$, if X is in $N(x)$ then all points y such that $X \in N(y)$ constitute the elements of $N(x)$.

Next let us define a special type of evaluation which maps propositional variables and their negations to the powerset of W. We denote the set of negated propositional variables as $\mathcal{P}^{\neg}$. Thus a *paraconsistent valuation* is a function $\vartheta\colon \mathcal{P} \cup \mathcal{P}^{\neg} \to \mathscr{P}(W)$. A *paraconsistent neighbourhood model* is a pair $\mathcal{M} = (\mathcal{F}, \vartheta)$, where $\mathcal{F}$ is a neighbourhood frame.

Definition 6.3. An $\mathbf{MML}_n^{\mathbf{MNT4}}$-model is a paraconsistent neighbourhood model satisfying the conditions **sm, nc, tc** and **4c**.

The paraconsistent truth relation is given by the following statements:

Definition 6.4. For an $\mathbf{MML}_n^{\mathbf{MNT4}}$-model $\mathcal{M} = (W, N, \vartheta)$, an $x \in W$, formulas φ, $\psi \in \mathscr{L}_{\mathrm{M}}$ and $p \in \mathcal{P} \cup \mathcal{P}^{\neg}$, all j, k ($1 \leqslant j, k \leqslant n$, $j \neq k$), the paraconsistent truth relation $\models^{\mathfrak{p}}$ is defined by the following clauses:

(1) $x \models^{\mathfrak{p}} p$ iff $x \in \vartheta(p)$,
(2) $x \models^{\mathfrak{p}} \varphi \wedge_j \psi$ iff $x \models^{\mathfrak{p}} \varphi$ and $x \models^{\mathfrak{p}} \psi$,
(3) $x \models^{\mathfrak{p}} \varphi \vee_j \psi$ iff $x \models^{\mathfrak{p}} \varphi$ or $x \models^{\mathfrak{p}} \psi$,
(4) $x \models^{\mathfrak{p}} \varphi \to_j \psi$ iff $x \not\models^{\mathfrak{p}} \varphi$ or $x \models^{\mathfrak{p}} \psi$,
(5) $x \models^{\mathfrak{p}} \varphi \leftarrow_j \psi$ iff $x \models^{\mathfrak{p}} \varphi$ and $x \not\models^{\mathfrak{p}} \psi$,
(6) $x \models^{\mathfrak{p}} \Box_j \varphi$ iff $\|\varphi\|_{\mathcal{M}} \in N(x)$,
(7) $x \models^{\mathfrak{p}} \Diamond_j \varphi$ iff $W \setminus \|\varphi\|_{\mathcal{M}} \notin N(x)$,
(8) $x \models^{\mathfrak{p}} \neg_j(\varphi \wedge_j \psi)$ iff $x \models^{\mathfrak{p}} \neg_j \varphi$ or $x \models^{\mathfrak{p}} \neg_j \psi$,
(9) $x \models^{\mathfrak{p}} \neg_j(\varphi \vee_j \psi)$ iff $x \models^{\mathfrak{p}} \neg_j \varphi$ and $x \models^{\mathfrak{p}} \neg_j \psi$,
(10) $x \models^{\mathfrak{p}} \neg_j(\varphi \to_j \psi)$ iff $x \models^{\mathfrak{p}} \neg_j \psi$ and $x \not\models^{\mathfrak{p}} \neg_j \varphi$,
(11) $x \models^{\mathfrak{p}} \neg_j(\varphi \leftarrow_j \psi)$ iff $x \models^{\mathfrak{p}} \neg_j \varphi$ or $x \not\models^{\mathfrak{p}} \neg_j \psi$,
(12) $x \models^{\mathfrak{p}} \neg_j \Box_j \varphi$ iff $W \setminus \|\neg_j \varphi\|_{\mathcal{M}} \notin N(x)$,
(13) $x \models^{\mathfrak{p}} \neg_j \Diamond_j \varphi$ iff $\|\neg_j \varphi\|_{\mathcal{M}} \in N(x)$,
(14) $x \models^{\mathfrak{p}} \neg_j \neg_j \varphi$ iff $x \models^{\mathfrak{p}} \varphi$,

(15) $x \models^{\mathfrak{p}} \neg_k(\varphi \wedge_j \psi)$ iff $x \models^{\mathfrak{p}} \neg_k\varphi$ and $x \models^{\mathfrak{p}} \neg_k\psi$,
(16) $x \models^{\mathfrak{p}} \neg_k(\varphi \vee_j \psi)$ iff $x \models^{\mathfrak{p}} \neg_k\varphi$ or $x \models^{\mathfrak{p}} \neg_k\psi$,
(17) $x \models^{\mathfrak{p}} \neg_k(\varphi \rightarrow_j \psi)$ iff $x \models^{\mathfrak{p}} \neg_k\psi$ or $x \not\models^{\mathfrak{p}} \neg_k\varphi$,
(18) $x \models^{\mathfrak{p}} \neg_k(\varphi \leftarrow_j \psi)$ iff $x \models^{\mathfrak{p}} \neg_k\varphi$ and $x \not\models^{\mathfrak{p}} \neg_k\psi$,
(19) $x \models^{\mathfrak{p}} \neg_k\Box_j\varphi$ iff $\|\neg_k\varphi\|_{\mathcal{M}} \in N(x)$,
(20) $x \models^{\mathfrak{p}} \neg_k\Diamond_j\varphi$ iff $W \setminus \|\neg_k\varphi\|_{\mathcal{M}} \notin N(x)$,
(21) $x \models^{\mathfrak{p}} \neg_k\neg_j\varphi$ iff $x \not\models^{\mathfrak{p}} \varphi$.

Finally, let us fix a bunch of standard semantic notions. Abusing notation we will write $x \in \mathcal{M}$ to mean that x is an element of the carrier set of $\mathcal{M}$.

Definition 6.5. A formula $\varphi \in \mathscr{L}$ (resp. $\varphi \in \mathscr{L}_{\mathrm{M}}$) is valid in an **MNT4**-model (resp. $\mathbf{MML}_n^{\mathbf{MNT4}}$-model) $\mathcal{M}$ if $\mathcal{M}, x \models \varphi$ (resp. $\mathcal{M}, x \models^{\mathfrak{p}} \varphi$) for all $x \in \mathcal{M}$. A formula φ is **MNT4**-valid (resp. $\mathbf{MML}_n^{\mathbf{MNT4}}$-valid) if it is valid in all **MNT4**-models (resp. $\mathbf{MML}_n^{\mathbf{MNT4}}$-models). In the sequel let $\mathbf{MNT4} \models \varphi$ (resp. $\mathbf{MML}_n^{\mathbf{MNT4}} \models^{\mathfrak{p}} \varphi$) denote **MNT4**-validity (resp. $\mathbf{MML}_n^{\mathbf{MNT4}}$-validity) for a formula φ.

6.3. Neighbourhood completeness via embedding

The aim of this section is to show that our system $\mathbf{MML}_n^{\mathbf{MNT4}}$ is complete with respect to the class of all paraconsistent neighbourhood models. This result can be regarded as a by-product of the embedding technique elaborated in previous subsections.

As a starting point, we relate the truth conditions of a formula φ (of the language $\mathscr{L}_{\mathrm{M}}$) at a point in an $\mathbf{MML}_n^{\mathbf{MNT4}}$-model and the truth conditions of its translation $f(\varphi)$ at a point in an **MNT4**-model.

Suppose that we have an $\mathbf{MML}_n^{\mathbf{MNT4}}$-model $\mathcal{M} = (W, N, \vartheta)$ at hand. It is sufficient to reconstruct a paraconsistent valuation ϑ into a standard valuation φ, defined on the same frame (W, N), and specifying for all $p \in \mathcal{P}$, $x \in W$ and $j \leqslant n$:

$$x \in \vartheta(p) \text{ iff } x \in \theta(p);$$
$$x \in \vartheta(\neg_j p) \text{ iff } x \in \theta(p^j).$$

Thereby we have constructed a neighbourhood model $\mathcal{M}' = (W, N, \theta)$. Let us check that this is an intended **MNT4**-model.

Lemma 6.1. *$\mathcal{M}' = (W, N, \theta)$ is an* **MNT4***-model. Moreover, for each formula φ of the language $\mathscr{L}_{\mathrm{M}}$ and for each $x \in W$,*

$$\mathcal{M}, x \models^{\mathfrak{p}} \varphi \textit{ iff } \mathcal{M}', x \models f(\varphi).$$

Proof. The proof is almost a straightforward induction argument. Let us consider some typical cases.

1. For a propositional variable or a negated propositional variable the statement follows directly from the definition of θ.
2. $\mathcal{M}, x \models^{\mathfrak{p}} \neg_j(\varphi \wedge_j \psi)$ iff $\mathcal{M}, x \models^{\mathfrak{p}} \neg_j \varphi$ or $\mathcal{M}, x \models^{\mathfrak{p}} \neg_j \psi$ (Definition 6.4) iff $\mathcal{M}', x \models f(\neg_j \varphi)$ or $\mathcal{M}', x \models f(\neg_j \psi)$ (induction hypothesis) iff $\mathcal{M}', x \models f(\neg_j \varphi) \vee f(\neg_j \psi)$ iff $\mathcal{M}', x \models f(\neg_j(\varphi \wedge_j \psi))$ (Definition 6.1).
3. $\mathcal{M}, x \models^{\mathfrak{p}} \neg_k(\varphi \wedge_j \psi)$ iff $\mathcal{M}, x \models^{\mathfrak{p}} \neg_k \varphi$ and $\mathcal{M}, x \models^{\mathfrak{p}} \neg_k \psi$ (Definition 6.4) iff $\mathcal{M}', x \models f(\neg_k \varphi)$ and $\mathcal{M}', x \models f(\neg_k \psi)$ (induction hypothesis) iff $\mathcal{M}', x \models f(\neg_k \varphi) \wedge f(\neg_k \psi)$ iff $\mathcal{M}', x \models f(\neg_k(\varphi \wedge_j \psi))$ (Definition 6.1).
4. $\mathcal{M}, x \models^{\mathfrak{p}} \Box_j \varphi$ iff $\|\varphi\|_{\mathcal{M}} \in N(x)$ iff $\|f(\varphi)\|_{\mathcal{M}'} \in N(x)$ (induction hypothesis) iff $\mathcal{M}', x \models \Box(f(\varphi))$ iff $\mathcal{M}', x \models f(\Box_j \varphi)$.
5. $\mathcal{M}, x \models^{\mathfrak{p}} \neg_j \Box_j \varphi$ iff $W \setminus \|\neg_j \varphi\|_{\mathcal{M}} \notin N(x)$ iff $W \setminus \|f(\neg_j \varphi)\|_{\mathcal{M}'} \notin N(x)$ (induction hypothesis) iff $\mathcal{M}', x \models \Diamond(f(\neg_j \varphi))$ iff $\mathcal{M}', x \models f(\neg_j \Box_j \varphi)$. ⊣

For the other direction of reconstructing models we stipulate the following which we get similar to the previous lemma:

Lemma 6.2. *For every* **MNT4***-model $\mathcal{M}' = (W, N, \theta)$ there exists an* $\mathbf{MML}_n^{\mathbf{MNT4}}$*-model $\mathcal{M} = (W, N, \vartheta)$ such that for every formula $\varphi \in \mathscr{L}_{\mathrm{M}}$, for all $x \in W$:*

$$\mathcal{M}', x \models f(\varphi) \textit{ iff } \mathcal{M}, x \models^{\mathfrak{p}} \varphi.$$

In the context of completeness we are interested in validity preservation which is the subject of the next lemma.

Lemma 6.3. *For every $\mathscr{L}_{\mathrm{M}}$ formula φ, φ is an* $\mathbf{MML}_n^{\mathbf{MNT4}}$*-valid iff $f(\varphi)$ is* **MNT4***-valid.*

Proof. For the *only if* part suppose that for some $\mathbf{MML}_n^{\mathbf{MNT4}}$-valid formula φ, some **MNT4**-model $\mathcal{M}'$ and $x \in \mathcal{M}'$, $\mathcal{M}', x \not\models f(\varphi)$. Then, by Lemma 6.2 there is an $\mathbf{MML}_n^{\mathbf{MNT4}}$-model refusing φ. For the other direction assume that for some φ there is an $\mathbf{MML}_n^{\mathbf{MNT4}}$-model falsifying it. Then by Lemma 6.1, $f(\varphi)$ is refused in some **MNT4**-model. ⊣

Theorem 6.2. *The calculus* $\mathbf{MML}_n^{\mathbf{MNT4}}$ *is sound and complete with respect to the class of all* $\mathbf{MML}_n^{\mathbf{MNT4}}$*-models.*

Proof. By Lemma 6.3, Theorem 6.1, and the completeness result for the calculus **MNT4**, we have the following sequence of equivalences: $\mathbf{MML}_n^{\mathbf{MNT4}} \models^{\mathfrak{p}} \varphi$ iff $\mathbf{MNT4} \models f(\varphi)$ iff $\mathbf{MNT4} \vdash \Rightarrow f(\varphi)$ iff $\mathbf{MML}_n^{\mathbf{MNT4}} \vdash \Rightarrow \varphi$ ⊣

7. Conclusion

We have introduced a new modal multilattice logic which is called $\mathbf{MML}_n^{\mathbf{MNT4}}$ and is based on modal multilattices with Tarski closure and interior operators. We have proved an algebraic completeness theorem and presented a sequent calculus for $\mathbf{MML}_n^{\mathbf{MNT4}}$ as well as a neighbourhood semantics for it. We have studied modal multilattices with Kuratowski and Halmos closure and interior operators and shown that the former structure determines the logic $\mathbf{MML}_n^{\mathbf{S4}}$ and the latter one determines the logic $\mathbf{MML}_n^{\mathbf{S5}}$. Future investigation may address modal multilattices with the other closure and interior operators.

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