Logic and **L**ogical **P**hilosophy
Volume 29 (2020), 223–237
DOI: 10.12775/LLP.2019.035

Janusz Ciuciura

On the System CB^1 and a Lattice of the Paraconsistent Calculi

Abstract. In this paper, we present a calculus of paraconsistent logic. We propose an axiomatisation and a semantics for the calculus, and prove several important meta-theorems. The calculus, denoted as CB^1, is an extension of systems PI, $\mathrm{C}_{\mathrm{min}}$ and B^1, and a proper subsystem of Sette's calculus P^1. We also investigate the generalization of CB^1 to the hierarchy of related calculi.

Keywords: paraconsistent logic; paraconsistency; hierarchy of the paraconsistent calculi

1. Introduction

One of the most commonly quoted definitions of paraconsistent logic runs as follows: a logic $\langle \mathcal{L}, \vdash \rangle$ is said to be *paraconsistent* if $\{\alpha, \neg\alpha\} \nvdash \beta$, for some formulas α, β. The definition is very general and covers a broad range of logics. Therefore, some authors have suggested that additional criteria should be taken into account when introducing a calculus of paraconsistent logic. It is worth mentioning three of them here: (a) the law of non-contradiction must not be a valid schema in any paraconsistent calculus; (b) a paraconsistent calculus should be rich enough to enable practical inference; and last but not least, (c) a paraconsistent calculus should have an intuitive justification.[1] Unfortunately, the latter two are rather vague and imprecise. They suffer from a lack of accuracy

[1] Criterion (a) was introduced by Newton C. A. da Costa [10, p. 498]. Criteria (b) and (c) were formulated by Stanisław Jaśkowski [12, the second English translation, p. 38]. More complex and up-to-date definitions of paraconsistency are proposed in [see esp. Chapter 2 of 2; and Chapter 1 of 4].

Received May 15, 2019. *Revised* November 12, 2019. *Published online* December 2, 2019

and open a wide field for speculation and conjecture. On the other hand, there are a few significant examples of paraconsistent calculi in which the law of non-contradiction has not been abandoned. Jaśkowski's discursive calculus and Asenjo–Tamburino's logic of antinomies may serve as good examples of this kind of calculi [see 9, pp. 52, 71–72].

The aim of this paper is to propose a calculus of paraconsistent logic which is intended to satisfy at least some of the requirements. The calculus, denoted as CB^1, arises as a result of the extension of the system $\mathrm{C_{min}}$ with the principle of weak explosion $\alpha \to (\neg\alpha \to (\neg\neg\alpha \to \beta))$ or the calculus B^1 with the law of double negation $\neg\neg\alpha \to \alpha$.[2] It can also be viewed as an extension of the propositional logic PI,[3] or as a proper subsystem of the calculus P^1 [16]. All of them form together a lattice of paraconsistent calculi. In addition, we will investigate the generalization of CB^1 to a hierarchy of related calculi.

2. Basic notation

Let Var denote a denumerable set of propositional variables: p, q, p_1, p_2, ... The set For of formulas is standardly defined using variables from Var and the symbols $\neg$, $\vee$, $\wedge$ and $\to$ for negation, disjunction, conjunction and implication, respectively. The connective of equivalence, $\alpha \leftrightarrow \beta$, is treated as an abbreviation for $(\alpha \to \beta) \wedge (\beta \to \alpha)$.

In For, we will consider axiomatic propositional calculi in a Hilbert-style formalization with (MP) as the only rule of interference: $\alpha \to \beta$, α / β. Such a calculus $\mathcal{C}$ is determined by its set of axioms $\mathrm{Ax}_{\mathcal{C}}$ which is included in For. For $\mathcal{C}$, any $\alpha \in$ For and any $\Gamma \subseteq$ For, we say that *α is provable from Γ within $\mathcal{C}$* (in symbols: $\Gamma \vdash_{\mathcal{C}} \alpha$) iff there is a finite sequence of formulas, $\beta_1, \beta_2, \ldots, \beta_n$ such that $\beta_n = \alpha$ and for each $i \leqslant n$,

[2] See [5, p. 31], [6, sections 1–3] and [9, pp. 80–84]. Note that $\mathrm{C_{min}}$ has been also considered in [17, pp. 18–19], under the abbreviation $\langle 13 \rangle$. A modern discussion on $\mathrm{C_{min}}$ can be found in [see 2, Section 7.4]. The calculus B^1 originally appeared in [7] as mbC^1. Unfortunately, the chosen abbreviation and narrative in the paper are a bit misleading, e.g. "mbC^1 [...] essentially coincides with mbC by Carnielli, Coniglio and Marcos" [17, p. 174]. As a result, one could conclude that $\mathrm{mbC}^1/\mathrm{B}^1$ was equivalent to mbC. Obviously, it is not the case and such a claim should be rejected [see 9, p. 140].

[3] For details, see [3]. Nowadays, the system PI is perhaps better known under the abbreviation CLuN. The calculi $\mathrm{C_{min}}$ and B^1 are not the only extensions of PI. Some other (not necessarily paraconsistent) extensions of PI, are, e.g., presented in [2, 11, 15, 17, 18].

either $\beta_i \in \Gamma$, or $\beta_i \in \mathrm{Ax}_{\mathcal{C}}$, or for some $j, k \leqslant i$, we have $\beta_k = \beta_j \to \beta_i$. A formula α is a thesis of $\mathcal{C}$ iff α is provable from $\emptyset$ within $\mathcal{C}$. Let $\mathrm{Th}(\mathcal{C})$ be the set of all theses of $\mathcal{C}$. Observe that $\mathcal{C}$ can be identified with the triple $\langle \mathrm{For}, \mathrm{Ax}_{\mathcal{C}}, \vdash_{\mathcal{C}} \rangle$, but $\mathcal{C}$ is determined by $\mathrm{Ax}_{\mathcal{C}}$. Also, it can be easily seen that $\vdash_{\mathcal{C}}$ is a finitary consequence relation satisfying Tarskian properties (reflexivity, monotonicity, transitivity).

LEMMA 2.1. *For every* $\Gamma, \Delta \subseteq \mathrm{For}$ *and* $\alpha, \beta \in \mathrm{For}$:

1. $\Gamma \vdash_{\mathcal{C}} \alpha$ *iff for some finite* $\Delta \subseteq \Gamma$, $\Delta \vdash_{\mathcal{C}} \alpha$.
2. *If* $\alpha \in \Gamma$, *then* $\Gamma \vdash_{\mathcal{C}} \alpha$.
3. *If* $\Gamma \subseteq \Delta$ *and* $\Gamma \vdash_{\mathcal{C}} \alpha$, *then* $\Delta \vdash_{\mathcal{C}} \alpha$.
4. *If* $\Delta \vdash_{\mathcal{C}} \alpha$ *and, for every* $\beta \in \Delta$ *such that* $\Gamma \vdash_{\mathcal{C}} \beta$, *then* $\Gamma \vdash_{\mathcal{C}} \alpha$.
5. *If* $\Gamma \cup \{\alpha\} \vdash_{\mathcal{C}} \beta$ *and* $\Delta \vdash_{\mathcal{C}} \alpha$, *then* $\Gamma \cup \Delta \vdash_{\mathcal{C}} \beta$; *in particular, if* $\Gamma \cup \{\alpha\} \vdash_{\mathcal{C}} \beta$ *and* α *is a thesis of* $\mathcal{C}$, *then* $\Gamma \vdash_{\mathcal{C}} \beta$.

Each calculus considered in this work, except for the calculi discussed in the last section, is expected to contain all axiom schemas of the positive fragment of Classical Propositional Calculus (CPC^+ for short), i.e., all instances of the following schemas:

$$\alpha \to (\beta \to \alpha) \tag{A1}$$

$$(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)) \tag{A2}$$

$$((\alpha \to \beta) \to \alpha) \to \alpha \tag{A3}$$

$$(\alpha \wedge \beta) \to \alpha \tag{A4}$$

$$(\alpha \wedge \beta) \to \beta \tag{A5}$$

$$\alpha \to (\beta \to (\alpha \wedge \beta)) \tag{A6}$$

$$\alpha \to (\alpha \vee \beta) \tag{A7}$$

$$\beta \to (\alpha \vee \beta) \tag{A8}$$

$$(\alpha \to \gamma) \to ((\beta \to \gamma) \to (\alpha \vee \beta \to \gamma)) \tag{A9}$$

Notice that if (A1), (A2) are theses of $\mathcal{C}$ and (MP) is the sole rule of inference, then the deduction theorem holds for $\mathcal{C}$, that is, for any $\Gamma \subseteq \mathrm{For}$ and $\alpha, \beta \in \mathrm{For}$, we have:

$$\Gamma \cup \{\alpha\} \vdash_{\mathcal{C}} \beta \text{ iff } \Gamma \vdash_{\mathcal{C}} \alpha \to \beta. \tag{DT}$$

If (A9) is a thesis of $\mathcal{C}$ then, for any $\Gamma, \Delta \subseteq \mathrm{For}$ and $\alpha, \beta, \gamma \in \mathrm{For}$, the following holds:

$$\text{if } \Gamma \cup \{\alpha\} \vdash_{\mathcal{C}} \gamma \text{ and } \Gamma \cup \{\beta\} \vdash_{\mathcal{C}} \gamma, \text{ then } \Gamma \cup \{\alpha \vee \beta\} \vdash_{\mathcal{C}} \gamma. \tag{Dis}$$

From Lemma 2.1(2) and (DT), it follows that

$$\alpha \to \alpha \tag{R}$$

is a thesis of $\mathcal{C}$.

From (DT), it also immediately follows that

$$(\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma)) \tag{PoC}$$

$$(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma)) \tag{HS}$$

$$(\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta) \tag{C}$$

are theses of $\mathcal{C}$.

For any calculi $\mathcal{C}_1$ and $\mathcal{C}_2$ (in For), we say that $\mathcal{C}_1$ is an *extension* of $\mathcal{C}_2$ iff $\mathrm{Th}(\mathcal{C}_2) \subseteq \mathrm{Th}(\mathcal{C}_1)$. We say that $\mathcal{C}_2$ is a *proper subsystem* of $\mathcal{C}_1$ (in symbols: $\mathcal{C}_2 \sqsubset \mathcal{C}_1$) iff $\mathrm{Th}(\mathcal{C}_2) \subseteq \mathrm{Th}(\mathcal{C}_1)$ and $\mathrm{Th}(\mathcal{C}_1) \nsubseteq \mathrm{Th}(\mathcal{C}_2)$.

Finally, let INT^+ denote the positive fragment of intuitionistic propositional calculus obtained from CPC^+ by dropping (A3), also known as Peirce's law.

3. The Paraconsistent calculus $\mathbf{CB^1}$. Syntax

The paraconsistent calculus CB^1 is defined, in a Hilbert-style formalization, by (MP), as the sole rule of inference, the axiom schemas (A1)–(A9) and the following ones involving negation:

$$\alpha \vee \neg\alpha \tag{ExM}$$

$$\neg\neg\alpha \to \alpha \tag{NN}$$

$$\alpha \to (\neg\alpha \to (\neg\neg\alpha \to \beta)) \tag{DS^2}$$

In the succeeding paragraphs, we consider several subsystems of CB^1, i.e.: PI (= CLuN), $\mathrm{C_{min}}$ and B^1. They are all defined by (MP), axiom schemas (A1)–(A9) and additionally:

- PI has the axiom (ExM),
- $\mathrm{C_{min}}$ contains the axioms (ExM) and (NN),
- B^1 has the axiom schemas (ExM) and (DS^2).

It is obvious that $\mathrm{C_{min}}$ and B^1 are extensions of PI, whereas CB^1 is an extension of $\mathrm{C_{min}}$, B^1 and PI. It is also known that: (i) (DS^2) and (NN) are not theses of PI; (ii) (NN) is not a thesis of B^1; (iii) (DS^2) is not a thesis of $\mathrm{C_{min}}$. Therefore, we have:

FACT 3.1. 1. $\mathrm{B}^1 \not\sqsubset \mathrm{C_{min}}$ *and* $\mathrm{C_{min}} \not\sqsubset \mathrm{B}^1$.
2. $\mathrm{PI} \sqsubset \mathrm{B}^1$, $\mathrm{PI} \sqsubset \mathrm{C_{min}}$, $\mathrm{B}^1 \sqsubset \mathrm{CB}^1$ *and* $\mathrm{C_{min}} \sqsubset \mathrm{CB}^1$.

Notice that from (DT), (A9), (R), (ExM) and (MP), we obtain:

$$(\neg\alpha \to \alpha) \to \alpha \tag{CM1}$$

$$(\alpha \to \neg\alpha) \to \neg\alpha \tag{CM2}$$

which means that (CM1) and (CM2) are theses of PI. Furthermore, from (DT), (DS^2), (NN), (HS), (PoC), (C), (CM2) and (MP), we receive:

$$(\alpha \to \neg\beta) \to ((\alpha \to \neg\neg\beta) \to \neg\alpha) \tag{NI}$$

Thus, this formula is a thesis of CB^1.

Fact 3.2. *The axioms* (NN) *and* (DS^2) *can be replaced by a single one*:

$$\neg\alpha \to (\neg\neg\alpha \to \beta). \tag{DSn}$$

Consequently, the calculus CB^1 *may as well be defined by the axioms* (A1)–(A9), (ExM), (DSn) *and* (MP) *as the sole rule of inference.*

Proof. To demonstrate that (DSn) is provable in CB^1, consider the sequence of formulas: $\neg\alpha$, $\neg\neg\alpha$; α, by (NN), $\neg\neg\alpha$ and (MP); β, by (DS^2), α, $\neg\alpha$, $\neg\neg\alpha$ and (MP); $\neg\alpha \to (\neg\neg\alpha \to \beta)$, by (DT).

Observe that (PoC) and (CM1) are provable from axioms of CPC$^+$, (ExM), (DSn) and (MP). Now, for (NN): Assume that $\neg\neg\alpha$. Then we have $\neg\alpha \to \alpha$, by (DSn), (PoC), the assumption and (MP). Now, apply (MP) to (CM1) and $\neg\alpha \to \alpha$, to get α. But this means that, by (DT), we obtain (NN). For (DS^2): It is enough to apply (MP) to (A1) and (DSn). ⊣

Theorem 3.3. *For* CB^1, *the following weaker variants of the indirect deduction theorem hold, where $\Gamma \subseteq$* For *and $\alpha, \beta \in$* For:

1. *If $\Gamma, \alpha \vdash_{\mathrm{CB}^1} \neg\beta$ and $\Gamma, \alpha \vdash_{\mathrm{CB}^1} \neg\neg\beta$, then $\Gamma \vdash_{\mathrm{CB}^1} \neg\alpha$.*
2. *If $\Gamma, \neg\alpha \vdash_{\mathrm{CB}^1} \neg\beta$ and $\Gamma, \neg\alpha \vdash_{\mathrm{CB}^1} \neg\neg\beta$, then $\Gamma \vdash_{\mathrm{CB}^1} \alpha$.*

Proof. *Ad* 1. Assume that $\Gamma, \alpha \vdash_{\mathrm{CB}^1} \neg\beta$ and $\Gamma, \alpha \vdash_{\mathrm{CB}^1} \neg\neg\beta$. Then, by (DT), we have $\Gamma \vdash_{\mathrm{CB}^1} \alpha \to \neg\beta$ and $\Gamma \vdash_{\mathrm{CB}^1} \alpha \to \neg\neg\beta$. Since (NI) is a thesis of CB^1, then $\Gamma \vdash_{\mathrm{CB}^1} \neg\alpha$.

Ad 2. Assume that $\Gamma, \neg\alpha \vdash_{\mathrm{CB}^1} \neg\beta$ and $\Gamma, \neg\alpha \vdash_{\mathrm{CB}^1} \neg\neg\beta$. Then, by 1, $\Gamma \vdash_{\mathrm{CB}^1} \neg\neg\alpha$. Since (NN) is an axiom of CB^1, we also have $\Gamma \vdash_{\mathrm{CB}^1} \alpha$. ⊣

There is an important point which has not been discussed yet, namely, whether CB^1 is a paraconsistent calculus. The fact below shows that this is indeed the case.

Fact 3.4. *The formulas*:

$$p \to (\neg p \to q) \tag{DS}$$

$$p \to (\neg p \to \neg q) \tag{DS$^{\neg}$}$$

$$\neg(p \wedge \neg p) \tag{NC}$$

$$p \to \neg\neg p \tag{NN*}$$

are not provable in CB^1. *Moreover, neither* $\{\alpha, \neg\alpha\} \vdash_{\mathrm{CB}^1} \beta$, *nor* $\{\alpha, \neg\alpha\} \vdash_{\mathrm{CB}^1} \neg\beta$, *nor* $\{\alpha \to \beta\} \vdash_{\mathrm{CB}^1} \neg\beta \to \neg\alpha$, *nor* $\{\neg\alpha \to \neg\beta\} \vdash_{\mathrm{CB}^1} \beta \to \alpha$ *hold.*[4]

Proof. Apply the matrix $\mathcal{M}^3 = \langle\{1,2,0\},\{1,2\},\neg,\wedge,\vee,\to\rangle$, where $\{1,2,0\}$ and $\{1,2\}$ are the sets of logical values and designated values, respectively; and $\neg$, $\wedge$, $\vee$, $\to$ are defined as follows:

$\to$	1	2	0
1	1	2	0
2	1	2	0
0	1	1	1

	$\neg$
1	2
2	0
0	2

$\wedge$	1	2	0
1	1	2	0
2	2	2	0
0	0	0	0

$\vee$	1	2	0
1	1	1	1
2	1	2	2
0	1	2	0

The truth tables for implication, conjunction and disjunction are isomorphic to the ones given by Asenjo and Tamburino [1, p. 18]. The truth table for negation seems to be pretty new. Note that each axiom schema of CB^1 is valid in the matrix $\mathcal{M}^3$ and (MP) preserves validity. To demonstrate that (DS), (DS$^{\neg}$), (NC) and (NN*) are not valid in $\mathcal{M}^3$, assign 1 to p in the formulas $\neg(p \wedge \neg p)$ and $p \to \neg\neg p$, respectively; 1 to p and 0 to q in $p \to (\neg p \to q)$; and 1 to p and 2 to q in (DS$^{\neg}$). Next, assign 1 to α and 0 to β in $\{\alpha, \neg\alpha\} \vdash_{\mathrm{CB}^1} \beta$; 1 to α and 2 to β in $\{\alpha, \neg\alpha\} \vdash_{\mathrm{CB}^1} \neg\beta$; 2 to α and 1 to β in $\{\alpha \to \beta\} \vdash_{\mathrm{CB}^1} \neg\beta \to \neg\alpha$; and finally, 0 to α and 1 to β in $\{\neg\alpha \to \neg\beta\} \vdash_{\mathrm{CB}^1} \beta \to \alpha$. ⊣

Now we can prove [for details, see 16 and 8]:

Fact 3.5. $\mathrm{CB}^1 \sqsubset \mathrm{P}^1$, *where* P^1 *is Sette's calculus.*

Proof. In [9, pp. 116–120], it is demonstrated that (A1)–(A9) and (ExM) are theses of P^1. In [8, p. 267], we prove that (DSn) is a thesis of P^1.

[4] The arguments given in Fact 3.4 might be expressed more concisely in a more advanced terminology. For instance, one could perceive that (a) the connective of $\neg$ is not explosive in CB^1 and $\{p, \neg p\} \nvdash_{\mathrm{CB}^1} \neg q$, then CB^1 is *strongly pre-$\neg$-paraconsistent*; (b) $\neg$ is *left-involutive* (but not *right-involutive*); (c) $\neg$ is not *contrapositive*; etc. [for details see 2, Chapter 2]. For the purpose of this paper, however, we have decided to use a simpler set of terms.

Notice that (MP) is the sole rule of inference of both calculi. Thus, all theses of CB^1 are provable in P^1.

Now we show that the following axiom of Sette's calculus P^1 is not provable in CB^1:

$$(p \to q) \to \neg\neg(p \to q) \tag{\$}$$

We apply the matrix $\mathcal{M}^3_\star = \langle\{1,2,0\},\{1,2\},\neg,\wedge,\vee,\to\rangle$, where $\{1,2,0\}$ and $\{1,2\}$ are the sets of logical values and designated values, respectively; and $\neg$, $\wedge$, $\vee$, $\to$ are defined as follows:

$\to$	1	2	0
1	1	2	0
2	1	2	0
0	1	1	1

	$\neg$
1	0
2	1
0	1

$\wedge$	1	2	0
1	1	1	0
2	1	1	0
0	0	0	0

$\vee$	1	2	0
1	1	1	1
2	1	1	1
0	1	1	0

All axioms of CB^1 are valid in $\mathcal{M}^3_\star$ and (MP) preserves validity. Now, assign the value 1 (or 2) to p and 2 to q in ($) to demonstrate that it is not valid in $\mathcal{M}^3_\star$. Thus, ($) is a not a thesis of CB^1. $\dashv$

Let us recall a few well-known facts. For this reason, they will be given without proofs.

FACT 3.6. 1. $\mathrm{INT}^+ \sqsubset \mathrm{CPC}^+ \sqsubset \mathrm{PI}$.
2. $\mathrm{PI} \sqsubset \mathrm{B}^1$ *and* $\mathrm{PI} \sqsubset \mathrm{C}_{\min}$.
3. $\mathrm{P}^1 \sqsubset \mathrm{CPC}$, *where* CPC *is the classical propositional calculus.*

As a summary of this section, let us note that the calculi can be represented by the lattice structure of Figure 1.

4. Bivaluational semantics for CB^1

In this section, we introduce a bivaluational semantics for CB^1. It can be easily obtained from the semantics proposed in [7].

DEFINITION 4.1. A CB^1-valuation is any function $v\colon \mathrm{For} \longrightarrow \{1,0\}$ satisfying, for any $\alpha,\beta \in \mathrm{For}$, the following conditions:

($\vee$) $v(\alpha \vee \beta) = 1$ iff $v(\alpha) = 1$ or $v(\beta) = 1$,
($\wedge$) $v(\alpha \wedge \beta) = 1$ iff $v(\alpha) = 1$ and $v(\beta) = 1$,
($\to$) $v(\alpha \to \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$,
($\neg$) if $v(\neg\alpha) = 0$ then $v(\alpha) = 1$,
($\neg\neg$) if $v(\neg\neg\alpha) = 1$ then $v(\neg\alpha) = 0$.

A formula α is a CB^1-*tautology* iff $v(\alpha) = 1$, for any CB^1-valuation v.

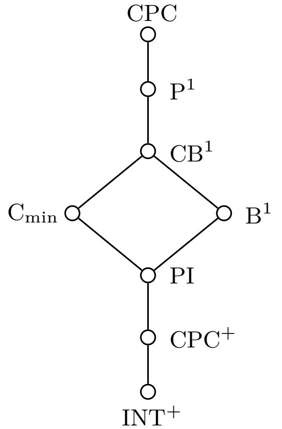

Figure 1. A lattice of the paraconsistent calculi

Definition 4.2. For all $\alpha \in$ For and $\Gamma \subseteq$ For, α is a semantic consequence of Γ (in symbols: $\Gamma \models_{\mathrm{CB}^1} \alpha$) iff for any CB^1-valuation v: if $v(\beta) = 1$ for any $\beta \in \Gamma$, then $v(\alpha) = 1$.

The soundness of CB^1 can be obtained in the standard way, by induction on the length of a derivation in CB^1:

Theorem 4.1. *If* $\Gamma \vdash_{\mathrm{CB}^1} \alpha$, *then* $\Gamma \models_{\mathrm{CB}^1} \alpha$.

For the proof of completeness $\vdash_{\mathrm{CB}^1}$, we use the method which is based on the notion of maximal non-trivial sets of formulas. We apply the technique described in [see 4, Section 2.2]. To begin with, let us recall some important definitions and results.

Let $\mathcal{C} = \langle \mathrm{For}, \mathrm{Ax}_{\mathcal{C}}, \vdash_{\mathcal{C}} \rangle$ be a calculus (satisfying Tarskian properties) and $\Delta \subseteq$ For. We say that Δ is a *closed theory of* $\mathcal{C}$ iff for any $\beta \in$ For: $\Delta \vdash_{\mathcal{C}} \beta$ iff $\beta \in \Delta$. We say that Δ is *maximal non-trivial with respect to* $\alpha \in$ For in $\mathcal{C}$ iff (a) $\Delta \nvdash_{\mathcal{C}} \alpha$ and (b) for any $\beta \in$ For, if $\beta \notin \Delta$ then $\Delta \cup \{\beta\} \vdash_{\mathcal{C}} \alpha$.

Lemma 4.2 (4, Lemma 2.2.5). *Every maximal non-trivial set with respect to some formula is a closed theory.*

Of course, Lemma 4.2 holds for CB^1, as well. Moreover, we have:

Lemma 4.3. *For any maximal non-trivial set* Δ *with respect to* α *in* CB^1 *the mapping* $v\colon \mathrm{For} \longrightarrow \{1, 0\}$ *defined, for any* $\delta \in$ For, *as* $(\star)$: $v(\delta) = 1$ *iff* $\delta \in \Delta$, *is a* CB^1*-valuation.*

Proof. We only show that the conditions for ($\neg$) and ($\neg\neg$) are fulfilled. The rest of the proof is analogous to that of Theorem 2.2.7 in [4].

Assume, for a contradiction, that $v(\neg\beta) = 0$ and $v(\beta) = 0$. Thus we have $\neg\beta \notin \Delta$ and $\beta \notin \Delta$, by ($\star$). Since Δ is a maximal non-trivial set with respect to α, then $\Delta \cup \{\beta\} \vdash_{\mathrm{CB}^1} \alpha$ and $\Delta \cup \{\neg\beta\} \vdash_{\mathrm{CB}^1} \alpha$. Consequently, $\Delta \cup \{\beta \vee \neg\beta\} \vdash_{\mathrm{CB}^1} \alpha$, by (Dis). Hence $\Delta \vdash_{\mathrm{CB}^1} \alpha$, by (ExM) and Lemma 2.1(5). Observe that Δ is a closed theory (see Lemma 4.2), so $\alpha \in \Delta$. But $\alpha \notin \Delta$ (by the main assumption). This yields a contradiction.

Assume, for a contradiction, that $v(\neg\neg\beta) = 1$ and $v(\neg\beta) = 1$. Then $\neg\neg\beta \in \Delta$ and $\neg\beta \in \Delta$, by ($\star$). Hence $\Delta \vdash_{\mathrm{CB}^1} \neg\neg\beta$ and $\Delta \vdash_{\mathrm{CB}^1} \neg\beta$, by Lemma 2.1(2). Notice that (DSn) is a thesis of CB^1, so we have $\Delta \vdash_{\mathrm{CB}^1} \alpha$. Since Δ is a closed theory, hence $\alpha \in \Delta$. But $\alpha \notin \Delta$, by the assumption. This results in a contradiction. $\dashv$

Note that the so-called Lindenbaum–Łoś theorem holds, for any finitary calculus $\mathcal{C} = \langle \mathrm{For}, \mathrm{Ax}_{\mathcal{C}}, \vdash_{\mathcal{C}} \rangle$:

Lemma 4.4 (4, Theorem 2.2.6; 14, Theorem 3.31). *For any $\Gamma \subseteq \mathrm{For}$ and $\alpha \in \mathrm{For}$ such that $\Gamma \nvdash_{\mathcal{C}} \alpha$, there is a maximal non-trivial set Δ with respect to α in $\mathcal{C}$ such that $\Gamma \subseteq \Delta$.*

Thus, the completeness of CB^1 follows:

Theorem 4.5. *For all $\Gamma \subseteq \mathrm{For}$ and $\alpha \in \mathrm{For}$: if $\Gamma \models_{\mathrm{CB}^1} \alpha$, then $\Gamma \vdash_{\mathrm{CB}^1} \alpha$.*

Proof. Assume that $\Gamma \nvdash_{\mathrm{CB}^1} \alpha$ and Δ is a maximal non-trivial set with respect to α in CB^1 such that $\Gamma \subseteq \Delta$ (see Lemma 4.4). Then $\alpha \notin \Delta$. Therefore, by Lemma 4.3, there is a valuation v such that $v(\alpha) = 0$ and $v(\beta) = 1$, for any $\beta \in \Delta$. Hence $\Gamma \not\models_{\mathrm{CB}^1} \alpha$. $\dashv$

5. Hierarchy of CB^m-calculi

Let $m \in \mathbb{N}$ and $\neg^m\alpha$ be an abbreviation for $\overbrace{\neg\neg\ldots\neg}^{m}\alpha$. The hierarchy is obtained by replacing (DSn) with the following schema, for any $m \in \mathbb{N}$:

$$\neg^m\alpha \to (\neg^{m+1}\alpha \to \beta) \qquad (\mathtt{DSn}^m)$$

More precisely, for any $m \in \mathbb{N}$, let CB^m result from CPC$^+$ by adding to it (ExM) and (DSn^m). For $m = 0$, the logic CB^m *collapses* into the classical propositional calculus.

As a semantics for CB^m, for each $m \in \mathbb{N}$, we use CB^m-valuations which are obtained from CB^1-valuations by replacing the condition $(\neg\neg)$ with a more general one, that is,

$(\neg^{m+1})$ if $v(\neg^{m+1}\alpha) = 1$ then $v(\neg^m\alpha) = 0$.

The semantic clauses for $(\vee)$, $(\wedge)$, $(\rightarrow)$ and for $(\neg)$ remain unchanged. The soundness of CB^m is obtained analogously to that of CB^1. For the completeness of CB^m, we need to modify the proof of Lemma 4.3, i.e.:

LEMMA 5.1. *For any maximal non-trivial set Δ with respect to α in* CB^m *the mapping* $v\colon \mathrm{For} \longrightarrow \{1,0\}$ *defined, for any* $\delta \in \mathrm{For}$, *as* $(\star)$: $v(\delta) = 1$ *iff* $\delta \in \Delta$, *is a* CB^m*-valuation.*

PROOF. Assume, for a contradiction, that $v(\neg^{m+1}\beta) = 1$ and $v(\neg^m\beta) = 1$. Then $\neg^{m+1}\beta \in \Delta$ and $\neg^m\beta \in \Delta$, by $(\star)$. Hence $\Delta \vdash_{\mathrm{CB}^1} \neg^{m+1}\beta$ and $\Delta \vdash_{\mathrm{CB}^1} \neg^m\beta$. The formula $(\mathtt{DSn}^m)$ is a thesis of CB^m, so it follows that $\Delta \vdash_{\mathrm{CB}^m} \alpha$. Since Δ is a closed theory (see Lemma 4.2), then $\alpha \in \Delta$. But $\alpha \notin \Delta$ (by the main assumption). This yields a contradiction. $\dashv$

Thus, we receive:

THEOREM 5.2. *For all* $m \in \mathbb{N}$, $\Gamma \subseteq \mathrm{For}$, $\alpha \in \mathrm{For}$: $\Gamma \vdash_{\mathrm{CB}^m} \alpha$ *iff* $\Gamma \models_{\mathrm{CB}^m} \alpha$.

Notice that each calculus in the hierarchy is essentially weaker than the preceding one(s), viz., $\mathrm{CB}^1 \sqsupset \mathrm{CB}^2 \sqsupset \mathrm{CB}^3 \sqsupset \cdots \sqsupset \mathrm{CB}^m \sqsupset \cdots$. The proof that $\mathrm{Th}(\mathrm{CB}^k) \subseteq \mathrm{Th}(\mathrm{CB}^m)$, for $k > m$, basically reduces to the observation that every instance of $(\mathtt{DSn}^k)$ is also an instance of $(\mathtt{DSn}^m)$. Consequently, it suffices to show that the following holds: if $k > m$, then

$$\neg^m p \rightarrow (\neg^{m+1} p \rightarrow q)$$

is not a thesis of CB^k. But this fact can be easily proved with the help of the completeness theorem for CB^k. There is a CB^k-valuation such that $v(\neg^m p) = 1 = v(\neg^{m+1} p)$ and $v(q) = 0$. This entails that:

FACT 5.3. *For any* $k, m \in \mathbb{N}$ *such that* $k > m$, *we have* $\mathrm{CB}^k \sqsubset \mathrm{CB}^m$.

Moreover, we obtain the following result:

FACT 5.4. *Let* $m, k \in \mathbb{N}$ *and* $m \geqslant k$, *then the formula* $\neg^{k-1} p \rightarrow \neg^{k+1} p$ *is not provable in any* CB^m*-calculus.*

PROOF. We have already noticed that $p \to \neg\neg p$ (i.e., $\neg^0 p \to \neg^2 p$) is not provable in CB^1 (and neither is in any CB^m-calculus weaker than CB^1). For the other cases, it suffices to apply the completeness theorem for CB^m. ⊣

At the end of this section, we state two simple facts about CB^m.

FACT 5.5. *For any $m \in \mathbb{N}$, enriching the set of axiom schemas of* CB^m *with the formula $\alpha \to \neg\neg\alpha$, results in the axiom system of* CPC.

In other words, for any $m > 0$, we need to prove that the axiom schemas of CPC^+, (ExM), ($\mathtt{DSn}^m$), ($\mathtt{NN}^\star$) and (MP), as the sole rule of inference, constitute an axiomatization of CPC.

PROOF. It immediately follows due to the fact that the schemas ($\mathtt{DSn}^m$) and ($\mathtt{NN}^\star$) are equivalent to (DS) in CPC; to put it more precisely, let CPC be defined by CPC^+, (ExM), (DS) and (MP). Then ($\mathtt{NN}^\star$) follows from the deduction theorem, (DS), (CM2) and (MP); ($\mathtt{DSn}^m$) is an instance of (DS). Now, for CPC being defined by CPC^+, (ExM), ($\mathtt{DSn}^m$), ($\mathtt{NN}^\star$) and (MP), assume that α and $\neg\alpha$. Let m be even (the proof for m being odd is similar). Then, by α, ($\mathtt{NN}^\star$) (applied $\frac{m}{2}$ times) and (MP), we receive $\neg^m\alpha$. Likewise, by $\neg\alpha$, ($\mathtt{NN}^\star$) and (MP), we get $\neg^{m+1}\alpha$, and finally β by ($\mathtt{DSn}^m$) and (MP). Hence, by (DT), we obtain: $\alpha \to (\neg\alpha \to \beta)$. ⊣

The proof of the following fact is analogous to the proof of Fact 5.5.

FACT 5.6. *For any $m > 1$, enriching the set of axiom schemas of any* CB^m*-calculus with the formula $\neg\alpha \to \neg\neg\neg\alpha$, results in obtaining* CB^1.

PROOF. By (DT), (DSn), (CM2) and (MP), we find that $\neg\alpha \to \neg\neg\neg\alpha$ is a thesis of CB^1. But it is not a thesis of any CB^m-calculus that is weaker than CB^1 (see Fact 5.4). Now it suffices to show that $\neg^m\alpha \to (\neg^{m+1}\alpha \to \beta)$, where $m > 1$, and $\neg\alpha \to \neg\neg\neg\alpha$ are equivalent to ($\mathtt{DS}^2$). ⊣

6. Final remarks

So far, every calculus that has been discussed here contains CPC^+ as its positive base. In this section, we will weaken the base to the positive fragment of intuitionistic propositional calculus. As a result, we will be able to enrich our discussion with a few interesting calculi among which Newton da Costa's calculus C_ω seems to be the most remarkable [see 6, 10, 13 and 9, Section 2.6].

We define, in a Hilbert-style formalization, the following calculi:

1. INTuN := INT^+ + (ExM),
2. C_ω := INTuN + (NN),
3. A^1 := CPC^+ + (DS^2).

The calculus C_ω is well-known in the literature. The calculi INTuN and A^1 are extremely weak and far less known than C_ω. Let us state a few facts about the calculi.

Fact 6.1. 1. $\mathrm{INT}^+ \sqsubset \mathrm{INTuN} \sqsubset \mathrm{C}_\omega \sqsubset \mathrm{C}_{\min}$.
2. $\mathrm{INTuN} \sqsubset \mathrm{PI} \sqsubset \mathrm{B}^1$.
3. $\mathrm{CPC}^+ \sqsubset \mathrm{A}^1 \sqsubset \mathrm{B}^1$.
4. *It is not the case that*
   (a) $\mathrm{C}_\omega \sqsubset \mathrm{PI}$ *or* $\mathrm{PI} \sqsubset \mathrm{C}_\omega$,
   (b) $\mathrm{C}_\omega \sqsubset \mathrm{A}^1$ *or* $\mathrm{A}^1 \sqsubset \mathrm{C}_\omega$,
   (c) $\mathrm{A}^1 \sqsubset \mathrm{PI}$ *or* $\mathrm{PI} \sqsubset \mathrm{A}^1$.

Proof. *Ad* 1. It is obvious that both $\mathrm{INT}^+ \sqsubset \mathrm{INTuN}$ and $\mathrm{C}_\omega \sqsubset \mathrm{C}_{\min}$. To prove that $\mathrm{INTuN} \sqsubset \mathrm{C}_\omega$, we should slightly modify the matrix $\mathcal{M}^3$ (see Fact 3.4), i.e., replace the truth table for negation with the three-valued table for the so-called cyclic (or rotary) negation:

	$\neg$
1	2
2	0
0	1

and assign 0 to p in $\neg\neg p \to p$ of the form (NN).

Ad 2. We have already noticed that $\mathrm{PI} \sqsubset \mathrm{B}^1$ (ct. Fact 3.6). Since $((\alpha \to \beta) \to \alpha) \to \alpha$ is unprovable in C_ω [see 10, Theorem 15, p. 501] and $\mathrm{INTuN} \sqsubset \mathrm{C}_\omega$, then $((p \to q) \to p) \to p$ of the form (A3) is not provable in INTuN, either.

Ad 3. The case $\mathrm{CPC}^+ \sqsubset \mathrm{A}^1$ is trivial. To show that $\mathrm{A}^1 \sqsubset \mathrm{B}^1$, apply the classical truth tables for implication, conjunction, and disjunction. Next, define negation as follows:

	$\neg$
1	0
0	0

and assign 0 to p in $p \vee \neg p$ of the form (ExM).

Ad 4a. Observe that $((p \to q) \to p) \to p$ of the form (A3) is not a thesis of C_ω (see above). To demonstrate that $\neg\neg p \to p$ of the form (NN)

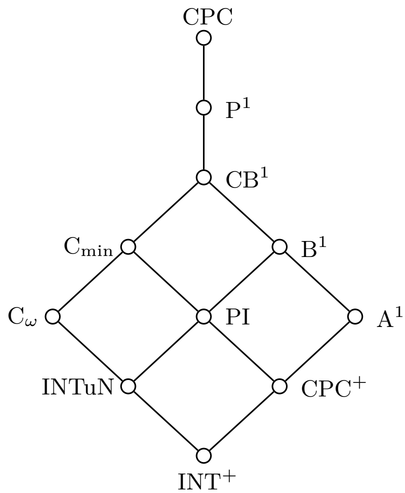

Figure 2. A lattice of the paraconsistent calculi

is not a thesis of PI, it suffices to apply the completeness theorem and semantics for the calculus.

Ad 4b. We need to show that either $\neg\neg p \to p$ of the form (NN) or $p \vee \neg p$ of the form (ExM) is not a theses of A^1 on the one hand, and $p \to (\neg p \to (\neg\neg p \to q))$ of the form (DS^2) is not a thesis of C_ω on the other. The former is obvious (cf. the item 3). The latter can be easily proved with the help of the completeness theorem for C_ω.

Ad 4c. Notice that $p \to (\neg p \to (\neg\neg p \to q))$ of the form (DS^2) is not a theorem of PI (by the completeness theorem and semantics for PI). To prove that $p \vee \neg p$ of the form (ExM) is not a thesis of A^1, it is enough to recall the two-valued matrix given in the item 3 and assign 0 to p in $p \vee \neg p$. ⊣

As a final remark, let us emphasise that all calculi presented in this paper fulfil the requirements for being considered as paraconsistent (at least in a broad sense of the term), and they form a more complex structure than that of Figure 1; see Figure 2.

Acknowledgements. I am very grateful to anonymous reviewers, Andrzej Pietruszczak, and Marek Nasieniewski for their helpful comments on an earlier version of this paper.

References

[1] Asenjo, F. G., and J. Tamburino, "Logic of antinomies", *Notre Dame Journal of Formal Logic* 16, 1 (1975): 17–44. DOI: 10.1305/ndjfl/1093891610

[2] Avron, A., O. Arieli and A. Zamansky, *Theory of Effective Propositional Paraconsistent Logics*, Studies in Logic, Mathematical Logic and Foundations, vol. 75, College Publications, 2018.

[3] Batens, D., "Paraconsistent extensional propositional logics", *Logique et Analys* 23, 90–91 (1980): 195–234.

[4] Carnielli, W., and M. E. Coniglio, *Paraconsistent Logic: Consistency, Contradiction and Negation*, Logic, Epistemology, and the Unity of Science, vol. 40, International Publishing, 2016. DOI: 10.1007/978-3-319-33205-5

[5] Carnielli, W., M. E. Coniglio and J. Marcos, "Logics of formal inconsistency", Chapter 1, pages 1–93, in D. M. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic*, 2nd edition, vol. 14, Springer, 2007. DOI: 10.1007/978-1-4020-6324-4_1

[6] Carnielli, W., and J. Marcos, "Limits for paraconsistent calculi", *Notre Dame Journal of Formal Logi* 40, 3 (1999): 375–390. DOI: 10.1305/ndjfl/1022615617

[7] Ciuciura, J., "Paraconsistent heap. A Hierarchy of mbCn-systems", *Bulletin of the Section of Logic* 43, 3/4 (2014): 173–182.

[8] Ciuciura, J., "Paraconsistency and Sette's calculus P1", *Logic and Logical Philosophy* 24, 2 (2015): 265–273. DOI: 10.12775/LLP.2015.003

[9] Ciuciura, J., *Hierarchies of the Paraconsistent Calculi* (in Polish), Wydawnictwo Uniwersytetu Łódzkiego, Łódź, 2018.

[10] da Costa, N. C. A., "On the theory of inconsistent formal systems", *Notre Dame Journal of Formal Logic* 15, 4 (1974): 497–510. DOI: 10.1305/ndjfl/1093891487

[11] da Costa. N. C. A., and J.-Y. Béziau, "Carnot's logic", *Bulletin of the Section of Logic* 22, 3 (1993): 99–105.

[12] Jaśkowski, S., "Rachunek zdań dla systemów dedukcyjnych sprzecznych" (in Polish), *Societatis Scientiarum Torunensis* Sect. A, I, 5 (1948): 57–77. First English translation: "Propositional calculus for contradictory deductive systems", *Studia Logica* 24 (1969): 143–157. Second one: "A propositional calculus for inconsistent deductive systems", *Logic and Logical Philosophy*, 7 (1999): 35–56. DOI: 10.12775/LLP.1999.003

[13] Loparić, A., "A semantical study of some propositional calculi", *The Journal of Non-Classical Logic* 3, 3 (1986): 73–95.

[14] Pogorzelski, W. A., and P. Wojtylak, *Completeness Theory for Propositional Logics*, Studies in Universal Logic, Birkhäuser, Basel, 2008. DOI: 10.1007/978-3-7643-8518-7

[15] Qingyu, Z., "A weak paraconsistent conditional logic", *Journal of Non-Classical Logic* 8, 1 (1991): 45–57.

[16] Sette, A. M., "On the propositional calculus P^1", *Mathematica Japonicae* 18, 3 (1973): 173–180.

[17] Tuziak, R., "Paraconsistent extensions of positive logic", *Bulletin of the Section of Logic* 25, 1 (1996): 15–20.

[18] Urbas, I., "A note on 'Carnot's logic'", *Bulletin of the Section of Logic* 23, 3 (1994): 118–125

Janusz Ciuciura
Department of Logic and Methodology of Science
Institute of Philosophy
Faculty of Philosophy and History
University of Łódź
Lindleya 3/5
90–131 Łódź, Poland
janusz.ciuciura@uni.lodz.pl