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On Logic of Strictly-Deontic Modalities. A Semantic and Tableau Approach

Abstract. Standard deontic logic (SDL) is defined on the basis of possible world semantics and is a logic of alethic-deontic modalities rather than deontic modalities alone. The interpretation of the concepts of *obligation* and *permission* comes down exclusively to the logical value that a sentence adopts for the accessible deontic alternatives. Here, we set forth a different approach, this being a logic which additionally takes into consideration whether sentences stand in relation to the normative system or to the system of values under which we predicate the deontic qualifications. By taking this aspect into account, we arrive at a logical system which preserves laws proper to a deontic logic but where the standard paradoxes of deontic logic do not arise. It is a logic of strictly-deontic modalities DR.

Keywords: deontic logic; deontic relationship; relating logic; relating semantics; quasi-deontic modalities; strictly-deontic modalities; tableau approach

1. Main goals and outline

In this paper we present a certain modification of a possible world semantics for standard deontic logic (SDL) where we will show the following:

1. How SDL is weakened but without the loss of important laws of deontic logic or the properties of the logical consequence relation.
2. How the problem of the relation between sentences under a given normative system (deontic relationship) is analyzed by specifying the truth-conditions for normative sentences and how, on this basis, quasi-deontic (alethic-deontic) modalities can be distinguished from strictly-deontic modalities.

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3. How various paradoxes of deontic logic can be solved and how, according to the notion of deontic relationship we introduce, the paradoxical formulas cannot be laws of the deontic logic.
4. How to construct a tableau-system for the logic.

The main goal of the paper is to improve SDL by the introduction of the notion of a deontic relationship and the distinction of the quasi-deontic (alethic-deontic) modalities $\Box, \Diamond$ from the strictly-deontic modalities O, P.

In Section 2 we discuss a problem of association between sentences and normative systems. We refer to some examples of sentences and normative systems where logical truth is not sufficient for an obligation while logical falsity is not sufficient for a prohibition. In this section some problems of SDL are discussed and the ideas of quasi-deontic and strictly-deontic modalities are presented. In Section 3 we introduce a language and semantics for the logic DR of strictly-deontic modalities. We also consider a language consisting of all deontic modalities, which enables us to define the logic $\mathrm{DR}^{\Box}$. In this section we define a formal notion of deontic relationship. Section 4 is entirely devoted to an analysis of the well known paradoxes of deontic logic and to a problem of the interpretation of logical connectives in a range of strictly-deontic modalities. In Section 5 we focus on laws of the logics DR and $\mathrm{DR}^{\Box}$ — a logic of strictly as well as quasi-modalities — and on possible extensions of DR. In this section we also analyse a rule of cautious monotonicity which in our opinion is a better tool for the logical investigation of deontic sentences, especially if the notion of a deontic relationship is considered. In Section 6 we propose a tableau approach to our proposed systems of logic, and finally in Section 7 we give a handful of intuitions to sum up our investigations.

2. Introduction

2.1. The main idea: sentences related by normative systems

In this paper, we start from the empirical observation that obligatory and permitted sentences have that status relative to a system of values or a normative system.[1] So there are no absolutely obligatory or absolutely permitted sentences. Therefore, when speaking of obligations,

[1] We shall use these terms interchangeably with this hope that the ethicists forgive us this simplification. Our concern is to go straight to logical matters but without losing sight of the motivations for deontic logics.

prohibitions and permissions, we do so with respect to a system of values that determines what is obligatory, permissible and prohibited.

This observation leads to the conclusion that the sentences that do not stand in a relation to a normative system are neither obligatory nor prohibited by it — they are simply neutral. While the sentences that are neutral with respect to a given normative system are undoubtedly permitted by it, as they cannot be prohibited, since they are neutral. Another observation makes us believe that no sentence which is obligatory relative to a given normative system can be neutral in relation to it. In particular, the normative system cannot oblige us to what it does not pertain to. Both observations compel us to add a new item to SDL in order to take account of their consequences of these obvious observations.

A normative system, in a simplified manner, may be understood as a collection of sentences that are somehow interlinked. For instance, in a normative system of Catholicism, the sentence describing a child birth and a sentence on the fact of its baptism are interlinked due to this system of values. By contrast, a sentence describing the birth of a child and a sentence stating that the child resides in Warsaw have no interconnection because the Catholic system of values has nothing to say about residing Warsaw — any such statement is neutral with respect to this system of values.

Let us in more detail consider the following example: the Highway Code which is in force in the countries of Europe (for short: $\mathbb{HC}$) — which is probably similar to the highway codes that are effective in other countries. Obviously, the $\mathbb{HC}$ is a normative system. Sentence:

(s1) *Jan is intoxicated*

stands in relation to $\mathbb{HC}$. It is specifically prohibited when the following sentence holds too:

(s2) *Jan is driving a car.*

In turn, (s2) is not only prohibited in conjunction with (s1), but also itself stands in relation to $\mathbb{HC}$.[2] Without the occurrence of (s1), it may be permitted if no other exclusive circumstances occur such as e.g. lack of a driving license. In turn, sentence:

(s3) *Jan enjoys eating sweets*

[2] After all $\mathbb{HC}$, inter alia, regulates the behaviours and actions of drivers.

is not only permitted by $\mathbb{HC}$, but also absolutely not related to this normative system. $\mathbb{HC}$ is about something else entirely. However, if we take account of another normative system, e.g. a normative system of an athlete who controls their weight, (s3) is related to this system and truth of (s3) may lead to some new deontic conclusions.

The main idea behind our study is to enhance standard deontic logic, based on possible world semantics, with an additional element: a relation between sentences with respect to a given normative system which forms the basis for making deontic qualifications and establishing logical consequences.

This seemingly slight modification leads to serious consequences for the system of logic. The article validates this approach, demonstrating that well-known paradoxes of deontic logic, such as Ross's paradox and the Good Samaritan paradox, do not occur in the logic we set forth (see Section 4). At the same time, our logic preserves properties and laws that are desirable for a deontic logic to have (see Section 5).

2.2. SDL, some of its problems, and possible worlds semantics

In SDL, that is the modal logic D, the concepts of obligation and permission correspond respectively to the concepts of necessity and possibility in an alethic logic. A technical or semantic assumption that distinguishes SDL in the family of all modal logics is the occurrence of axiom (D), which corresponds to the serial relation of accessibility across worlds.

> [...] the 'standard semantics' [i.e. possible worlds semantics] of deontic logic [...] gives an intuitively plausible account of the meanings of simple deontic sentences when the deontic alternatives to a given world u are taken to be worlds (or situations) in which everything that is obligatory at u is the case; they are worlds in which all obligations are fulfilled. Hence, the worlds related to a given world u by R [accessibility relation] may be termed deontically perfect or ideal worlds (relative to u). (Hilpinen, 2001, p. 163)

The deontic alternatives are different possible variants of the actual world in which simultaneously occur desirable deontic values from the viewpoint of a normative system. Consequently, what is obligatory is to occur in all such worlds, whereas what is permitted is to occur in at least one of them.

> These deontic alternatives are also 'deontically perfect worlds' of sorts: all obligations, both these that obtain in the actual world and those

> that would obtain in such an alternative possible world, are assumed to be fulfilled in each of them. (Hintikka, 1969, p. 189)

Let us indicate, however, that in such worlds there may also be sentences that are true that will not be obligations of a given normative system; in other words, sentences that are neutral rather than obligatory. For instance, in all deontic alternatives it could be that nobody smokes, but it does not have to mean that smoking is forbidden (see Wansing, 1998, pp. 195–196). What if, in any deontic alternative, any of our alteregos drinks milk daily? Can we then claim that it is obligatory that we drink a glass of milk daily? Or maybe we should consider obligations as something different from truths which are not related to a given normative system (see Solt, 1984, pp. 350–351).

So how should we separated those sentences that are true and related to a given normative system from those that are true though neutral? How can we express in a formal way that something is related or neutral with respect to a given normative system? The standard approach does not allow us to answer this question. And in case of many non-standard approaches the problem is not specified or at least not in the presented form. In the case of our logic, the distinction we need will be available.

What is more — as we know very well — the standard approach leads to various paradoxes, widely described in the literature on the subject (see Hilpinen, 2001, pp. 163–167; Hilpinen and McNamara, 2013, pp. 58–64; Carmo and Jones, 2002, pp. 268–270).[3] The source of some of them is the closure of the obligation operator O under the relation of logical consequence also known as the monotonicity rule.

> A first group of paradoxes has its origin in the closure of the O-operator under logical consequence (that is, in the fact that SDL, like any normal modal logic, contains the (RM)-rule: if $\vdash A \to B$, then $\vdash \mathrm{O}A \to \mathrm{O}B$). (Carmo and Jones, 2002, p. 268)

Another problem is the closure of tautologies under the operator O (necessitation rule), which causes that any tautology is always obligatory (cf. Wright, 1980, p. 402).[4]

[3] Solving deontic paradoxes is a common motivation for the development of new approaches. For example, Kulicki and Trypuz (2015, p. 1241) write: "it is also shown that the existing definitions of obligation in these systems are unacceptable due to their non-intuitive interpretation or paradoxical consequences".

[4] Such closure together with axiom (K), classical logic and the detachment rule,

> A second problem of SDL has to do with the O-necessitation rule itself, according to which any tautology (more generally, any theorem) is obligatory, which is incompatible with the idea that obligations should be possible to fulfill and possible to violate.
>
> (Carmo and Jones, 2002, p. 270)

This property is also strange for a slightly different reason, namely that the sentences that make up a given tautology may stand in no relation to the normative system which produces the grounds on which we consider a given obligation. For instance, as per the above-mentioned $\mathbb{HC}$, the sentence:

(s4) *Jan is a dark-haired man or Jan is not a dark-haired man*

is not obligatory in any case. So, even though this sentence is true in each possible world and thus also in any world in which the $\mathbb{HC}$ is absolutely respected, it specifies no obligation with respect to $\mathbb{HC}$ at all. So, the concept of obligation with which we are dealing in SDL, is too broad. Logical truths do not have to be obligatory, which is always the case in SDL.

Looking further ahead, no countertautology in SDL is permitted, although even simple sentences forming parts of one of the countertautologies may be absolutely neutral with respect to the normative system under which we predicate the permission. This is perhaps a more striking concern.

Even if a given sentence must not occur for logical reasons (is self-contradictory), how could it not be permitted by any normative system? Most of the normative systems lead to a prohibition of only such sentences that interfere with the deontic values of those systems. The prohibition of self-contradiction is a property at the very outside, which can be borne by a normative system used by logicians in their metalogical considerations. Still, logicians too require some relation of the sense or contents between sentences they consider. Let us check the following countertautology:

(s5) *Jan is a dark-haired man and Jan is not a dark-haired man.*

In terms of SDL, (s5) is not permitted and is even prohibited — as in each of the possible worlds it is false. But from the viewpoint of the majority of normative systems, this sentence exceeds the scope of their

obviously enables the introduction of (RM). Hence, it allows a derivation of the same paradoxes which we get due to (RM) (cf. Carmo and Jones, 2002, pp. 268–270).

operation. Let us therefore repeat the question: how can it be generally prohibited?

For example, how would $\mathbb{HC}$ prohibit the occurrence of (s5)? This sentence — howsoever logically impossible and therefore false — is absolutely permitted from the viewpoint of $\mathbb{HC}$. Would the occurrence of (s5) — even though it cannot occur for logical reasons — violate any of $\mathbb{HC}$'s provisions? No, it would not, because $\mathbb{HC}$ does not regulate hair colour. The permissibility of (s5) from the viewpoint of $\mathbb{HC}$ follows from the absence of a relation between this sentence and $\mathbb{HC}$ or its property of being neutral with respect to $\mathbb{HC}$.

To sum up, neither is truth in all possible worlds a sufficient condition for an obligation to hold (although probably it is a necessary condition), nor is falsity in all worlds a sufficient condition for a prohibition to hold (although it is also probably a necessary condition). These are among the postulates met by the logic we set forth. Thus, the strictly-deontic modalities that we shall define below include the issue of the truth of the deontic alternatives but also the relations with other sentences.

The dominance of possible worlds semantics in the research on the philosophical logic seems to be coming to an end. This is apparent, for example, in the context of deontic logic. However, it does not mean that SDL and its extensions are completely ignored in the descriptions of the new types of deontic logics. Very often new logics have been compared to the old ones in terms of axioms, rules of inference and metalogical properties. One of the common features of many of the new proposals is that they endeavour to improve SDL in some chosen way, such as by removing paradoxes. Other common features include, for example: (i) making use of the concept of the classical relation of logical consequence, e.g. in order to determine the meaning of the obligation operator; (ii) attempting to provide standards with a direct representation in the model; and (iii) justifying of a logic by analysing various specific problems related to obligations and prohibitions often drawn from everyday life. A good example of an approach featuring these properties is the non-monotonic deontic logic of Horty (1997) which makes use of Reiter's default theory on the basis of deontic logic. Such a solution is particularly interesting in the context of the problem of normative conflicts. Another interesting example is the input/output logic featured in (Makinson and van der Torre, 2000) or the logic of imperatives and imperative structures, analysed in detail by Hansen (2001). It is worth noting that Hansen refers in his studies not only to SDL, but also to many non-standard deontic

logics, exploring, among other things, the possibility of expressing such logics by means of the imperative structures he analyses.

In the context of our deliberations, it is worth stressing the use of the classical relation of consequence. It is, after all, clear that involving such a relation in research into the relationship between standards and their consequences entails the risk of irrelevance. Obviously, we could modify various innovative approaches, replacing the classical relation with one taken from some other relevant logic. However, what do we actually mean by relevance in such a case? Will such meaning be coincident with the concept of deontic relationship we have introduced? Will such relevance be more than just so-called derivational utility? The article proposes a different approach. We start with some examples, drawing on intuitions and recognitions of relevance (or better a relationship) from the viewpoint of the normative system. We will then recast them in a formal system. To this end, we use the common semantics of possible worlds. It is a good idea to develop a new approach on the basis of a familiar one. Possible world semantics is familiar and well-understood and so it will prove easier to develop our approach by enrich such a semantics. This translates, for example, into an analysis of the problems of decidability, axiomatization or how to introduce other syntactic approaches such as tableau systems or sequent calculus. A modification of possible worlds semantics is only the beginning and the initial test of the formal analysis of the deontic relationship we describe. Having mastered the method of enriching the relational structures with possible worlds by a family of additional relations and undertaken the development of a proof theory for the logic in the light of a consideration of more complex structures, we can attempt to analyse the problem of the deontic relationship on the basis of various innovative approaches.

2.3. Sentences related by normative system — technical solution

In order to take account of the aspect of standing in a relation to a normative system, we complement possible worlds semantics with additional elements: relations between sentences which hold according to the normative system from the viewpoint of which we establish obligations and permissions. In our approach, to be related to a normative system (to be deontically related) will mean that there is a relation between some sentences according to a given normative system. The worlds accessible from a given world must therefore, not only within the philosophical in-

terpretation, but above all within the technical implementation, comply with the rules of the normative system from the viewpoint of which we ascribe deontic properties to sentences.

The standard model of deontic logic is a triple $\langle W, Q, v\rangle$, where W is a set of possible worlds, Q is the accessibility (serial) relation for deontic logic, and v assigns a set of possible worlds to propositional letters. Our logic enhances this with the addition of a new element to produce a quadruple: $\langle W, Q, v, \{\mathrm{R}_w\}_{w\in W}\rangle$.

The set $\{\mathrm{R}_w\}_{w\in W}$ is a family of indexed by possible worlds binary relations, defined on the Cartesian product of the set of formulas. Therefore, for each possible world $w \in W$, there is exactly one such relation R_w. The relation will be the technical representation of the fact that the two sentences are related in a given world. We call this relation a *relating relation*. This relation enables us to express that two sentences, from the point of view of the given normative system, are connected (cf. Jarmużek, 2020; Jarmużek and Klonowski, 2020).

Let us emphasize that our semantics does not feature a normative system as such but is influenced by one through the way in which relates sentences in individual worlds. Such a system might be an agent who issues obligations and prohibitions. Whereas in an epistemic logic the agent knows that p, in a deontic logic an agent obliges that p. In both cases, the agents' point of view should be taken into account.

Let us now note several historical facts pertaining to the relation defined on the Cartesian product of formulas, i.e. the relating relation and logics defined by means of valuations and relating relations, i.e. relating logics (cf. Jarmużek, 2020; Jarmużek and Klonowski, 2020). The idea underlying a semantics based on the binary relation defined on a set of formulas probably has its origin in (Walton, 1979; Epstein, 1979). An example of its application may be the analysis of the content relationships which is the foundation of the so-called relatedness logics and dependence logics defined by Epstein with particular conditions imposed on models (see Epstein, 1990, pp. 61–84, 115–143). A more general approach — without assumptions imposed on the relating relation — was proposed in (Jarmużek and Kaczkowski, 2014). This work also suggested some philosophical interpretations of relating relations; for example, causal, temporal and analytical ones (see Jarmużek and Kaczkowski, 2014; Jarmużek, 2020; Jarmużek and Klonowski, 2020). It should, however, be stressed that a casual interpretation of the relating relation, as one of the many philosophical interpretations of such relations, was also suggested

by Walton (1979, p. 131). The aim of these papers was to examine various propositional logics that consider both the logical properties of sentences and their non-logical connections, the latter permitting a richer understanding of relationships of content between them.[5]

In this work, we propose interpreting the relating relation as relating sentences with respect to a normative system and a range of deontic alternatives. Another novelty of our approach is that we mix the relating relation with possible world semantics. It is a further step in the applications of this technique. It is clear that it can be used to reinforce/improve any logics based on possible worlds semantics (see also Jarmużek and Malinowski, 2019b).

The introduction to the model of representations of the deontic relations between sentences $\{\mathrm{R}_w\}_{w\in W}$ produces a major change in truth-conditions for obligations and permissions. In the proposed approach, a sentence is obligatory if it occurs in all accessible possible worlds and stands in relation to a normative system, which means that a proper relation holds between some sentences from the viewpoint of the system. In turn, a sentence is permitted when in an accessible possible world it is true or is not related to the normative system, being neutral.

2.4. From deontic necessity to obligation

The general idea behind deontic necessity and possibility, i.e. quasi-deontic modalities, is as follows. The deontic necessity of a sentence implies its occurrence in each deontic alternative. Even so, it does not have to stand in these worlds in a relationship to a given normative system. In turn, a sentence is deontically possible when it is true in a deontic alternative.

In order to better clarify the idea of the deontic necessities and possibilities, let us again consider the example of $\mathbb{HC}$ as a normative system. Although this system — at least in its European variants — obliges the passengers and the driver to fasten seatbelts, it does not order, in the sense of obligation, that there exist places where seatbelts can be purchased. Furthermore, seatbelts can be damaged and need repair and it is likewise not ordered that there be places where this can happen. Of course, $\mathbb{HC}$ does not order the existence of points of sale or distribution of car seatbelts or parts thereof. A sentence asserting the existence of a

[5] It is worth saying that the idea has also been used to emulate the semantics for connexive logics (see Jarmużek and Malinowski, 2019a).

place where seatbelts can be purchased must undeniably be true in any world in which $\mathbb{HC}$ is in force, although $\mathbb{HC}$ does not oblige the existence of such a place. Therefore, a sentence expressing the availability of seatbelts must be deontically necessary. Where $\mathbb{HC}$ is in force, seatbelts must be things that can be acquired, even though this is not the "must" of an obligation.

Still, in the context of $\mathbb{HC}$, not everything permitted by this code must be deontically possible. As already established, $\mathbb{HC}$ allows for what (s5) claims because this sentence is neutral with respect to $\mathbb{HC}$. Although this sentence is permitted with regard to $\mathbb{HC}$, it is not alethically possible — it is a counter-tautology after all. But it is not deontically possible either. In no possible world in which $\mathbb{HC}$ holds to the same extent as in the actual world it is true.

So, sentences deontically necessary and deontically possible are sentences that, even though they do not have to be related to normative system, have logical value dependant on the states of worlds which form the deontic alternatives.

Assuming that in the model $\langle W, Q, v, \{\mathrm{R}_w\}_{w\in W}\rangle$ the relation Q is of a serial nature, an assumption we must surely make, the modalities $\Box$ and $\Diamond$, which stand for deontic necessity and possibility respectively, correspond to the standard interpretation of deontic modalities in logic D, they behave as such. We claim that so far SDL has analysed the deontic modalities which corresponded to our weak modalities $\Box$, $\Diamond$ wrongly treating them as O, P. It is for this reason that the so-called paradoxes of deontic logic have arisen.

Meanwhile, modalities $\Box$, $\Diamond$ are just a step towards the real deontic modalities O, P. Therefore, in the logic we shall develop, we will consider these two types of modalities although, ultimately, our logic DR is a logic of sentences built by means of the obligation operator O and the permission operator P.

3. Language and semantics

In this section, we will define the language in which we will develop our deontic logic. This will be a richer language as we want to develop a modal logic for two types of modality. Furthermore, we will also introduce and portray a function of formula demodalization and introduce the concept of a deontic relationship. Thanks to them we will be able to

provide the basic semantic concepts for the logic DR and for the modal logic that employs them.

3.1. Language

Let us now define two sets of formulas. They will enable us to examine the relations between the two types of modality we mentioned. The smallest of the sets is a set on which we will ultimately define our DR logic.

The most general of the sets of formulas contains expressions composed of: propositional letters p_1, p_2, p_3, ..., classical connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$, deontic connectives (or: deontic operators, deontic modalities) $\Box$, $\Diamond$, O, P and brackets), (. A set of propositional letters is denoted by Var. A set of formulas is defined in the standard way.

DEFINITION 3.1. *A set of formulas* $\mathsf{For}^{\Box}$ (resp. For) is the smallest set X such that:

- $\mathsf{Var} \subseteq X$;
- if $A, B \in X$ and $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, then $(A * B) \in X$;
- if $A \in X$ and $* \in \{\Box, \Diamond, \mathrm{O}, \mathrm{P}, \neg\}$ (resp. $* \in \{\mathrm{O}, \mathrm{P}, \neg\}$), then $* A \in X$.

In the cases where this does not lead to ambiguities, we will omit external brackets for ease of reading along with internal ones, taking account of the strength of binding of Boolean and modal connectives. Finally, we will use p, q, r etc. rather than p_1, p_2, p_3 etc.

Let us note that a subset of each of the distinguished sets is a set F of formulas of Classical Propositional Logic (conditions 1–2 and 3 with negation only). Obviously we have $\mathsf{F} \subset \mathsf{For} \subset \mathsf{For}^{\Box}$.

3.2. Demodalization

Let us now prepare for the technical representation of the fact that sentences stand in a deontic relationship or that they are deontically related. To this end, we will introduce the concept of the demodalization of formulas, this being the idea of stripping them of their modal properties.

We assume that relations between sentences *modulo* a given normative system do not pertain to their modal nature, but only to the factual content they bear. Therefore, the relating relation of two formulas A, B, in a deontic context, is a matter of their content irrespective of their modal features. Therefore, we assume that sentences:

(s6) *Jan is a driver*
(s7) *It is deontically necessary/possible that Jan is a driver*
(s8) *It is obligatory/permitted that Jan is a driver*

may equally be related to a sentence:

(s9) *Jan is sober*

under $\mathbb{HC}$. This is because what is important is the positive relation between the common, content-type component of (s6)–(s9), that is *de facto* (s6) and (s9). Therefore, we assume — in accordance with the classical view — that the modalities do not change the content of sentences, but only indicate their *modus* or manner of occurrence. The content of a sentence is what remains after the modality gets removed.

We also expand our view to include the negation. We will naturally also consider it as a modality.[6] So let us consider a sentence:

(s10) *It is not the case that Jan is a driver.*

We assume that (s10) is also in a relation to (s9), as long as (s6) is in relation to (s9), which we have admitted after all. For, even though (s10) states that Jan is not a driver, in terms of content it is related to the sobriety/insobriety of Jan. Both these issues undoubtedly lie within $\mathbb{HC}$'s scope, even if we assume that the relation of (s10) is negative in nature, whereas the relation of (s6) is positive, in a sense that the latter exposes Jan to sanctions while the former does not.

Let us note that the concepts of demodalization and deontic relationship that we will introduce only tell us about potential relations between sentences. In fact, these relations will follow from the premises adopted by the inferences along with the ascribed modalities of obligation and permission.

Although demodalization for the formulas from set $\mathsf{For}^{\Box}$ may be further discussed, our ultimate aim here is the definition of logic DR on the set of formulas For. Therefore ultimately, demodalization only pertains to obligation and permission.

Generally, we shall understand demodalization as defined below.

Definition 3.2 (Demodalization of formula). Let $\mathbf{d}\colon \mathsf{For}^{\Box} \longrightarrow \mathsf{F}$ be a function such that for any $A \in \mathsf{For}^{\Box}$ we put:

- if $A \in \mathsf{Var}$, then $\mathbf{d}(A) = A$;

[6] Negation is treated in modal logic as a modality (see Chellas, 1980, p. 29).

- if $A := B * C$, for some $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, then $\mathbf{d}(A) = \mathbf{d}(B) * \mathbf{d}(C)$;
- if $A := * B$, for some $* \in \{\Box, \Diamond, \mathrm{O}, \mathrm{P}\neg\}$, then $\mathbf{d}(A) = \mathbf{d}(B)$.[7]

A *demodalization* of formula A is formula $\mathbf{d}(A)$.

A demodalization of a given formula is its modality-free and negation-free aspect that takes account of its structure and content. It therefore contains just propositional letters and Boolean connectives and so in practice represents a simple or complex sentence stating something about the world. Notice that demodalization is an idempotent operator.

3.3. Semantics

The semantics we offer for our DR logic is an extension of possible worlds semantics with a family of relating relations indexed by possible worlds, introduced in Section 2. Of course, it shall take account of the entire set of formulas $\mathsf{For}^{\Box}$, with all modalities that occur in formulas from the set $\mathsf{For}^{\Box}$. But the crucial fragment will specify the logic DR that is of interest here.

3.3.1. Deontic relationship

As we have seen, an important concept in our philosophical analysis is that of a deontic relationship, which establishes whether a given sentence is or is not neutral. In formal terms, the deontic relationship shall be specified in the following way:

DEFINITION 3.3 (Deontic relationship). Let $\langle W, Q, v, \{\mathrm{R}_w\}_{w \in W}\rangle$ be a model and $w \in W$. Formula A is *deontically related in* w (for short: $\mathbf{r}_w(A)$) iff the following conditions hold:

1. if $A \in \mathsf{Var}$, then $\forall_{u \in W} \exists_{B \in \mathsf{For}^{\Box}} (\mathrm{R}_u(A, \mathbf{d}(B))$ or $\mathrm{R}_u(\mathbf{d}(B), A))$
2. if $A := B * C$, for some $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, then $\mathrm{R}_w(\mathbf{d}(B), \mathbf{d}(C))$
3. if $A := * B$, for some $* \in \{\Box, \Diamond, \mathrm{O}, \mathrm{P}, \neg\}$, then $\mathbf{r}_w(\mathbf{d}(B))$.

The property of being deontically related means that in world w a given sentence — simple or complex — stands in a relation to the normative system from which we predicate obligation or permission. We will cover all the cases present in the above definition in turn. Let us, however, start from the fact that although the deontic relationship is

[7] Demodalization is similar to erasure transformation defined for example in (Chellas, 1980, pp. 22–23), except for case of demodalization of negation that is also a modality (see also Jarmużek and Malinowski, 2019b).

defined on all formulas, ultimately it refers to sentences without modalities — they are demodalized by function **d**, provided they do not contain any modalities at all. The deontic modalities are qualifications we only ascribe to such sentences.

For the sentences featuring a structure of propositional letters $A \in \mathsf{Var}$, we assume that $\mathbf{r}_w(A)$ occurs when this sentence in each deontically possible world is related to a demodalised sentence or a demodalised sentence is related to it. In other words, a simple sentence does not have to be related to another, distinct sentence in order to be considered suitable for deontic qualifications. As an example, let us consider a sentence:

(s11) *Jan drives a car.*

Sentence (s11) is always in a relation to $\mathbb{HC}$. Due to normative system $\mathbb{HC}$, this sentence always stands in relation with something, because it constitutes one of the central reference points of the deontic qualifications in $\mathbb{HC}$. Let us also notice that if a propositional letter is related in some world it is related in any world. By definition 3.3, therefore, we have the following fact.

Fact 3.4. *Let* $\langle W, Q, v, \{\mathrm{R}_w\}_{w\in W}\rangle$ *be a model and* $A \in \mathsf{Var}$. *Then* $\mathbf{r}_w(A)$, *for some* $w \in W$, *iff* $\mathbf{r}_u(A)$, *for any* $u \in W$.

Another case is the case of complex sentences containing at least one binary connective. Then we have: $\mathbf{r}_w(A)$ iff $\mathrm{R}_w(\mathbf{d}(B), \mathbf{d}(C))$, if $A := B * C$, for some $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. Therefore, a sentence that consists of two sentences combined by a Boolean connective is deontically related iff, following demodalization, these two sentences stand in a deontic relation. So, for example, the sentence:

(s12) *Jan drives a car and Jan is intoxicated*

is deontically related by virtue of $\mathbb{HC}$ since it relates (s11) with the sentence:

(s13) *Jan is intoxicated.*

The sentence:

(s14) *If Jan drives a car, then Jan has two daughters*

is not deontically related by virtue of $\mathbb{HC}$, as $\mathbb{HC}$ does not relate (s11) with the sentence:

(s15) *Jan has two daughters.*

Notice that if a formula $A * B$ with a Boolean connective $*$ as a main connective is deontically related, then any other formula $A \star B$ with any Boolean connective $\star$ as a main connective is deontically related.

When it comes, in turn, to sentences with negation, we assume that $\mathbf{r}_w(\neg A)$ iff $\mathbf{r}_w(\mathbf{d}(\neg A))$ iff $\mathbf{r}_w(\mathbf{d}(A))$ meaning that a negated sentence is deontically related by virtue of a given normative system iff if the sentence with no negation and no modalities is deontically related. So, for instance, sentence (s11) is deontically related by virtue of normative system $\mathbb{HC}$ in world w exactly when its negation is deontically related by virtue of this system, i.e. the sentence:

(s16) *Jan is not a driver.*

Of course, as we will see later, it does not mean either that both these sentences are obligatory or that they are permitted. But both remain in a relation to $\mathbb{HC}$: they are deontically related according to it.

The last of the cases is the case of a sentence preceded by a modality. Obviously, the property of the deontic relationship $\mathbf{r}_w(A)$ iff $\mathbf{r}_w(\mathbf{d}(A))$, if $A := * B$, for some $* \in \{\Box, \Diamond, \mathrm{O}, \mathrm{P}\}$, forces the modalities to be removed as the deontic relations cover sentences without modal qualifications. Justification for this was given in Section 3.2 devoted to the function of demodalization.

As a sort of metalanguage denotation, we adopt a tilde operator. It shall mean a metalanguage negation which acts classically. So, for instance, expressions $\mathbf{r}_w(A)$ and $\sim\sim\mathbf{r}_w(A)$ are equivalent, while $\sim\mathbf{r}_w(A)$ contradicts them.

Let us remark that in our approach the notion of a relationship is understood in a slightly different way than in mainstream relevance logic (see Read, 2012, pp. 114–130). Our notion is to some extent close to the idea of a meaning relationship, since sentences stand in some of relation in virtue of their content from the point of view of a given normative system. This raises the important question of whether there are any connections between our approach and approaches presented in, for example (Lokhorst, 2006, 2008; Stelzner, 1992; Cheng and Tagawa, 2002). It is an interesting issue but it is not one that we will address in this article. At least three differences, however, should be emphasized. First, our approach is different, since we start with a semantic investigation and claim that a deontic relationship is a semantic one. Second, we try to introduce the idea of a sentence being related to a normative system directly by means of a relating relation. Third, we do not want

to explore a relevance-logic concept of entailment but rather focus on deontic relations between sentences.

3.3.2. Truth conditions

Let us now go to the definition of the truth conditions for sentences constructed by considered logical connectives. These conditions shall assign a different meaning to the formulas constructed using quasi-deontic modalities $\Box, \Diamond$, and strictly-deontic modalities O, P.

Definition 3.5 (Truth in a model). Let $\mathfrak{M} = \langle W, Q, v, \{\mathrm{R}_w\}_{w \in W} \rangle$ be a model and $w \in W$. For propositional letters and classical connectives we put standart conditions, while for modalities as follows:

$$\begin{aligned}
\mathfrak{M}, w \models \Box A &\iff \forall_{u \in W}(wQu \Rightarrow \mathfrak{M}, u \models A),\\
\mathfrak{M}, w \models \Diamond A &\iff \exists_{u \in W}(wQu \,\&\, \mathfrak{M}, u \models A),\\
\mathfrak{M}, w \models \mathrm{O}A &\iff \forall_{u \in W}(wQu \Rightarrow \mathfrak{M}, u \models A \,\&\, \mathbf{r}_u(A)),\\
\mathfrak{M}, w \models \mathrm{P}A &\iff \exists_{u \in W}(wQu \,\&\, (\mathfrak{M}, u \models A \text{ or } \sim \mathbf{r}_u(A))).
\end{aligned}$$

For any $X \subseteq \mathsf{For}^{\Box}$, in the case $\forall_{A \in X}\ \mathfrak{M}, w \models A$, we shall write $\mathfrak{M}, w \models X$. Moreover, for any $A \in \mathsf{For}^{\Box}$, in a standard way, $X \models^{\Box} A$ iff for any $\mathfrak{M}$ and $w \in W$: if $\mathfrak{M}, w \models X$ then $\mathfrak{M}, w \models A$. Having narrowed down the set of formulas, in accordance with the observation $\mathsf{For} \subset \mathsf{For}^{\Box}$, we shall obtain various relations of consequence. The relation defined on set $\wp(\mathsf{For}) \times \mathsf{For}$, we shall denote as $\models$. The logic $\models$ of strict deontic modalities O and P we shall name DR, while by $\mathrm{DR}^{\Box}$ we shall specify the multimodal logic $\models^{\Box}$, of strictly-deontic modalities O, P and quasi-deontic modalities $\Box$ and $\Diamond$.

Under the observations that $\mathsf{F} \subset \mathsf{For} \subset \mathsf{For}^{\Box}$ and definition 3.5 of truth in a model the following relation occurs: $\models \subset \models^{\Box}$. What is more, our logics are closed under substitution. Therefore, $\mathrm{DR}^{\Box}$ also includes laws of DR, but expressed in a richer language. Similarly DR and $\mathrm{DR}^{\Box}$ include all formulas of For and $\mathsf{For}^{\Box}$ respectively which are instances of laws of Classical Propositional Logic.

4. Paradoxes of deontic logic

In this section, we will take a closer look at some paradoxes of deontic logic. We will explain why these formulas should not be tautologies and we will prove that they indeed are not tautologies of DR. When

analysing various paradoxes, to better illustrate the selected problems, we will again refer to $\mathbb{HC}$.

The paradoxes form a group of formulas which are often cited as problematic by various experts (see, e.g., Carmo and Jones, 2002, pp. 268–277; Hilpinen and McNamara, 2013, pp. 58–97; Hilpinen, 2001, pp. 163–173; Wright, 1980, pp. 415–418). We skip, however, the famous riddle of Chisholm, other paradoxes showing problems of representation of conditional obligations and paradoxes concerning situations where obligations conflict. These issues require a separate study. We also leave aside the paradoxes that would require us to extend our language with new connectives (e.g. epistemic ones).

In addition to the standard paradoxes discussed in the literature, we will also present several problematic formulas which are not tautologies of DR, even though they are laws of SDL. It is far from certain that such formulas should be tautologies of the smallest deontic logic — we believe they should not be. Nevertheless, we will show how to extend DR in order to demonstrate their tautological nature under the approach we propose. Furthermore, we will also examine Gödel's (necessitation) rule for O.

4.1. Ross's paradox

Let us begin with the Ross's paradox (see, e.g., Ross, 1941; Carmo and Jones, 2002, pp. 268–269; Hilpinen and McNamara, 2013, pp. 63; Hilpinen, 2001, pp. 165–166), where we have a formula of the following form:

$$\mathrm{O}A \to \mathrm{O}(A \vee B). \tag{RP}$$

In order to see the problem, consider the following instance of (RP): if it is obligatory that the driver stops at a red light, then it is obligatory either that a driver stops at a red light or doesn't stop at a stop sign.

In other words (RP) permits us to conclude that the disjunction of what is obligatory by a given system is obligatory by this system. But that one of disjuncts of an disjunction is deontically related does not warrant that the disjunction itself be deontically related. For this reason, (RP) should not be a tautology of DR.

Let us consider a model $\mathfrak{M}$ defined as follows: $W = \{w_1\}$, $Q = \{\langle w_1, w_1\rangle\}$, $\mathrm{R}_{w_1} = \{\langle p, p\rangle\}$ and for every $A \in \mathsf{Var}$ we put:

$$v(A) = \begin{cases} \{w_1\}, & \text{if } A = p \\ \varnothing, & \text{if } A \neq p. \end{cases}$$

Hence $\mathfrak{M}, w_1 \models p$. We have $\mathrm{R}_{w_1}(p,p)$, so $\forall_{w\in W}\exists_{C\in \mathsf{For}}(\mathrm{R}_w(p,\mathbf{d}(C))$ or $\mathrm{R}_w(\mathbf{d}(C),p))$, and hence $\mathbf{r}_{w_1}(p)$. Thus $\forall_{w\in W}(Q(w_1,w) \Longrightarrow (\mathfrak{M}, w \models p$ and $\mathbf{r}_w(p)))$, so $\mathfrak{M}, w_1 \models \mathrm{O}p$. We also have $\sim \mathrm{R}_{w_1}(p,q)$, so $\sim \mathbf{r}_{w_1}(p\vee q)$. Thus $\exists_{w\in W}(Q(w_1,w)$ and $(\mathfrak{M}, w \not\models p \vee q$ or $\sim \mathbf{r}_w(p \vee q)))$, so $\mathfrak{M}, w_1 \not\models \mathrm{O}(p \vee q)$. Therefore, (RP) is not a tautology of DR.

4.2. Free choice permissions paradox

By the free choice permissions paradox (see, e.g., Carmo and Jones, 2002, p. 269; Hilpinen and McNamara, 2013, pp. 61–63; Hilpinen, 2001, pp. 166–167) we mean a formula of the following form:

$$\mathrm{P}(A \vee B) \to \mathrm{P}A \wedge \mathrm{P}B. \tag{FCPP}$$

In this case we consider a formula that is not a law of SDL.

At first glance, formula (FCPP) seems to state something that we might intuitively agree on. For example, we are aware that an emergency vehicle is permitted to violate a road traffic regulation. In particular, it is permitted for an ambulance with the siren on to break a red light or to not stop at a stop sign. This means that an ambulance with the siren on can ignore both a red light and a stop sign. In this case, we understand the permission as something more than a lack of prohibition. This is because we are dealing here with something explicitly authorised by a normative system. Such permissions obviously entail the lack of prohibition. However, the lack of prohibition itself does not necessarily mean something has been authorized. Let us therefore simply stay with the assumption that what is not prohibited is permitted. Then, e.g. the following disjunction is permitted: you will break a red light or drink some water, because in some deontically possible world, in a deontic alternative, this sentence is true or its disjuncts are not deontically related. Despite this fact, it would be wrong to conclude that breaking a red light is permitted — at least in the case of a regular road user — and drinking water is permitted. Thus we see that when adopting a weaker sense of permission, (FCPP) is not true.

Let us consider a model $\mathfrak{M}$ defined as follows: $W = \{w_1\}$, $Q = \{\langle w_1, w_1\rangle\}$, $\mathrm{R}_{w_1} = \{\langle p,p\rangle\}$ and $v(\mathsf{Var}) = \{\varnothing\}$. We have $\sim \mathrm{R}_{w_1}(p,q)$, so $\sim \mathbf{r}_{w_1}(p \vee q)$. Thus $\exists_{w\in W}(Q(w_1,w)$ and $(\mathfrak{M}, w \models p \vee q$ or $\sim \mathbf{r}_w(p \vee q)))$, so $\mathfrak{M}, w_1 \models \mathrm{P}(p \vee q)$. Moreover $\mathrm{R}_{w_1}(p,p)$, so $\forall_{w\in W}\exists_{C\in \mathsf{For}}(\mathrm{R}_w(p,C)$ or $\mathrm{R}_w(C,p))$, and thus $\mathbf{r}_{w_1}(p)$. So we have $\forall_{w\in W}(Q(w_1,w) \Longrightarrow$

$(\mathfrak{M}, w \not\models p$ and $\mathbf{r}_w(p)))$, and thus $\mathfrak{M}, w_1 \not\models \mathrm{P}p$. Hence $\mathfrak{M}, w_1 \not\models \mathrm{P}p \wedge \mathrm{P}q$. Therefore, (FCPP) is not a tautology of DR.

4.3. The Good Samaritan paradox

The Good Samaritan paradox (see, e.g., Prior, 1954; Carmo and Jones, 2002, p. 269; Hilpinen and McNamara, 2013, pp. 63–64), we call a formula of the following form:

$$\mathrm{O}(A \wedge B) \to \mathrm{O}B. \tag{GSP}$$

Formula (GSP) according to (Hilpinen and McNamara, 2013, pp. 63–64) would allow, for example, that if assistance should to be provided to someone who has been robbed, then it is obligatory for this person to be robbed. Here's another example. Assume that someone is obliged to drive in a built-up area where a speed limit of maximum 50km per hour is in force. Therefore, the person is obliged to drive in a built-up area and not faster than 50km per hour. But this implies the person was obliged to drive not faster than 50km per hour. Obviously, the consequent of this implication is not unconditionally true.

In the case of (GSP) the problem of how to understand the logical connectives occurring within the scope of deontic operators is evident. For SDL, we understand these connectives extensionally. In our examples, we can use conjunction both to express that the clauses are true and that they are interrelated. More specifically, it allows us to state that two situations only produce an obligatory state of affairs when an appropriate interconnection is present.

The above examples show that conjunction can be used to describe a complex situation in which not everything that takes place is obligatory, but it only provides a context or background for what is obligatory. Because of this (GSP) should not be a tautology of DR.

Let us consider model $\mathfrak{M}$ defined as follows: $W = \{w_1, w_2\}$, $Q = \{\langle w_1, w_1\rangle, \langle w_2, w_2\rangle\}$, $\mathrm{R}_{w_1} = \{\langle p, q\rangle\}$, $\mathrm{R}_{w_2} = \varnothing$ and for any $A \in \mathsf{Var}$:

$$v(A) = \begin{cases} \{w_1\}, & \text{if } A \in \{p, q\} \\ \varnothing, & \text{if } A \notin \{p, q\}. \end{cases}$$

Hence $\mathfrak{M}, w_1 \models p$ and $\mathfrak{M}, w_1 \models q$, so $\mathfrak{M}, w_1 \models p \wedge q$. We have $\mathrm{R}_{w_1}(p, q)$, so $\mathbf{r}_{w_1}(p \wedge q)$. Thus $\forall_{w \in W}(Q(w_1, w) \Longrightarrow (\mathfrak{M}, w \models p \wedge q$ and $\mathbf{r}_w(p \wedge q)))$, so $\mathfrak{M}, w_1 \models \mathrm{O}(p \wedge q)$. Moreover $\exists_{w \in W} \forall_{C \in \mathsf{For}}(\sim \mathrm{R}_w(q, \mathbf{d}(C))$ and

$\sim \mathrm{R}_w(\mathbf{d}(C), q))$, thus $\sim \mathbf{r}_{w_1}(q)$. Hence $\exists_{w \in W}(Q(w_1, w)$ and $(\mathfrak{M}, w \not\models q$ or $\sim \mathbf{r}_w(q)))$, so $\mathfrak{M}, w_1 \not\models \mathrm{O}q$. Thus, (GSP) is not a tautology of DR.

4.4. The paradoxes of derived obligations

From among the known problems for SDL, let us also consider the paradoxes of derived obligations, which form deontic counterparts of the strict implication paradoxes (see, e.g., Prior, 1958), meaning the formulas of the following forms:

$$\mathrm{O}\neg A \to \mathrm{O}(A \to B), \tag{P1}$$

$$\mathrm{O}B \to \mathrm{O}(A \to B). \tag{P2}$$

Formula (P1) states that:

> the doing of what is forbidden commits us to the doing of anything whatsoever. (Stealing, e.g. — supposing that to be forbidden — commits us to committing adultery; and also of course, to not committing adultery.) (Prior, 1954, p. 64)

Whereas (P2) states that:

> if the omission of any act is not permitted, i.e. if the act is obligatory, then we are 'committed' to it by any act whatsoever. (Prior, 1954, p. 64)

Formulas (P1) and (P2) suggest one more thing, namely that what is prohibited and what is obligatory are always parts of a more complex conditional obligation. But this does not appear to be justified since it rules out the existence of absolute prohibitions and obligations that are not parts of any more complex obligations. For example in the case of the Ten Commandments, we have the prohibition against killing without any specification of sanctions that will result if we break it as well as an unconditional obligation to respect our parents but not specification of any conditions under which we have to have respect our parents.

We can therefore state that both (P1), and (P2) should not always be true. On the other hand, it is worth analysing logics for which modifications of the given formulas would be tautologies. Such logics could describe systems in which any content of a non-complex obligation or prohibition would at the same time form a component of some complex obligation or prohibition.

In order to demonstrate that (P1) is not a tautology of DR, let us consider model $\mathfrak{M}$ defined as follows: $W = \{w_1\}$, $Q = \{\langle w_1, w_1\rangle\}$,

$\mathrm{R}_{w_1} = \{\langle p, p\rangle\}$ and $v(\mathsf{Var}) = \{\varnothing\}$. We have $\mathfrak{M}, w_1 \not\models p$, so $\mathfrak{M}, w_1 \models \neg p$. Moreover $\mathrm{R}_{w_1}(p, p)$, so $\forall_{w\in W}\exists_{C\in\mathsf{For}}(\mathrm{R}_w(p, \mathbf{d}(C))$ or $\mathrm{R}_w(\mathbf{d}(C), p))$. Thus $\mathbf{r}_{w_1}(p)$, so $\mathbf{r}_{w_1}(\neg p)$. So $\forall_{w\in W}(Q(w_1, w) \Longrightarrow (\mathfrak{M}, w \models \neg p$ and $\mathbf{r}_w(\neg p)))$, and hence $\mathfrak{M} \models \mathrm{O}\neg p$. We have $\sim\mathrm{R}_{w_1}(p, q)$, so $\sim\mathbf{r}_{w_1}(p \to q)$. Hence $\exists_{w\in W}(Q(w_1, w)$ and $(\mathfrak{M}, w \not\models p \to q$ or $\sim\mathbf{r}_w(p \to q)))$, so $\mathfrak{M}, w_1 \not\models \mathrm{O}(p \to q)$. Therefore, (P1) is not a tautology of DR.

In order to demonstrate that (P2) is not a tautology of DR let us consider model $\mathfrak{M}$ defined as follows: $W = \{w_1\}$, $Q = \{\langle w_1, w_1\rangle\}$, $\mathrm{R}_{w_1} = \{\langle q, q\rangle\}$ and for every $A \in \mathsf{Var}$ we put:

$$v(A) = \begin{cases} \{w_1\}, & \text{if } A = q \\ \varnothing, & \text{if } A \neq q. \end{cases}$$

Hence $\mathfrak{M}, w_1 \models q$. We have $\mathrm{R}_{w_1}(q, q)$, so $\forall_{w\in W}\exists_{C\in\mathsf{For}}(\mathrm{R}_w(q, \mathbf{d}(C))$ or $\mathrm{R}_w(\mathbf{d}(C), q))$, thus $\mathbf{r}_{w_1}(q)$. Hence $\forall_{w\in W}(Q(w_1, w) \Longrightarrow (\mathfrak{M}, w \models q$ and $\mathbf{r}_w(q)))$, so $\mathfrak{M} \models \mathrm{O}q$. Moreover $\sim\mathrm{R}_{w_1}(p, q)$, so $\sim\mathbf{r}_{w_1}(p \to q)$. Hence $\exists_{w\in W}(Q(w_1, w)$ and $(\mathfrak{M}, w \not\models p \to q$ or $\sim\mathbf{r}_w(p \to q)))$, so $\mathfrak{M}, w_1 \not\models \mathrm{O}(p \to q)$. Therefore, (P2) is not a tautology of DR.

4.5. Commutative, associative laws and distribution of O

In the case of DR, the following formulas are not tautologies:

$$\mathrm{O}(A \wedge B) \to \mathrm{O}(B \wedge A) \tag{4.1}$$
$$\mathrm{P}(A \wedge B) \to \mathrm{P}(B \wedge A) \tag{4.2}$$
$$\mathrm{O}(A \vee B) \to \mathrm{O}(B \vee A) \tag{4.3}$$
$$\mathrm{P}(A \vee B) \to \mathrm{P}(B \vee A). \tag{4.4}$$

Let us consider model $\mathfrak{M}$ defined as follows: $W = \{w_1\}$, $Q = \{\langle w_1, w_1\rangle\}$, $\mathrm{R}_{w_1} = \{\langle p, q\rangle\}$ and $v(\mathsf{Var}) = \{W\}$. We notice that the occurrence of relation $\mathrm{R}_{w_1}(p, q)$ does not have to entail $\mathrm{R}_{w_1}(q, p)$. Therefore, $\mathbf{r}_{w_1}(p \wedge q)$ and $\sim\mathbf{r}_{w_1}(q \wedge p)$. So (1) fails. Similarly we can falsify the rest of the formulas.

There is, however, a valid question as to whether formulas of forms (4.1)–(4.4) should be tautologies of the minimal deontic logic or, to put it another way, whether the relating relation should be symmetrical in a deontic context. The question is how to understand conjunction and disjunction in the range of deontic operators. In the case of temporal and causal relations, interchangeability is not desirable. For instance, it

may be obligatory for Jan to graduate and find a job without it being obligatory for Jan to find a job and graduate (a similar example could be found for the permission operator).

However, if we treat conjunction in the scope of the deontic operators only extensionally, DR will be too weak. We therefore propose extensions to take account of the symmetry.

In the case of DR the following formulas are also not tautologies:

$$\mathrm{O}(A \wedge (B \wedge C)) \leftrightarrow \mathrm{O}((A \wedge B) \wedge C), \tag{4.5}$$

$$\mathrm{P}(A \wedge (B \wedge C)) \leftrightarrow \mathrm{P}((A \wedge B) \wedge B), \tag{4.6}$$

$$\mathrm{O}(A \vee (B \vee C)) \leftrightarrow \mathrm{O}((A \vee B) \vee C), \tag{4.7}$$

$$\mathrm{P}(A \vee (B \vee C)) \leftrightarrow \mathrm{P}((A \vee B) \vee B). \tag{4.8}$$

Let us consider model $\mathfrak{M}$ defined as follows: $W = \{w_1\}$, $Q = \{\langle w_1, w_1\rangle\}$, $\mathrm{R}_{w_1} = \{\langle p, q \wedge r\rangle\}$ and $v(\mathsf{Var}) = \{W\}$. We notice that the occurrence of relation $\mathrm{R}_{w_1}(p, q \wedge r)$ does not have to entail $\mathrm{R}_{w_1}(p \wedge q, r)$. Therefore $\mathbf{r}_{w_1}(p, q \wedge r)$ and $\sim \mathbf{r}_{w_1}(p \wedge q, r)$. So (5) fails. Similarly we can falsify the rest of formulas.

Here is another interesting question: should formulas of forms (4.5)–(4.8) be tautologies of the minimal deontic logic — i.e., should the deontic relation be associative in some sense? Again, the question is how we understand conjunction and disjunction in the range of the deontic operators. For instance, one could believe that it is permitted for Jan to be a driver, and then to be intoxicated, but sober up; while it is not permitted for Jan to be a driver and to be intoxicated, and then sober up. So, sentences under the deontic operators may be as well understood as standing in temporal-casual relations. Furthermore, a sentence $(A \wedge B)$ within a sentence $((A \wedge B) \wedge C)$ in the range of a deontic operator may state something beyond the extensional conjunction of A and B, as there must have been a reason for the brackets being such as they are. The issues waits for further philosophical discussion.

One again, if we treat conjunction and disjunction under the deontic operators only extensionally, then DR is too weak. We will late propose an extension which will be sensitive to the property of associativity, and thus formulas (4.5)–(4.8) shall become logical laws.

In addition, in the case of DR, axiom (K) for operator O is not a tautology:

$$\mathrm{O}(A \to B) \to (\mathrm{O}A \to \mathrm{O}B).$$

Initially, we find it reasonable as the fact that sentence A and sentence $A \to B$ are deontically related to a normative system does not have to imply that sentence B alone is such.

Let us consider a model $\mathfrak{M}$ defined as follows: $W = \{w_1, w_2\}$, $Q = \{\langle w_1, w_1\rangle, \langle w_2, w_1\rangle\}$, $\mathrm{R}_{w_1} = \{\langle p, q\rangle\}$, $\mathrm{R}_{w_2} = \{\langle p, p\rangle\}$ and $v(\mathsf{Var}) = \{W\}$. Then $\mathfrak{M}, w_1 \models p$ and $\mathfrak{M}, w_1 \models q$. Moreover, $\mathrm{R}_{w_1}(p, q)$, so $\mathbf{r}_{w_1}(p \to q)$. Thus we have $\forall_{w \in W}(Q(w_1, w) \Longrightarrow (\mathfrak{M}, w \models p \to q$ and $\mathbf{r}_w(p \to q)))$. Thus $\mathfrak{M}, w_1 \models \mathrm{O}(p \to q)$. Moreover, $\forall_{w \in W} \exists_{C \in \mathsf{For}}(\mathrm{R}_w(p, C)$ or $\mathrm{R}_w(C, p))$, so $\mathbf{r}_{w_2}(p)$. Therefore we have $\forall_{w \in W}(Q(w_1, w) \Longrightarrow (\mathfrak{M}, w \models p$ and $\mathbf{r}_w(p)))$. Consequently, $\mathfrak{M}, w_1 \models \mathrm{O}p$. Let us note that $\exists_{w \in W} \forall_{C \in \mathsf{For}}$ $(\sim \mathrm{R}_w(q, C)$ and $\sim \mathrm{R}_w(C, q))$. Thus $\sim \mathbf{r}_{w_2}(q)$. Therefore we have $\exists_{w \in W}$ $(Q(w_1, w)$ and $(\mathfrak{M}, w \not\models q$ or $\sim \mathbf{r}_w(q)))$. So $\mathfrak{M}, w_1 \not\models \mathrm{O}q$. Therefore $\mathfrak{M}, w_1 \not\models \mathrm{O}(p \to q) \to (\mathrm{O}p \to \mathrm{O}q)$.

We see that the occurrence of $\mathbf{r}_{w_1}(p \to q)$ and $\mathbf{r}_{w_1}(p)$ does not have to guarantee $\mathbf{r}_{w_1}(q)$. But if someone considers formula of the form of (K) intuitive in the deontic contexts, he should consider an appropriate extension of DR. The details of such an extension are provided in one of the further sections.

4.6. Gödel's rule

Gödel's rule also does not hold for O. For instance, $\models_{\mathrm{DR}} p \vee \neg p$, but $\not\models_{\mathrm{DR}} \mathrm{O}(p \vee \neg p)$. Obviously, in each model $\mathfrak{M} \models p \vee \neg p$, but for some $\mathfrak{M} \not\models \mathrm{O}(p \vee \neg p)$.

Let us consider model $\mathfrak{M}$ defined as follows: $W = \{w_1\}$, $Q = \{\langle w_1, w_1\rangle\}$, $\mathrm{R}_{w_1} = \varnothing$ and $v(\mathsf{Var}) = \{\varnothing\}$. Consequently, $\sim \mathrm{R}_{w_1}(p, \neg p)$ and $\sim \mathbf{r}_{w_1}(p \vee \neg p)$. Therefore we have $\exists_{w \in W}(Q(w_1, w)$ and $(\mathfrak{M}, w \not\models p \vee \neg p$ or $\sim \mathbf{r}_w(p \vee \neg p)))$. Thus $\mathfrak{M}, w_1 \not\models \mathrm{O}(p \vee \neg p)$, so $\not\models \mathrm{O}(p \vee \neg p)$.

4.7. Summary of paradoxes and puzzles

The above considerations show that DR is appropriately weak and does not allow for the undesirable laws. It is a natural starting point as the logic determined for all classes of models in the above-defined sense. Therefore, we find DR as the basic deontic logic of strictly-deontic modalities. For the analysis of stronger logical relations between deontic sentences, we may use its extensions, the details of which we will cover later.

5. Some laws of logic DR and DR$^{\Box}$

Logic DR$^{\Box}$ contains modalities $\Box$, $\Diamond$, O, P. In our article we consider modalities $\Box$, $\Diamond$ in order to show that they are only quasi-deontic modalities, incorrectly interpreted as proper deontic modalities, and to present some of their relations with obligations and permissions. Ultimately, a deontic logic should only contain obligation and permission operators, this logic is the logic DR, which is a sublogic of DR$^{\Box}$.

5.1. Laws of DR$^{\Box}$

Let us go to the issue of laws of our logics. Quasi-deontic modalities $\Box$ and $\Diamond$ behave as in the modal logic D. So Gödel's rule holds: if $\models A$, then $\models \Box A$. In addition Aristotelian laws hold: $\Box A \leftrightarrow \neg\Diamond\neg A$, $\Diamond A \leftrightarrow \neg\Box\neg A$, axiom (K): $\Box(A \to B) \to (\Box A \to \Box B)$, and of course axiom (D): $\Box A \to \Diamond A$. Quasi-deontic modalities are therefore standard deontic modalities, and their logic is SDL.

Let us now take a look at some bridging laws between the quasi-deontic modalities $\Box$, $\Diamond$ and the strictly-deontic modalities O and P. It is obvious that what is obligatory is also deontically necessary in DR$^{\Box}$:

$$\mathrm{O}A \to \Box A$$

Let $\mathfrak{M} = \langle W, Q, v, \{\mathrm{R}_w\}_{w\in W}\rangle$ be a model and $w \in W$. Assume that $\mathfrak{M}, w \models \mathrm{O}A$. Let $u \in W$ and $Q(w,u)$. Then $\mathfrak{M}, u \models A$. Therefore $\forall_{u\in W}(Q(w,u) \Longrightarrow \mathfrak{M}, u \models A)$, that is $\mathfrak{M}, w \models \Box A$. However in the logic DR$^{\Box}$ the opposite dependence does not occur, that is: $\not\models \Box A \to \mathrm{O}A$. For, what is deontically necessary, occurring in each deontic alternative, does not need to be obligatory itself; it may be simply an inevitable result of what is obligatory.

But there is a dependence saying that what is deontically possible is also permitted:

$$\Diamond A \to \mathrm{P}A \tag{5.1}$$

Let $\mathfrak{M} = \langle W, Q, v, \{\mathrm{R}_w\}_{w\in W}\rangle$ be a model and $w \in W$. Assume that $\mathfrak{M}, w \models \Diamond A$. Therefore for some $u \in W$, $Q(w,u)$ and $\mathfrak{M}, u \models A$. And thus $\mathfrak{M}, u \models A$ or $\sim \mathbf{r}_u(A)$. Therefore $\exists_{u\in W}(Q(w,u)$ and $(\mathfrak{M}, u \models A$ or $\sim \mathbf{r}_u(A)))$. Thus $\mathfrak{M}, w \models \mathrm{P}A$. However, in the logic DR$^{\Box}$ the opposite dependence does not occur that is: $\not\models \mathrm{P}A \to \Diamond A$. For, what is permitted does not have to be deontically possible; it may be simply deontically irrelevant, even if for instance logically contrary.

From the previous section, we know that for the modality O axiom (K) does not hold. Instead we have weaker and more reasonable forms of this axiom based on deontic necessity collaborating with obligation.

$$\mathrm{O}(A \to B) \to (\mathrm{O}A \to \Box B)$$

Let $\mathfrak{M} = \langle W, Q, v, \{\mathrm{R}_w\}_{w \in W}\rangle$ be a model and $w \in W$. Assume that $\mathfrak{M}, w \models \mathrm{O}(A \to B)$ and $\mathfrak{M}, w \models \mathrm{O}A$. Therefore $\mathfrak{M}, w \models \mathrm{O}A$ and $\mathfrak{M}, w \models \mathrm{O}(A \to B)$. Let $u \in W$ and $Q(w, u)$. Then $\mathfrak{M}, u \models A$ and $\mathfrak{M}, u \models A \to B$. Thus $\mathfrak{M}, u \models B$. Therefore $\forall_{u \in W}(Q(w, u) \Longrightarrow \mathfrak{M}, u \models B)$, that is $\mathfrak{M}, w \models \Box B$.

$$\Box(A \to B) \to (\mathrm{O}A \to \Box B) \tag{5.2}$$

Let $\mathfrak{M} = \langle W, Q, v, \{\mathrm{R}_w\}_{w \in W}\rangle$ be a model and $w \in W$. Assume that $\mathfrak{M}, w \models \Box(A \to B)$ and $\mathfrak{M}, w \models \mathrm{O}A$. Therefore $\mathfrak{M}, w \models \mathrm{O}A$ and $\mathfrak{M}, w \models \Box(A \to B)$. Let $u \in W$ and $Q(w, u)$. Then under $\mathfrak{M}, u \models A$ and $\mathfrak{M}, u \models A \to B$. Thus $\mathfrak{M}, u \models B$. Therefore $\forall_{u \in W}(Q(w, u) \Longrightarrow \mathfrak{M}, u \models B)$, that is $\mathfrak{M}, w \models \Box B$.

Moreover, we have the following interesting law:

$$(\mathrm{O}A \wedge \Diamond B) \to \Diamond(A \wedge B) \tag{5.3}$$

Let $\mathfrak{M} = \langle W, Q, v, \{\mathrm{R}_w\}_{w \in W}\rangle$ be a model and $w \in W$. Assume that $\mathfrak{M}, w \models \mathrm{O}A$ and $\mathfrak{M}, w \models \Diamond B$. Therefore $\mathfrak{M}, u \models A$, for some $u \in W$ such that $Q(w, u)$ and $\mathfrak{M}, u \models B$. Then $\mathfrak{M}, u \models A \wedge B$. Thus $\mathfrak{M}, w \models \Diamond(A \wedge B)$. As a consequence of laws (5.1) and (5.3) we have:

$$(\mathrm{O}A \wedge \Diamond B) \to \mathrm{P}(A \wedge B). \tag{5.4}$$

Law (5.3) and (5.4) state that an obligatory sentence and deontically possible sentence are jointly possible and jointly permitted. This way we can combine sentences under the modalities P and $\Diamond$.

5.2. Laws of DR

We now consider the logic DR, which means that we will be restricted to the modalities O and P. In the subsection on the paradoxes of deontic logic, we indicated many non-intuitive or somewhat problematic formulas along with Gödel's rule that are not satisfied in DR. So let us now cover some important laws which make our logic a good starting point for the development of deontic logic.

DR is weaker than D, but it is incomparable with normal modal logic K. We already know that for O, neither Gödel's rule holds, nor axiom (K). Nevertheless, under the definition of truth in a model 3.5 in logic DR Aristotelian laws hold: $\mathrm{O}A \leftrightarrow \neg\mathrm{P}\neg A$, $\mathrm{P}A \leftrightarrow \neg\mathrm{O}\neg A$. Of course, for this logic we have an important for deontic logic axiom (D) for strictly-deontic operators: $\mathrm{O}A \rightarrow \mathrm{P}A$, due to a serial relation in the models. We also have a kind of limited rule of monotonicity (RM). We can call it the *cautious rule of monotonicity*.

For any formula A of DR we introduce a function s of structural similarity that preserves brackets, but removes the main logical connectives in formula $\mathbf{d}(A)$. By definition 3.2, formula $\mathbf{d}(A)$ does not consist of negation and any modalities. The function s removes the other main connectives.

Definition 5.1 (Structural similarity). Let $\mathsf{s}\colon \mathsf{For} \longrightarrow \mathsf{Var} \cup (\mathsf{F} \times \mathsf{F})$ be a function such that for any $A \in \mathsf{For}$ we put:

- if $A \in \mathsf{Var}$, then $\mathsf{s}(A) = A$,
- if $A := B * C$, for some $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, then $\mathsf{s}(A) = \langle \mathbf{d}(B), \mathbf{d}(C)\rangle$,
- if $A := *B$, for some $* \in \{\mathrm{O}, \mathrm{P}, \neg\}$, then $\mathsf{s}(A) = \mathsf{s}(\mathbf{d}(B))$.

Formulas A and B are *structurally similar* iff $\mathsf{s}(A) = \mathrm{s}(B)$.

For example, $\mathsf{s}(\mathrm{O}(p \vee q)) = \mathsf{s}(\mathrm{O}\neg(\neg p \wedge \neg q))$, while $\mathsf{s}(((p \vee q) \vee r)) \neq \mathsf{s}((p \vee (q \vee r)))$, although both pairs of sentences are logically equivalent in DR, as can be easily checked.

Fact 5.2. *Let* $A, B \in \mathsf{For}$, $\langle W, Q, v, \{\mathrm{R}_w\}_{w\in W}\rangle$ *be a model and* $w \in W$. *Then if* $\mathsf{s}(A) = \mathsf{s}(B)$ *then* $(\mathbf{r}_w(A)$ *iff* $\mathbf{r}_w(B))$.

Proof. Let us take two formulas A, B, model $\langle W, Q, v, \{\mathrm{R}_w\}_{w\in W}\rangle$, $w \in W$ and assume that $\mathsf{s}(A) = \mathsf{s}(B)$.

Expressions $\mathsf{s}(A)$, $\mathsf{s}(B)$: either (a) can be propositional letters or (b) can have form $\mathsf{s}(A) = \langle \mathbf{d}(A_1), \mathbf{d}(A_2)\rangle$ and $\mathsf{s}(B) = \langle \mathbf{d}(B_1), \mathbf{d}(B_2)\rangle$, for some formulas A_1, A_2, B_1, B_2, by the definition of structural similarity 5.1. When (a) holds the thesis is obviously satisfied. In turn, if (b) holds, then $\mathbf{d}(A_1) = \mathbf{d}(B_1)$ and $\mathbf{d}(A_2) = \mathbf{d}(B_2)$, and thus also $\mathbf{r}_w(A)$ iff $\mathbf{r}_w(B)$, by the definition of a deontic relationship 3.3, point (2). ⊣

So by fact 5.2 and definition 3.5, we have the limited rule of monotonicity:

$$\text{If } \mathsf{s}(A) = \mathsf{s}(B), \text{ then } \frac{\models_{\mathrm{DR}} A \rightarrow B}{\models_{\mathrm{DR}} \mathrm{O}A \rightarrow \mathrm{O}B} \qquad \text{(O-CRM)}$$

By $\mathrm{P}A \leftrightarrow \neg\mathrm{O}\neg A$, some classical inferences, and (O-CRM), we get a version for P:

$$\text{If } \mathsf{s}(A) = \mathsf{s}(B), \text{ then } \frac{\models_{\mathrm{DR}} A \to B}{\models_{\mathrm{DR}} \mathrm{P}A \to \mathrm{P}B} \qquad \text{(P-CRM)}$$

These formulas with one of the rules seem to be good candidates for an axiomatization of DR. As we have seen, DR is an example of a weak modal logic, one that prevents the generation of puzzles and paradoxes, but is at the same time strong enough to permit desirable inferences in deontic logics. That is why it is a better initial, minimal deontic logic than SDL.

5.3. Extensions of logic DR

Logic DR is a starting point for further extensions of deontic logic. We can restrict the models of DR using various conditions in order to accommodate new laws that we find important. Let us introduce some of the many possible extensions on offer. We leave the problem of dependence of conditions for further examination, since subsets of the conditions together with the basic notions can be logically dependent.

5.3.1. Distribution of O

Even though we do not believe that axiom (K) should hold someone could argue the opposite. Therefore, to validate it, we might consider a class of models $\langle W, Q, v, \{\mathrm{R}_w\}_{w\in W}\rangle$, where the following condition is to be satisfied for any $A, B \in \mathsf{For}$ and $w \in W$:

$$(\mathbf{r}_w(A \to B) \text{ and } \mathbf{r}_w(A)) \Longrightarrow \mathbf{r}_w(B), \qquad \text{(Dis)}$$

Application of this condition guaranties the tautological nature of formulas having (K)'s form. Let $\mathfrak{M} = \langle W, Q, v, \{\mathrm{R}_w\}_{w\in W}\rangle$ be a model that satisfies (Dis) and $w \in W$. Assume that $\mathfrak{M}, w \models \mathrm{O}(A \to B)$ and $\mathfrak{M}, w \models \mathrm{O}A$. Then $\forall_{u\in W}(Q(w,u) \Longrightarrow (\mathfrak{M}, u \models A \to B \text{ and } \mathbf{r}_u(A \to B)))$, and $\forall_{u\in W}(Q(w,u) \Longrightarrow (\mathfrak{M}, u \models A \text{ and } \mathbf{r}_u(A)))$. Let us take any $t \in W$ and assume that $Q(w,t)$. Then $\mathfrak{M}, t \models A \to B$, $\mathbf{r}_t(A \to B)$, $\mathfrak{M}, t \models A$ and $\mathbf{r}_t(A)$. And if $\mathfrak{M}, t \models A \to B$ and $\mathfrak{M}, t \models B$, so $\mathfrak{M}, t \models B$. Furthermore, if $\mathbf{r}_t(A \to B)$ and $\mathbf{r}_t(A)$, so $\mathbf{r}_t(B)$. Therefore $\mathfrak{M}, w \models \mathrm{O}B$. If (Dis) is satisfied, then we also get $\mathrm{O}(A \leftrightarrow B) \to (\mathrm{O}A \to \mathrm{O}B)$ and $\mathrm{O}(A \wedge B) \to (\mathrm{O}A \to \mathrm{O}B)$ as tautologies, by definition 3.3.

5.3.2. Aggregation laws for O

If we want to have the laws of aggregation as tautologies, then we may consider a class of models $\langle W, Q, v, \{\mathrm{R}_w\}_{w\in W}\rangle$ such that for any $A, B \in$ For and $w \in W$:

$$(\mathbf{r}_w(A) \text{ and } \mathbf{r}_w(B)) \Longrightarrow \mathbf{r}_w(A \wedge B) \qquad (\mathrm{Ag}_1)$$
$$\mathbf{r}_w(A) \Longrightarrow \mathbf{r}_w(A \vee B) \qquad (\mathrm{Ag}_2)$$
$$\mathbf{r}_w(B) \Longrightarrow \mathbf{r}_w(A \vee B). \qquad (\mathrm{Ag}_3)$$

Again, by the definition of a deontic relationship 3.3, in the class of models such that (Ag_3) is satisfied formula $\mathrm{O}B \to \mathrm{O}(A \to B)$ is also a tautology.

5.3.3. Commutative laws for O and P

If we would like to have the commutative laws as tautologies, we can assume a class of models $\langle W, Q, v, \{\mathrm{R}_w\}_{w\in W}\rangle$ such that for any $A, B \in$ For, $w \in W$ and $* \in \{\wedge, \vee\}$:

$$\mathbf{r}_w(A * B) \Longrightarrow \mathbf{r}_w(B * A). \qquad (\mathrm{Com}*)$$

Once again by the definition of a deontic relationship 3.3, if ($\mathrm{Com}*$) is satisfied we also get $\mathrm{O}(A \leftrightarrow B) \to \mathrm{O}(B \leftrightarrow A)$ and $\mathrm{P}(A \leftrightarrow B) \to \mathrm{P}(B \leftrightarrow A)$ as tautologies.

By the considerations that we presented in context of the Good Samaritan paradox, it might be said that the main constituents of complex formulas in the scope of deontic modalities can be understood as if they were in temporal or causal relations. If we do not want to accept such an interpretation, we can assume the symmetry of the relevant deontic relations.

5.3.4. Associative laws for O and P

If we want to have the associative laws as deontic laws, we may take a class of models $\langle W, Q, v, \{\mathrm{R}_w\}_{w\in W}\rangle$ such that for any $A, B, C \in$ For, $w \in W$ and $* \in \{\wedge, \vee\}$:

$$\mathbf{r}_w(A * (B * C)) \Longrightarrow \mathbf{r}_w((A * B) * C) \qquad (\mathrm{Asc}_1*)$$
$$\mathbf{r}_w((A * B) * C) \Longrightarrow \mathbf{r}_w(A * (B * C)) \qquad (\mathrm{Asc}_2*)$$

The conditions guarantee that formulas like $\mathrm{P}(A \to (B \wedge C)) \to \mathrm{P}((A \wedge B) \to C)$ become tautologies, by definition 3.3.

6. Tableau systems of quasi- and strictly-deontic modalities

We shall now outline the tableau approach to our logics. We will be governed here by a strategy adopted in (Jarmużek, 2014) which introduced a formalized tableau theory derived from some modal logics. Let us, however, not focus too much on the formal aspects but stress instead the crucial points which determine the completeness of the tableau approach related to the semantically designated consequence relation.

For this purpose, we shall need a new language: a language of tableau proofs. As we know, a tableau proof for a logic determined by a possible world semantics is usually carried out in a set of formulas with labels that are natural numbers. We would like to call them *indexes*. However, in our context we must extend the tableau language even more. Let $\mathbb{N}$ be the set of natural numbers. A set of tableau expressions Ex is a union of the following sets:

- $\{irj : i, j \in \mathbb{N}\}$,
- $\mathsf{For} \times \mathbb{N}$,
- $\mathsf{For} \times \{i^{\circ} : i \in \mathbb{N}, \circ \in \{+, -\}\}$,
- $\{(A, B) : A, B \in \mathsf{For}\} \times \{i^{\circ} : i \in \mathbb{N}, \circ \in \{-, +\}\}$.

Where possible, we will omit the angle brackets and so, for example, instead of $\langle A, i\rangle$, we will just write A, i etc. Let us now explain what the particular expressions are intended to encode. Expressions of the form irj naturally encode in the tableau language an accessibility relation between worlds i and j. The expression A, i traditionally encodes that a formula A is true at a world denoted by i.

The remaining expressions are novel. They do not represent logical values of formulas at worlds, but generally just state that a formula in a given world i is (i^{+}) or is not (i^{-}) deontically related. So, for example, in the third case we have A, i^{+} (resp. A, i^{-}) which means A is (resp. is not) deontically related in deontic alternative i. Due to the complexity of formulas in our proofs we assume also the last case, so expressions like $(A, B), i^{+}$ (resp. $(A, B), i^{-}$) which means that A and B are (resp. are not) related by relating relation in deontic alternative i. The last two cases are the result of the definition of a deontic relationship 3.3 we formerly introduced and discussed.

Now, all tableau proofs are carried out in language Ex. *A tableau inconsistent set of expressions* (that closes a given branch) comprises at least one of the following:

1. at least one of the pairs:
   - A, i and $\neg A, i$
   - A, i^+ and A, i^-
   - $(A, B), i^+$ and $(A, B), i^-$, for some $A, B \in \mathsf{For}$ and $i \in \mathbb{N}$
2. A, i^+ and A, j^-, for some $A \in \mathsf{Var}$ and $i, j \in \mathbb{N}$.

The justification for the last point is by fact 3.4, since the relevance of propositional letters is inherited in all deontic worlds in a model.

Finally, we say that a set of tableau expressions is *tableau consistent* iff it is not a tableau inconsistent set.

6.1. Tableau rules for $\mathrm{DR}^{\Box}$ and DR

We propose a set of tableau rules for $\mathrm{DR}^{\Box}$ (and thus also for DR). Let us go to the tableau rules. For the formulas with main Boolean connectives, we shall assume the standard tableau rules. We do not need to list or elaborate on them as they have been thoroughly examined in many papers (for example here Goré, 1999; Jarmużek, 2014; Priest, 2008; Pietruszczak and Jarmużek, 2018).

The quasi-deontic modalities $\Box$ and $\Diamond$ behave like modalities in modal logic D. So we assume the following standard tableau rules for them:

$$(\mathrm{R}_{\neg\Box})\ \frac{\neg\Box A, i}{\Diamond\neg A, i} \qquad (\mathrm{R}_{\neg\Diamond})\ \frac{\neg\Diamond A, i}{\Box\neg A, i} \qquad (\mathrm{R}_{\Box})\ \frac{\Box A, i \quad irj}{A, j} \qquad (\mathrm{R}_{\Diamond})\ \frac{\Diamond A, i}{\begin{array}{l} irj \\ A, j \end{array}}$$

where rule $(\mathrm{R}_{\Diamond})$ is obviously limited by the clause that index j is new on the branch.

Since accessibility relation Q in a model is supposed to be serial, we assume also a rule:

$$(\mathrm{R}_{\mathrm{ser}})\ \frac{}{irj}$$

where i previously appeared on the branch and j is new.

Now we introduce tableau rules for the strictly-deontic modalities: O, P. The strictly-deontic modalities include an aspect of a deontic relationship. We propose the following tableau rules:

$$(\mathrm{R}_{\neg\mathrm{O}})\ \frac{\neg\mathrm{O}A, i}{\mathrm{P}\neg A, i} \qquad (\mathrm{R}_{\neg\mathrm{P}})\ \frac{\neg\mathrm{P}A, i}{\mathrm{O}\neg A, i}$$

Rules ($R_{\neg O}$) and ($R_{\neg P}$) are self-explanatory, since strictly-deontic modalities are interdefiniable in terms of negation. So we will concentrate on the introduction and explanation of the rules for O and P without negation.

$$(R_O)\ \frac{\begin{array}{l} OA, i \\ irj \end{array}}{\begin{array}{l} A, j \\ A, j^+ \end{array}} \qquad (R_P)\ \frac{PA, i}{\begin{array}{l|l} irj & irj \\ A, j & A, j^- \end{array}}$$

Rule (R_P) is obviously limited by the clause that index j is new on the branch. Rule (R_O) expresses the semantic idea that what is obligatory is not only true at any accessible deontic alternative, but is also deontically related. By contrast rule (R_P) says that what is permitted at least at one accessible deontic alternative is true or is not deontically related there. So, an application of tableau rule (R_P) generates two branches.

Now we present the rules that are responsible for a transformation of expressions that encode information about deontic relationships in deontic alternatives. We can say that taken together, they form some kind of tableau calculus of being deontically related. For the main unary connectives we have:

$$(R^\circ_*)\ \frac{*A, i^\circ}{A, i^\circ}$$

where $* \in \{\neg, \Box, \Diamond, P, O\}$ and $\circ \in \{+, -\}$. Tableau rule (R°_*) removes external unary connectives (all kinds of modalities, including negation), preserving simultaneously the fact that the proposition is (is not) in a deontic relationship. Next for the main binary connectives we have the rule:

$$(R^\circ_\star)\ \frac{A \star B, i^\circ}{(A, B), i^\circ}$$

where $\star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$ and $\circ \in \{+, -\}$. Tableau rule ($R^\circ_\star$) removes a main binary connective $\star$ in a formula, since according to 3.3 and 3.2 the deontic relationship does not depend on a main connective, but on whether its components are deontically related, after demodalization.

That is why, finally, we have a tableau rule that corresponds to the demodalization of internal expressions. Let us assume that for any formulas A and $*B$, where $* \in \{\neg, \Box, \Diamond, P, O\}$ formula $A(*B/B)$ is the

result of replacement of all occurrences of formula $*B$ in formula A by formula B. Now, we can introduce the last basic tableau rule:

$$(\mathrm{R}^{\circ}_{\mathrm{rep}*})\ \frac{(A,B),i^{\circ}}{(A(*C/C),B(*C/C)),i^{\circ}}$$

where $* \in \{\neg,\Box,\Diamond,\mathrm{P},\mathrm{O}\}$ and $\circ \in \{+,-\}$. Using tableau rule $(\mathrm{R}^{\circ}_{\mathrm{rep}*})$ we can ultimately remove all internal occurrences of modalities and negation and reveal a pure relation (or lack of it) between formulas A and B according to definition 3.3. Let us notice again that rules (R°_{*}), $(\mathrm{R}^{\circ}_{\star})$ and $(\mathrm{R}^{\circ}_{\mathrm{rep}*})$ do not transform the truth-value properties of expressions in a model, but structures of expressions that are related in given worlds which are independent of logical values at these worlds.

In the subsequent sections of the paper we will extend the set of tableau rules and prove metalogical properties of $\mathrm{DR}^{\Box}$. Of course, modalities $\Diamond,\Box$ are identical to the modalities of SDL. But the collection of the tableau rules includes rules for DR, so we have a quite general starting point, including our strictly-deontic modalities P, O, however. Consequently, we adopt the following tableau rules:

1. tableau rules for Boolean connectives,
2. tableau rules that are intended for DR: $(\mathrm{R}_{\mathrm{ser}})$ (because we still need a serial accessibility relation), $(\mathrm{R}_{\neg\mathrm{O}})$, $(\mathrm{R}_{\neg\mathrm{P}})$, $(\mathrm{R}_{\mathrm{O}})$, $(\mathrm{R}_{\mathrm{P}})$, (R°_{*}), $(\mathrm{R}^{\circ}_{\star})$, $(\mathrm{R}^{\circ}_{\mathrm{rep}*})$,
3. tableau rules for $\Diamond,\Box$: $(\mathrm{R}_{\neg\Diamond})$, $(\mathrm{R}_{\neg\Box})$, $(\mathrm{R}_{\Diamond})$, $(\mathrm{R}_{\Box})$ — they are intended for extension of DR to $\mathrm{DR}^{\Box}$.

The set of all of these tableau rules is denoted by $\mathsf{TR}^{\Box}$ (its subset is the set of tableau rules for DR — the rules from points 1 and 2, which obviously is a keynote of our paper — and can be denoted as TR).

6.2. Tableau rules for extensions of $\mathrm{DR}^{\Box}$ and DR

Here we propose some tableau rules for extensions of $\mathrm{DR}^{\Box}$ and DR given in Section 5.3. If we opt for axiom (K), we assume the rule:

$$(\mathrm{R}_{\mathrm{Dis}})\ \frac{\begin{array}{c}(A,B),i^{+}\\ A,i^{+}\end{array}}{B,i^{+}}$$

Tableau rule $(\mathrm{R}_{\mathrm{Dis}})$ enables to expresses that the second component of some formula is deontically related, if this formula and its first component also have this property.

In order to prove the aggregation laws we can adopt the following rules:

$$(\mathrm{R}_{\mathrm{Ag}_1})\ \frac{A, i^+ \quad B, i^+}{(A, B), i^+} \qquad (\mathrm{R}_{\mathrm{Ag}_2})\ \frac{A, i^+}{(A, B), i^+} \qquad (\mathrm{R}_{\mathrm{Ag}_3})\ \frac{B, i^+}{(A, B), i^+}$$

In case of rule $(\mathrm{R}_{\mathrm{Ag}_2})$ and $(\mathrm{R}_{\mathrm{Ag}_3})$ formula B and A respectively, previously appeared on the branch as subformulas. Such a restriction allows us to make some tableau proofs shorter, since we can use formulas which have already been introduced in the proof.

The rule for the commutativity law is of the following form:

$$(\mathrm{R}_{\mathrm{Com}})\ \frac{(A, B), i^+}{(B, A), i^+}$$

Using tableau rule $(\mathrm{R}_{\mathrm{Com}})$ we can easily express using the tableau approach, the symmetry of the deontic relationship, which obviously corresponds to the idea of commutativity.

Finally we have two rules for the associative laws:

$$(\mathrm{R}_{\mathrm{Asc}_1 *})\ \frac{(A, (B * C)), i^+}{((A * B), C), i^+} \qquad (\mathrm{R}_{\mathrm{Asc}_2 *})\ \frac{((A * B), C), i^+}{(A, (B * C)), i^+}$$

where $* \in \{\wedge, \vee\}$. Tableau rules $(\mathrm{R}_{\mathrm{Asc}_1 *})$ and $(\mathrm{R}_{\mathrm{Asc}_2 *})$ enable us to rearrange brackets in case of formulas build by binary operators according to the associative laws.

6.3. Completeness result

For simplicity's sake, let us call the expressions in a tableau rule numerator *input*, while those in denominator *output*. Some rules, e.g. $(\mathrm{R}_{\mathrm{P}})$ and some of those for the Boolean connectives (like negation of conjunction etc.) may have more than one output and they generate more than one branch.

Let us now introduce two concepts which are important for the tableau issues.

DEFINITION 6.1 (Set of indexes). Let $X \subseteq \mathsf{Ex}$. By function $\mathsf{Ind}\colon X \longrightarrow \wp(\mathbb{N})$ we mean a mapping satisfying condition, for all $i, j \in \mathbb{N}$, $A, B \in \mathsf{For}$ and $\circ \in \{-, +\}$:

- if $X = \{irj\}$ then $\mathsf{Ind}(X) = \{i, j\}$,
- if $X = \{\langle A, i\rangle\}$ then $\mathsf{Ind}(X) = \{i\}$,

- if $X = \{\langle A, i^{\circ}\rangle\}$ then $\mathsf{Ind}(X) = \{i\}$,
- if $X = \{\langle (A, B), i^{\circ}\rangle\}$ then $\mathsf{Ind}(X) = \{i\}$,
- $\mathsf{Ind}(X) = \bigcup\{\mathsf{Ind}(y) \colon y \in X\}$.

Function Ind collects indexes contained in expressions from a given subset of Ex.

Now, we can extend in a certain sense the concept of truth in a model from the formulas to all expressions from Ex.

Definition 6.2 (Model suitable to a set of expressions). Let $\mathfrak{M} = \langle W, Q, \{\mathrm{R}_w\}_{w\in W}, v\rangle$ be a model and $X \subseteq \mathsf{Ex}$. Model $\mathfrak{M}$ is *suitable to* X iff there exists a function f from the set of indexes contained in expressions from X to W, i.e. $f \colon \mathsf{Ind}(X) \longrightarrow W$ such that, for any $A, B \in \mathsf{For}$ and $i, j \in \mathbb{N}$:

- if $irj \in X$ then $Q(f(i), f(j))$,
- if $\langle A, i\rangle \in X$ then $\mathfrak{M}, f(i) \models A$,
- if $\langle A, i^{+}\rangle \in X$ then $\mathbf{r}_{f(i)}(A)$,
- if $\langle A, i^{-}\rangle \in X$ then $\sim\mathbf{r}_{f(i)}(A)$,
- if $\langle (A, B), i^{+}\rangle \in X$ then $\mathrm{R}_{f(i)}(\mathbf{d}(A), \mathbf{d}(B))$,
- if $\langle (A, B), i^{-}\rangle \in X$ then $\sim\mathrm{R}_{f(i)}(\mathbf{d}(A), \mathbf{d}(B))$.

Making use of the concept of a suitable model and conducting an inspection of the provided tableau rules, we are able to demonstrate that if model $\mathfrak{M}$ of a given type, fulfilling some of conditions introduced in Section 5.3 is suitable for a set of expressions $X \subseteq \mathsf{Ex}$, then the application of a selected tableau rule for the relevant conditions extends the set X with new expressions for which $\mathfrak{M}$ is still suitable.

Let $\mathcal{C}$ be a set of conditions introduced in Section 5.3 and $\mathsf{T}_{\mathcal{C}}$ be a set of tableau rules introduced in Section 6.2. For convenience with formulation of the further theorems, we introduce function $\tau \colon \mathcal{C} \to \mathsf{T}_{\mathcal{C}}$ such that for any $(X) \in \mathcal{C}$ we put $\tau((X)) = (\mathrm{R}_X)$. For example, rules $(\mathrm{R}_{\mathrm{Dis}})$, $(\mathrm{R}_{\mathrm{Com}*})$ are assigned, respectively, to conditions (Dis) and (Com$*$), we have $\tau(\text{(Dis)}) = (\mathrm{R}_{\mathrm{Dis}})$ and $\tau(\text{(Com}*\text{)}) = (\mathrm{R}_{\mathrm{Com}*})$, etc.

Let us establish the following:

Lemma 6.3. *Let*:

- $X \subseteq \mathsf{Ex}$,
- $\mathfrak{M} = \langle W, Q, v, \{\mathrm{R}_w\}_{w\in W}\rangle$ *be a model satisfying conditions contained in some* $Y \subseteq \mathcal{C}$,
- $\mathfrak{M}$ *be suitable to* X.

If any tableau rule from $\mathsf{TR}^{\Box}$ *or* $\tau(Y)$ *has been applied to set* X*, then* $\mathfrak{M}$ *is suitable to the union of* X *and at least one output obtained through the application of this rule.*

PROOF. Assume all the hypotheses. Let $f\colon \mathsf{Ind}(X) \longrightarrow W$ be a function as in definition 6.2. For cases of applications of rules from set $\mathsf{TR}^{\Box}$, except rules $(\mathrm{R_O})$, $(\mathrm{R_{\neg O}})$ $(\mathrm{R_P})$, $(\mathrm{R_{\neg P}})$, (R°_{*}), $(\mathrm{R}^{\circ}_{\star})$ and $(\mathrm{R}^{\circ}_{\mathrm{rep}*})$, the proof is standard (see Goré, 1999; Jarmużek, 2014; Priest, 2008; Pietruszczak and Jarmużek, 2018).

Suppose $(\mathrm{R_O})$ has been applied to X. Then $\langle \mathrm{O}A, i\rangle \in X$ and $irj \in X$ and we get output $\langle A, j\rangle$, $\langle A, j^{+}\rangle$. Since the model is suitable to X, we have $\mathfrak{M}, f(i) \models \mathrm{O}A$ and $Q(f(i), f(j))$. Thus, by definition 3.5, also for the output $\mathfrak{M}, f(j) \models A$ and $\mathbf{r}_j(A)$.

Suppose $(\mathrm{R_{\neg O}})$ has been applied to X. Then $\langle \neg\mathrm{O}A, i\rangle \in X$ and we get output $\langle \mathrm{P}\neg A, i\rangle$. Since the model is suitable to X, we have $\mathfrak{M}, f(i) \models \neg\mathrm{O}A$. Thus, by definitions 3.5 and 3.3, $Q(f(i), j)$ and $\mathfrak{M}, j \models \neg A$ or $\sim\mathbf{r}_j(\neg A)$, for some $j \in W$. Hence, by definition 3.5, $\mathfrak{M}, f(i) \models \mathrm{P}\neg A$.

Suppose $(\mathrm{R_P})$ has been applied to X. Then $\langle \mathrm{P}A, i\rangle \in X$ and we get two outputs irj, $\langle A, j\rangle$ and irj, $\langle A, j^{+}\rangle$, where j is a new index. Since the model is suitable to X, we have $\mathfrak{M}, f(i) \models \mathrm{P}A$. Thus, by definition 3.5, $Q(f(i), u)$ and either $\mathfrak{M}, u \models A$ or $\sim\mathbf{r}_u(A)$, for some $u \in W$. Let $g\colon \mathsf{Ind}(X) \cup \{j\} \longrightarrow W$ be a function such that, for any $x \in \mathsf{Ind}(X) \cup \{j\}$ we put:

$$g(x) = \begin{cases} f(x), & \text{if } x \neq j \\ u, & \text{if } x = j. \end{cases}$$

Hence, for the outputs, $Q(g(i), g(j))$ and we have that either $\mathfrak{M}, g(j) \models A$ or $\sim\mathbf{r}_{g(j)}(A)$.

Suppose $(\mathrm{R_{\neg P}})$ has been applied to X. Then $\langle \neg\mathrm{P}A, i\rangle \in X$ and we get output $\langle \mathrm{O}\neg A, i\rangle$. Since model is suitable to X, we have $\mathfrak{M}, f(i) \models \neg\mathrm{P}A$. Thus, by definition 3.5, if $Q(f(i), j)$, then $\mathfrak{M}, j \models \neg A$ and $\mathbf{r}_j(\neg A)$, for any $j \in W$. Hence, for the output by definition 3.5 and by the definition of a deontic relationship 3.3, $\mathfrak{M}, f(i) \models \mathrm{O}\neg A$.

Suppose (R°_{*}) (resp. $(\mathrm{R}^{\circ}_{\star})$), where $* \in \{\neg, \Box, \Diamond, \mathrm{O}, \mathrm{P}\}$ (resp. $\star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$) and $\circ \in \{+, -\}$, has been applied to X. Then $\langle *A, i^{\circ}\rangle \in X$ (resp. $\langle A \star B, i^{\circ}\rangle \in X$) and we get output $\langle A, i^{\circ}\rangle$ (resp. $\langle (A, B), i^{\circ}\rangle$). Since model is suitable to X, we have $\mathbf{r}_{f(i)}(*A)$ (resp. $\mathbf{r}_{f(i)}(A \star B)$), if $\circ = +$ and $\sim\mathbf{r}_{f(u)}(*A)$ (resp. $\sim\mathbf{r}_{f(i)}(A \star B)$), if $\circ = -$. By the definition of a deontic relationship 3.3, $\mathbf{r}_{f(i)}(A)$ iff $\mathbf{r}_{f(i)}(\mathbf{d}(A))$, for any $A \in \mathsf{For}$

(resp. $\mathbf{r}_{f(i)}(A \star B)$ iff $\mathrm{R}_{f(i)}(\mathbf{d}(A), \mathbf{d}(B))$, for any $A, B \in \mathsf{For}$). Thus, for output, $\mathbf{r}_{f(i)}(A)$ (resp. $\mathrm{R}_{f(i)}(\mathbf{d}(A), \mathbf{d}(B))$, if $\circ = +$ and $\sim \mathbf{r}_{f(i)}(A)$ (resp. $\sim \mathrm{R}_{f(i)}(\mathbf{d}(A), \mathbf{d}(B))$), if $\circ = -$.

Suppose $(\mathrm{R}^{\circ}_{\mathrm{rep}*})$, where $* \in \{\neg, \Box, \Diamond, \mathrm{O}, \mathrm{P}\}$ and $\circ \in \{+, -\}$, has been applied to X. Hence $\langle (A, B), i^{\circ} \rangle \in X$ and we get output $\langle (A(*C/C), B(*C/C)), i^{\circ} \rangle$. Since model is suitable to X, $\mathrm{R}_{f(i)}(\mathbf{d}(A), \mathbf{d}(B))$, if $\circ = +$ and $\sim \mathrm{R}_{f(i)}(\mathbf{d}(A), \mathbf{d}(B))$, if $\circ = -$. Notice that, by definition 3.2, $\mathbf{d}(A) = \mathbf{d}(A(*C/C))$. Thus, for the output, $\mathrm{R}_{f(i)}(\mathbf{d}(A), \mathbf{d}(B))$, if $\circ = +$ and $\sim \mathrm{R}_{f(i)}(\mathbf{d}(A(*C/C)), \mathbf{d}(B(*C/C)))$, if $\circ = -$.

Suppose that a tableau rule $(\mathrm{R}_C) \in \tau(Y)$ has been applied to X and $\mathfrak{M}$ is such that condition (C) is satisfied. Hence, the input of (R_C) must be in X and $\mathfrak{M}$ is suitable to this input. Thus, by condition (C), $\mathfrak{M}$ must be also suitable to an output of (R_C). Let us consider two examples, since the remaining ones are similar.

Suppose $(\mathrm{R}_{\mathrm{Dis}})$ has been applied to X. Hence $\langle (A, B), i^{+} \rangle \in X$, $\langle A, i^{+} \rangle \in X$ and we have got output $\langle B, i^{+} \rangle$. Since the model is suitable to X, $\mathrm{R}_{f(i)}(\mathbf{d}(A), \mathbf{d}(B))$ and $\mathbf{r}_{f(i)}(A)$. Hence, for output by condition (Dis), $\mathbf{r}_{f(i)}(B)$.

Suppose $(\mathrm{R}_{\mathrm{Ag}_1})$ has been applied to X. Hence $\langle A, i^{+} \rangle \in X$, $\langle B, i^{+} \rangle \in X$ and we have got output $\langle (A, B), i^{+} \rangle$. Since the model is suitable to X, $\mathbf{r}_{f(i)}(A)$ and $\mathbf{r}_{f(i)}(B)$. Hence, for output by condition (Ag_1), $\mathbf{r}_{f(i)}(A \wedge B)$. By the definition of a deontic relationship 3.3 $\mathrm{R}_{f(i)}(\mathbf{d}(A), \mathbf{d}(B))$. ⊣

The proof of completeness of our tableau methods in relation to the presented semantics still requires a fact in the opposite direction. Let us introduce the concept of a model produced by a set of expressions.

Definition 6.4 (Model generated by a branch). Let $X \subseteq \mathsf{Ex}$. Set $AT(X)$ is defined as follows: $x \in AT(X)$ iff one of the following conditions holds:

- $x \in X \cap \{irj \colon i, j \in \mathbb{N}\}$,
- $x \in X \cap (\mathsf{Var} \times \mathbb{N})$,
- $x \in X \cap \mathsf{Var} \times \{i^{\circ} : i \in \mathbb{N}, \circ \in \{+, -\}\}$,
- $x \in X \cap \{\langle (A, B), i^{+} \rangle \colon A, B \in \mathsf{For}, i \in \mathbb{N}\}$.

Model $\langle W, Q, v, \{\mathrm{R}_w\}_{w \in W} \rangle$ is *generated by* X iff

1. $W = \{i \colon i \in \mathsf{Ind}(AT(X)\} \cup \{\omega\}$, where $\omega \notin \mathbb{N}$,
2. Q is the smallest relation Y such that $\{\langle i, j \rangle : irj \in AT(X)\} \subseteq Y$ and $\langle \omega, \omega \rangle \in Y$,

3. for every $A \in \mathsf{Var}$, $i \in W$: $i \in v(A)$ iff $\langle A, i\rangle \in AT(X)$,
4. for every $i \in W$, R_i is a minimal relation that satisfies the conditions:
   a. $\{\langle \mathbf{d}(A), \mathbf{d}(B)\rangle : \langle (A, B), i^+\rangle \in AT(X)\} \subseteq \mathrm{R}_i$,
   b. for every $j \in W$, if for some $x \in \mathsf{Var}$, $\langle x, j^+\rangle \in AT(X)$, then there is such $B \in \mathsf{For}$ that $\mathrm{R}_i(x, \mathbf{d}(B))$ or $\mathrm{R}_i(\mathbf{d}(B), x)$,
5. for every $i \in W$, if for some $x \in \mathsf{Var}$, $\langle x, i^-\rangle \in AT(X)$, then for no $B \in \mathsf{For}$: $\mathrm{R}_\omega(x, \mathbf{d}(B))$ or $\mathrm{R}_\omega(\mathbf{d}(B), x)$.

Let us cast a bit of light on the definition of generated model 6.4. By the minimality of R_i, if such a model exists, there is only one such model.

Point 1 of the definition indicates a set of worlds that is determined by indexes that occur in a branch, but with one exception. We add world ω to set W to have a world at which we can verify the fact that $\langle x, i^-\rangle$, for $x \in \mathsf{Var}$, if it is needed. We will examine the details about ω and $\langle x, i^-\rangle$ later.

Point 2 without doubt defines the accessibility relation Q. Also point 3 is natural: only the propositional letters that occur in X with an index i are true at world i in the model.

However, we must consider some nuances concerning relation R_i. We see it must be a minimal relation that fulfills some constraints.

According point 4a, for all worlds i in the model, relation R_i collects all such pairs $\langle \mathbf{d}(A), \mathbf{d}(B)\rangle$ that $\langle (A, B), i^+\rangle$ belongs to X. This also seems self-explanatory. A more controversial condition may be point 4b, though. If for some $x \in \mathsf{Var}$, $\langle x, j^+\rangle \in AT(X)$, then there must be such formula $B \in \mathsf{For}$ that $\mathrm{R}_i(x, \mathbf{d}(B))$ or $\mathrm{R}_i(\mathbf{d}(B), x)$. Notice that it is possible under some condition.

Assume that some $x \in \mathsf{Var}$, $\langle x, j^+\rangle \in AT(X)$. If there is such formula $B \in \mathsf{For}$ that $\langle (x, \mathbf{d}(B)), i^+\rangle \in X$ or $\langle (\mathbf{d}(B), x), i^+\rangle \in X$, then point 4b is satisfied by 4a. However, if there is no such formula, we must take some B that is not a member of X. It is absolutely possible if X is a finite set (or at least finite in a sense we will define few paragraphs later: formula-finite). Then we take any formula B that is not a part of any tableau expression in X as a subformula and we define $\mathrm{R}_i(x, \mathbf{d}(B))$. Consequently, point 4b is satisfied, but let us observe that if some formula B is not employed in the expressions that are in X (it is not necessary, but sufficient).

Finally, point 5 says that if $\langle x, i^-\rangle \in AT(X)$, for $x \in \mathsf{Var}$, then we take world ω and put that for no $B \in \mathsf{For}$: $\mathrm{R}_\omega(x, \mathbf{d}(B))$ or $\mathrm{R}_\omega(\mathbf{d}(B), x)$,

which means letter x is not deontically related in world i. It may happen that world ω is unneeded since for some other world letter x might be not related to any formula. However, world ω generally make us sure that the condition for propositional letters from definition 3.3 is not satisfied.

Before we proceed we must explain one thing. Since we have a serial relation of accessibility and as a consequence tableau rule ($\mathrm{R}_{\mathrm{ser}}$), so no complete tableau proof can be of a finite length. Nevertheless, even infinite proofs can include a finite number of formulas from For as parts of more complex expressions. For formal reasons we will introduce a special function that picks out all formulas that are parts of any tableau expression.

Definition 6.5. Let $X \subseteq \mathsf{Ex}$. By function $\phi\colon X \longrightarrow \wp(\mathsf{For})$ we mean a mapping satisfying condition, for all $i, j \in \mathbb{N}$, $A, B \in \mathsf{For}$ and $\circ \in \{-, +\}$:

- if $X = \{irj\}$, then $\phi(X) = \emptyset$,
- if $X = \{\langle A, i\rangle\}$ then $\phi(X) = \{A\}$,
- if $X = \{\langle A, i^{\circ}\rangle\}$ then $\phi(X) = \{A\}$,
- if $X = \{\langle (A, B), i^{\circ}\rangle\}$ then $\phi(X) = \{A, B\}$,
- $\phi(X) = \bigcup\{\phi(y)\colon y \in X\}$.

Function ϕ collects formulas contained in expressions from a given subset of Ex.

Now, let $X \subset \mathsf{Ex}$. We define X to be a *formula-finite* set of expressions iff $\phi(X)$ is a finite subset of For. It is obvious that any $Y \subseteq \mathsf{Ex}$ may be at the same time infinite as well as a formula-finite set of expressions.

Assume that we have a set of tableau rules $\mathsf{T} = \mathsf{TR}^{\Box} \cup \tau(Y)$, for some set of conditions $Y \subseteq \mathcal{C}$. $\mathsf{TR}^{\Box}$. Now, if we take a formula-finite set of expressions $X \subseteq \mathsf{Ex}$ such that:

- it is closed under all rules from T, for all expressions from X to which one of the rules is applicable, there exists at least one output in X,
- X is a tableau consistent set of expressions,

then there is a model $\mathfrak{M}$ generated by set X. It is a model for $\mathrm{DR}^{\Box}$, but we will show how to convert it into a model for conditions $Y \subseteq \mathcal{C}$. Therefore, we have one more proposition.

Lemma 6.6. *Let X be such a formula-finite subset of* Ex *that*:

- *X is a tableau consistent set*
- *X is closed under $\mathsf{TR}^{\Box} \cup \tau(Y)$, for some set of conditions $Y \subseteq \mathcal{C}$.*

Then there is a model $\mathfrak{N}$ *such that*:

1. *for any* $\langle A, i\rangle \in X \cap (\mathsf{For} \times \mathbb{N})$, $\mathfrak{N}, i \models A$,
2. *model* $\mathfrak{N}$ *is such that conditions contained in* Y *are satisfied.*

PROOF. Assume all the hypotheses. Since X is tableau consistent and formula-finite, so there is a generated model $\mathfrak{M} = \langle W, Q, v, \{\mathrm{R}_w\}_{w\in W}\rangle$, according to definition 6.4.

First, notice that for all $i \in W$, for all propositional letters $A \in \mathsf{Var}$:

(a1) if expression $\langle A, i^+\rangle$ belongs to X, then $\mathbf{r}_i(A)$
(a2) if expression $\langle A, i^-\rangle$ belongs to X, then $\sim \mathbf{r}_i(A)$.

We have this by the definition of a generated model 6.4 and definition of a deontic relationship 3.3. Also for all $i \in W$, A, $B \in \mathsf{For}$, if an expression:

(b1) $\langle (A, B), i^+\rangle$ belongs to X, then $R_i(\mathbf{d}(A), \mathbf{d}(B))$
(b2) $\langle (A, B), i^-\rangle$ belongs to X, then $\sim R_i(\mathbf{d}(A), \mathbf{d}(B))$,

which is a consequence of the definition of a generated model 6.4.

Now we can consider any formula $A \in \mathsf{For}$, such that for some $i \in W$, $\langle A, i^+\rangle \in X$.

If $A \in \mathsf{Var}$, then $\mathbf{r}_i(A)$. If A is more complex, then since X is closed under tableau rules (so among others rules: (R°_*), $(\mathrm{R}^\circ_\star)$ and $(\mathrm{R}^\circ_{\mathrm{rep}*})$), A is reduced to a propositional letter or expression $\langle (\mathbf{d}(B), \mathbf{d}(C)), i^+\rangle$. But in both cases $\mathbf{r}_i(A)$, by (a1) or (b1) respectively. In a similar way we prove that $\sim \mathbf{r}_i(A)$ for any $A \in \mathsf{For}$, such that $i \in W$ and $\langle A, i^-\rangle \in X$. So, finally we state that for any $A \in \mathsf{For}$ and $i \in W$:

(α^+) if $\langle A, i^+\rangle$ belongs to X, then $\mathbf{r}_i(A)$
(α^-) if $\langle A, i^-\rangle$ belongs to X, then $\sim \mathbf{r}_i(A)$.

Now, we can conduct a proof by induction on the complexity of expressions contained in X.

The initial step is for letters and negated letters. Let $\langle A, i\rangle \in X \cap (\mathsf{Var} \times W)$, for some $i \in \mathbb{N}$. Then by the definition of v in a generated model 6.4, $i \in v(A)$, and so $\mathfrak{M}, i \models A$. Let $\langle \neg A, i\rangle \in X \cap (\mathsf{For} \times W)$, for some $i \in \mathbb{N}$, where $A \in \mathsf{Var}$. Since X is tableau consistent, so $\langle A, i\rangle \notin X$. Then, by definition 6.4, we have $i \notin v(A)$; and so $\mathfrak{M}, i \not\models A$. In consequence $\mathfrak{M}, i \models \neg A$.

The inductive step. We assume that for any expression $\langle A, i\rangle \in X \cap (\mathsf{For} \times W)$, where A is of the complexity n, for some $n \in \mathbb{N}$, $\mathfrak{M}, i \models A$. We will consider cases of formula B of the complexity $n + 1$.

Again the cases for classical connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ and $\Box$, $\Diamond$, $\neg\Box$, $\neg\Diamond$ are obvious and have been thoroughly examined (see Goré, 1999; Jarmużek, 2014; Priest, 2008; Pietruszczak and Jarmużek, 2018). So we concentrate on new cases for: O, P, $\neg$O, and $\neg$P and the rules for the additional conditions.

Let $B := \langle \mathrm{O}A, i\rangle \in X \cap (\mathsf{For} \times W)$, for some $A \in \mathsf{For}$ and $i \in \mathbb{N}$. Since X is closed under tableau rules $\mathsf{TR}^{\Box}$, it is also closed under ($\mathrm{R}_{\mathrm{ser}}$). Hence, there is at least one $j \in \mathbb{N}$ such that $irj \in X$. So by (R_{O}) the set of all expressions of the form: $\langle A, k\rangle$ and $\langle A, k^{+}\rangle$ that belong to X, for some $irk \in X$, is non-empty. By the construction of a model, $Q(i, k)$, 6.4, for any such $k \in \mathbb{N}$, that $irk \in X$. However, since by the induction hypothesis $\mathfrak{M}, k \models A$ and $\mathbf{r}_k(A)$ by (α^{+}), therefore $\mathfrak{M}, i \models \mathrm{O}A$.

Let $B := \langle \mathrm{P}A, i\rangle \in X \cap (\mathsf{For} \times W)$, for some $A \in \mathsf{For}$ and $i \in \mathbb{N}$. Since X is closed under tableau rules $\mathsf{TR}^{\Box}$, it is also closed under (R_{P}). Hence, to X belongs (1) either a pair irj, $\langle A, j\rangle$, or (2) a pair irj, $\langle A, j^{-}\rangle$, where $j \in \mathbb{N}$. In both cases we have $Q(i, j)$, by the construction of a model 6.4. If (1) happens, then by the induction hypothesis $\mathfrak{M}, j \models A$, and $\mathfrak{M}, i \models \mathrm{P}A$. If (2) happens, then $\sim \mathbf{r}_j(A)$ by (α^{-}), and $\mathfrak{M}, i \models \mathrm{P}A$.

Let $B := \langle \neg\mathrm{O}A, i\rangle \in X \cap (\mathsf{For} \times W)$, for some $A \in \mathsf{For}$ and $i \in \mathbb{N}$. Since X is closed under tableau rules $\mathsf{TR}^{\Box}$, it is also closed under ($\mathrm{R}_{\neg\mathrm{O}}$), then to X belongs $\langle \mathrm{P}\neg A, i\rangle$. By Aristotelian law 5.2, $\neg\mathrm{O}A$ is logically equivalent to $\mathrm{P}\neg A$. But formula $\neg A$ is of complexity n. So, with reference to case we considered for $\langle \mathrm{P}\neg A, i\rangle$, for any $i \in W$, we have $\mathfrak{M}, i \models \mathrm{P}\neg A$, and so for $\mathfrak{M}, i \models \neg\mathrm{O}A$. Analogous reasoning applies to the case $B := \langle \neg\mathrm{P}A, i\rangle \in X \cap (\mathsf{For} \times W)$, for some $A \in \mathsf{For}$ and $i \in \mathbb{N}$.

The model we have generated is sound with respect to the empty set of conditions $Y \subseteq \mathcal{C}$. However, Y may be non-empty. For example, we assume that Y includes (Dis) or (Ag_2). The remaining cases are similar.

Let us analyse the case (Dis). Let $\mathbf{r}_i(A \rightarrow B)$ and $\mathbf{r}_i(A)$, for some $i \in W$ in model $\mathfrak{M}$. So, by the construction of model 6.4 and the definition of a deontic relationship 3.3, $\langle (A, B), i^{+}\rangle$ and $\langle A, i^{+}\rangle \in X$. Since X is closed under tableau rule ($\mathrm{R}_{\mathrm{Dis}}$), so $\langle B, i^{+}\rangle \in X$, and by (α^{+}), $\mathbf{r}_i(B)$, which means that (Dis) is satisfied by $\mathfrak{M}$.

Now, we examine case of (Ag_2). Let $\mathbf{r}_i(A)$, for some $i \in W$ in model $\mathfrak{M}$. So, by the construction of model 6.4 and the definition of a deontic relationship 3.3, $\langle A, i^{+}\rangle \in X$. If we take such a formula B that $B \in \phi(X)$ then by ($\mathrm{R}_{\mathrm{Ag}_2}$), $\langle (A, B), i^{+}\rangle \in X$, and by the construction of model 6.4 and the definition of a deontic relationship 3.3, $\mathbf{r}_i(A \vee B)$. If we take such a formula B that $B \notin \phi(X)$, then we just add to R_i in the model

pair $\langle A, B\rangle$, which by the definition of a deontic relationship 3.3 means that $\mathbf{r}_i(A \vee B)$. The extension of relation R_i converts model $\mathfrak{M}$ into new model $\mathfrak{N}$. Model $\mathfrak{N}$ satisfies the thesis of our lemma. ⊣

We have finally arrived a theorem on the completeness of tableau and relating semantics for the deontic models we have discussed.

THEOREM 6.7 (Completeness theorem). *Let* $Y \subseteq \mathcal{C}$. *Let* $\models \subseteq \mathsf{P}(\mathsf{For}) \times \mathsf{For}$ *be the consequence relation defined by a class of models designated by the set of conditions* Y. *Then for any* $X \subseteq \mathsf{For}$, $A \in \mathsf{For}$ *the following facts are equivalent*:

1. $X \models A$,
2. *there is a finite subset* $Z \subseteq X$ *and index* $i \in \mathbb{N}$ *such that each closure of the set* $\{\langle B, i\rangle : B \in Z \cup \{\neg A\}\}$ *under the set of tableau rules* $\mathsf{TR}^{\Box} \cup \tau(Y)$ *is a tableau inconsistent set of expressions.*

PROOF. Let us adopt the assumptions. In the proof of the theorem, we make use of the prior propositions. For (1) ⇒ (2), Lemma 6.6 is sufficient. In turn, for (2) ⇒ (1), Lemma 6.2 is sufficient. ⊣

6.4. Termination problem

One of the goals of our work was to present the method of automated reasoning for the logic DR. In the paper we have introduced a tableau system to determine whether a given formula is a tautology. However, as we noted above, the straightforward applications of the defined rules do not exclude infinite tableau proofs. Nevertheless, due the inclusion DR ⊆ D tableau methods accepted for the logic D — in the case where the problem of infinite proofs does not occur — can be appropriately used for the logic DR. This is because: (a) from the point of view of possible world semantics the frames of DR are identical to the frames of D, (b) relation R in a given world always requires a finite decomposition of formulas only in that world, since the formulas consist of finite number of subformulas. So the problem of termination is reduced to the strictly modal part, which for logic D is obviously solved.

This also applies to the problem of computation analysis provided by tableaus which, however, goes beyond the scope of our work since it demands a computational interpretation of proofs (cf. Goré, 1999, p. 305).

Regardless of an adaptation of well-known tableau methods used in the analysis of the logic D, below we define modifications of rules that are responsible for infinite proofs. Of course, the problem appears only in the case of open complete tableaus.

First we modify the rule for seriality. We assume the following modification:

$$(\mathrm{R}_{\mathrm{ser}})\ \frac{A, i}{irj}$$

where j is new and for any $k \in \mathbb{N}$, irk did not previously appear on the branch.[8]

Let us notice that having made this modification, a model read off from a complete, open branch might be unsuitable. But it is not a problem. We can expand the model, assuming that a world with no accessible world has access to itself.

Similarly, we can modify the elimination rules for operators P and $\Diamond$ by assuming the analogous modification:

$$(\mathrm{R}_{\mathrm{P}})\ \frac{\mathrm{P}A, i}{\begin{array}{c|c} irj & irj \\ A, j & A, j^{-} \end{array}} \qquad (\mathrm{R}_{\Diamond})\ \frac{\Diamond A, i}{\begin{array}{c} irj \\ A, j \end{array}}$$

where j is new and for any $k \in \mathbb{N}$, irk, A, k together did not previously appear on the branch.[9]

7. A handful of intuitions to sum up

In our proposed approach, the formulas beyond the reach of modalities O and P behave purely extensionally, in accordance with their Boolean meaning. But their inclusion in the scope of the obligation and permission operators somewhat alters this meaning. They become more intensional — their truth or falsity turns on more than just the logical values they possess in appropriate possible worlds. Another important thing is the component of the content relation with the deontic system under which we predicate obligations or permissions.

[8] This modification was suggested in (Jarmużek, 2014, 2013). Note that in all modifications discussed here the clauses refer to branches, since some metatheory of tableau proofs in the cited works has been developed.

[9] The modification for $\Diamond$ was first proposed in (Jarmużek, 2014, 2013) and it has been rewritten for P here.

In turn, operators $\Box$ and $\Diamond$ in no way alter the meanings of formulas that occur within their scope. While at the same time we expect this from these formulas. Most (perhaps all?) deontic paradoxes stem from the fact that the extensional functors, formally speaking, at the philosophical level are intensional. We can resolve these problems by introducing strictly-deontic modalities.

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