Hans-Peter Leeb

States of Affairs as Structured Extensions in Free Logic

Abstract. The search for the extensions of sentences can be guided by Frege's "principle of compositionality of extension", according to which the extension of a composed expression depends only on its logical form and the extensions of its parts capable of having extensions. By means of this principle, a strict criterion for the admissibility of objects as extensions of sentences can be derived: every object is admissible as the extension of a sentence that is preserved under the substitution of co-extensional expressions. The question is: what are the extensions of elementary sentences containing empty singular terms, like 'Vulcan rotates'. It can be demonstrated that in such sentences, states of affairs as structured objects (but not truth-values) are preserved under the substitution of co-extensional expressions. Hence, such states of affairs are admissible (while truth-values are not) as extensions of elementary sentences containing empty singular terms.

Keywords: states of affairs; structured objects; non-existence; extension; extensionality; free logic

1. Introduction

Originally my paper [10] and the present paper were intended to be parts of a single essay dealing with some foundational questions of free logic, such as: How can the extensionality of free logic be secured? What are the extensions of sentences in free logic if they have an extension? It turned out that the answers to these questions are connected with each other. After some discussion with Karel Lambert, who asked me to reformulate his Non-Extensionality-Argument in terms of states of

affairs, I decided to split the essay up into its two parts because they have different arguments at their core and pursue different aims. In what follows, at first the differences between these two papers are explained and afterwards their similarities.

The first paper, [10], has at its core the reformulation of Lambert's Non-Extensionality-Argument in terms of states of affairs. The aim of the paper is twofold: (i) to reconcile Quine's requirement of extensionality as a criterion of adequacy for a scientific language with the conviction that such a language must contain empty singular terms and (ii) to prove that elementary sentences containing empty singular terms are extensional in the substitutivity sense if it is assumed that the extensions of such sentences are states of affairs that have holes which are marked by hole-marker objects. A sentence is extensional in the substitutivity sense if and only if it is always possible to substitute co-extensional expressions for each other within this sentence without changing its extension. Extensionality in the substitutivity sense is proven by first proving extensionality in the substitutivity-salvo-statu-rerum sense (the Latin phrase 'salvo statu rerum' means 'without changing the state of affairs'), where a sentence is extensional in the substitutivity-salvo-statu-rerum sense if and only if it is extensional in the substitutivity sense and states of affairs are the extensions of sentences. The extensionality results proven in [10] with respect to such elementary sentences containing empty singular terms apply only to *single*-domain-based negative and positive free logic. Five assumptions are needed to prove these results (the principle of compositionality is not among them). Furthermore, it is demonstrated that the fact that elementary sentences containing empty singular terms are extensional in the substitutivity-salvo-statu-rerum sense implies that such sentences are also extensional in Quine's truth-value-related sense given those five assumptions. This connection between extensionality in the substitutivity-salvo-statu-rerum sense and extensionality in Quine's truth-value-related sense is proven by using the basic idea of state-of-affairs semantics, according to which a sentence is true if and only if it describes a state of affairs that obtains. It is demonstrated that this connection holds quite generally for all kinds of logics that can be interpreted by means of a state-of-affairs semantics, and which turn out to be extensional in the substitutivity-salvo-statu-rerum sense vis-à-vis their state-of-affairs-semantic interpretation. Furthermore, some metaphysical questions concerning states of affairs which involve hole-marker objects are discussed, e.g., how the unity of such states of affairs comes

about. [10] also contains the first part of my analysis of the notion of extensionality.

By contrast, the present paper has at its core Lambert's original version of his Non-Extensionality-Argument; thus, no reformulation of this argument in terms of states of affairs is undertaken. Furthermore, the paper modifies Lambert's Non-Extensionality-Argument in order to achieve a certain aim, that is, to prove the following result: Whereas structured objects are admissible as extensions of sentences in free logic, truth-values are not, because the assumption that structured objects are the extensions of sentences in free logic leads to extensionality in the substitutivity sense, but the assumption that truth-values are such extensions does not. In order to derive the criterion of admissibility needed to prove this result, the principle of compositionality — understood as a substitution thesis — is used (this principle plays no role in [10]). In short, an object is admissible as the extension of a sentence if and only if the assumption that this object is the extension of the sentence makes the sentence extensional in the substitutivity sense. Thus, an object is *non*-admissible as the extension of a sentence if and only if the assumption that this object is the extension of the sentence makes the sentence *non*-extensional in the substitutivity sense. The *non*-admissibility of truth-values as extensions of sentences in free logic then follows from the assumption that truth-values are the extensions of sentences and some further assumptions (using my modification of Lambert's argument) since elementary sentences containing empty singular terms turn out to be *non*-extensional in the substitutivity sense under these assumptions. By contrast, the admissibility of structured objects as extensions of sentences in free logic then follows from the assumption that structured objects are the extensions of sentences and some further assumptions (using my proofs for certain extensionality results) because elementary sentences containing empty singular terms turn out to be extensional in the substitutivity sense under these assumptions. Furthermore, the extensionality results proven in this paper hold for *dual*-domain-based positive free logic as well, and it also contains the second part of my analysis of the notion of extensionality.

The similarities between the two papers are due to the fact that to achieve their aims as described above, various notions of extensionality in the substitutivity sense are needed, though different things are said about these notions in both papers. Furthermore, certain extensionality results must be proven in both papers as well. In this respect, the

scope of the present paper extends the scope of the paper [10] since the former additionally contains an extensionality proof for dual-domain-based positive free logic whereas the latter does not. And the paper [10] extends the scope of the present paper because it contains the proof that extensionality in the substitutivity-salvo-statu-rerum sense implies extensionality in Quine's truth-value-related sense. Nevertheless, what both papers have in common is that they contain similar extensionality proofs for single-domain-based negative and positive free logic: The extensionality proofs in [10] aim at proving extensionality in the substitutivity sense via the intermediate step of first proving extensionality in the substitutivity-salvo-statu-rerum sense. On the other hand, the extensionality proofs in the present paper aim at proving extensionality in the substitutivity sense directly without such an intermediate step by using structured objects as extensions, that is, by using structured extensions. The additional assumption that these structured objects might well be viewed as abstract states of affairs is helpful when it comes to specifying the properties of those structured objects. Since strictly spoken the same properties can also be specified just by stipulation, this additional assumption is not needed for the extensionality proofs of the present paper. Thus, the proofs under consideration are only similar, not identical. Overall, the differences between the two papers far outweigh their similarities which are founded in the nature of the subject.

2. Admissibility of objects as extensions of sentences and extensionality in the substitutivity sense

In philosophical logic it is mostly uncontested that the extensions of singular terms are individuals, and that the extensions of n-ary general terms are sets of ordered n-tuples of individuals. The question of what the extensions of sentences are (if they have an extension) was, however, answered differently during the early era of philosophical logic. While Frege [3] considered *truth-values* to be extensions, Carnap's early semantics held (in Wittgenstein's terminology) *states of affairs* as extensions [2]. To decide which answer is right, fundamental *metasemantic* principles will be used in this paper. These principles will deal quite generally with the extensions of sentences.

With Frege, the *principle of compositionality of extension* plays the role of a guiding principle in the search for the extensions of sentences.

It states that the extension of a composed expression depends only on its logical form and the extensions of its parts capable of having extensions. Against any semantics for free logic that are not compositional, one might raise the objection that common truth-value semantics for classical logic are compositional. However, the difference between free logic and classical logic is minimal: whereas free logic allows for empty singular terms, classical logic does not, and the former does so in order to make the hidden existence assumptions of the latter explicit. To avoid this theoretical objection, I will proceed from the principle of compositionality when developing semantics for free logic and investigating the question of what the extensions of sentences are. Regarded as a *substitution* thesis, this principle states the following:

(C) It is always possible to substitute expressions that have the same extension (that is, co-extensional expressions) for each other within all sentences *salva extensione*.

Observe that it is not yet settled what objects the extensions of sentences are; if this question was already settled, then this principle could not guide the search for such extensions without already having supposition about what one is looking for. As long as this principle is guiding the search for such extensions, it will impose restrictions on what objects are admissible as extensions. Thus, (C) suggests the following *criterion* for the admissibility of objects as extensions:

(Cr.AO) An object o is *admissible* as the extension of a sentence S (due to (C)) if and only if o is always preserved under the substitution of co-extensional expressions within S.[1] Hence, an object is *not admissible* as the extension of a sentence S if and only if it is not always preserved under the substitution of co-extensional expressions within S.

There is a relationship between the admissibility of objects as extensions of sentences and the notion of extensionality in the substitutivity-salva-extensione sense. In order to spell out this relationship, I must

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[1] That is, if and only if for all substitution-results T and expressions A, B: if A is co-extensional to B and o is the extension of S, then o is also the extension of T. Let the quantifier in 'for all substitution-results T' range over the results of substituting one or more occurrences of an expression A in a sentence S by an expression B of the same syntactical category as A.

first introduce this notion of extensionality in the substitutivity sense.
It can be defined as a property of sentences, as follows:

(Df.SE) A sentence S is *extensional* in the substitutivity-salva-extensione
sense $:\Leftrightarrow$[2] it is always possible to substitute co-extensional ex-
pressions for each other within S salva extensione.[3]

Hence, a sentence is *non-extensional* in the substitutivity-salva-exten-
sione sense if and only if it is not always possible to substitute co-
extensional expressions for each other within the sentence salva exten-
sione. The question of what kinds of objects (truth-values, states of af-
fairs, etc.) the extensions of sentences are is left open. Thus, extensions
are considered in a general and neutral way, and with disregard for spe-
cific choices of certain kinds of objects as extensions of sentences. When
developing formal semantics, one might well say that the set-theoretical
objects chosen as representatives for the extensions of sentences are not
yet philosophically interpreted as representatives for truth-values, states
of affairs, or whatever. Additionally, the use of the term *co-extensionality*
of two expressions means that they have the same extension. Extension-
ality in the substitutivity-salva-extensione sense has, accordingly, three
characteristic features: the notion of extension, the (equivalence) relation
of co-extensionality, and the concept of substitutivity-salva-extensione,
which can be reduced to the relation of co-extensionality.

Now one can define two meanings of co-extensionality.

(Df.WC) An expression A is *weakly co-extensional* to an expression B
$:\Leftrightarrow$ for all objects o_1, o_2 (if o_1 is the extension of A and o_2 is
the extension of B, then $o_1 = o_2$).

By contrast,

(Df.SC) an expression A is *strongly co-extensional* to an expression B
$:\Leftrightarrow$ A is weakly co-extensional to B and there is an object o_1
such that o_1 is the extension of A and there is an object o_2 such
that o_2 is the extension of B.

Hence, strong co-extensionality implies weak co-extensionality, but not
vice versa. Extensionality in the substitutivity sense can then be spec-
ified as in (Df.SE) above. Since every property of extensionality in the

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[2] Read the metalinguistic sign ':⇔' as 'is definitional equivalent to'.

[3] That is, if and only if for all substitution-results T and expressions A, B: if A
is co-extensional to B, then S is co-extensional to T.

substitutivity sense can be defined by means of one of these two relations of co-extensionality, one can accordingly distinguish two meanings of extensionality, as well as of non-extensionality in the substitutivity-salva-extensione sense. The reason for doing so is that a sentence might be non-extensional in one of these two meanings, while being extensional in the other. Thus, due to the ambiguity of co-extensionality, (Df.SE) is a definition-schema, that is, a schema to construct several explicit definitions which have analogous forms.

Furthermore, a language is extensional in the substitutivity-salva-extensione sense if and only if it is always possible to substitute co-extensional expressions for each other within all sentences of this language salva extensione. Hence, the substitution thesis (C) expresses the full extensionality of a language.

Finally, the point of the relationship between the admissibility of objects as extensions of sentences and extensionality in the substitutivity-salva-extensione sense can be spelled out as follows: if an object is assumed to be the extension of a sentence and this sentence turns out to be extensional itself, then that object is admissible as its extension; if this sentence turns out to be non-extensional, then that object is not admissible as its extension, and one must look for another object to achieve compositionality (C).

3. Truth-values are not admissible as extensions of elementary sentences containing empty singular terms

Free logic is a logic that is free of existence assumptions with respect to its singular and general terms, and in which the quantifiers are understood in the same way as they would be in classical logic (that is, they are read as 'all existing things' and 'some existing things') [8, p. 124], [7, p. 258], [6, pp. 41, 59], [11, p. 2]. If one accepts logic as the ideal tool of philosophical analysis, then one should also expect it to be free of existence assumptions and independent of empirical facts. For this reason, free logic attempts to free classical logic from existence assumptions by making these existence assumptions explicit. This process, however, brings about new problems (such as the extension problem) because free logic allows for empty singular terms. An *empty* singular term is a singular term that either refers to nothing (that is, is irreferential) or to something non-existent (that is, is referential). There are, nevertheless,

several good reasons for the admittance of such terms. For instance, the following reason: singular terms that might be empty are necessary in rendering classical logic's existence assumptions (with respect to singular terms) explicit.

In what follows, I will examine the *problem of extension* by asking what the extensions of elementary sentences containing empty singular terms are if they have an extension. Although this problem was not examined by Lambert, I will modify and use his original Non-Extensionality-Argument [9, pp. 22–24], [8, pp. 95–97, 109f.], [5, pp. 278f., 282], [4, pp. 257–259]. My *modification of the Non-Extensionality-Argument* (abbreviated: 'MNEA') is based on various assumptions which are different from Lambert's assumptions. The use to which the MNEA is put is also different from the use of the original argument: I will use the MNEA to demonstrate that elementary sentences containing empty singular terms are non-extensional in the substitutivity-salva-extensione sense if truth-values are the extensions of such sentences. Hence, in light of the strict criterion for the admissibility of objects as extensions (CR.AO), truth-values are not admissible as the extensions of elementary sentences containing empty singular terms, and one must look for different objects that might serve as the extensions of such sentences.

Quine's conception of extensionality is summarized by Lambert in the following citation:[4]

> A statement is SV-extensional according to the salva veritate substitution conception just in case singular terms co-referential with a statement's constituent singular terms, predicates co-extensive with the statement's constituent predicates, and statements co-valent with a statement's constituent statement(s), substitute in that statement salva veritate.

Quine's conception of SV-extensionality contains neither the notion of extension nor the assumption that truth-values are the extensions of sentences. However, in another conception of SV-extensionality, both can be added. The result is a conception of SV-extensionality that is open to different choices of objects as extensions of sentences. Under this conception, one can make important statements concerning extensionality in the substitutivity sense which, otherwise, could not be made.

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[4] [8, p. 107], [7, p. 274], [13, p. 151]. Following Lambert and Quine, the expressions capable of having extensions are confined in the paper to singular and general terms (or predicates) and sentences.

For this reason, the MNEA does not proceed from Quine's conception (as did Lambert's original argument), but from the following conception of truth-value-as-extension-related extensionality:

(SV) A sentence S is extensional in the substitutivity-salva-veritate sense if and only if it is always possible to substitute co-extensional singular and general terms (or predicates) and sentences for each other within S without changing the truth-value *as extension*.[5]

(SV) expresses a conception of extensionality in the substitutivity-salva-veritate sense in which the notion of extension and the assumption that truth-values are the extensions of sentences are added. For this reason, the latter assumption can be replaced with different assumptions about the kinds of objects that are the extensions of sentences (e.g., states of affairs). This would not be possible under Quine's conception of extensionality in the substitutivity-salva-veritate sense since the latter refrains from adding the notion of extension. The complex notion of extensionality in the sense of (SV) can now be decomposed into its two components as follows: a sentence S is extensional in the substitutivity-salva-veritate sense if and only if for all substitution-results T and expressions A, B: if A is co-extensional to B, then S has the same extension as T (that is, S is co-extensional to T), and truth-values are the extensions of sentences. Observe that the left conjunct in the right part of this biconditional is the definiens of (Df.SE). Thus, this conception of *truth-value-as-extension-related* extensionality can be defined as follows:

(Df.SV) A sentence S is *extensional* in the substitutivity-salva-veritate sense $:\Leftrightarrow S$ is extensional in the substitutivity-salva-extensione sense and truth-values are the extensions of sentences.

Due to this decomposability of extensionality in the sense of (SV), the MNEA is based on the definition-schema (Df.SE) for extensionality in the substitutivity-salva-extensione sense.

Usually, modal, epistemic and time-logical presuppositions are considered to be sources for non-extensionality in the substitutivity-salva-veritate sense. Without such presuppositions, I will argue on a *metasemantic* level that elementary sentences containing empty singular terms

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[5] That is, if and only if for all substitution-results T and expressions A, B: if A is co-extensional to B, then S has the *same* truth-value as *extension* as T.

are non-extensional in the substitutivity-salva-extensione sense if truth-values are the extensions of sentences. The MNEA is based on the following seven assumptions:

I. The extensions of sentences are specified as follows: (T) Truth-values are the extensions of sentences.

II. The definition-schema (Df.SE) for extensionality in the substitutivity-salva-extensione sense is accepted.

III. Quine's theory of predication [13, 96, 105, 175] in *Word and Object* is accepted. According to Quine, the logical form of a predication depends on how its truth-value is determined [8, 104]. Quine's definition qualifies all elementary sentences containing empty singular terms as predications without settling their truth-values.

IV. The *classical principle of term abstraction* is accepted. It can be formulated as axiom-schema as follows: $(\Delta) \, (\Delta y A)a \leftrightarrow A(a/y)$. This principle states that two sentences with the forms $(\Delta y A)a$ and $A(a/y)$ always have the same truth-value. In the MNEA, it is assumed that this principle is valid for all predications which do not imply the existence of their alleged subjects — as, for example, in the intuitively true sentence, 'Sherlock Holmes is a thing y such that y is fictitious'. This sentence is true because Sherlock Holmes does not exist; if he did exist, then it would not be true, but false. Hence, its truth does not imply the existence of its alleged subject.

V. It is assumed that general terms can be complex and

VI. that the following sentence,

(1) Vulcan is a thing y such that y exists,

is false.

VII. The three general terms, 'a thing y such that y rotates', 'a thing y such that y rotates and y exists', and 'a thing y such that y rotates if y exists' are strongly co-extensional because they are true (or false) of the same individuals, namely every existing individual that rotates (or does not rotate). The MNEA contains at this point the existence assumption that these three general terms are only applied to existing individuals but not to non-existent ones. Applying these three general terms to non-existing individuals would allow such non-existents in their extensions. Hence, these three general terms were no longer strongly co-extensional.

According to Quine's theory of predication, the following predication,

(2) Vulcan is a thing y such that y rotates,

is either *true*, *false*, or *truth-valueless*. One can now demonstrate that the truth-value of the predication (2) is changed if co-extensional general terms are substituted for each other within (2).

Since it is unclear which truth-values elementary sentences containing empty singular terms like (2) have, the following three cases must be distinguished:

A. First case:

Assume that the predication (2) is *true*. Replacing the term 'a thing y such that y rotates' in (2) with the co-extensional general term 'a thing y such that y rotates and y exists' results in the predication

(3) Vulcan is a thing y such that y rotates and y exists.

Since Quine's theory of predication does not settle the truth-value for such predications containing empty singular terms, the truth-value of (3) is unknown, at least as long as only (3) is taken into consideration. For this reason, the classical principle of term abstraction is needed. According to this principle (and the rules of the propositional calculus), the predication (3) always has the same truth-value as the sentence

(4) Vulcan is a thing y such that y rotates and Vulcan is a thing y such that y exists.

(4) is false because it has the false predication (1) as a conjunct, and a conjunction is false if one of its conjuncts is false. Therefore (3) is false.

B. Second case:

Assume that the predication (2) is *false*. Replacing the term 'a thing y such that y rotates' in (2) with the co-extensional general term 'a thing y such that y rotates if y exists' results in the predication

(5) Vulcan is a thing y such that y rotates if y exists.

Once again and for the same reason, the truth-value of (5) is unknown, at least as long as only (5) is taken into consideration. According to the classical principle of term abstraction, the predication (5) always has the same truth-value as the sentence

(6) If Vulcan is a thing y such that y exists, then Vulcan is a thing y such that y rotates.

(6) is true because it has the false predication (1) as its antecedent, and a material conditional is true if its antecedent is false. So (5) is true.

C. Third (and final) case:

Assume that the predication (2) is *truth-valueless*. Now, as in the first case, the relevant result of substitution (3) is false.

Thus, independent of whether the predication (2) '$\Delta y(y\ rotates)$ *Vulcan*' has a truth-value (and if yes, which one?), if we substitute the general term '$\Delta y(y\ rotates)$' in (2) by strongly co-extensional general terms, we change the truth-value of (2). Moreover, due to the assumption (T) that truth-values are the extensions of sentences, these substitutions also change the truth-value as extension of (2). So, since truth-values are identified with the extensions of sentences, changing the truth-value also changes the extension. Therefore, it is not always possible to substitute co-extensional general terms for each other within (2) without changing the truth-value as extension and, consequently, without changing the extension, that is, (2) is non-extensional in the substitutivity-salva-extensione sense.

To summarize (given the seven assumptions above), the MNEA demonstrates that elementary sentences containing empty singular terms are non-extensional in the substitutivity-salva-veritate sense (Df.SV). It also demonstrates that such sentences are non-extensional in the substitutivity-salva-extensione sense. To achieve the latter result, it is necessary to add to the premises the assumption (T) that truth-values are the extensions of sentences, and to proceed from (Df.SE). A consequence of these two assumptions (together with the other five assumptions) is that elementary sentences containing empty singular terms turn out to be non-extensional in the substitutivity-salva-extensione sense. However, as discussed in section 2, whatever objects the extensions of sentences are, they must be preserved under the substitution of co-extensional expressions, otherwise they are not admissible as extensions. Therefore, if free logic proceeds from these other five assumptions of the MNEA, then truth-values are not admissible (in the strict sense of (Cr.AO)) as the extensions of sentences in free logic. This result is unavoidable as long as the search for the extensions of sentences in free logic is guided by two convictions: the substitution thesis (C) which justifies the strict criterion (Cr.AO) for the admissibility of objects as extensions, and assumptions two to seven of the MNEA. The additional assumption of truth-values as extensions of sentences then implies unavoidably the non-extensionality

in the substitutivity-salva-extensione sense of elementary sentences con-
taining empty singular terms, and thus the non-admissibility of truth-
values as their extensions.

Hence, the following questions arise: is it possible to secure the ex-
tensionality in the substitutivity-salva-extensione sense of elementary
sentences containing empty singular terms — and if yes, how? What
objects are admissible as the extensions of such sentences in free logic?

4. Structured objects as extensions of sentences

The critical point in the MNEA seems to be the interplay between the
assumption (T), according to which truth-values are the extensions of
sentences, and the classical principle of term abstraction (Δ), saying that
two sentences of the forms $(\Delta yA)a$ and $A(a/y)$ always have the same
truth-value. Due to the latter principle (and some rules of the proposi-
tional calculus), the predication (3) '$\Delta y(y\ rotates \wedge y\ exists)Vulcan$' and
the conjunction (4) '$\Delta y(y\ rotates)Vulcan \wedge \Delta y(y\ exists)Vulcan$' always
have the same truth-value. However, if truth-values are the extensions
of sentences and these two sentences always have the same truth-value,
then they also always have the same extension (namely, the truth-value
that they have in common). If one proceeds from the classical principle
of term abstraction and assumes that truth-values are the extensions of
the sentences (3) and (4), then these sentences have the same object
as extension because they have their truth-value in common (and also,
analogously, for (5) and (6)).

However, do the predication (3) and the conjunction (4) really have
the same object as extension if both have an extension?

Consider analogously the two *true* sentences:

(7) $\Delta y(y\ rotates \wedge y\ exists)Mars$,
(8) $\Delta y(y\ rotates)Mars \wedge \Delta y(y\ exists)Mars$.

Observe that the sentences (7) and (8) are both true (both even have the
same truth-value, namely, True). But do, for this reason, both sentences
already have the same object as extension if they have an extension (e.g.,
True)?

Assume that the extensions of sentences can have an internal struc-
ture, that is, can be in a certain sense complex. To determine this
internal structure, one must consider the extensions of the descriptive

parts of a sentence that are capable of having extensions, and also the logical form of the sentence.

If one considers the extensions of the singular and general terms occurring in (7), then one might find it more plausible to assume that

the extension of '$\Delta y(y\ rotates \wedge y\ exists)Mars$' = the structured object that Mars lies in the set of rotating and existing individuals of the domain.

Furthermore, if one considers the extensions of the two sub-sentences of (8), then one should be more convinced by the assumption that

the extension of '$\Delta y(y\ rotates)Mars \wedge \Delta y(y\ exists)Mars$' = the structured object that Mars lies in the set of rotating individuals of the domain, and Mars lies in the set of existing individuals of the domain.

Therefore, the two true sentences (7) and (8) have different objects as extensions. While (7) has the extension that Mars lies in a certain set, (8) has the extension that Mars lies in a first set, and that Mars also lies in a second one. Therefore, if one considers the extensions of the sub-expressions of sentences of the two forms $(\Delta yA)a$ and $A(a/y)$, and if two sentences of these forms have an extension, then these two sentences generally have different objects as extensions.

From the perspective of structured objects, the interplay in the MNEA between the classical principle of term abstraction and the assumption (T), according to which truth-values are the extensions of sentences, is problematic because, then, sentences which actually have (from that perspective) different structured objects as extensions have, nevertheless, the same extension — only because one is assuming that they have truth-values as extensions.

5. State-of-affairs-semantic and formal-ontological assumptions

My approach to securing the extensionality in the substitutivity-salva-extensione sense of elementary sentences containing empty singular terms presupposes some state-of-affairs-semantic and formal-ontological assumptions. Assume that sentences can describe states of affairs understood as *structured objects*, and that the extensions of sentences are the states of affairs they describe. The basic idea of state-of-affairs-semantics is that a sentence is true if and only if it describes a state of affairs that obtains. A state of affairs is a way in which things behave or do not be-

have (or might behave or might not behave) in the world. Furthermore, a state of affairs obtains (holds) if and only if things in the world behave in such a way as is demonstrated in that state of affairs. Since my approach to securing the extensionality in the substitutivity-salva-extensione sense of elementary sentences containing empty singular terms also needs a *criterion* of identity for states of affairs, I state it as follows:

(ID) Two states of affairs are *identical* if and only if they involve the same objects and have the same necessary and sufficient condition of obtaining.

The states of affairs that are described by elementary sentences containing irreferential singular terms have, in a certain sense, "holes", since irreferential singular terms do not contribute an individual which such states of affairs could involve. In order to "mark" the holes in such states of affairs, I introduce non-existing objects. In this way, one can formulate the necessary and sufficient conditions of obtaining of such states of affairs. I propose three formal-ontological interpretations of states of affairs involving non-existents, such as the state of affairs that Vulcan rotates. Furthermore, I will formulate the necessary and sufficient condition of obtaining of such states of affairs in the framework of a model for free logic [12, pp. 1032f.], [11, pp. 13f.], [1, pp. 280–282].

(i) In state-of-affairs-semantics for negative free logic, I introduce non-elements of the domain (e.g., the domain itself, its power set, the power set of the power set of the domain, etc.) to represent non-existents as "hole-markers". Such hole-markers are not the extensions of *irreferential* singular terms; they only mark the holes in states of affairs where an object is missing due to an irreferential singular term. Since only *referential* empty singular terms have extensions that together form an outer domain of non-existents, such hole-markers together do not form an outer domain of non-existents as extensions for *irreferential* singular terms. The sentence 'Vulcan rotates' describes, then, a state of affairs that involves a non-existent object (as represented by a non-element of the domain), and the set of rotating individuals from this domain. Such a state of affairs obtains (in a model) if and only if the chosen non-element of the domain is an element of the set of rotating individuals from that domain (which is, of course and as intended, never the case if one assumes that no set has itself as member).[6] Assume that the two

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[6] This assumption guarantees that all elementary sentences containing empty singular terms are false if one assumes bivalence.

states of affairs described by two elementary sentences containing the same irreferential singular term involve the same non-existent object (as represented by the same non-element of the domain) as hole-marker.

(ii) In dual-domain-based state-of-affairs-semantics for positive free logic, the elements of the outer domain can be interpreted as non-existents. The sentence 'Vulcan rotates' describes, then, a state of affairs that involves the non-existent individual Vulcan (that is, an element of the outer domain), referred to by 'Vulcan', and the set of rotating individuals from the union set of inner domain and outer domain. Such a state of affairs obtains (in a model) if and only if the chosen element of the outer domain is an element of the set of rotating individuals from the union set of inner domain and outer domain (which may or may not be the case).

(iii) In single-domain-based state-of-affairs-semantics for positive free logic, I introduce sets of subsets of the domain to represent the being-so[7] of non-existent objects as hole-markers. Once again, such hole-markers are not the extensions of irreferential singular terms but only mark the holes in states of affairs where an object is missing due to an irreferential singular term. The sentence 'Vulcan rotates' describes, then, a state of affairs that involves the being-so of Vulcan (as represented by a set of subsets of the domain) and the set of rotating individuals from this domain. Such a state of affairs obtains (in a model) if and only if the set of rotating individuals of the domain is an element of the chosen set of subsets of that domain (which may or may not be the case). Once again it is assumed that the two states of affairs described by two elementary sentences containing the same irreferential singular term involve the same being-so (as represented by the same set of subsets of the domain) as hole-marker.

6. States of affairs are admissible as extensions of elementary sentences containing empty singular terms

In order to secure the extensionality in the substitutivity-salva-extensione sense of elementary sentences containing empty singular terms, it is assumed that (S) structured objects (called *abstract states of affairs*) are the extensions of sentences. Furthermore, it is stipulated that these

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[7] The *being-so* of an object might by understood as a set of subsets of the domain, or as a set of constitutive (nuclear) properties, or as a complex or combination of constitutive (nuclear) properties.

structured objects have the properties specified in section 5. The basic idea of securing the extensionality in the substitutivity-salva-extensione sense of an elementary sentence such as

(2) $\Delta y(y\ rotates)Vulcan$

(containing an empty singular term such as 'Vulcan') can be formulated as follows:

6.1. Basic idea

The elementary sentence (2) has one of the following structured objects, under one of the three formal-ontological interpretations (i)–(iii), as extension (see section 5):

 (i) the structured object that Vulcan lies in the set of rotating individuals of the domain (where Vulcan is represented by a non-element of that domain), or

 (ii) the structured object that Vulcan lies in the set of rotating individuals which is a subset of the union set of inner domain and outer domain (where Vulcan is an element of the outer domain), or

(iii) the structured object that the set of rotating individuals of the domain lies in the being-so of Vulcan (where the being-so of Vulcan is represented by a set of subsets of the domain).

Substitute in (2) expressions that are *strongly* co-extensional to the empty singular term 'Vulcan', or to the general term '$\Delta y(y\ rotates)$', or to the elementary sentence (2). Demonstrate that the structured object which is the extension of (2), under one of the three formal-ontological interpretations, and that the structured object which is the extension of the relevant result of substituting co-extensional expressions, involve the same objects. Demonstrate that both structured objects have the same necessary and sufficient condition of obtaining. Then both structured objects are identical according to the above criterion of identity for structured objects (ID). Hence, it is always possible to substitute strongly co-extensional expressions for each other within (2) without changing the structured object as *extension*. Therefore, the elementary sentence (2) '$\Delta y(y\ rotates)Vulcan$' is extensional in the substitutivity-salva-extensione sense. Since the same applies to other elementary sentences containing empty singular terms, all such sentences are extensional in the substitutivity-salva-extensione sense.

6.2. The case of co-extensional singular terms

We must now consider empty singular terms that are co-extensional to 'Vulcan'. An empty singular term can be either irreferential or referential.

  (i) In negative free logic, all empty singular terms are irreferential.
 (ii) In dual-domain-based positive free logic, all empty singular terms are referential.
(iii) In single-domain-based positive free logic, all empty singular terms are irreferential.

All irreferential singular terms are weakly co-extensional because they have no extension. Replacing in (2) the irreferential singular term 'Vulcan' with a *different* weakly co-extensional singular term, such as the irreferential singular term 'Selena', might result in a predication which describes (e.g., according to the formal-ontological interpretation (iii)) something that is different from that which (2) describes. For example,

> the extension of '$\Delta y(y\ rotates)Selena$' = the structured object that the set of rotating individuals of the domain lies in the being-so of Selena,

where the being-so of Selena is represented by a set of subsets of the domain that is possibly different from the set of subsets of the domain which represents the being-so of Vulcan; thereby Selena is yet another non-existent planet of our planetary system. Therefore, such irreferential singular terms endanger extensionality in the substitutivity-salva-extensione sense as long as it is based on *weak* co-extensionality. However, irreferential singular terms cannot be *strongly* co-extensional to the irreferential singular term 'Vulcan', because they have no extension. Hence, one cannot replace in (2) the irreferential singular term 'Vulcan' with a singular term that is strongly co-extensional to 'Vulcan', for there is no such strongly co-extensional irreferential singular term. Therefore, such irreferential singular terms cannot endanger extensionality in the substitutivity-salva-extensione sense as long as it is based on *strong* co-extensionality. For this reason, extensionality in the substitutivity-salva-extensione sense must be based on strong co-extensionality.

Replacing in (2) the referential singular term 'Vulcan' with singular terms that are strongly co-extensional to 'Vulcan' results in sentences that have as extensions, under the second formal-ontological interpretation, the structured object in 6.1.(ii). Hence, (2) and the relevant results

of substitution have the same structured object as extension. In single-domain-based positive or negative free logic, such substitutions are not possible because all empty singular terms are irreferential.

6.3. The case of co-extensional elementary sentences

Recall the assumption that (2) has an extension. However, there are not any sentences without extension that could be strongly co-extensional to (2); the former have no extensions, whereas the latter has an extension. Therefore, such sentences without extension cannot endanger extensionality in the substitutivity-salva-extensione sense as long as it is based on strong co-extensionality.

Replacing in (2) the elementary sentence (2) with sentences that are strongly co-extensional to (2) results in sentences that have as extensions, under one of the three formal ontological interpretations, either the structured object in 6.1.(i), or the structured object in 6.1.(ii), or the structured object in 6.1.(iii). Hence, (2) and the relevant results of substitution have the same structured object as extension.

6.4. The case of co-extensional general terms

Observe that general terms always have an extension if there is an empty set. Hence, co-extensionality of general terms is always their *strong* co-extensionality. Assume that '$(\Delta y A)$' is a general term that is co-extensional to the general term '$\Delta y(y\ rotates)$'. Replacing in (2) the general term '$\Delta y(y\ rotates)$' with the co-extensional general term '$(\Delta y A)$' results in the following elementary sentence:

(9) $(\Delta y A)Vulcan$.

When considering the substitution of co-extensional general terms in (2), one must distinguish three cases.

(i) In negative free logic, all empty singular terms are irreferential. The extension of an elementary sentence containing an irreferential singular term, such as 'Vulcan', is a structured object that involves a non-existent object (as represented by a non-element of the domain) as hole-marker (this nonexistent object is *not* the extension of 'Vulcan'), and the extension of the general term that is part of this elementary sentence. Assume that the extensions of two elementary sentences containing the

same irreferential singular term involve the same non-existent object as hole-marker. Then

> the extension of '$\Delta y(y\ rotates)Vulcan$' = the structured object that the non-existent object Vulcan lies in the set of rotating individuals of the domain (where Vulcan is represented by a non-element of the domain).

And

> the extension of '$(\Delta yA)Vulcan$' = the structured object that the non-existent object Vulcan lies in the set which is the extension of 'ΔyA' (where Vulcan is represented by the same non-element of the domain).

Observe that (due to the co-extensionality of both general terms) both sets, which are their extensions, are identical. Hence, the two predications (2) and (9) have as extensions structured objects that involve the same non-existent object (namely, Vulcan), and also identical subsets of the domain. Both structured objects have the same necessary and sufficient condition of obtaining, namely, the condition that the same non-element of the domain is an element of identical subsets of that domain. Therefore, the two predications (2) and (9) have the same structured object as extension if they have an extension, namely, the structured object that the same hole-marker, that is, the same non-existent object (Vulcan) (as represented by the same non-element of the domain), lies in identical subsets of that domain.

(ii) In dual-domain-based positive free logic all empty singular terms, such as 'Vulcan', are referential. The extension of an elementary sentence containing a referential empty singular term, such as 'Vulcan', is a structured object that involves an element of the outer domain (which is the extension of 'Vulcan'), plus the extension of the general term that is part of this elementary sentence. Hence, the two predications (2) and (9) have as extensions structured objects that involve the same element of the outer domain as the extension of 'Vulcan' (namely, Vulcan), and identical subsets of the union set of inner domain and outer domain. Both structured objects have the same necessary and sufficient condition of obtaining, namely, the condition that the extension of 'Vulcan' is an element of identical subsets of the union set of inner domain and outer domain. Therefore, the two predications (2) and (9) have the same structured object as extension if they have an extension, namely, the

structured object that the extension of 'Vulcan' lies in identical subsets of the union set of inner domain and outer domain.

(iii) In single-domain-based positive free logic, all empty singular terms are irreferential. The extension of an elementary sentence containing the irreferential singular term 'Vulcan' is a structured object that involves the being-so of Vulcan as hole-marker (which is *not* the extension of 'Vulcan'), and the extension of the general term that is part of this elementary sentence. Assume that the extensions of two elementary sentences containing the same irreferential singular term involve the same being-so of a non-existent object as hole-marker. Then

the extension of '$\Delta y(y\ rotates)Vulcan$' = the structured object that the set of rotating individuals of the domain lies in the being-so of Vulcan (where the being-so of Vulcan is represented by a set of subsets of the domain).

And

the extension of '$(\Delta y A)Vulcan$' = the structured object that the set which is the extension of '$\Delta y A$' lies in the being-so of Vulcan (where the being-so of Vulcan is represented by the same set of subsets of the domain as before).

Observe that (due to the co-extensionality of both general terms) both subsets of the domain, which are their extensions, are identical. Hence, the two predications (2) and (9) have as extensions structured objects that involve identical subsets of the domain, and the same being-so of Vulcan (as represented by the same set of subsets of the domain). Both structured objects have the same necessary and sufficient condition of obtaining, namely, the condition that identical subsets of the domain are an element of the same set of subsets of the domain. Therefore, the two predications (2) and (9) have the same structured object as extension if they have an extension, namely, the structured object that identical subsets of the domain lie in the same hole-marker, that is, the same being-so of Vulcan (as represented by the same set of subsets of the domain).

In light of the three cases (i)–(iii), it is always possible to substitute co-extensional general terms for each other within the predication (2) '$\Delta y(y\ rotates)Vulcan$' salva extensione.

6.5. Conclusion and consequences

As demonstrated in the three cases 6.2–6.4, it is always possible to substitute strongly co-extensional expressions for each other within (2) '$\Delta y(y \; rotates)Vulcan$' salva extensione. Therefore, (2) turns out to be extensional in the substitutivity-salva-extensione sense if one assumes that (2) has as extension the structured object that Vulcan rotates (as interpreted under one of the three formal-ontological interpretations), and if one proceeds from strong co-extensionality. Whatever kind of object this might be, (2) turns out to be extensional in the substitutivity-salva-extensione sense if one assumes that (2) has this object as extension. This object is consequently admissible, in the strict sense of (Cr.AO) (see section 2), as the extension of (2). For this reason, it is not necessary to assume that in free logic elementary sentences containing empty singular terms have no extensions. It is only necessary to abandon the assumption (T) that truth-values are the extensions of sentences, and to replace it with the assumption (S) that structured objects are the extensions of sentences. One might well view these structured objects as *abstract states of affairs*, thus favoring Carnap's answer to the initial question.

Finally, observe that my way of securing extensionality in the substitutivity-salva-extensione sense also works for empty singular terms that are irreferential: assuming that extensionality in the substitutivity-salva-extensione sense should be based on strong co-extensionality does not imply that irreferential (empty) singular terms are treated as being referential. A non-existent object or its condition of being-so, respectively, is introduced in order to mark the hole in a state of affairs where an object is, due to an irreferential singular term, missing. Irreferential singular terms neither refer to these non-existent objects nor to their condition of being-so. Such non-existent objects (and their condition of being-so) are merely formal-ontological devices to indicate the holes in such states of affairs. Since only *referential* empty singular terms have extensions that together form an outer domain of non-existents, such hole-markers together do not form an outer domain of non-existents as extensions for *irreferential* singular terms. The unity of a state of affairs that involves a non-existent object, and that also involves the extension of a unary general term (that is, a property), comes about through the logical possibility of the predication of the latter to the former, or through the logical possibility of the predication of the former's being-so to the latter.

7. Summary

The question of what the extensions of sentences are was answered differently during the early era of philosophical logic. While Frege considered truth-values to be extensions, Carnap's early semantics held states of affairs as extensions. With Frege, the principle of compositionality of extension plays the role of a guiding principle in the search for the extensions of sentences. Regarded as a substitution thesis, this principle (C) states that it is always possible to substitute expressions that have the same extension (that is, co-extensional expressions) for each other within all sentences salva extensione. (C) suggests that an object is admissible as the extension of a sentence if and only if it is always preserved under the substitution of co-extensional expressions within this sentence. Thus, if an object is assumed to be the extension of a sentence and this sentence turns out to be extensional itself, then that object is admissible as its extension; if this sentence turns out to be non-extensional, then that object is not admissible as its extension, and one must look for another object to achieve compositionality (C).

In the MNEA, the assumption (T), according to which truth-values are the extensions of sentences, implies (together with the other six assumptions) that the predication (2) '$\Delta y(y\ rotates)Vulcan$' is non-extensional in the sense of (Df.SV), and hence non-extensional in the substitutivity-salva-extensione sense. This is why truth-values are not admissible as the extensions of elementary sentences containing empty singular terms in free logic.

By contrast, I have argued that such sentences are extensional in the substitutivity-salva-extensione sense if one assumes that structured objects (called 'abstract states of affairs'), according to the three formal-ontological interpretations, are the extensions of sentences, and if one proceeds from strong co-extensionality. Hence, states of affairs understood as structured objects are admissible as the extensions of elementary sentences containing empty singular terms in free logic.

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Hans-Peter Leeb
Department of Philosophy
University of Salzburg, Austria
hanspeter.leeb@aon.at