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Sorites, Curry and Suitable Models

Abstract. In this paper we present two new approaches for dealing with semantic paradoxes and soritical predicates based on fuzzy logic. We show that both of them have conceptual advantages over the more traditional Łukasiewicz approach, and that the second one even avoids standard proofs of ω-inconsistency.

Keywords: paradoxes; vagueness; truth; ω-inconsistency; Łukasiewicz logic

1. Introduction

Usually, different kinds of logics are used to deal with semantic paradoxes on the one and with the sorites paradox on the other. The main reason usually raised against this discrepancy has to do with the *Principle of Uniform Solution* [19], which states that kindred paradoxes should be solved the same way. In this vein, for instance Priest in [21, 20] argues that the Liar and vagueness-related paradoxes share the same form. More specifically, these antinomies fit the so-called *Inclosure Schema* (see [19] for details) and so he proposes to deal with both of them by adopting a dialetheic solution.

There are also some other authors who, even though believe that these paradoxes should be considered two different kinds of phenomena, they still sympathize with the need to block them by rejecting the same classical principles in both cases — for instance, Field [9], who abandons

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Excluded Middle, and some conditional contraction, or Cobreros et al. [5], who abandon the structural rule of Cut.[1]

Our proposal fits better with this second group, in that we also believe that the sorites and the semantic paradoxes may cause the failure of the same principles, although for different reasons. But, contrary to them, we think this difference should somehow be reflected in the semantics of the language. Thus, in this paper we propose a model theory suitable for dealing non-trivially with both vague vocabulary and which is rich enough to distinguish between them.

Our starting point will be one of the most prominent candidates in the literature, which is the infinitely many-valued Łukasiewicz logic, given that it is indeed possible to build a consistent theory of truth over this logic (as proved by [23] and [13]), and also to provide a nice explanation for the cases of vagueness. However, consistency is not enough for a theory to be completely satisfactory. Even though the theory has a model, it lacks the intended one, in what concerns the semantic part of the language. On the one hand, the theory is ω-inconsistent, and on the other one, some paradoxical sentences have to necessarily be considered truer/falser than others — which we find undesirable. Nonetheless, we still think that Łukasiewicz logic provides a good solution to the sorites and the phenomenon of vagueness related with it.

Thus, we will propose a subsystem of the infinitely many-valued Łukasiewicz logic, *Type-2 Interval Fuzzy Logic* (**T2**) (introduced by [7]). These models, we think, fare better as they allow finer distinctions regarding the assignment of truth values to paradoxical sentences. At the same time, we get to keep the solution to vagueness as an inheritance from Łukasiewicz logic. However, we still have the problem of ω-inconsistency. To resolve this issue, we will define a new concept which is that of a suitable model as a refinement of the concept of intended interpretation. We will show that although the theory of truth and vagueness based on **T2** cannot have the latter, it does have the former. So, we consider this system to be a good step forward and we will end by sketching some possible routes to avoid these problems.

It is worth mentioning that we here take a semantic standpoint. Proof-theoretic approaches, which are quite common when dealing with

[1] Although it is not the aim of this paper to argue against these theories, it is worth noting that there is no definitive consensus in favor of any of them. In other words, here we adopt a positive attitude, putting the criticism aside.

the Liar and related paradoxes, are not as useful in some sub-classical first order theories, when no complete axiomatization is possible (Łukasiewicz logic is Π_2). Algebraic approaches, on the other hand, can be very fruitful from a technical point of view, but sometimes there is no structure that can be regarded as an intuitive semantics for the language. However, we believe that formal semantics can shed philosophical light over natural language, and that this is especially important in the case of formalising vague vocabulary. Hence, when we talk about a given logic, we will mean the arguments determined by a relation between models of some kind, for a given language.

We assume familiarity with the use of corner quotes $\ulcorner\ \urcorner$, overline notation to indicate that $\overline{n}$ is the numeral of n, and with dot notation (see [14], for details).

The rest of this paper is organized as follows: in Section 2, we present the paradoxes we want to tame. In Section 3, we introduce the approach based on Łukasiewicz logic and we argue against it. In Section 4, we propose a new approach based on Interval-Type-2 Fuzzy Logic (**T2**), and then we prove the limitative result of ω-inconsistency. We end by suggesting another theory based on Interval-Type-2 Fuzzy Logic (**T2***) that, in virtue of its conditional, avoids the usual proof of ω-inconsistency. Finally, in Section 5, we conclude with some remarks.

2. Paradoxes

The sorites paradox arises in languages containing vague predicates, understood as those which satisfy what Wright [24, p. 334] calls *Tolerance*:

> F is *tolerant* with respect to [a concept] Φ if there is also some positive degree of change in respect of Φ insufficient ever to affect the justice with which F is applied to a particular case.

The problem comes when one faces a soritical sequence, i.e. a series of objects $a_1, \dots, a_n$ ranging from clearly-F to clearly not-F, and where each object is just marginally Φ-different from the ones nearby in the sequence. Presumably then, the sentences Fa_1, $\neg Fa_n$ and $\forall i(Fa_i \to Fa_{i+1})$ should all be true, but they imply a contradiction just by Modus Ponens and Universal Instantiation. So the two horns of the dilemma are: to either reject the major premise, thus abandoning Tolerance, or weaken the logic, giving up at least one of the most seemingly uncontroversial rules of inference.

Semantic paradoxes, on the other hand, arise when we have some kind of semantic vocabulary (for the sake of simplicity, we will just deal with the truth predicate) and a language in which we can state its own theory of syntax. For then we will have to assign some truth value to sentences such as the following:

(**Liar**) This sentence is false.

(**Curry**) If this sentence is true, then everything is.

However, assuming they are true leads to contradiction, as well as assuming they are false does, and so there is no stable assignment of truth values to either of them. If our theory of syntax is, for instance, Peano Arithmetic (**PA**), this means we cannot add a truth predicate 'Tr' to the language, satisfying the induction schema and the unrestricted T-schema[2]:

(**T-schema**) $\mathrm{Tr}\ulcorner\phi\urcorner \leftrightarrow \phi$

If we did, assuming classical logic, the resulting theory would be inconsistent, since we can prove by the fixed-point theorem the existence of a sentence λ equivalent to its own falsity, as well as a sentence κ, equivalent to the one saying that the truth of κ implies everything. In general, we understand semantically paradoxical (sets of) sentences in the following way[3]:

Definition 2.1. A (**PA**-consistent) set of sentences Γ stated in the language of **PA** augmented with a truth predicate 'Tr' is semantically paradoxical if $\mathbf{PA} \cup \Gamma \cup \{\mathrm{Tr}\ulcorner\gamma\urcorner \leftrightarrow \gamma : \gamma \in \Gamma\}$ lacks a classical model.

In this sense, sentences such as the Liar or Curry count as paradoxical and those like the Truth-teller:

(**Truth-teller**) This sentence is true.

do not.

A common feature behind the two kinds of paradoxes is that they rely on the perhaps unwarranted assumption that there are just two

[2] In other words, formulas possibly containing the predicate 'Tr'.

[3] This definition is actually a very simple characterisation of what a paradoxical set is. In particular, strictly speaking, it overgenerates, since sets such as $\{\mathrm{Tr}\ulcorner 0 = 1\urcorner\}$ turn out to be paradoxical. One way of avoiding this is to consider that *inconsistent with* **PA** includes not only the arithmetically inconsistent sentences, but also each of their truth predications. For our purposes, we will ignore these cases and focus only on the truly problematic ones, such as the Liar or the Curry sentence.

semantic statuses available to evaluate sentences expressing propositions. Thus, it seems natural to give up bivalence and adopt instead some type of plurivalent logic. Amid them, the Łukasiewicz family stands out in virtue of their strong conditionals, which allow us to have identity as a logical theorem, Modus Ponens as a rule and the T-schema as valid in a theory of truth.

The problem is that all finitely valued semantics face immediate revenge paradoxes (see for instance [1]). The Liar can be generalized to sentences saying they do not have a designated value, and the sorites by appealing to higher order vagueness: just as one hair cannot make any predication of baldness go from true to false, it cannot make it switch from true to indeterminate either.

Of special interest to us will be the generalization of the Curry paradox via the following sequence of sentences:

1. $\kappa_1 \leftrightarrow (\mathrm{Tr}\ulcorner\kappa_1\urcorner \to \bot)$
2. $\kappa_2 \leftrightarrow (\mathrm{Tr}\ulcorner\kappa_2\urcorner \to (\mathrm{Tr}\ulcorner\kappa_2\urcorner \to \bot))$
3. $\kappa_3 \leftrightarrow (\mathrm{Tr}\ulcorner\kappa_3\urcorner \to (\mathrm{Tr}\ulcorner\kappa_3\urcorner \to (\mathrm{Tr}\ulcorner\kappa_3\urcorner \to \bot)))$

$$\vdots$$

In many non-classical logics (other than the n-Łukasiewicz family), since the conditional is weakened[4], it is possible to assign to each κ_n the same value as the conditional to which it is equivalent. But, as it was aforementioned, we do not want to weaken the conditional too much, and this set of sentences is inconsistent with any finite-valued Łukasiewicz logic[5]. Nonetheless, it is indeed consistent with any infinitely-valued Łukasiewicz logic, which is why we want to focus on systems based on sub-logics of it.

At this point, one question that might arise is: what kind of solution to the paradoxes are we looking for? In other words, what kind of interpretations do we consider to be adequate for both the vague and semantic predicates? A natural characterisation of an intended model might be:

[4] We are referring to the most usual solutions to semantic paradoxes, based on Strong Kleene matrices (e.g. see [18] in the paraconsistent case and [16, 9], in the paracomplete case).

[5] For those readers who are familiar with the Łukasiewicz family of logics, $Ł_n$ is inconsistent with κ_{n+1}, for any n. So no finite many-valued logic of the Łukasiewicz family will be consistent with the above sequence (see [1] for details).

Definition 2.2. A model M for a theory containing Peano arithmetic, vague vocabulary and a truth predicate is intended if the following conditions hold:

1. It is an extension of the standard model of arithmetic.[6]
2. It satisfies all of the instances of the T-schema and does not establish any order among semantically paradoxical sentences (it does not make any paradoxical sentence truer/falser than another).
3. If F is a vague predicate, a_1 is the name of an object that is clearly-F, a_n is the name of an object that is clearly not-F, and $a_1 \ldots a_n$ is a soritical sequence of objects, then Fa_1 is true in the model, Fa_n is false in the model, and for any $i \in \{1, \ldots, n-1\}$, Fa_i is at least as true as Fa_{i+1} in the model.

Regarding this definition, to have an intended model for a theory is to be able to capture the intuitive interpretation of the sentences. But even though it is not always possible to have the intended model, not all of the unintended interpretations are equally erroneous. In particular, the following is a more relaxed condition:

Definition 2.3. A model M for a theory containing Peano arithmetic, vague vocabulary and paradoxical sentences is suitable if it satisfies at least one of the above conditions.

The first — trivial — thing to note is that for a model to be intended implies that it is suitable. Secondly, although our final aim is to provide an intended interpretation to the complete set of sentences, we consider a model-theory which provides a suitable model to be better than another one lacking even that. And also, we consider a model satisfying more of these conditions to be more suitable than another one satisfying less.

3. The problem with the solution based on Łukasiewicz logic

In this section, we argue against the solution of the paradoxes based on Łukasiewicz logic. For that, we first introduce the logic and then show the limitations of this approach.

[6] A model M is an extension of the standard model of the arithmetic if restricted to arithmetical vocabulary is the standard model.

3.1. Łukasiewicz Logic

Let $\mathcal{L}_{\mathbf{PA}}$ be a first order language with the signature $\Sigma = \{s, +, \times, =, \overline{0}\}$, where $=$ is a binary relation, $+$ and $\times$ are two place functors, s is a one place functor and $\overline{0}$ is a name; and the logical operator names $\Theta = \{\bot, \neg, \wedge, \rightarrow, \exists\}$. In order to interpret the language, we need to define a logical matrix and a class of models. Regarding the matrix: the set of truth values is the closed interval $[0, 1]$ of real numbers; the set of designated values is just $\{1\}$; the propositional operations $\{\mathrm{bot}_Ł, \mathrm{neg}_Ł, \mathrm{conj}_Ł, \mathrm{cond}_Ł\}$ of arity 0, 1, 2 and 2 over $[0, 1]$:

- $\mathrm{bot}_Ł = 0$
- $\mathrm{neg}_Ł(a) = 1 - a$
- $\mathrm{conj}_Ł(a, b) = \sqcap(a, b)$, where $\sqcap := \min$
- $\mathrm{cond}_Ł(a, b) = \begin{cases} 1 & \text{if } a \leq b \\ 1 - a + b & \text{if } a > b \end{cases}$

and the distribution function $\mathrm{exist}_Ł$ over $2^{[0,1]} \rightarrow [0, 1]$ for the quantifier are defined as follows:

- $\mathrm{exist}_Ł(A) = \sup A$, where A is a subset of $[0, 1]$.

A (standard)[7] Łukasiewicz-model $M_Ł$ is a structure $\langle D, A\rangle$, where D is a non-empty set and A is an interpretation function such that:

- $a^A \in D$ for each constant a,
- $f^A = D^n \rightarrow D$ for each n-ary functor f,
- $F^A = D^n \rightarrow [0, 1]$ for each n-ary predicate F.

The value of a term $t_{[\alpha]}$ relative to a variable assignment α can be defined inductively as usual. Finally, a valuation $v_{Ł,\alpha}$ relative to a model $M_Ł$ and a variable assignment α is determined by the operations previously defined:

- $v_{Ł,\alpha}(Ft_1, \ldots, t_n) = F^A(\langle t_1^{A[\alpha]} \ldots t_n^{A[\alpha]}\rangle)$
- $v_{Ł,\alpha}(\bot) = \mathrm{bot}_Ł$
- $v_{Ł,\alpha}(\phi \wedge \psi) = \mathrm{conj}_Ł(v_{Ł,\alpha}(\phi), v_{Ł,\alpha}(\psi))$
- $v_{Ł,\alpha}(\phi \rightarrow \psi) = \mathrm{cond}_Ł(v_{Ł,\alpha}(\phi), v_{Ł,\alpha}(\psi))$
- $v_{Ł,\alpha}(\neg\phi) = \mathrm{neg}_Ł(v_{Ł,\alpha}(\phi))$
- $v_{Ł,\alpha}(\exists x\phi) = \mathrm{exist}_Ł(\{v_{Ł,\alpha x}(\phi) : \alpha x \text{ is an } x\text{-variant of } \alpha\})$

[7] Here and throughout the article, by *Łukasiewicz logic* we mean the logic characterised only by the infinite-valued semantics over the real interval $[0, 1]$ with the specified Łukasiewicz truth functions. This is usually called *the standard semantics* of Łukasiewicz logic. We do not take into account the general semantics, based on MV-algebras.

We will call the arguments determined by the class of such models, together with the relation of 1-preservation $Ł_\infty$. Other connectives definable from these are, for instance, the (strong) conjunction (fusion), weak and strong disjunction (fission)[8].

When it comes to the sorites, building a Łukasiewicz-model allows us to set the interpretation of the predicate in question to reflect the degree to which each object possesses the property. Because of the way the valuation function is defined, truth turns out to be a matter of degree too. Worries about sharp boundaries then tend to vanish, since we can in some sense capture the smooth transition from one object in the sequence to the next one. And even though the paradoxical reasoning is deemed unsound — as it would be if we were using classical models — that does not mean we have to reject the inductive premise, because it is just slightly untrue, and certainly much truer than its negation.

On the other hand, with respect to the semantic paradoxes, Restall [22] and Hájek et al. [13] proved that if we extend $\mathcal{L}_{\mathbf{PA}}$ to $\mathcal{L}^+_{\mathbf{PA}}$ with a truth predicate, the closure under Łukasiewicz consequence of the axioms of **PA**[9] plus all instances of the T-schema has a Łukasiewicz model. We will call such a theory $\mathbf{PA}^+_{Ł}$. More specifically, in every $\mathbf{PA}^+_{Ł}$-model, the Liar and Curry sentences (mentioned in the last section) receive value 1/2. More interesting is the case of the infinite sequence of paradoxical sentences $\{\kappa_i\}_{i\in\mathbb{N}}$:

Remark 3.1. Take the set $\{\kappa_i\}_{i\in\mathbb{N}}$, with each κ_i as defined in Section 2. For every $\mathbf{PA}^+_{Ł}$-model, and for every m, $v_{Ł}(\kappa_m) = m/(m+1) \in [0,1]$.

As a consequence of the above, note the following:

Remark 3.2. For any natural number m, and for any $\mathbf{PA}^+_{Ł}$-model, $v_{Ł}(\kappa_m) < v_{Ł}(\kappa_{m+1})$. In other words, $\{v_{Ł}(\kappa_i)\}_{i\in\mathbb{N}}$ is an increasing sequence with 1 as its least upper bound.

This last remark will play a crucial role in the following section, where we will argue against $\mathbf{PA}^+_{Ł}$.

[8] We have not followed the usual way of defining the conditional as the residuum of a continuous t-norm (strong conjunction), because we will focus on the conditional (see, e.g., [12]).

[9] We are assuming the formulation of arithmetic with an induction rule schema, instead of an axiom schema (see [22, p. 306] for details). Note that the set of Łukasiewicz consequences of the axioms of **PA** is the same as the set of classical consequences of these axioms (see for instance [22, p. 2]).

3.2. Trouble for Łukasiewicz logic

From our point of view, Łukasiewicz semantics offers one of the most satisfactory frameworks for dealing with vague vocabulary. If so, a solution to the semantic paradoxes would have to somehow accommodate these sort of models. As we will see, this cannot be as straightforward as one would like. There are two main problems: one is more of a conceptual issue, while the other one is strictly of technical nature.

Let us start with the first one. We want to consider the formal semantics as models, not only in the sense of them being useful interpretations (for instance, of show consistency), but in a quite stronger one, the same sense we find for instance in Goguen [10], Edgington [8] or Cook [6]. Just as a scale model of a building represents the building, our semantics are meant to represent those of natural language, and in particular, real numbers are used in place of *verities*, which are the semantic values that relate to English sentences. Łukasiewicz models are used both by vagueness-theorists and — although much less frequently — by truth-theorists. The issue here is that even if the models are isomorphic, this does not necessarily mean that, from a conceptual point of view, they are characterising the same semantic interpretation. Here, we follow Cook:

> Sometimes theorists will introduce a particular many-valued logic, demonstrate how this logic, and the semantics built upon it, solves both the Liar paradox and the sorites paradox, and then conclude that this logic is the correct logic. The problem, however, is that the relations between statements and the world represented by the additional truth values when the logic is applied to languages involving the semantic paradoxes might be a different from the relations that those same truth values represent when applied to languages involving vague expressions. [6, p. 188]

The first problem with the solution based on Łukasiewicz logic is then that we find the semantic and the vague interpretations to be indeed mutually incompatible.

In the case of vagueness, the usual understanding of the semantic values is that of degrees of truth: the maximum represents truth, the minimum represents falsehood, with all the intermediate values representing a smooth transition between them. The interpretation of the connectives is then somewhat justified by the fact that they are natural extensions of the crisp operations to the case of degrees of truth: it seems intuitive, for instance, to consider that a sentence $\neg\phi$ is as true as ϕ is false, and vice versa.

In the case of theories of truth, though, it is in fact important *not to consider* values between 0 and 1 as degrees of truth, but instead as just different indeterminacies. Otherwise, we would have to say that a Curry sentence κ_n is falser than a Curry sentence κ_{n+1}, and it is not easy to understand what would that mean (see Remark 3.2).

According to Definition 2.3, although Łukasiewicz logic can accommodate a suitable interpretation for the vague vocabulary, it does not have a suitable interpretation for the truth predicate.

The second problem with Łukasiewicz logic has nothing to do with vagueness, but only with the truth predicate. We say that a theory is ω-inconsistent if there is a formula $\phi(x)$, such that in every model and for every numeral $\overline{n}$, $\phi(\overline{n})$ is not true, but $\exists x\phi(x)$ is true[10]. It means that each model of the theory is non-standard (it has non-standard numbers in its domain, in this case, satisfying the existential formula). Unfortunately, $\mathbf{PA}^{+}_{Ł}$, although satisfiable, is in fact ω-inconsistent [22, 13]. See for instance Field [9], Barrio [2] and Barrio and Picollo [3], Barrio and Da Ré [4], among others who argue against ω-inconsistent theories[11].

More recently, Bacon [1], in order to proof-theoretically derive the proof of ω-inconsistency, showed that it is sufficient to have the following two rules:

($\exists$ 1) if $\phi \vdash \psi$ then $\exists x\phi \vdash \exists x\psi$
($\exists$ 2) $\phi \to \exists x\psi \vdash \exists x(\phi \to \psi)$

Both rules are valid in Łukasiewicz logic and the second one relies on the provability of the axiom of prelinearity (also valid in Łukasiewicz logic): $\vdash (\phi \to \psi) \vee (\psi \to \phi)$. In the next section, we will explore sublogics of Łukasiewicz where prelinearity fails. However, as we will prove in the case of the logic **T2**, failure of prelinearity is not sufficient for avoiding ω-inconsistency.

Before going into our proposal, it is important to mention that some authors have tried to find sublogics of Łukasiewicz logic suitable to support a theory that deals with semantic paradoxes. For instance, we

10 Although, usually, the notion of ω-inconsistency is syntactic, here we define it semantically.

11 However, there are some authors that still defend ω-inconsistent theories of truth, such as Yatabe [25], and Gupta [11].

dismiss the logic developed in [17] by Pailos and Rosenblatt, because the conditional there presented is non-deterministic. Therefore, it is very weak compared with the logic that we will favour here. Also, the authors did not try to give an account of the sorites problem. On the other hand, in order to avoid the derivation of ω-inconsistency, Bacon in [1] mentions two logics: BCKN and BCKD. Nevertheless, although these logics are weaker than $Ł_\infty$, it is really straightforward to check that the semantics that he presents for each logic are incompatible with the truth predicate.[12]

4. A new approach

In this section, we will focus on *Type-2 Interval Fuzzy Logic* (**T2**). We roughly follow one of the logics presented in [7].[13]

4.1. Type-2 Interval Fuzzy Logic (T2)

This new semantics is built upon a class of models which are, in a sense, fuzzier than the Łukasiewicz ones. Instead of taking the set $[0, 1]$ as the set of semantic values, the logic **T2** is characterised by taking the set of its closed intervals $L^I = \{[a, b] :< a, b >\in [0, 1]^2 \text{ and } a \leq b\}$ whereas the designated value is the interval [1,1] and the propositional operations and the distribution function of the existential quantifier will be now defined as follows:[14]

- $\text{neg}_2([a, b]) = [1 - b, 1 - a]$,
- $\text{cond}_2([a, b], [c, d]) = [\sqcap\{1, 1 + c - a, 1 + d - b\}, \sqcap\{1, 1 + d - a\}]$,
- $\text{conj}_2([a, b], [c, d]) = [\sqcap\{a, c\}, \sqcap\{b, d\}]$,
- $\text{bot}_2 = [0, 0]$,

[12] For BCKD, the point was made by Pailos and Rosenblatt in [17]. In the case of BCKN, Bacon offers a semantics via a truth table for the conditional. Although these semantics invalidate the rule (∃2) and the axiom of prelinearity mentioned before, they still validate the particular instance required for the ω-inconsistency proof (for details of the semantics of BCKN, see [1]).

[13] As far as we know, there is not in the literature any attempt to connect these logics with semantic vocabulary. It is worth noting that here we will focus on the more natural logic presented in [7], although the limitative result can be extended to the rest of the logics straightforwardly.

[14] Here, as in Section 3.1, we will only consider **T2** as the logic characterised by the semantics over the closed intervals of the real unit $[0, 1]$ with truth functions derived from the standard semantics.

- $\mathrm{exist}_2(A) = [\sup\{a : [a,b] \in A\}, \sup\{b : [a,b] \in A\}]$, where A is a subset of L^I,

where $\sqcap$ and sup are the same meet and join operations as in Section 3.1.

Type-2 models are defined exactly as before, but with type-2 characteristic functions for the predicates. Valuations v_2 relative to those models can be defined as usual. We will call the class of all type-2-valid arguments *Type-2 Interval Fuzzy Logic* or **T2** for short. Accordingly, $\mathbf{PA}_2^+$ will be the theory obtained by closing the axioms of **PA** and the instances of the T-schema under **T2**-consequence.[15]

As we saw, we need to represent two sorts of values: indeterminacies and degrees of truth. In order to do this, we distinguish between two types of members of the space. Degenerate intervals, which are those containing only one point, represent degrees of truth and their order is the natural one corresponding to the order between real numbers. On the other hand, non-degenerate ones represent modes of indeterminacy and they are not ordered.[16]

Formally, the structure of both subsets has some elements meant to be taken seriously, and some that are not. Degenerate intervals represent real verities in terms of how many of them are, their order, their density, and some of the distances between them (we follow Cook [6]).

What about the representativeness of non-degenerate ones? Their standard interpretation in these kind of semantics is meant to be epistemic: a sentence assigned the value $[0.3, 0.5]$ has a value between those endpoints, but we do not know which one. In the same vein, one can go for a more semantic flavor, and say that any value in the interval is admissible. We do not want to adopt any of these sorts of interpretations, though. If a sentence gets assigned an interval instead of a precise degree of truth, it means that something in the process of its evaluation has failed, and there is no deep difference between one indeterminacy and another one besides the fact that they need to be different for technical reasons.

This leads to a very well-known problem with the logic-as-modelling view we are following. As in any model, there are some features of the space of values which do not correspond with any natural language phenomenon. For instance, just like the size of a model of a building is

[15] Note that the closure of the axioms of **PA** under Łukasiewicz consequence coincides with its closure under **T2**-consequence.

[16] And so, the axiom of prelinearity fails in **T2**.

not intended to represent the size of the actual building, the absolute precision of real numbers does not reflect a property that real verities have. Can we identify the non-artefactual features though? Rosanna Keefe in [15] — when arguing against Łukasiewicz models — claims that obtaining all acceptable valuations by linear transformations is the way to make sense of the idea that the exact number corresponding to a sentences is not meant to be taken seriously: even though ϕ gets the value 0.567 in the model, it could just as easily have been evaluated as 0.568, and so on, within some range. The problem is that if in addition of demanding linear transformations, you expect all suitable models to keep the endpoints fixed — as it is expected, if some sentences are true or false no matter what — then the only available transformation is identity.

However, are transformations actually an accurate guide to which element are representative? That the size of a scale model of, say, the MOMA is not representative can indeed be stated by affirming that any transformation that preserves everything but size is equally good. But there are some properties that, although invariant through transformations, are not to be regarded as representative. Take for instance the property of not being located where the MOMA is. Every model will necessarily share that property, but of course, the MOMA does not have the property of not being located where it in fact is.

Hence, since remaining fixed — even if necessary — it certainly is not sufficient, we are left with no clue as to which elements of the model should be taken seriously and which ones should not. The answer is, we think, to give up the idea that the representative elements can be pinpointed by a mathematically rigorous concept, and to adhere to Cook's proposal that the relation between the formal semantics and the natural language will always be stated in a vague language [6].

Leaving this problem aside, we can now ponder how the desired model would have to look like. The idea is that, now that we have values representing modes of indeterminacy, we can use them to evaluate paradoxical sentences, whereas the non-paradoxical ones — among them, in particular, sentences involving vague vocabulary — receive a degenerate interval.

As an example, take **PA** with the T-schema for the Liar and the Curry sequence we considered above. This theory has a suitable model, in which λ gets value $[0, 1]$, κ_2 gets value $[1/3, 1]$, κ_3 gets value $[2/4, 1]$, and in general, for any κ_i, its value is $[(i-1)/(i+1), 1]$. Sentences like the Truth-teller may be considered either true, false, or any other semantic value one pleases, and the choice between these options involves metaphysical and semantical commitments we will not try to settle here. Thus, we will let our models roam free with respect to them. Accommodating vagueness, on the other hand, is an easy task. We just interpret soritical predicates by means of characteristic functions giving only degenerate intervals as their values. Finally, extending these models to one which is a suitable for the whole theory — that is, with the full T-schema, and all paradoxical sentences inside — is an open problem that we leave for future work.[17] This would complete our goal of finding a model which is suitable both for the semantic and the vague parts of the language.

4.2. Some limitative results

The first thing to notice is that the interaction between these two fragments of the language is not always as expected. In fact, some sentences containing a mixed vocabulary could be evaluated in a dubious way. For example, given a semantically paradoxical sentence that takes a non degenerate value, e.g. $[1/3, 1]$ in some suitable model, its conjunction with a vague sentence with value $[0.2, 0.2]$ would also receive the degenerate interval $[0.2, 0.2]$ as its semantic value. This may count as an undesirable behaviour on the part of the models, although intuitions are not as clear in these cases as they may be in the pure ones. Be this as it may, this problem goes beyond the original task of finding suitable models in the sense of Definition 2.3. Next, we will present what we take as the biggest problem for **T2**.

As in the case of the theory built over Łukasiewicz logic, the main issue here is that models of the theory built over **T2** cannot be extended to the intended ones, given that the same ω-inconsistency phenomenon arises in $\mathbf{PA}_2^+$, as we will prove in the rest of this section.

The hope of there being standard **T2**-models might be grounded on the fact that in **T2** the prelinearity axiom is not valid $((\phi \to \psi) \vee (\psi \to$

[17] Proving a result like this is not such an easy task, given that we cannot simply use Brouwer's fixed point theorem (as in [13]), since our space of values (the one lacking degenerated intervals other than $[0, 0]$ and $[1, 1]$) does not satisfy the initial conditions for its application. However, it is worth noting that it is closed under the operations corresponding to the connectives — in particular, the conditional — which is also a necessary condition for the general result.

ϕ)).[18] And — recall — prelinearity is needed in order to prove the rule ($\exists$2). Unfortunately, this is not enough to guarantee the existence of a standard model: $\mathbf{PA}_2^+$ is just as ω-inconsistent as $\mathbf{PA}_{\text{Ł}}^+$. Before proving the main theorem, we need first to prove the following two lemmas:

LEMMA 4.1. *Let f be a function such that $f(\overline{0}, x) = x \dot{\rightarrow} \dot{\perp}$ and $f(\overline{n+1}, x) = x \dot{\rightarrow} f(\overline{n}, x)$* [19]. *For any γ and n, if $v_2(\gamma) = [a, b]$ then $v_2(\mathrm{Tr}(f(\overline{n}, \ulcorner\gamma\urcorner)))= [\sqcap\{1, n + 1 - (na + b)\}, \sqcap\{1, (n + 1)(1 - a)\}]$.*

PROOF. We will prove this lemma by induction over n.

Base case $v_2(f(\overline{0}, \ulcorner\gamma\urcorner)) = v_2(\mathrm{Tr}(\ulcorner\gamma\urcorner \dot{\rightarrow} \dot{\perp})) = \mathrm{cond}_2([a, b], [0, 0]) = [\sqcap\{1, 1 - b\}, \sqcap\{1, 1 - a\}]$, because $1 - b \leq 1 - a$. And it is easy to check that this satisfies the lemma.

Inductive step: Let us assume that the lemma holds for $n = h$. We need to show that it also holds for $n = h + 1$. Using the hypothesis, we have that $v_2(\mathrm{Tr}(f(\overline{h}, \ulcorner\gamma\urcorner))) = [\sqcap\{1, h+1-(ha+b)\}, \sqcap\{1, (h+1)(1-a)\}]$. And we know by definition of f that $v_2(\mathrm{Tr}(f(\overline{h+1}, \ulcorner\gamma\urcorner)))= v_2(\mathrm{Tr}(\ulcorner\gamma\urcorner \dot{\rightarrow} f(\overline{h}, \ulcorner\gamma\urcorner)))$.

Hence, $v_2(\mathrm{Tr}(f(\overline{h+1}, \ulcorner\gamma\urcorner))) = \mathrm{cond}_2([a, b], [\sqcap\{1, h + 1 - (ha + b)\}, \sqcap\{1, (h+1)(1-a)\}])$. Thus we have to check four cases according to the possible semantic value of each infimum:

Case 1. $v_2(\mathrm{Tr}(f(\overline{h+1}, \ulcorner\gamma\urcorner))) = \mathrm{cond}_2([a, b], [1, 1])$.

It is easy to check that a conditional with a true consequent is true, so $\mathrm{cond}_2([a, b], [1, 1]) = [1, 1]$. Now, we need to prove that $[1, 1] = [\sqcap\{1, (h+2) - ((h + 1)a + b)\}, \sqcap\{1, (h + 2)(1 - a)\}]$, that is we need to prove the pointwise identity. For the first endpoint, by hypothesis of the case, $h + 1 - (ha + b) \geq 1$, and since $1 \geq a$, by arithmetic we have that $h + 2 \geq 1 + (h + 1)a + b$. Therefore, $1 = \sqcap\{1, h + 2 - ((h + 1)a - b)\}$. For the second point, by hypothesis of the case we have that $(h+1)(1-a) \geq 1$. Again, since $1 \geq a$, by arithmetic we know that $(h+1)(1-a)+(1-a) \geq 1$. Then also $1 = \sqcap\{1, (h + 2)(1 - a)\}$.

Case 2. $v_2(\mathrm{Tr}(f(\overline{h+1}, \ulcorner\gamma\urcorner))) = \mathrm{cond}_2([a, b], [1, (h + 1)(1 - a)])$
By hypothesis of this case, $1 > (h + 1)(1 - a)$, and then by definition of interval, $(h+1)(1-a) = 1$. But these conditions are mutually exclusive, so this case cannot occur.

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[18] To see this, take for instance a model where ϕ gets value [0.2, 0.8] and ψ gets value $[0.3, 0.7]$

[19] For readability, we omit the dot notation for the symbol that represents the function f in the theory.

Case 3. $v_2(\mathrm{Tr}(f(\overline{h+1}, \ulcorner\gamma\urcorner))) = \mathrm{cond}_2([a,b],[h+1-(ha+b),1])$.

By definition, $\mathrm{cond}_2([a,b],[h+1-(ha+b),1]) = [\sqcap\{1, h+2-((h+1)a+b), 2-b\}, \sqcap\{1, 2-a\}]$. Given that $a \leq b \leq 1$ and then $2-a \geq 2-b \geq 1$, we have that $\mathrm{cond}_2([a,b],[h+1-(ha+b),1]) = [\sqcap\{1, h+2-((h+1)a+b)\}, 1]$. Now, it remains to check that in this case $1 \leq (h+2)(1-a)$. But this is trivial, given that, by hypothesis of the case, $1 \leq (h+1)(1-a)$ and $1-a \geq 0$. Thus, $\mathrm{cond}_2([a,b],[h+1-(ha+b),1]) = [\sqcap\{1, h+2-((h+1)a+b)\}, \sqcap\{1,(h+2)(1-a)\}]$.

Case 4. $v_2(\mathrm{Tr}(f(\overline{h+1}, \ulcorner\gamma\urcorner))) = \mathrm{cond}_2([a,b],[h+1-(ha+b), h+1(1-a)])$.

In this case, it is enough to apply the definition of the valuation of the conditional. $\mathrm{cond}_2([a,b],[h+1-(ha+b),(h+1)(1-a)]) = [\sqcap\{1, h+2-((h+1)a+b)\}, \sqcap\{1,(h+2)(1-a)\}]$. ⊣

LEMMA 4.2. *Let ϕ be such that $\vDash_{\mathbf{PA}_2^+} \phi \leftrightarrow \exists x Tr(f(x, \ulcorner\phi\urcorner))$, where f is the function defined in Lemma 4.1. For each n, we have that $v_2(\mathrm{Tr}(f(\overline{n}, \ulcorner\phi\urcorner))) \neq [1,1]$.*

PROOF. Suppose, on a contrary, that there is a k such that $v_2(\mathrm{Tr}(f(\overline{k}, \ulcorner\phi\urcorner))) = [1,1]$.Then, by definition of the existential, $v_2(\exists x \mathrm{Tr}(f(x, \ulcorner\phi\urcorner))) = [1,1]$. Let $v_2(\phi) = [a,b]$. Thus, by definition of the biconditional $[a,b] = [1,1]$. Nonetheless, by the previous lemma $v_2(\mathrm{Tr}(f(\overline{k}, \ulcorner\phi\urcorner))) = [\sqcap\{1, k+1-(ka+b)\}, \sqcap\{1,(k+1)(1-a)\}]$. Therefore, $v_2(\mathrm{Tr}(f(\overline{k}, \ulcorner\phi\urcorner))) = [\sqcap\{1,0\}, \sqcap\{1,0\}]$, which contradicts the initial assumption. ⊣

Finally, with these results we can prove what we wanted:

THEOREM 4.3. *$\mathbf{PA}_2^+$ is ω-inconsistent.*

PROOF. We will argue by reductio. Let ϕ be as in Lemma 4.2 and suppose $v_2(\phi) = [a,b] \neq [1,1]$. Because of the definition of interval, this implies that $a < 1$. Also, thanks to Lemma 4.1, we know that for any n, $v_2(\mathrm{Tr}(f(\overline{n}, \ulcorner\phi\urcorner)))=[\sqcap\{1, n+1-(na+b)\}, \sqcap\{1,(n+1)(1-a)\}]$. But because of Lemma 4.2 and the definition of an interval, we know the first endpoint cannot be 1. Thus, $v_2(\mathrm{Tr}(f(\overline{n}, \ulcorner\phi\urcorner))) = [n+1-(na+b), \sqcap\{1,(n+1)(1-a)\}]$. But for the same reason, it must be the case that $n+1-(na+b) < 1$ and so, by arithmetic: $n(1-a) < b$. But no matter how small $1-a$ is, there will be a number n such that $1 < n(1-a)$ and thus $1 < b$. Contradiction. Hence, $v_2(\phi) = [1,1]$. But since Lemma 4.2 shows that the theory does not force any of the instances of the existential, this suffices to show that it is ω-inconsistent. ⊣

This shows that **T2** fares no better than $Ł_\infty$ when it comes to providing a standard model for arithmetic.[20]

4.3. Some possible routes to ω-consistency

Improving upon **T2** to capture the intended interpretation of the whole set of sentences of the theory is a challenging task. Thus, our purpose in this section is mainly exploratory, and we do not wish to claim that the proposal hereafter presented is definitive in any sense. We will play around with a possible modification and show that the theory built over this other logic has a model where the value of the sentence ϕ presented in the last proof is different from [1,1], avoiding the derivation of ω-inconsistency.

Let $\mathbf{T2}^\star$ have — as in the case of **T2** — the set of all closed intervals of $[0,1]$ as its set of truth values, $[1,1]$ as its only designated value, and the same propositional operations and distribution function for the quantifier as before, with the exception of the conditional, which will be characterised the following way:

$$\mathrm{cond}_2^\star([a,b],[c,d]) = \begin{cases} \mathrm{cond}_2([a,b],[c,d]) & \text{if } [a,b]\cap[c,d]=\emptyset \text{ or} \\ & [a,b]=[c,d] \\ [(a+c)/2,(b+d)/2] & \text{otherwise} \end{cases}$$

This new conditional works exactly as **T2**'s, for the cases where the two intervals share either no point or all of them. This allows us to retain the Łukasiewicz-like behavior for degenerate intervals, and also identity as a law, which was one of the original motivating features. It is worth noting that there are an infinite number of ways to modify the conditional, avoiding the derivation of ω-inconsistency. We just chose one among them, which we found simple enough.

This faces us with a more general problem, which concerns the criteria for choosing the *right* truth-functional operations in the case of fuzzy theories of truth. It seems that quite often, the justification for the interpretation of the connectives is just transposed from theories of

[20] As an anonymous referee has pointed out, in [13] the authors show that (what we call here) $\mathbf{PA}^+_{Ł}$ lacks a standard model even for non-standard semantics, i.e. for any linear MV-algebra. One could ask whether a similar point holds for $\mathbf{PA}^+_2$. The proof of the Theorem 4.3 (and the previous results) strongly depends on the standard semantics, so whether or not it can be generalised is left for future work.

vagueness, since it is not common to find in the literature an explanation other than “they generalise the two-valued case”. Nevertheless, although this generalisation is obvious for three-valued theories, it is not so when we have infinite values; much less if they are taken to be — as we think they have to — just different modes of indeterminacy, and not genuine degrees of truth.

Our proposal is to be liberal, and allow connectives to be legitimately interpreted by any function, as long as they behave properly, in the case that they satisfy some desirable principles or rules. If so, even if at first sight, the conditional of $\mathbf{T2}^{\star}$ could seem sort of artificial, it actually complies with a lot of the standard requirements of a conditional. Some of these nice properties are, for instance:

1. (*Modus Ponens*) $\phi, \phi \rightarrow \psi \vDash_{\mathbf{T2}^{\star}} \phi$
2. (*Identity*) $\vDash_{\mathbf{T2}^{\star}} \phi \rightarrow \phi$
3. (*Contraposition*) $\phi \rightarrow \psi \vDash_{\mathbf{T2}^{\star}} \neg\psi \rightarrow \neg\phi$
4. (*Contraposition′*) $\neg\psi \rightarrow \neg\phi \vDash_{\mathbf{T2}^{\star}} \phi \rightarrow \psi$
5. (*De Morgan*) $\neg(\phi \wedge \neg\psi) \vDash_{\mathbf{T2}^{\star}} \phi \rightarrow \psi$[21]
6. It is Łukasiewicz-normal, in the sense that, when the intervals are degenerate, the output is the interval-equivalent of the Łukasiewicz conditional.

From this list, it is possible to conclude that, despite its non-standard features, this conditional is still a strong contender among the non-classical ones, and that it behaves in a very natural way.

As in the case of $\mathbf{T2}$, we will also refer by $\mathbf{PA}_2^{+\star}$ to $\mathbf{PA}^{+\star}$ closed under $\mathbf{T2}^{\star}$ consequence. This theory still has a suitable model in which λ gets value $[0, 1]$, κ_2 gets value [1/3, 1], κ_3 gets value [2/4, 1], and in general, for every κ_i, its value is [(i-1)/(i+1), 1].

However, we do not know if the theory has an intended model, since the usual proof of the absence of such a model cannot go through. Take the function defined in Lemma 4.1: $f(\overline{0}, x) = x \dot{\rightarrow} \dot{\perp}$ and $f(\overline{n+1}, x) = x \dot{\rightarrow} f(\overline{n}, x)$, and let again ϕ be the sentence such that $\vDash_{\mathbf{PA}_2^{+\star}} \phi \leftrightarrow \exists x \mathrm{Tr}(f(x, \ulcorner\phi\urcorner))$. We have seen that in the case of $\mathbf{PA}_2^{+}$, for every model, $v_M(\phi) = [1, 1]$, leading to ω-inconsistency. However, in $\mathbf{PA}_2^{+\star}$ that is not the case. Simply to illustrate this point, let M be a model such that $v_M(\phi) = [0.3, 0.8]$. It is easy to check that $v_M(\phi \leftrightarrow \exists x \mathrm{Tr}(f(x, \ulcorner\phi\urcorner))) =$

[21] However, the other direction does not hold, i.e. $\phi \rightarrow \psi \nvDash_{\mathbf{T2}^{\star}} \neg(\phi \wedge \neg\psi)$. We'd like to thank an anonymous referee for pointing this out.

$[1, 1]$, even though ϕ does not have the designated value. If the value of each $\mathrm{Tr}(f(\overline{i}, \ulcorner\phi\urcorner))$ is $[a_i, bi]$, then we have the following:

- $[a_0, b_0] = v_M(\mathrm{Tr}f(\overline{0}, \ulcorner\phi\urcorner)) = v_M(\phi \dot{\rightarrow} \dot{\bot}) = [0.2, 0.7]$,
- for any $k > 0$, $0.2 < a_k < a_{k+1} < \sup\{a_i : i \in \omega\} = 0.3$,
- for any $k > 0$, $0.7 < b_k < b_{k+1} < \sup\{b_i : i \in \omega\} = 0.8$.

Informally, this means that, with each iteration of the conditional, the first point increases towards 0.3, and the second one increases towards 0.8. So, as $v_M(\exists x\mathrm{Tr}(f(x, \ulcorner\phi\urcorner))$ is the supremum pointwise of the above sequence of values, $v_M(\exists x\mathrm{Tr}(f(x, \ulcorner\phi\urcorner)) = [0.3, 0.8]$. So, neither ϕ nor $\exists x\mathrm{Tr}(f(x, \ulcorner\phi\urcorner)$ receive a designated value in every model (of course the biconditional still holds since is an instance of the diagonal lemma), and thus they cannot be used to show ω-inconsistency any longer.

5. Conclusion

In this paper, we introduced two new approaches for dealing with semantic paradoxes and soritical predicates, one based on Type-2 Interval Fuzzy Logic and one based on Type-2* Interval Fuzzy Logic. First, we characterised the concept of *intended model* for a theory containing Peano arithmetic, vague and semantic vocabulary as an interpretation that: (a) is an extension of the standard model of arithmetic; (b) satisfies all of the instances of the T-schema and does not establish any order among paradoxical sentences (it does not make any paradoxical sentence truer/falser than another), and finally (c) if F is a vague predicate, a_1 is the name of an object that is clearly-F, a_n is the name of an object that is clearly not-F, and $a_1 \ldots a_n$ is a soritical sequence of objects, then Fa_1 is true in the model, Fa_n is false in the model, and for each $i \in \{1, \ldots, n-1\}$, Fa_i is at least as true as Fa_{i+1}.

We also defined the concept of *suitable model* for a theory containing Peano arithmetic, vague and semantic vocabulary as a partial adequacy criterion, where an interpretation is suitable when it satisfies at least some of the above requirements. Then, we introduced the solution based on Łukasiewicz logic and showed its limitations.

Next, we presented Type-2 Interval Fuzzy Logic and we showed that although it is ω-inconsistent, it is possible to build a theory for some paradoxical sentences (such as the Liar and the sequence of Curry sentences) and vague vocabulary with a suitable interpretation. Finally, we presented Type-2* Interval Fuzzy Logic whose models are more suitable than those of its competitors, and also avoids the usual proof of ω-inconsistency (although we have not proved that $\mathbf{T2}^{\star}$ is ω-consistent). This work is exploratory and we hope further research will enlighten the problems investigated.

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