



Alexandra Pavlova

Dialogue Games for Minimal Logic

Abstract. In this paper, we define a class of dialogue games for Johansson’s minimal logic and prove that it corresponds to the validity of minimal logic. Many authors have stated similar results for intuitionistic and classical logic either with or without actually proving the correspondence. Rahman, Clerbout and Keiff [17] have already specified dialogues for minimal logic; however, they transformed it into Fitch-style natural deduction only. We propose a different specification for minimal logic with the proof of correspondence between the existence of winning strategies for the Proponent in this class of games and the sequent calculus for minimal logic.

Keywords: dialogue logic; sequent calculi; minimal logic

Introduction

In the present paper, we discuss some issues related to *dialogue games* and Johansson’s minimal logic. By *dialogue games* we understand the concept of dialogue the logic of P. Lorenzen and K. Lorenz defining validity. Johansson’s minimal logic is quite well studied in the literature; however, we propose a dialogue characterisation for it and prove the relevant correspondence result. Major work has been done to prove the correspondence between dialogue games and sequent calculi or natural deduction. Several authors proposed their proofs for the intuitionistic dialogues and the corresponding notion of intuitionistic validity, such as Fermüller [6], Felscher [5], Sørensen and Urzyczyn [19]. One should mention that Rahman, Clerbout and Keiff [17] published their result for minimal logic represented by Fitch-style natural deduction and a corresponding dialogue logic which they define differently from this paper. In the paper [16], a paraconsistent tableaux out of minimal is been

developed. This paraconsistent logic is based on dialogical minimal logic enriched by indexes for each bottom “which characterises univocally the statement it belongs to”. Recently, an elegant version of a proof (of the correspondence result between the sequent calculus and dialogue logic) has been proposed for both intuitionistic and classical logic by Alama, Knoks and Uckelman [1]. They used a variant of the sequent calculus system GKcp [20] for classical propositional logic.

We define a class of dialogue games for minimal logic and a corresponding sequent calculus. Minimal propositional logic can be obtained by rejecting not only the classical *law of excluded middle* (as intuitionistic logic does) but also *the principle of explosion* (*ex falso quodlibet*) $A, \neg A \vdash B$, where B is arbitrary. Thus, in the first section of the paper we define a sequent calculus for minimal logic as an intuitionistic calculus (like LJ of Gentzen) but without *the right weakening* (WR) of the form:

$$\frac{\Gamma \longrightarrow \emptyset}{\Gamma \longrightarrow \mathfrak{D}} \text{ WR}\emptyset$$

where \mathfrak{D} is an arbitrary formula. It is easy to see that this rule corresponds to the Gentzen NJ rule of the form: $\frac{\perp}{\mathfrak{D}}$, as $\Gamma \longrightarrow \emptyset$ means $\Gamma \longrightarrow \perp$ [7, 8]. We also justify the correspondence between the minimal sequent calculus $G_3^{\text{min}a}$ and the minimal natural deduction calculus NM.¹

In the second section, we define a minimal dialogue game as an intuitionistic game where the Proponent cannot leave any attack from the Opponent without a defence. For the minimal logic, there are no exceptions as one can use *bottom* as a defence against an attack on negation.

Finally, in the third section, we come up with a proof of the correspondence between the winning strategies for the Proponent in that class of games and validity in minimal propositional logic. In our proof, we use a modified version of Kleene’s intuitionistic system G_{3a} [11] without structural rules. The axiom now has the following form: $\mathfrak{A}, \Gamma \longrightarrow \Theta, \mathfrak{A}$. Furthermore, the inference rules are modified in such a way that we keep the main formulae.

The motivation for this work has several aspects. The first one represents a purely technical interest in exploring possible dialogue characterisations of various logics. Given the many different ways to define intuitionistic and classical dialogues and to prove the relevant correspondence result, we are interested in setting up a minimal dialogue system.

¹ We use Gentzen-style Natural Deduction.

Though there already exists a variant of minimal dialogue logic [17], we have come up with a different specification independently². Moreover, we give a proof of the correspondence result with respect to the sequent calculus, not a Fitch-style natural deduction (as in [17]). This sheds light on the nature of sequent calculus rules as well. The second aspect of our motivation centres around the relation between minimal and intuitionistic logics which constitute a part of research on their modal versions. The meaning of the structural and logical rules is one of the questions that occur as a result of our correspondence proof. The last aspect places the present result in a more general and broad framework.

1. Sequent Calculus for Minimal Logic $G_3^{\min a}$

First, we introduce the sequent calculus for the minimal logic $G_3^{\min a}$. Our new calculus is based on the Kleene logic $G_3 a$. Johansson has first proposed a sequent calculus for his minimal logic LM where he also rejects the rule of weakening on the right and restricts the number of the formulas in the succedent to exactly one formula [10]. Natural deduction and a sequent calculus for the minimal logic can also be found in [20]. However, for our purposes, we need a modified version of the calculus. We have chosen this system proposed by Kleene because it does not have separate structural rules [11] and we can get rid of the formulae that is not used in the above part of the inference. Thus, as we cannot repeat an attack on negation or implication, we should not keep them in the antecedent of the sequence. In this logic, we restrict the usage of the *right weakening* structural rule (WR). We formulate the rules of our system using a meta-language.

DEFINITION 1. The language \mathfrak{L}_{\min} is defined in BNF-style as follows³:

$$p \mid \mathfrak{A} \mid \neg \mathfrak{A} \mid \mathfrak{A} \wedge \mathfrak{B} \mid \mathfrak{A} \vee \mathfrak{B} \mid \mathfrak{A} \supset \mathfrak{B} \mid \perp$$

We also make use of the meta-language sign of “syntactic consequence” (or the “sequent sign”) \longrightarrow .

² We mostly use the terminology specified by Krabbe, cf. [12] which is different from that of [17]; for instance, we do not make use of a dialogue “*history*”.

³ We use Gothic letters to indicate meta-language variables and capital Greek letters to refer to sets of formulae.

We define a formula by recursive induction. Now we shall define the rules of inference for the minimal sequent calculus $G_3^{\text{min}a}$.

DEFINITION 2. The axiom of the system $G_3^{\text{min}a}$ is

$$\mathfrak{A}, \Gamma \longrightarrow \Theta, \mathfrak{A} \quad (\text{Ax})$$

where \mathfrak{A} is atomic and $\Theta = \emptyset^4$, thus we can reformulate it as follows:

$$\mathfrak{A}, \Gamma \longrightarrow \mathfrak{A} \quad (\text{Ax}^{\text{Int}})$$

There are no structural rules, but we keep the main formula in the premisses. We use Kleene's G_3a system as our basis. This system is different from the G_3 one in that it is possible to omit arbitrary formula in an antecedent or a succedent of a sequence. The only restriction that we make will be specified below.

DEFINITION 3. Given that Γ and Θ are multisets of formulae, the system $G_3^{\text{min}a}$ has the following rules of inference:

$$\frac{\mathfrak{A}, \Gamma \longrightarrow \mathfrak{B}}{\Gamma \longrightarrow \mathfrak{A} \supset \mathfrak{B}} \supset^{S+}$$

$$\frac{\mathfrak{A} \supset \mathfrak{B}, \Gamma \longrightarrow \mathfrak{A} \text{ and } \mathfrak{B}, \mathfrak{A} \supset \mathfrak{B}, \Gamma \longrightarrow \Theta}{\mathfrak{A} \supset \mathfrak{B}, \Gamma \longrightarrow \Theta} \supset^{A+}$$

$$\frac{\Gamma \longrightarrow \mathfrak{A} \text{ and } \Gamma \longrightarrow \mathfrak{B}}{\Gamma \longrightarrow \mathfrak{A} \wedge \mathfrak{B}} \wedge^{S+}$$

$$\frac{\mathfrak{A}, \mathfrak{A} \wedge \mathfrak{B}, \Gamma \longrightarrow \Theta \text{ or } \mathfrak{B}, \mathfrak{A} \wedge \mathfrak{B}, \Gamma \longrightarrow \Theta}{\mathfrak{A} \wedge \mathfrak{B}, \Gamma \longrightarrow \Theta} \wedge^{A+}$$

$$\frac{\Gamma \longrightarrow \mathfrak{A} \text{ or } \Gamma \longrightarrow \mathfrak{B}}{\Gamma \longrightarrow \mathfrak{A} \vee \mathfrak{B}} \vee^{S+}$$

$$\frac{\mathfrak{A}, \mathfrak{A} \vee \mathfrak{B}, \Gamma \longrightarrow \Theta \text{ and } \mathfrak{B}, \mathfrak{A} \vee \mathfrak{B}, \Gamma \longrightarrow \Theta}{\mathfrak{A} \vee \mathfrak{B}, \Gamma \longrightarrow \Theta} \vee^{A+}$$

$$\frac{\mathfrak{A}, \Gamma \longrightarrow \perp}{\Gamma \longrightarrow \neg \mathfrak{A}} \neg^{S+} \quad \frac{\neg \mathfrak{A}, \Gamma \longrightarrow \mathfrak{A}}{\neg \mathfrak{A}, \Gamma \longrightarrow \perp} \neg^{A+}$$

where, for all rules, the succedent should contain exactly one formula.⁵

⁴ This restriction is used both for minimal and intuitionistic calculi, but not for the classical calculus.

⁵ For intuitionistic logic it would be that Θ contains at most one formula; classical logic does not have any restrictions on Θ .

Multiset Γ contains all the subformulae of the derived formula that are asserted in the antecedents of the sequents that constitute the inference. This corresponds to the *minimal rule* for dialogue logic that we discuss in section 2. However, as we have chosen the sequent calculus without separate structural rules, we modify the rules by imposing above the restriction on the succedent. Furthermore, even though we have listed the negation as a primitive, we could have defined it as $\neg\mathfrak{A} := \mathfrak{A} \supset \perp$. Thus, we can see that our rules for negation are just a special case of rules for implication:

$$\frac{\frac{\mathfrak{A}, \Gamma \longrightarrow \perp}{\Gamma \longrightarrow \mathfrak{A} \supset \perp} \supset^{S+}}{\Gamma \longrightarrow \neg\mathfrak{A}} \text{def} \quad \frac{\frac{\mathfrak{A} \supset \perp, \Gamma \longrightarrow \mathfrak{A} \quad \mathfrak{A} \supset \perp, \Gamma, \perp \longrightarrow \perp^{Ax.}}{\mathfrak{A} \supset \perp, \Gamma \longrightarrow \perp} \supset^{A+}}{\neg\mathfrak{A}, \Gamma \longrightarrow \perp} \text{def}$$

We provide our proof of the correspondence result in section 3 for the sequent calculus with branching in \neg^{A+} , but, as one can see, it applies to the variant without branching as well.

It is easy to see why this calculus is minimal. In Gentzen style natural deduction the minimal system is set up by the rules of the intuitionistic one without *ex falso quodlibet*, i.e.

$$NM = NJ - \frac{\perp}{\mathfrak{D}}$$

We argue that in a Gentzen-style sequent calculus the rule of *ex falso quodlibet* is represented by a particular case of *the right weakening* (WR) structural rule

$$\frac{\Gamma \longrightarrow \emptyset}{\Gamma \longrightarrow \mathfrak{D}} \text{WR}\emptyset$$

Whereas the general form of the rule is as follows:⁶

$$\frac{\Gamma \longrightarrow \Theta}{\Gamma \longrightarrow \Theta, \mathfrak{D}} \text{WR}$$

We make use of Gentzen’s proof of the equality of $LJ \equiv NJ \equiv LHJ$ [7].⁷ According to his proof, all these three systems are equivalent, so (WR \emptyset) is equivalent to the rule *ex falso quodlibet*. We can transform the *ex falso quodlibet* into (WR \emptyset) by the following procedure:

⁶ Though one should not forget about the restrictions on Θ for intuitionistic logic: Θ should be empty to implement this rule there. Thus, in minimal logic, one cannot apply the right weakening at all.

⁷ By LHJ we understand Heyting calculus for intuitionistic logic that Gentzen makes use of (still it is represented in a tree-form).

1. we write formulae⁸ of the rule *ex falso quodlibet* into the succedent of the sequence;
2. in the antecedent of the sequence we write all the premises that the formula is dependent on;
3. then we change the constant \perp to $\mathfrak{A} \wedge \neg\mathfrak{A}$.

So we get the following rule transformation:

$$\frac{\perp}{\mathfrak{D}} \quad \Longrightarrow \quad \frac{\longrightarrow \mathfrak{A} \wedge \neg\mathfrak{A}}{\longrightarrow \mathfrak{D}^*}$$

According to the rules of LJ this can be transformed as follows:

$$\frac{\frac{\Gamma \longrightarrow \mathfrak{A} \wedge \neg\mathfrak{A} \quad \mathfrak{A} \wedge \neg\mathfrak{A} \longrightarrow}{\Gamma \longrightarrow \mathfrak{D}^*} \text{ (WR)}}{\Gamma \longrightarrow \mathfrak{A} \wedge \neg\mathfrak{A}} \text{ (cut)}$$

Then we add the derivation of the sequent which has the formula $\mathfrak{A} \wedge \neg\mathfrak{A}$ as an antecedent:

$$\frac{\frac{\frac{\frac{\mathfrak{A} \longrightarrow \mathfrak{A}}{\mathfrak{A} \wedge \neg\mathfrak{A} \longrightarrow \mathfrak{A}} \text{ (UEA)}}{\neg\mathfrak{A}, \mathfrak{A} \wedge \neg\mathfrak{A} \longrightarrow} \text{ (NEA)}}{\mathfrak{A} \wedge \neg\mathfrak{A}, \mathfrak{A} \wedge \neg\mathfrak{A} \longrightarrow} \text{ (UEA)}}{\mathfrak{A} \wedge \neg\mathfrak{A} \longrightarrow} \text{ (CL)}}{\frac{\Gamma \longrightarrow \mathfrak{A} \wedge \neg\mathfrak{A} \quad \mathfrak{A} \wedge \neg\mathfrak{A} \longrightarrow}{\Gamma \longrightarrow \mathfrak{D}^*} \text{ (WR)}} \text{ (cut)}$$

However, for LM⁹ to be equal to NM, we should add there the bottom (\perp) as a constant. It is important because in minimal logic the condition of *uniqueness of negation* is not satisfied [15]. Following [15], we understand uniqueness as follows:

if two n -ary operators \dagger and \dagger^* are governed by the same inference rules, then for all A_1, \dots, A_n , $\dagger(A_1, \dots, A_n)$ and $\dagger^*(A_1, \dots, A_n)$ are interdeducible — i.e., $\dagger(A_1, \dots, A_n) \dashv\vdash \dagger^*(A_1, \dots, A_n)$ — using imperatively at least one of the rules of \dagger or \dagger^* and, when needed, the reflexivity axiom rule to close the derivation. No other rules are admitted.

Nevertheless, if we use the standard Gentzen-style rules for LM (i.e. just forbidding the use of the *right weakening rule*) without \perp sign as a special

⁸ Gentzen calls them *H*-formulae.

⁹ LM stays for Gentzen style minimal sequent calculus.

constant, we get a system that is stronger than minimal logic. We can illustrate that by adding a new unary operator \neg^* that would have the same rules as the standard \neg , namely:

$$\frac{\mathfrak{A}, \Gamma \longrightarrow}{\Gamma \longrightarrow \neg \mathfrak{A}} \neg^{S+*} \qquad \frac{\neg \mathfrak{A}, \Gamma \longrightarrow \mathfrak{A}}{\neg \mathfrak{A}, \Gamma \longrightarrow} \neg^{A+*}$$

Then we would be able to deduce $\neg \phi \supset \neg^* \phi$ (or, equally, $\neg \phi \Vdash \neg^* \phi$) as follows:

$$\frac{\frac{\frac{\phi \longrightarrow \phi}{\neg \phi, \phi \longrightarrow}}{\neg \phi \longrightarrow \neg^* \phi}}{\longrightarrow \neg \phi \supset \neg^* \phi} \qquad \frac{\frac{\frac{\phi \longrightarrow \phi}{\neg^* \phi, \phi \longrightarrow}}{\neg^* \phi \longrightarrow \neg \phi}}{\longrightarrow \neg^* \phi \supset \neg \phi}}{\longrightarrow \neg \phi \supset \neg^* \phi}$$

However, if we add a bottom sign, we can get a separate bottom for each negation type; for instance, for \neg^* we would get \perp^* .

Then we can transform the right weakening rule in Gentzen style into 2.42 $\neg \mathfrak{A} \supset (\mathfrak{A} \supset \mathfrak{B})$ of *LHJ* (for details cf. [7]). It is easy to see that axiom $\neg \mathfrak{A} \supset (\mathfrak{A} \supset \mathfrak{B})$ of *LHJ* can be easily derived as a theorem in *NJ*:

$$\frac{\frac{\frac{[\mathfrak{A}] \quad [\neg \mathfrak{A}]}{\perp} \quad \perp}{\mathfrak{B}} \quad \perp}{\mathfrak{A} \longrightarrow \mathfrak{B}} \quad (\supset^+, 2)}{\neg \mathfrak{A} \longrightarrow (\mathfrak{A} \longrightarrow \mathfrak{B})} \quad (\supset^+, 1)$$

Thus, the calculus *LM* with the bottom sign is minimal. No weakening on the right is permitted because of the restriction on the number of formulae in the succedent. However, we use a modified variant of *LM* calculus, i.e. G_3^a calculus. Therefore, in this section, we have provided a system G_3^{min} of sequent calculus suitable for the proof in section 3.

2. Minimal Dialogue Logic

2.1. The system D^{min}

We introduce a dialogue interpretation for the minimal logic that we call D^{min} . We base our system on the Intuitionistic Dialogue Logic as presented by Krabbe in [12]. Within the framework of dialogue logic, we

set up validity in the spirit of operationalism.¹⁰ Dialogue is a two-player game about some formula with the *Proponent*¹¹ (*P*) whose task is to defend the formula in question and the *Opponent* (*O*) who is responsible for giving a counterexample to the formula, thus showing that it is not generally valid.

Normally, there are two levels of rules in dialogue logic:

I. *Logical rules* define the possible types of attacks and defences for each type of formula, i.e. containing specific logical operators as principal ones. These rules show us in an abstract form which formulae can be criticised (through an attack) and defended and how (according to the logical form of a formula in question). Thus, logical rules define the meaning of logical operators;

II. *Structural rules* define the general course of a game and its organisation. These rules define the exact protocol of communication, i.e. when each player can move and what type of move he is allowed to make.

Structural rules function on the global level, defining which sequences of dialogical moves will count as legal dialogues. [1]

DEFINITION 4. By an attack we understand a move of a player *Y* against one of the propositions of the player *X* that can be executed in one of the following ways:

1. in the form of a request to make an assumption; or
2. in affirming a proposition (as in the case of negation).

We also should define some basic notions of the dialogue approach.

DEFINITION 5. By an attack we understand a move of a player *Y* against one of the propositions of the player *X* that can be executed in one of the following ways:

1. in the form of a request to make an assumption; or
2. in affirming the contrary (as in the case of material implication and negation).

DEFINITION 6. By a defence we understand a response to an attack. Defence is always performed in the form of affirming a relevant proposition.

¹⁰ Though several contemporary researchers suppose that dialogue logic rather represents a type of semantics.

¹¹ We consider it as a type of semantics: the Proponent's task is to build an arbitrary model.

Here we shall define the logical rules and the structural rules of our D^{\min} system. We assume that the language of D^{\min} ($\mathfrak{L}_{D^{\min}}$) is similar to the \mathfrak{L}_{\min} , but does not contain a sequence sign.

DEFINITION 7 (Logical Rules). The system $D^{\min a}$ has the following logical rules:

Connective	Attack	Defence
$X-! - \mathfrak{A} \wedge \mathfrak{B}$	$Y-? - \wedge_L$	$X-! - \mathfrak{A}$
	$Y-? - \wedge_R$	$X-! - \mathfrak{B}$
$X-! - \mathfrak{A} \vee \mathfrak{B}$	$Y-? - \vee$	$X-! - \mathfrak{A}$
		$X-! - \mathfrak{B}$
$X-! - \mathfrak{A} \supset \mathfrak{B}$	$Y-! - \mathfrak{A}$	$X-! - \mathfrak{B}$
$X-! - \neg \mathfrak{A}$	$Y-! - \mathfrak{A}$	$X-! - \perp$
$X-! - \forall \mathfrak{r} \mathfrak{A}(\mathfrak{r})$	$Y-? - \forall \mathfrak{r} / \mathfrak{n}$	$X-! - \mathfrak{A}[\mathfrak{n} / \mathfrak{r}]$
$X-! - \exists \mathfrak{r} \mathfrak{A}(\mathfrak{r})$	$Y-? - \exists \mathfrak{r}$	$X-! - \mathfrak{A}[\mathfrak{n} / \mathfrak{r}]$

where \mathfrak{A} and \mathfrak{B} are metavariables, X and Y variables for players (with P and O being the precise roles) with $X \neq Y$, $!$ and $?$ are used to represent assertion and demand respectively. In the case of conjunction and universal quantification, the choice is made by the attacker (in the table depicted by Y), whereas in case of disjunction and existential quantification the choice is made by the defender (in the table depicted by X).

Let us now define the structural rules of the dialogue logic $D^{\min a}$.

DEFINITION 8. A dialogue is a sequence of attacks and defences obeying the structural and logical rules that begins with a finite (possibly empty) multiset Π of formulae that are initially granted by O ¹² and a finite (nonempty) multiset Δ of formulae that are initially disputed by O .

Here we follow [1]:

Formulae that have been initially granted by O can be attacked by P at any time, and formulae that are initially disputed by O can be asserted as a defence by P at any time. In the case where Δ is a singleton, we can understand the game as beginning with an assertion of Δ by P , with the first move then being an attack on Δ by O . [1]

¹² In our account of the correspondence between dialogue logic and game-theoretical semantics (*GTS*) we call dialogues with $\Pi \neq \emptyset$ *dialogues with hypotheses*.

DEFINITION 9 (Structural rules). The system D^{min}_a has the following structural rules:

Start. The first move of the dialogue is carried out by O and consists of an attack on (the unique) initially disputed formula \mathfrak{A} .¹³

Alternation. Moves strictly alternate between players O and P .

Atom. P can affirm atomic formulae, including \perp , iff they were previously stated by O .

D11. If it is X 's turn and there is more than one attack by Y that X has not yet defended, only the most recent one may be defended.¹⁴

D12. Any attack may be defended at most once.¹⁵

Attack-rule. O can attack P 's one and the same formula only once, whereas P can attack O 's formula at most twice.¹⁶

Due to the introduction of the defence for negation and the presence of the atom rule, we do not need any additional structural rules. However, one could introduce the following *minimal rule* for the system without \perp : *For each attack players must provide a defence.*¹⁷ Nevertheless, such a system would be in between minimal and intuitionistic propositional logic as discussed in section 1. In this dialogue system, the minimal rule is redundant.

Now we shall specify the rules governing the end of the game. Unlike [1], we do it in a more traditional way.

DEFINITION 10 (Ending). The game ends if and only if the player whose turn it is to move has no legal move to make.

To talk about the logical value of the formula we still need the winning conditions.

¹³ We count as a zero step the one where P proposes a formula for the dispute.

¹⁴ This is a rule for minimal and intuitionistic logic only. In classical logic, any attack can be defended.

¹⁵ This rule is applicable only for minimal and intuitionistic logics, but in classical logic, P can repeat his defences.

¹⁶ Here we rely on the works of Clerbout [3, 4]

¹⁷ Including the attack against the negation \neg , as, in this paper, we have introduced a rule for defence against this attack.

DEFINITION 11 (Winning conditions). The game ends with P winning iff it is O 's turn and she has no possible move left to make and P has satisfied the *minimal rule* (cf. Definition 9).

The game ends with O winning iff it is P 's turn and she has no possible move left to make or if P has not satisfied the *minimal rule* (cf. Definition 9).

Each round in the dialogue consists of a round number (represented by Roman numerals), a move¹⁸ of at least one and at most two players (an assertion of either a formula / one of the symbolic attacks), a stance (either attack or defence), and a reference (a natural number referring to the indices of previous moves of the game). A round consists of an attack by X and a defence by Y , or just an attack.

As we were claiming that dialogue logic can express logical validity (if $\Pi = \emptyset$), we should specify the conditions for that. A single won game shows us a possible interpretation satisfying the formula, thus giving us some information about its satisfiability. To express validity we need to know that all games can be won by P no matter what O does, thus we need a notion of a winning strategy. We take the definitions for the dialogue tree from [1], though we can not make use of their definition of a winning strategy. A player has a winning strategy if following that strategy guarantees that he wins the game no matter what other player's moves are.

DEFINITION 12 (Active Formula). In the first round, the initially disputed formula is active. In all other rounds, the active formula is the last formula asserted by P that O is required to attack if it exists; if it does not, then it is the formula that O has most recently attacked.

DEFINITION 13 (Dialogue Sequent). In a given round, we write $\Pi \longrightarrow \mathfrak{A}$ to indicate that the formulae in Π ¹⁹ have been granted by O and that \mathfrak{A} is the active formula; we call this a dialogue sequent.²⁰

DEFINITION 14. A dialogue tree T for a dialogue sequent $\Pi \longrightarrow \mathfrak{A}$ is a rooted directed tree whose nodes are rounds in a dialogue game such that every branch of T is a dialogue with initially granted formulas Π and initially disputed formula \mathfrak{A} .

¹⁸ Each move of the players has a number (Arabic numerals).

¹⁹ It is important to notice that dialogue sequent contains all the formulae granted by O and only one formula granted by P .

²⁰ We take the notions of an active formula and dialogue sequent from [1].

DEFINITION 15 (Winning Strategy). A finite dialogue tree T is a winning strategy τ for X if and only if each branch that is the result of X 's choice ends with the move of X , i.e. player Y has no possible move to make.

To clarify the notion of the winning strategy we shall refer to the following passage from [1]:

Note that the dialogue tree is just a representation of the game in extensive form and that a winning strategy is a *subtree* of the entire dialogue tree that can be understood as a procedure that P can follow to win the game no matter how O responds to P 's moves.

It is easy to see that the last move of the Proponent should be a defence because otherwise (if it were an attack) the Opponent would have been able to perform a defence against this attack and thus the Proponent would not be the last to move. From that we conclude that the last move of the Proponent is closing the last open round because all other rounds should be closed due to the minimal rule and to the fact that Opponent would be able to perform a defence if there were an open round where the Proponent attacked one of the Opponent's formulae. Hence, we can represent winning strategies of the Proponent as a set of dialogues where branching represents choices made by the Opponent.

2.2. Examples

Here we provide an example of a D^{\min} dialogue $\mathcal{D}(A)$ for the valid formula $A := \mathfrak{A} \supset \neg\neg\mathfrak{A}$ and later we shall discuss why this game is deterministic by providing an extensive form for this formula. In this particular play of

Round	Opponent	Proponent
0		(0) $\mathfrak{A} \supset \neg\neg\mathfrak{A}$
I	(1) \mathfrak{A} (<i>Att. 0</i>)	(2) $\neg\neg\mathfrak{A}$ (<i>Def. 1</i>)
II	(3) $\neg\mathfrak{A}$ (<i>Att. 2</i>)	(6) \perp (<i>Def. 5</i>)
III	(5) \perp (<i>Def. 4</i>)	(4) \mathfrak{A} (<i>Att. 3</i>)

Table 1. An example: $\mathfrak{A} \supset \neg\neg\mathfrak{A}$

the dialogue, \mathbf{P} wins. However, it just shows us one course of the game, whereas, in order to speak about validity, we need to find out whether there is a winning strategy for the Proponent.

We can easily see that the dialogue for the formula $\mathfrak{A} \supset \neg\neg\mathfrak{A}$ is deterministic as no other moves are available for any of the players. It

is easy to notice that it is a winning strategy for P . To anticipate our claim that we make in section 3 we provide a sequent version G_3^{\min} for the same formula:

$$\frac{\frac{\frac{\frac{\text{Ax}}{\neg\mathfrak{A}, \mathfrak{A} \longrightarrow \mathfrak{A}}{\neg\mathfrak{A}, \mathfrak{A} \longrightarrow \perp} (\neg^{A+})}{\mathfrak{A} \longrightarrow \neg\neg\mathfrak{A}} (\neg^{S+})}{\longrightarrow \mathfrak{A} \supset \neg\neg\mathfrak{A}} (\supset^{S+})}{\frac{\frac{\text{Ax}}{\neg\mathfrak{A}, \mathfrak{A} \longrightarrow \mathfrak{A}}}{\neg\mathfrak{A}, \mathfrak{A}, \perp \longrightarrow \perp} (\neg(\supset)^{A+})}{\frac{\neg\mathfrak{A}, \mathfrak{A} \longrightarrow \perp}{\mathfrak{A} \longrightarrow \neg\neg\mathfrak{A}} (\neg(\supset)^{S+})}{\longrightarrow \mathfrak{A} \supset \neg\neg\mathfrak{A}} (\supset^{S+})}$$

This example shows that in the case of determined dialogues, the transformations allowed in the sequent calculus in question are determined as well.

To see the difference between this game and a non-deterministic one let us consider another example of the dialogue $\mathcal{D}(A)$ for the formula $A := \neg(\mathfrak{A} \wedge \mathfrak{B}) \supset (\mathfrak{A} \supset \neg\mathfrak{B})$. First, we show up a table for one of the winning games (in according with one of the winning strategies for P in this game) of the Proponent:

Round	Opponent	Proponent
0		(0) $\neg(\mathfrak{A} \wedge \mathfrak{B}) \supset (\mathfrak{A} \supset \neg\mathfrak{B})$
I	(1) $\neg(\mathfrak{A} \wedge \mathfrak{B})$ (<i>Att. 0</i>)	(2) $\mathfrak{A} \supset \neg\mathfrak{B}$ (<i>Def. 1</i>)
II	(3) \mathfrak{A} (<i>Att. 2</i>)	(4) $\neg\mathfrak{B}$ (<i>Def. 3</i>)
III	(5) \mathfrak{B} (<i>Att. 4</i>)	(8) \perp (<i>Def. 5</i>)
IV	(7) \perp (<i>Def. 6</i>)	(6) $\mathfrak{A} \wedge \mathfrak{B}$ (<i>Att. 1</i>)
V	(9) \wedge_L (<i>Att. 6</i>)	(10) \mathfrak{A} (<i>Def. 9</i>)

Table 2. An example: $\neg(\mathfrak{A} \wedge \mathfrak{B}) \supset (\mathfrak{A} \supset \neg\mathfrak{B})$

The tree on figure 1 shows the P 's winning strategy for: $\neg(\mathfrak{A} \wedge \mathfrak{B}) \supset (\mathfrak{A} \supset \neg\mathfrak{B})$. A strategy for P is constituted by plays such that each different play is triggered by some choice of O . In other words a strategy for P includes at most one choice by P but incorporates all possible choices by O . Different choices by P trigger different strategies. P might have more than one way to respond that will lead to P 's win, thus triggering different winning strategies for P . P has a winning strategy iff

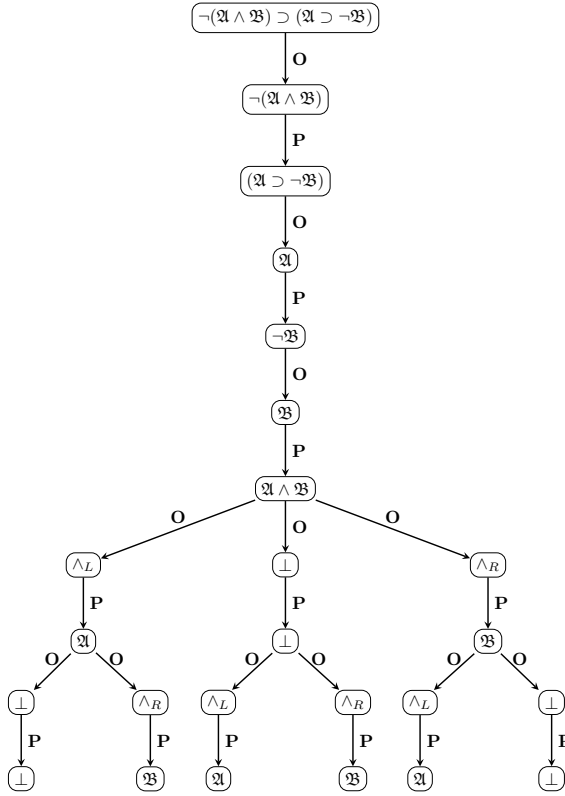


Figure 1. The **P**'s winning strategy for: $\neg(A \wedge B) \supset (A \supset \neg B)$

she can win (choosing one way to respond) against all possible choices by **O**. In Table 2, although there are multiple strategies for **P**, only the one depicted by the tree is winning.

Finally, as we are approaching section 3, we provide a sequent version G_3^{\min} for the same formula:

$$\begin{array}{c}
 \frac{Ax}{\mathfrak{B}, \mathfrak{A}, \neg(\mathfrak{A} \wedge \mathfrak{B}) \longrightarrow \mathfrak{A}} \quad \frac{Ax}{\mathfrak{B}, \mathfrak{A}, \neg(\mathfrak{A} \wedge \mathfrak{B}) \longrightarrow \mathfrak{B}} \\
 \hline
 \frac{\mathfrak{B}, \mathfrak{A}, \neg(\mathfrak{A} \wedge \mathfrak{B}) \longrightarrow \mathfrak{A} \wedge \mathfrak{B}}{\mathfrak{B}, \mathfrak{A}, \neg(\mathfrak{A} \wedge \mathfrak{B}) \longrightarrow \perp} \quad (\neg^{A+}) \\
 \frac{\mathfrak{B}, \mathfrak{A}, \neg(\mathfrak{A} \wedge \mathfrak{B}) \longrightarrow \perp}{\mathfrak{A}, \neg(\mathfrak{A} \wedge \mathfrak{B}) \longrightarrow \neg \mathfrak{B}} \quad (\neg^{S+}) \\
 \frac{\mathfrak{A}, \neg(\mathfrak{A} \wedge \mathfrak{B}) \longrightarrow \neg \mathfrak{B}}{\neg(\mathfrak{A} \wedge \mathfrak{B}) \longrightarrow \mathfrak{A} \supset \neg \mathfrak{B}} \quad (\supset^{S+}) \\
 \frac{\neg(\mathfrak{A} \wedge \mathfrak{B}) \longrightarrow \mathfrak{A} \supset \neg \mathfrak{B}}{\longrightarrow \neg(\mathfrak{A} \wedge \mathfrak{B}) \supset (\mathfrak{A} \supset \neg \mathfrak{B})} \quad (\supset^{S+})
 \end{array}$$

$$\begin{array}{c}
 \frac{\frac{Ax}{\mathfrak{B}, \mathfrak{A}, \neg(\mathfrak{A} \wedge \mathfrak{B}) \longrightarrow \mathfrak{A}}{\mathfrak{B}, \mathfrak{A}, \neg(\mathfrak{A} \wedge \mathfrak{B}) \longrightarrow \mathfrak{A} \wedge \mathfrak{B}}}{\mathfrak{B}, \mathfrak{A}, \neg(\mathfrak{A} \wedge \mathfrak{B}) \longrightarrow \mathfrak{A} \wedge \mathfrak{B}} \quad \frac{\frac{Ax}{\mathfrak{B}, \mathfrak{A}, \neg(\mathfrak{A} \wedge \mathfrak{B}) \longrightarrow \mathfrak{B}}{\mathfrak{B}, \mathfrak{A}, \neg(\mathfrak{A} \wedge \mathfrak{B}) \longrightarrow \mathfrak{A} \wedge \mathfrak{B}}}{\mathfrak{B}, \mathfrak{A}, \neg(\mathfrak{A} \wedge \mathfrak{B}) \longrightarrow \mathfrak{A} \wedge \mathfrak{B}} \quad (\wedge^{S+}) \quad \frac{\frac{Ax}{\mathfrak{B}, \mathfrak{A}, \neg(\mathfrak{A} \wedge \mathfrak{B}), \perp \longrightarrow \perp}}{\mathfrak{B}, \mathfrak{A}, \neg(\mathfrak{A} \wedge \mathfrak{B}), \perp \longrightarrow \perp}}{\mathfrak{B}, \mathfrak{A}, \neg(\mathfrak{A} \wedge \mathfrak{B}), \perp \longrightarrow \perp}} \quad (\neg / \supset^{A+}) \\
 \frac{\frac{\frac{\mathfrak{B}, \mathfrak{A}, \neg(\mathfrak{A} \wedge \mathfrak{B}) \longrightarrow \perp}{\mathfrak{A}, \neg(\mathfrak{A} \wedge \mathfrak{B}) \longrightarrow \neg \mathfrak{B}}}{\neg(\mathfrak{A} \wedge \mathfrak{B}) \longrightarrow \mathfrak{A} \supset \neg \mathfrak{B}}}{\longrightarrow \neg(\mathfrak{A} \wedge \mathfrak{B}) \supset (\mathfrak{A} \supset \neg \mathfrak{B})}}{\longrightarrow \neg(\mathfrak{A} \wedge \mathfrak{B}) \supset (\mathfrak{A} \supset \neg \mathfrak{B})}} \quad (\neg / \supset^{S+}) \quad (\supset^{S+}) \quad (\supset^{S+})
 \end{array}$$

3. The Correspondence between $G_3^{\min}a$ and D^{\min}

In this section we argue that a derivation in a sequent calculus $G_3^{\min}a$ can be seen as representing some winning strategies for the Proponent in the minimal dialogue logic; thus, in the case of a universally valid formula (this means that each branch of the tree ends up with an axiom), it encodes winning strategies for the Proponent in a dialogue without hypotheses. We can show this by providing an effective algorithm that transforms a dialogue into a branch of a derivation in sequent calculus.

THEOREM 1 (Minimal validity). *Let A be any formula of propositional logic. The following conditions are equivalent:*

1. *There is a winning strategy for the Proponent in the dialogue $\mathcal{D}(A)$;*
2. *There exists a $G_3^{\min}a$ derivation of the formula A (i.e., $\Gamma \longrightarrow A$, where Γ is empty).*

Furthermore, there exists an algorithm turning the Proponent's winning strategy into the $G_3^{\min}a$ derivation and vice versa.

This theorem deals only with validity. We, however, would like to show that any derivation in $G_3^{\min}a$ can be transformed into the minimal dialogue logic D^{\min} and *vice versa*. Thus, we reformulate our theorem so that it can deal with any derivation (not necessarily a theorem).

THEOREM 2 (Correspondence result). *Every winning strategy τ for $\mathcal{D}(A, \Gamma)$ (i.e., for a dialogue with initially disputed formula A , where the Opponent initially grants the formulae in the multiset Γ) can be transformed into a $G_3^{\min}a$ derivation of $\Gamma \longrightarrow A$ and a derivation in $G_3^{\min}a$ can be transformed into a set of winning strategies for the Proponent.*

PROOF. We prove Theorem 2 in two steps, by establishing two lemmata.

LEMMA 1. *Every winning strategy τ for $\mathcal{D}(A, \Gamma)$ (i.e., for a dialogue with initially disputed formula A , where the Opponent initially grants the formulae in the multiset Γ) can be transformed into a $G_3^{\min}a$ derivation of $\Gamma \longrightarrow A$*

Now let us show that there exists an algorithm that transforms any particular dialogue into a branch of a sequent calculus. We shall use our table representation of dialogues to be able to make use of the notion of rounds. Each round represents a sequence so that we can build sequences from the table with the 0-round representing the formula \mathfrak{A} to

be deduced (i.e., $\Gamma \longrightarrow \mathfrak{A}$). As the Opponent cannot repeat his attacks on previously attacked formulae of the Proponent, we do not keep the formulae whose subformulae have already been asserted in the succedent of the sequence.²¹ On the contrary, we keep formulae that were asserted by the Opponent because the Proponent has a right to repeat his attack on a formula of the Opponent.²² In what follows we shall use P for the Proponent and O for the Opponent.

PROOF. Consider an arbitrary winning strategy τ for the P .

Claim 1. For every round of the dialogue $\mathcal{D}(A, \Gamma)$ there is a $G_3^{\text{min}a}$ deduction of the sequent corresponding to the dialogue sequent at this round $\Theta \longrightarrow \mathfrak{A}_i$, which consists of $\Theta = \Gamma \cup \{\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_n\}$, where $\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_n$ are subformulae of A asserted by O and \mathfrak{A}_i represents a subformula of A asserted by P . Note that A_i cannot be empty.

PROOF. We prove this claim by induction on the depth d of τ .

The base case: $d = 1$. In this case the game terminates at round #1. So in this round the P moves last according to the winning conditions 11. Thus, the P has asserted an atomic formula \mathfrak{A} because otherwise (if the formula asserted by the P at the round #1 were not atomic) the O would have been able to move by attacking a complex formula. As \mathfrak{A} is atomic, the O must have already granted \mathfrak{A} (according to the *atom rule*, Definition 9) either with his move in the first round or in Γ . Thus, our dialogue sequent at round #1 is $\mathfrak{A}, \Gamma \longrightarrow \mathfrak{A}$ (or simply $\Gamma \longrightarrow \mathfrak{A}$ if $\mathfrak{A} \in \Gamma$).

We provide an example of such a formula whose dialogue ends in round #1: $A := p \supset p$:

Round	Opponent	Proponent
0	-	(0) $p \supset p$
I	(1) p (Att.0)	(2) p (Def.1)

Table 3. An example: $p \supset p$

We can easily transform this into the $G_3^{\text{min}a}$ deduction. According to our algorithm we assign a sequence to each round. In our example we

²¹ This requirement corresponds to the restrictions imposed on the succedent in intuitionistic logic, where we cannot have more than one formula in the succedent.

²² This corresponds to the rules of $G_3^{\text{min}a}$ where we keep all the formulae in the antecedent.

have only two rounds²³, so we associate the sequence $\Gamma \rightarrow A$ with the 0-round (in our example it is $\rightarrow p \supset p$ which is associated with the 0-round where the P states formula $p \supset p$ with $\Gamma = \emptyset$) and the sequence $A, \Gamma \rightarrow A$ to the first round (in our example it is $p \rightarrow p$ which is associated with the first round and the statement p representing the attack in the form of assertion by the O and the defence p of the P). Thus, we get the following $G_3^{\text{min}a}$ deduction:

$$\frac{p \rightarrow_I p}{\rightarrow_0 p \supset p} (\supset^{S+})$$

The induction hypothesis: we assume that the claim holds for all $n \leq d$ and show that it also holds at the $d + 1$ step.

The inductive step: we proceed by analysing the cases according to the form of the formula that is defended or attacked by P .

1. P defends $\mathfrak{A} \wedge \mathfrak{B}$. So our previous dialogue sequent has the following form: $\Theta \rightarrow \mathfrak{A} \wedge \mathfrak{B}$. So O can attack $\mathfrak{A} \wedge \mathfrak{B}$ by either $? - \wedge_L$ or $? - \wedge_R$. So if O makes the move $? - \wedge_L$, then at some point P will have to assert \mathfrak{A} (according to the minimal rule), and if O makes the move $? - \wedge_R$, then P will have to assert \mathfrak{B} .

(a) If \mathfrak{A} (or \mathfrak{B} accordingly) is atomic, then our sequent has the form $C, \Theta \rightarrow C$ (where C is atomic formula $\mathfrak{A} := C$, or $\mathfrak{B} := C$, and $C \in \Theta$), and we have reached the end as there should already be an assertion of C made by O in order for P to be able to assert C (cf. the base case). By induction, we get that all the previous steps d can be transformed into the sequent form, thus the whole dialogue can be transformed into $G_3^{\text{min}a}$ deduction.

(b) If \mathfrak{A} (and \mathfrak{B}) is not atomic, then we get a dialogue of the form $\Theta \rightarrow \mathfrak{A}$ ($\Theta \rightarrow \mathfrak{B}$) and then we continue our derivation with the formula \mathfrak{A} (\mathfrak{B}). According to our inductive step there already exists a derivation of \mathfrak{A} (\mathfrak{B}), and thus our transition from round d to round $d + 1$ can be transformed into the \wedge^{S+} rule of $G_3^{\text{min}a}$, which he chooses himself.

2. P attacks $\mathfrak{A} \wedge \mathfrak{B}$. Then the previous dialogue sequent has the form $\Theta, \mathfrak{A} \wedge \mathfrak{B} \rightarrow \mathfrak{D}$ (where \mathfrak{D} cannot be empty). This case is analogous to the cases 1a and 1b.

3. P defends $\mathfrak{A} \vee \mathfrak{B}$. Thus, the previous dialogue sequent has the following form: $\Theta \rightarrow \mathfrak{A} \vee \mathfrak{B}$. O has only one possible attack $? - \vee$. The P has two possible defences: \mathfrak{A} or \mathfrak{B} . Thus, in the next step we get either

²³ In each dialogue for a formula representing the *base case* there are only two rounds, i.e. 0-round and round #1.

$\Theta \longrightarrow \mathfrak{A}$ or $\Theta \longrightarrow \mathfrak{B}$; thus, this dialogue sequent can be transformed into the $G_3^{\text{min}a}$ sequences:

$$\frac{\Theta \longrightarrow \mathfrak{A}}{\Theta \longrightarrow \mathfrak{A} \vee \mathfrak{B}} \quad \text{or} \quad \frac{\Theta \longrightarrow \mathfrak{B}}{\Theta \longrightarrow \mathfrak{A} \vee \mathfrak{B}}$$

Either way the game proceeds on the dialogue sequent $\Theta \longrightarrow \mathfrak{A}$ or $\Theta \longrightarrow \mathfrak{B}$ (and so does the $G_3^{\text{min}a}$ inference). And according to our inductive step there exists a $G_3^{\text{min}a}$ inference for the dialogue $\mathcal{D}(A, \Gamma)$ (with the strategy τ of the length D), where $A = \mathfrak{A}$ or $A = \mathfrak{B}$. We have shown that this also hold for the dialogue $\mathcal{D}(\mathfrak{A} \vee \mathfrak{B}, \Gamma)$ (for τ of the length $d + 1$).

4. P attacks $\mathfrak{A} \vee \mathfrak{B}$. Then the previous dialogue sequent has the form $\Theta, \mathfrak{A} \vee \mathfrak{B} \longrightarrow \mathfrak{D}$ (where \mathfrak{D} cannot be empty). This case is analogous to case 3.

5. P defends $\neg\mathfrak{A}$. So at the previous round the dialogue sequent has the form $\Theta \longrightarrow \neg\mathfrak{A}$ (step $d + 1$). The attack on the formula $\neg\mathfrak{A}$ represents an assertion of \mathfrak{A} . There is only one possible defence of the formula in question the next round has the form $\mathfrak{A}, \Theta \longrightarrow \perp$. So the game proceeds on the formulae asserted by O (at d). However, as we have our claim to be true at point d , then it is true for $d + 1$ as we can transform our step into the following minimal inference:

$$\frac{\mathfrak{A}, \Theta \longrightarrow \perp}{\Theta \longrightarrow \neg\mathfrak{A}}$$

6. P attacks $\neg\mathfrak{A}$. Here we get the previous sequent $\neg\mathfrak{A}, \Theta \longrightarrow \perp$. This case is analogous to 5. Thus, we have our claim proven for case d and by the induction step and some reasoning similar to 5 we get that our claim is sound for the case $d + 1$ because we have the following transition:

$$\frac{\Theta \longrightarrow \mathfrak{A}}{\neg\mathfrak{A}, \Theta \longrightarrow \perp}$$

7. P defends $\mathfrak{A} \supset \mathfrak{B}$. At the previous round we get the dialogue sequent $\Theta \longrightarrow \mathfrak{A} \supset \mathfrak{B}$. As P defends the formula, this means that O has already attacked it. The only way to attack this formula is to state \mathfrak{A} . As we are in the minimal logic, all attacks should be defended (in other words, the rounds should be closed). And the only way to defend the formula in question against the attack is to assert \mathfrak{B} . Thus, at d we get to the dialogue sequent $\mathfrak{A}, \Theta \longrightarrow \mathfrak{B}$. We have proven our claim for d , so

by induction step, we get that it should be true at $d + 1$ by showing the resulting inference:

$$\frac{\mathfrak{A}, \Theta \longrightarrow \mathfrak{B}}{\Theta \longrightarrow \mathfrak{A} \supset \mathfrak{B}}$$

8. P attacks $\mathfrak{A} \supset \mathfrak{B}$. This case is analogous to 7 but has some additional restrictions. We should keep in mind that we keep $\mathfrak{A} \supset \mathfrak{B}$ as P can attack several times the formulae accepted by O .²⁴ Thus we are tempted to write the following inference:²⁵

$$\frac{\mathfrak{B}, \mathfrak{A} \supset \mathfrak{B}, \Theta \longrightarrow \mathfrak{A}}{\mathfrak{A} \supset \mathfrak{B}, \Theta \longrightarrow \mathfrak{D}}$$

where \mathfrak{D} is a formula. However, in this case our deduction is not minimal because this transformation implies the WR-rule:

$$\frac{\frac{\mathfrak{B}, \mathfrak{A} \supset \mathfrak{B}, \Theta \longrightarrow \mathfrak{A}}{\mathfrak{A} \supset \mathfrak{B}, \Theta \longrightarrow} (\supset^{A+})}{\mathfrak{A} \supset \mathfrak{B}, \Theta \longrightarrow \mathfrak{D}} (\text{WR})$$

One should keep in mind that \mathfrak{D} cannot be empty according to the restriction on the succedent in the minimal calculus. Furthermore, we would not be able to transform the whole dialogue into any $G_3^{\min}a$ derivation as we have to avoid violation of the intuitionistic condition. Consider the following example of the dialogue for the scheme of the form $(\neg\mathfrak{A} \supset \neg\mathfrak{B}) \supset (\mathfrak{B} \supset \neg\neg\mathfrak{A})$:

Round	Opponent	Proponent
0	-	(0) $(\neg\mathfrak{A} \supset \neg\mathfrak{B}) \supset (\mathfrak{B} \supset \neg\neg\mathfrak{A})$
I	(1) $\neg\mathfrak{A} \supset \neg\mathfrak{B}$ (<i>Att.</i> 0)	(2) $\mathfrak{B} \supset \neg\neg\mathfrak{A}$ (<i>Def.</i> 1)
II	(3) \mathfrak{B} (<i>Att.</i> 2)	(4) $\neg\neg\mathfrak{A}$ (<i>Def.</i> 3)
III	(5) $\neg\mathfrak{A}$ (<i>Att.</i> 4)	(14) \perp (<i>Def.</i> 5)
IV	(7) $\neg\mathfrak{B}$ (<i>Def.</i> 6)	(6) $\neg\mathfrak{A}$ (<i>Att.</i> 1)
V	(13) \perp (<i>Def.</i> 8)	(8) \mathfrak{B} (<i>Att.</i> 7)
VI	(9) \mathfrak{A} (<i>Att.</i> 6)	(12) \perp (<i>Def.</i> 9)
VII	(11) \perp (<i>Def.</i> 10)	(10) \mathfrak{A} (<i>Att.</i> 5)

If we try to transform this dialogue we will get the following sequent derivation:

²⁴ This condition holds for all formulae asserted by O , and thus, for cases 2, 4, 5.

²⁵ Some comments related to this case will be given later.

$$\begin{array}{c}
 \frac{\mathfrak{A}, \mathfrak{B}, \neg\mathfrak{A} \supset \neg\mathfrak{B} \longrightarrow_7 \mathfrak{A}, \mathfrak{B}}{\mathfrak{A}, \neg\mathfrak{A}, \mathfrak{B}, \neg\mathfrak{A} \supset \neg\mathfrak{B} \longrightarrow_6 \mathfrak{B}} \quad (\neg^{A+}) \\
 \frac{\mathfrak{A}, \neg\mathfrak{A}, \mathfrak{B}, \neg\mathfrak{A} \supset \neg\mathfrak{B} \longrightarrow_6 \mathfrak{B}}{\neg\mathfrak{A}, \mathfrak{B}, \neg\mathfrak{A} \supset \neg\mathfrak{B} \longrightarrow_5 \neg\mathfrak{A}, \mathfrak{B}} \quad (\neg^{S+}) \\
 \frac{\neg\mathfrak{A}, \mathfrak{B}, \neg\mathfrak{A} \supset \neg\mathfrak{B} \longrightarrow_5 \neg\mathfrak{A}, \mathfrak{B}}{\neg\mathfrak{B}, \neg\mathfrak{A}, \mathfrak{B}, \neg\mathfrak{A} \supset \neg\mathfrak{B} \longrightarrow_4 \neg\mathfrak{A}} \quad (\neg^{A+}) \\
 \frac{\neg\mathfrak{B}, \neg\mathfrak{A}, \mathfrak{B}, \neg\mathfrak{A} \supset \neg\mathfrak{B} \longrightarrow_4 \neg\mathfrak{A}}{\neg\mathfrak{A}, \mathfrak{B}, \neg\mathfrak{A} \supset \neg\mathfrak{B} \longrightarrow_3 \perp} \quad (\supset^{A+}) \\
 \frac{\neg\mathfrak{A}, \mathfrak{B}, \neg\mathfrak{A} \supset \neg\mathfrak{B} \longrightarrow_3 \perp}{\mathfrak{B}, \neg\mathfrak{A} \supset \neg\mathfrak{B} \longrightarrow_2 \neg\neg\mathfrak{A}} \quad (\neg^{S+}) \\
 \frac{\mathfrak{B}, \neg\mathfrak{A} \supset \neg\mathfrak{B} \longrightarrow_2 \neg\neg\mathfrak{A}}{\neg\mathfrak{A} \supset \neg\mathfrak{B} \longrightarrow_1 \mathfrak{B} \supset \neg\neg\mathfrak{A}} \quad (\supset^{S+}) \\
 \frac{\neg\mathfrak{A} \supset \neg\mathfrak{B} \longrightarrow_1 \mathfrak{B} \supset \neg\neg\mathfrak{A}}{\longrightarrow_0 \neg\mathfrak{A} \supset \neg\mathfrak{B} \supset \mathfrak{B} \supset \neg\neg\mathfrak{A}} \quad (\supset^{S+})
 \end{array}$$

We can see that at the round 5 and the corresponding sequence that the P has two assertions that can be attacked and, thus, there are two formulae in the succedent of the sequence # 5 which violates the intuitionistic rule for $G_3^{\min a}$. If we add the step between # 4 and # 5 that eliminates $\neg\mathfrak{A}$, then our sequence will cease to be minimal (as it would contain WR) and it would not correspond to the dialogue in question.

Given these considerations, we argue that the attack of P on $\mathfrak{A} \supset \mathfrak{B}$ should be transformed into the branching corresponding to the rule \supset^{A+} . We provide the following justification. An implication $\mathfrak{A} \supset \mathfrak{B}$ says that given \mathfrak{A} we can deduce \mathfrak{B} (or in the standard BHK²⁶ interpretation we say that the proof of $\mathfrak{A} \supset \mathfrak{B}$ is a function f that converts a proof of \mathfrak{A} into a proof of \mathfrak{B}), so if it is stated by some player X , for the player Y to challenge it, the player Y should state the antecedent \mathfrak{A} and the player X should state then the succedent \mathfrak{B} . However, this is not the whole story. We understand the statements that are affirmed by O to be the premises, and the formulae stated by P to be the consequents of those premises. Nevertheless, when O states $\mathfrak{A} \supset \mathfrak{B}$ he claims the existence of a transition from \mathfrak{A} as a premiss to \mathfrak{B} as the consequence. Hence, if we just wrote the sequence corresponding to the attack and defence as $\mathfrak{B}, \Gamma \longrightarrow \mathfrak{A}$, we would claim that \mathfrak{A} is the consequence of $\mathfrak{B} \cup \Gamma$, which, in the case of $\Gamma = \emptyset$ is false. Even if it is not true, still we claim that \mathfrak{B} is the consequence of \mathfrak{A} (and if we claim that \mathfrak{A} is the consequence of $\mathfrak{B} \cup \Gamma$ we cannot quite surely tell which it depends on: \mathfrak{B} or Γ). To clarify, we just say that the P affirms \mathfrak{A} which might depend on some previous assertions of the O represented by Θ and the O can defend it by stating some \mathfrak{B} that does not imply \mathfrak{A} (i.e., as a separate branch),

²⁶ BHK here stands for Brouwer–Heyting–Kolmogorov. This interpretation is also called the *realisability interpretation* and deals with the notion of *realisability* proposed by Stephen Kleene.

but which can imply a statement previously asserted by P . Thus we get the following branching representation:

$$\frac{\mathfrak{A} \supset \mathfrak{B}, \Theta \longrightarrow \mathfrak{A} \quad \text{and} \quad \mathfrak{B}, \mathfrak{A} \supset \mathfrak{B}, \Theta \longrightarrow \mathfrak{A}_i}{\mathfrak{A} \supset \mathfrak{B}, \Theta \longleftarrow \mathfrak{A}_i} \quad \square$$

We have shown that every round can be transformed into a sequent of a derivation. Thus, if we transform all dialogue winning strategies for the formula in question, we will get a $G_3^{\text{min}a}$ deduction of the formula. The branching of \wedge^{S+} and \vee^{A+} in $G_3^{\text{min}a}$ represents choices of the O that influence the strategy of the P . If we transform our strategies as proposed in our claim, we shall see that each winning strategy ends up with the *axiom* of $G_4^{\text{min}a}$. This is the case because O has no moves to make if and only if all the formulae asserted and not yet attacked by P are atomic (otherwise O could use the corresponding logical rule). For P to assert an atomic formula, it should be already stated by O ; thus we get the *axiom* of the from: $\mathfrak{D}, \Theta \longrightarrow \mathfrak{D}$. Thus all winning strategies (for dialogues $\mathcal{D}(A, \Gamma)$) for a formula in question can be transformed into valid $G_3^{\text{min}a}$ deductions. \square

LEMMA 2. *Every $G_3^{\text{min}a}$ derivation of $\Gamma \longrightarrow A$ can be transformed into a set of winning strategies $T = \tau_1, \dots, \tau_n$ (where n is the number of winning strategies for the P) for $\mathcal{D}(A, \Gamma)$ (i.e., for a dialogue with initially disputed formula A , where the O initially grants the formulae in the multiset Γ).*

PROOF. One difficulty that we can come across here is how to read the order of moves from the given sequent. One way to solve it would be to modify the sequent calculus to introduce a *regulation*, i.e., labels for players' moves in accordance with the *Structural Rules* of the dialogue. It can be done by alternating the labels between the antecedent and the succedent for the \supset, \neg rules. As for the rules \wedge^{S+} we can read that O attacks the conjunction stated by P by selecting the conjunct that P should later state (i.e. which is in the succedent of the upper sequent of the rule application) and *vice versa* for \wedge^{A+} . The application of the rule of \vee^{S+} can be read as an attack by O on the disjunction stated by P and it is P who chooses the concrete disjunct to state (i.e. which is in the succedent of the upper sequent of the rule application) and *vice versa* for the rule \vee^{A+} . The only difficulty arises when in the sequent there is an atom in the succedent which was not yet stated in the antecedent. This can be read as postponing of a defence of an attack by P so that

P states them later to close the round (as to win P should answer all attacks by O) when O will state the atom. This is guaranteed to happen because if P has a winning strategy, then the formula is valid and this the derivation should end with axioms, and as there is no weakening on the right, the atom in question should be at some point stated in the antecedent (i.e. by the O). As one can see, there can be several possible variants of the sequence of moves (for instance, either to attack or to defend a formula) which is why a derivation without *regulation* can decode more than one winning strategy, but at least one can always be received by just specifying that P should always first defend herself against an attack if that is possible according to the *Structural Rules*.

To prove Lemma 2 we show that each rule of $G_3^{\text{min}a}$ can be transformed into a corresponding dialogue rule. Each $G_3^{\text{min}a}$ corresponds to two rounds of the dialogue where round $\# m$ represents the initial formula and round $\# m + 1$ contains the attack (according to the logical rules) and defences. To transform our $G_3^{\text{min}a}$ rules into dialogue rules we shall map the antecedent of the sequent to the O column and the succedent of the sequent to the P column respectively. In the O column, we write only the formula that is being attacked only. All other formulae in the antecedent of the sequence represent the previous assertions of the O that can still be attacked by the P . Let us show this correspondence for each of the rule independently:

1. Let us start with the rule $\neg^{\text{S}+}$:

$$\frac{\mathfrak{A}, \Gamma \longrightarrow \perp}{\Gamma \longrightarrow \neg\mathfrak{A}} \neg^{\text{S}+}$$

This can be mapped into the rule for attacking the negation (the attack is effectuated by O , so $\neg\mathfrak{A}$ is asserted by P). We associate the formula $\neg\mathfrak{A}$ with the P column $\# m$ and \mathfrak{A} with the O column $\# m + 1$. Γ represents O 's assertions made in rounds $0 \leq j \leq m$. So, we get:

Round	Opponent	Proponent
$m - 1$
m	Γ	$\neg\mathfrak{A}$
$m + 1$	\mathfrak{A} (<i>Att.</i>)	\perp
$m + 2$

2. Consider the rule for negation introduction in the antecedent \neg^{A+} :

$$\frac{\neg\mathfrak{A}, \Gamma \longrightarrow \mathfrak{A}}{\neg\mathfrak{A}, \Gamma \longrightarrow \Theta} \neg^{A+}$$

where Θ is empty ($\Theta = \emptyset$). Here we keep formula $\neg A$ because keep all the formulae asserted in the antecedent (which corresponds to the fact that P can attack the same formula several times). Thus we get the following dialogue rule:

Round	Opponent	Proponent
$m - 1$	Γ	\dots
m	$\neg\mathfrak{A}$	\mathfrak{A}_i
$m + 1$	\perp	\mathfrak{A} (<i>Att.</i>)
$m + 2$	\dots	\dots

where \mathfrak{A}_i might be empty or represent an attack on the previous formula.

3. In the case of disjunction in the succedent \vee^{S+} it is the one deducing the formula who chooses a disjunct (it is sufficient for one disjunct to satisfy the formula) and thus it is the P who chooses in the dialogue.

$$\frac{\Gamma \longrightarrow \mathfrak{A} \text{ or } \Gamma \longrightarrow \mathfrak{B}}{\Gamma \longrightarrow \mathfrak{A} \vee \mathfrak{B}} \vee^{S+}$$

Round	Opponent	Proponent
$m - 1$	\dots	\dots
m	Γ	$\mathfrak{A} \vee \mathfrak{B}$
$m + 1$	$? - \vee$	\mathfrak{A}
$m + 2$	\dots	\dots

or

Round	Opponent	Proponent
$m - 1$	\dots	\dots
m	Γ	$\mathfrak{A} \vee \mathfrak{B}$
$m + 1$	$? - \vee$	\mathfrak{B}
$m + 2$	\dots	\dots

4. In the case of disjunction in the antecedent \vee^{A+} , both conjuncts should satisfy the formula (thus there is branching in G_3^{min} derivation), and thus, it is the O who chooses in the dialogue. Hence, for the formula to be minimally valid, the P should have a winning strategy for both disjuncts.

$$\frac{\mathfrak{A}, \mathfrak{A} \vee \mathfrak{B}, \Gamma \longrightarrow \Theta \text{ and } \mathfrak{B}, \mathfrak{A} \vee \mathfrak{B}, \Gamma \longrightarrow \Theta}{\mathfrak{A} \vee \mathfrak{B}, \Gamma \longrightarrow \Theta} \vee^{A+}$$

Round	Opponent	Proponent
$m - 1$	Γ	\dots
m	$\mathfrak{A} \vee \mathfrak{B}$	\mathfrak{A}_i
$m + 1$	\mathfrak{A}	$? - \vee$
$m + 2$	\dots	\dots

and

Round	Opponent	Proponent
$m - 1$	Γ	\dots
m	$\mathfrak{A} \vee \mathfrak{B}$	\mathfrak{A}_i
$m + 1$	\mathfrak{B}	$? - \vee$
$m + 2$	\dots	\dots

5. The case for conjunction in the succedent \wedge^{S+} is inverted with respect to the case of disjunction 3. Here both conjuncts should satisfy the formula, and thus, there is branching. It is reflected in the dialogue by the fact that it is the O who chooses the conjunct. Hence, the P should have a winning strategy against any attack of O , i.e., for all the conjuncts she asserts.

$$\frac{\Gamma \longrightarrow \mathfrak{A} \text{ and } \Gamma \longrightarrow \mathfrak{B}}{\Gamma \longrightarrow \mathfrak{A} \wedge \mathfrak{B}} \wedge^{S+}$$

Round	Opponent	Proponent
$m-1$
m	Γ	$\mathfrak{A} \wedge \mathfrak{B}$
$m+1$	$? - \wedge_L$	\mathfrak{A}
$m+2$

and

Round	Opponent	Proponent
$m-1$
m	Γ	$\mathfrak{A} \wedge \mathfrak{B}$
$m+1$	$? - \wedge_R$	\mathfrak{B}
$m+2$

6. The case of conjunction in the antecedent \wedge^{A+} is inverted with respect to the disjunction 4. If something follows from at least one conjunct, then it follows from the conjunction. Here the P decides (chooses the appropriate strategy) which conjunction she will make use of. However, we notice in mind that we keep the conjunction itself so that P can attack it once again with a different conjunct if she needs to.

$$\frac{\mathfrak{A}, \mathfrak{A} \wedge \mathfrak{B}, \Gamma \longrightarrow \Theta \text{ or } \mathfrak{B}, \mathfrak{A} \wedge \mathfrak{B}, \Gamma \longrightarrow \Theta}{\mathfrak{A} \wedge \mathfrak{B}, \Gamma \longrightarrow \Theta} \wedge^{A+}$$

Round	Opponent	Proponent
$m-1$	Γ	...
m	$\mathfrak{A} \wedge \mathfrak{B}$	\mathfrak{A}_i
$m+1$	\mathfrak{A}	$? - \wedge_L$
$m+2$

or

Round	Opponent	Proponent
$m-1$	Γ	...
m	$\mathfrak{A} \wedge \mathfrak{B}$	\mathfrak{A}_i
$m+1$	\mathfrak{B}	$? - \wedge_R$
$m+2$

The case of implication in succedent \supset^{S+} is quite straightforward. We have already discussed it:

$$\frac{\mathfrak{A}, \Gamma \longrightarrow \mathfrak{B}}{\Gamma \longrightarrow \mathfrak{A} \supset \mathfrak{B}} \supset^{S+}$$

Round	Opponent	Proponent
$m-1$
m	Γ	$\mathfrak{A} \supset \mathfrak{B}$
$m+1$	\mathfrak{A} (<i>Att.</i>)	\mathfrak{B} (<i>Def.</i>)
$m+2$

7. We have already discussed the case of \supset^{A+} while discussing *claim 1*. This particular case of branching in $G_3^{\text{min}a}$ deduction does not correspond to the branching of strategies. Here both branches belong to one strategy and one dialogue. However, as we have no branching in our dialogue here we just associate two branches of this rule to one round. We associate the formula in the succedent with the P column and the formulae in the antecedent with the O column:

$$\frac{\mathfrak{A} \supset \mathfrak{B}, \Gamma \longrightarrow \mathfrak{A} \quad \text{and} \quad \mathfrak{B}, \mathfrak{A} \supset \mathfrak{B}, \Gamma \longrightarrow \Theta}{\mathfrak{A} \supset \mathfrak{B}, \Gamma \longrightarrow \Theta} \supset^{A+}$$

Round	Opponent	Proponent
$m - 1$	Γ	\dots
m	$\mathfrak{A} \supset \mathfrak{B}$	Θ
$m + 1$	\mathfrak{B} (<i>Def.</i>)	\mathfrak{A} (<i>Att.</i>)
$m + 2$	\dots	\dots

With this rule, we can face the problem of the order of steps after branching (how we can identify the order based on the tree deduction), as in our example for $(\neg\mathfrak{A} \supset \neg\mathfrak{B}) \supset (\mathfrak{B} \supset \neg\neg\mathfrak{A})$. To identify the order we should make use of the rule forbidding P to assert atomic formulae that have not been yet asserted by O .

Why do we claim that for a valid derivation all rounds in the corresponding dialogue will be closed (if it is possible according to the rules)? This is guaranteed by the **minimal rule** of the dialogue (cf. Definition 9).

Furthermore, each valid derivation ends up with the *axiom* (with one formula in the succedent). This guarantees that the dialogue has a winning strategy because it ends with the atomic formula asserted by P . It is so because O cannot attack atomic formulae (so there is nothing left for her to attack), and P could not assert it before O asserts the corresponding atomic formula.

We have shown that each $G_3^{\text{min}a}$ deduction rule corresponds to a particular dialogue rule and each valid derivation ends with an *axiom* that guarantees a win for the P . \square

4. Conclusion

In this paper, we have achieved several goals. First of all, we have constructed two minimal systems, namely a variant of the minimal sequent calculus $G_3^{\text{min}a}$ and a minimal dialogical logic D^{min} . The main result that

we have shown in the present paper is our proof of the correspondence result between the systems $G_3^{\min a}$ and D^{\min} . This technical result gives rise to a range of questions for further research. Among philosophical questions that arise from the present paper, one can mention the relation between the properties of negation and the players' obligation to defend against all attacks.

Another possible direction of the research lies in the domain of logical games. By logical games here we understand two types of games: the Dialogue logic of Paul Lorenzen and Kuno Lorenz and Game-Theoretical Semantics (GTS) proposed by Jaakko Hintikka and developed by Gabriel Sandu. Dialogue logic and Game-Theoretical Semantics (GTS) are believed to define different types of truth: the former establishing validity and the later handling truth in a model [14, 17]. However, a correspondence between those two types of games affirming the existence of an algorithm permitting us to transform a winning strategy for Eloise (i.e., the Proponent) in Game-Theoretical Semantics into the corresponding one for the Proponent in a dialogue with hypotheses [18] and *vice versa* has been shown. Aimed at achieving this result some changes were proposed for the intuitionistic and classical dialogues as defined in [12, 13] adjusting them to a model. We have discussed possible ways of building minimal dialogue logic. Nevertheless, the question related to the study of GTS with respect to minimal dialogues remains intriguing.

Acknowledgement. We would like to thank Alberto Naibo from the Université Paris I Panthéon-Sorbonne and the Department of Logic of Saint Petersburg State University for fruitful discussions, comments and help. Without their invaluable help and friendly assistance, this paper would not be in its present shape. We are also thankful to the anonymous reviewers for valid feedback and suggestions for improving the paper. The reported study was funded by RFBR according to the research project n. 18-311-00363. The project's name is: "Truth, cognitive agents and information update in non-classical logics".

References

- [1] Alama. J., A. Knoks and S.L. Uckelman, "Dialogue games for classical logic", pages 82–86 in M. Giese and R. Kuznets (eds.), *TABLEAUX 2011: Workshops, Tutorials, and Short Papers*, 2011.

- [2] Alama, J., and S.L. Uckelman, “What is dialogical about dialogical logic?”, pages 207–222 in H. J. Ribeiro (ed.), *Inside Arguments: Logic and the Study of Argumentation*, Newcastle: Cambridge Scholars Publication, 2012.
- [3] Clerbout, N., “First-order dialogical games and tableaux”, *Journal of Philosophical Logic* 43, 4 (2014): 785–801. DOI: [10.1007/s10992-013-9289-z](https://doi.org/10.1007/s10992-013-9289-z)
- [4] Clerbout, N., “Finiteness of plays and dialogical problem of decidability”, *IfCoLog Journal of Logics and their Applications* 1, 1 (2014): 115–130.
- [5] Felscher, W., “Dialogues, strategies, and intuitionistic provability”, *Annals of Pure and Applied Logic* 28 (1985): 217–254. DOI: [10.1016/0168-0072\(85\)90016-8](https://doi.org/10.1016/0168-0072(85)90016-8)
- [6] Fermüller, C. G., “Parallel dialogue games and hypersequents for intermediate logics”, pages 48–64 in M. C. Mayer and F. Pirri (eds.), *TABLEAUX 2003 Automated Reasoning with Analytic Tableaux and Related Methods*, 2003. DOI: [10.1007/978-3-540-45206-5_7](https://doi.org/10.1007/978-3-540-45206-5_7)
- [7] Gentzen, G., and K. Erich, “Untersuchungen über das logische Schließen. I”, *Mathematische Zeitschrift* 39, 2 (1934): 176–210.
- [8] Gentzen, G., and Karl Erich, “Untersuchungen über das logische Schließen. II”, *Mathematische Zeitschrift* 39, 3 (1935): 405–431.
- [9] Hintikka, J., *The Principles of Mathematics Revisited*, Cambridge: Cambridge University Press, 1996. DOI: [10.1017/CB09780511624919](https://doi.org/10.1017/CB09780511624919)
- [10] Johansson, I., “Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus”, *Compositio Mathematica* 4 (1937): 119–136.
- [11] Kleene, S. C., *Introduction to Metamathematics*, the Netherlands, 1952.
- [12] Krabbe, E. C. W., “Dialogue logic”, pages 665–704 in D.M. Gabbay and J. Woods (eds.), *Handbook of the History of Logic*, Vol. 7, New York: Elsevier, 2006. DOI: [10.1016/S1874-5857\(06\)80035-X](https://doi.org/10.1016/S1874-5857(06)80035-X)
- [13] Lorenzen, P., and K. Lorenz, *Dialogische Logik*, Wissenschaftliche Buchgesellschaft, Darmstadt, 1978.
- [14] Pavlova, A. M., “Truth in dialogue logic and game-theoretical semantics (GTS)”, *Logical Investigations* 21, 2 (2015): 107–133.
- [15] Naibo, A., and P. Mattia, “Are uniqueness and deducibility of identicals the same?”, *Theoria* 81, 2 (2015): 143–181. DOI: [10.1111/theo.12051](https://doi.org/10.1111/theo.12051)
- [16] Rahman, S., and W. A. Carnielli, “The dialogical approach to paraconsistency”, *Synthese* 125 (2000): 201–232. DOI: [10.1023/A:1005294523930](https://doi.org/10.1023/A:1005294523930)

- [17] Rahman, S., N. Clerbout, and L. Keiff, “On dialogues and natural deduction”, pages 301–355 in G. Primeiro and S. Rahman (eds.), *Acts of Knowledge: History, Philosophy and Logic*, London: College Publications, 2009.
- [18] Rahman, S., and T. Tulenheimo, “From games to dialogues and back: Towards a general frame for validity”, in *Games: Unifying logic, Language and Philosophy*, Springer, 2009. DOI: [10.1007/978-1-4020-9374-6_8](https://doi.org/10.1007/978-1-4020-9374-6_8)
- [19] Sørensen, M.H., and P. Urzyczyn, “Sequent calculus, dialogues, and cut elimination”, pages 253–261 in *Reflections on Type Theory, λ -Calculus, and the Mind*, Universiteit Nijmegen, 2007.
- [20] Troelstra, A.S., and H. Schwichtenberg, *Basic Proof Theory*, 2nd edn, Cambridge University Press, 2000. DOI: [10.1017/CB09781139168717](https://doi.org/10.1017/CB09781139168717)

ALEXANDRA M. PAVLOVA
funded by FWF project W1255-N23
Technische Universität Wien
Vienna, Austria
alexandra@logic.at

International Laboratory for Logic, Linguistics and Formal Philosophy
National Research University Higher School of Economics
Russian Federation
pavlova.alex22@gmail.com