Logic and Logical Philosophy
Volume 29 (2020), 161–187
DOI: 10.12775/LLP.2019.030

Claudio E. A. Pizzi

Explicit Conditionals in the Framework of Classical Conditional Logic

Abstract. The paper proposes a first approach to systems whose language includes two primitives ($>_+$ and $>_-$) as symbols for factual and counterfactual conditionals which are explicit, i.e. that are stated jointly with the truth or falsity of the antecedent clause. In systems based on this language, here called 2-conditional, the standard corner operator may be defined by (Def $>$) $A > B := (A >_+ B) \vee (A >_- B)$, while in classical conditional systems one could introduce the two symbols for explicit conditionals by the definitions (Def $>_+$) $A >_+ B := A \wedge (A > B)$ and (Def $>_-$) $A >_- B := \neg A \wedge (A > B)$. Two 2-conditional systems, V^2 and VW^2, are axiomatized and proved to be definitionally equivalent to the monoconditional systems V and VW. A third system VWTr^2 is characterized by an axiom stating the transitivity of factual conditionals and is shown to be distinct from V^2, from VW^2 and from the 2-conditional version of Lewis' well-known system VC, here named VC^2. The same may be said for a fourth system $\mathrm{VW}^2\Diamond\pm$, based on an axiom inderivable in VC^2: $\Diamond(A >_+ B) \supset \Diamond(\neg A >_- \neg B)$. VC^2 contains what is here called a "semi-collapse" of the operator $>_+$ and it is argued that it is inadequate as a logic for both factual and counterfactual conditionals. The last section shows that several different definitions of the corner operators in terms of $>_+$ and $>_-$ may be introduced as an alternative to (Def $>$).

Keywords: indicative conditionals; counterfactual conditionals; modality; transitivity of conditionals; centering condition

§1. A common feature of the languages of so-called classical conditional logic is that they exhibit one and only one non truth-functional primitive, here symbolized by $>$, aimed at providing a formal representation of ordinary language conditionals. A well-known peculiarity of ordinary conditionals is however that their grammatical and formal properties

Received April 14, 2019. *Revised* May 22, 2019. *Published online* July 15, 2019

depend on the logical relation between their antecedent and the Background Knowledge BK, i.e. on the fact that their antecedent, provided it has a known truth-value, may be or not be in conflict with some element of BK. Extracting a conditional from its context and symbolizing it as $A > B$ says nothing about its being or not being a contrary-to-fact conditional, so that one cannot know if it has or lacks the distinctive properties of such class of conditionals.[1]

A good representation of the ambiguity of unqualified conditionals is offered by an equivalence which is valid in every classical conditional logic, being simply an instance of a truth-functional tautology:

$$A > B \text{ if and only if } \big(A \wedge (A > B)\big) \vee \big(\neg A \wedge (A > B)\big) \tag{$\star$}$$

An appropriate way to read $A > B$ is "A conditionally implies B". The two disjuncts on the right are disambiguated conditionals, i.e. conditionals conjoined with a statement which informs the listener about the truth or the falsity of their antecedent. Even if each of the disjuncts has the logical form of a conjunction, we will call them here *explicit conditionals*. More specifically, $A \wedge (A > B)$ represents what Goodman in [1947, p. 114] calls a *factual* conditional and reads "*since* A, B",[2] while the wff $\neg A \wedge (A > B)$ symbolizes an explicit counterfactual conditional. $\neg A \wedge (A > B)$ should be read "it is not the case that A, and A conditionally implies B" or, more colloquially, "If it were the case that A — which is not — it would be the case that B". The incidental phrase is normally omitted in ordinary language since in most western languages it is usually suggested by the conversational context or by the use of subjunctive mood in place of the indicative, which by contrast is typically used in factual conditionals.[3]

[1] The literature on the topic of the different properties of conterfactuals, subjunctive and so called "indicative" conditionals is huge. For two useful surveys see [Arlò-Costa, 2007] and [Edgington, 2014]. For a commented bibliography see the site https://philpapers.org/browse/logic-of-conditionals.

[2] In [1993, p. 67] J. Corcoran lists eight ordinary expressions which are usually used to assert jointly the truth of the antecedent and the implication from A to B. The most common beyond "since A, B" are "A, therefore B" and "A, so B". Goodman in [1947] seems to believe that every counterfactual conditional may be converted into a factual one by applying contraposition. For some considerations on this question see [Pizzi, 2013].

[3] This does not mean that the subjunctive mood in the antecedent is in itself a mark of counterfactuality. The subjunctive future, for instance, is not logically

In what follows a treatment of conditionals will be given by representing explicit factual and counterfactual conditionals by means of two distinct primitive conditional operators, symbolized by $>_+$ and $>_-$. Logics with such properties will be called *2-conditional*, while standard conditional logics will be called *monoconditional*. A possible advantage of 2-conditional logics is that they are a tool to throw light on the logical relations between factual and counterfactual conditionals, which are obscured if the basic language contains only an unqualified conditional operator. Other advantages are the following:

(a) Having at our disposal two conditional primitives helps conceiving axioms for conditional systems which up to now have not been taken in consideration in the literature. Furthermore, it may allow establishing non-specular axiomatic properties for $>_+$ and $>_-$.

(b) A 2-conditional logic for $>_+$ and $>_-$ is not necessarily one-one translatable into a monoconditional logic via the equivalence ($\star$) above. It may be of some interest to see how a corner operator endowed with properties which are different from the ones mirrored in the equivalence ($\star$) may be introduced by definition in 2-conditional logics.

§2. Let us take as a starting point the language of classical conditional logic consisting of an infinite set of variables for wffs A, B, C, . . . , with or without numerical subscripts, and of the primitive symbols (,), $\bot$, $\supset$, $>$. The rules of formation and of elimination of parentheses are submitted to usual conventions. The auxiliary symbols $\neg$, $\wedge$, $\vee$, $\equiv$, $\top$ are defined as usual. Two useful auxiliary symbols are the following:

(Def $\asymp$) $\quad A \asymp B \;:=\; (A > B) \wedge (B > A)$
(Def $\Supset$) $\quad A \Supset B \;:=\; \neg(A > \neg B)$

distinguishable from the indicative future, since in both cases the antecedent, when it is contingent, has a truth value which is unknown at the time of the utterance. So one could maintain that in such cases $A > B$ is improperly classified as factual or counterfactual, contrary to what is apparently stated in the disjunction in ($\star$); we could call it *afactual* or *neutral-to-fact* conditional. For such conditionals it is intuitive to accept the contraction law $(A > (A > B)) \equiv (A > B)$. Let us now consider the conditional theorem $(A > B) \equiv ((A > A) \wedge (A > B))$. In every regular conditional logic which contains also the contraction law the wff $(A > A) \wedge (A > B)$ is equivalent to $(A > A) \wedge (A > (A > B))$ and to $A > (A \wedge (A > B))$. Such second order formula is then equivalent to $A > B$, so also to ($\star$): $(A \wedge (A > B)) \vee (\neg A \wedge (\neg A > B))$. This suggests that an afactual conditional may be intended as a special kind of second order conditional which has a factual conditional as a consequent, and the disjunction in ($\star$) may be read as having this meaning.

Parentheses will be omitted around wffs of form $A > B$, $A \asymp B$, $A \Supset B$ when no ambiguity will arise.

In the line of [Lewis, 1973] it is possible to define two distinct modal operators for what Lewis calls *outer* and *inner* necessity respectively:

(Def$\Box$) $\Box A \ := \ \neg A > \bot$
(Def$\boxdot$) $\boxdot A \ := \ \top > A$

The following system is the minimal system V of conditional logic as axiomatized in [Nute, 1980, pp. 128–129]. V is equivalent to the system C0 in [Lewis, 1971]. In what follows the terms "axiom" and "theorem" will be used in place of "axiom schema" and "theorem schema".

PCT	The set of all truth-functional tautologies
ID	$A > A$
MOD	$\Box A \supset B > A$
CSO	$A \asymp B \supset (A > C \supset B > C)$
CV	$(A > B \wedge A \Supset C) \supset ((A \wedge C) > B)$

Rules:

MP	Modus Ponens for $\supset$
RCK	From $\vdash (A_1 \wedge \cdots \wedge A_n) \supset B$ infer $\vdash (C>A_1 \wedge \cdots \wedge C>A_n) \supset C>B$, for any $n > 0$
Eq	Replacement of proved material equivalents

Notice that the following rule:

RCE From $\vdash A \supset B$ infer $\vdash A > B$

is derivable as a special case of RCK for $n = 1$ and $C = A = A_1$, by using ID and MP.

A consequence of the definition of the box-operator as $\Box A \ := \ \neg A > \bot$ is the following. $\neg A \supset \bot$ and $\top \supset A$ are both equivalent to A, $\neg A \prec \bot$ (where $\prec$ is the symbol for strict implication, i.e. $A \prec B \ := \ \Box(A \supset B)$) and $\top \prec A$ are both equivalent to $\Box A$ (i.e. to $\neg A > \bot$); however, they are not equivalent to $\top > A$ (i.e. to $\boxdot A$), since $>$ is not in general a contraposable operator. The given definition of $\Box$ allows proving that the $\Box$-fragment of V is the minimal normal system K, while the $\boxdot$-fragment of V is the nonnormal system D2 of Lemmon.[4]

[4] The first result is stated in [Lewis, 1973, p. 137]. D2 is a non-normal system which is like KD with the only difference that the necessitation rule is replaced by the weaker $A \supset B / \Box A \supset \Box B$ [see Lewis, 1973, p. 142].

Let us now introduce the following auxiliary symbols for explicit conditionals of various sorts.

(Def $>_+$)	$A >_+ B \;:=\; A \wedge (A > B)$
(Def $>_-$)	$A >_- B \;:=\; \neg A \wedge (A > B)$
(Def $\Supset_+$)	$A \Supset_+ B \;:=\; A \wedge \neg(A > \neg B)$
(Def $\Supset_-$)	$A \Supset_- B \;:=\; \neg A \wedge \neg(A > \neg B)$
(Def $\asymp_+$)	$A \asymp_+ B \;:=\; A >_+ B \wedge B >_+ A$
(Def $\asymp_-$)	$A \asymp_- B \;:=\; A >_- B \wedge B >_- A$

Parentheses will be omitted around formulas $A \# B$, where $\#$ is one of the symbols introduced in the preceding definitions, when no ambiguity arises.

Remark 1. Lewis reads $A \Supset B$ as "If it were the case that A it *might* be the case that B". So $A \Supset_+ B$ may be read as "Since A is true, B might be true" and $A \Supset_- B$ may be read as "If A were true — which is not — B might be true". One could be willing to remark that an alternative definition of the dual of $>_+$ could be $\neg(A >_+ \neg B)$, i.e. $\neg(A \wedge A > \neg B)$ or equivalently $A \supset \neg(A > \neg B)$, which means "either A is false or $A \Supset B$ is true": an expression lacking any interesting meaning since it is implied by the simple $\neg A$.

Remark 2. In the present paper no treatment is offered of explicit conditionals in which the truth or falsity of their consequent, but not of the antecedent, is explicitly stated: specifically $(A{>}B){\wedge}B$ and $(A{>}B){\wedge}\neg B$. The latter could be read "If it were the case that A it would be the case that B, which is not the case". Note that this statement in system V does not imply that A is false, but implies it in stronger systems (see §4). The formula $(A > B) \wedge B$ has been read in various ways. Pollock in [1975, p. 56] reads it as "B, even if A", while Pizzi in [1980, p. 84] reads it as "B, as if A". "Even if A, B" in [Goodman, 1947] is intended as stating something whose form is $\neg(A > \neg B)$, i.e. $A \Supset B$.

The formula $(A > B) \wedge A \wedge B$, i.e. $A >_+ B \wedge B$, asserts that B is true since A is true, and may be interpreted as implying that the really happened fact B is explained in terms of the really happened fact A.

The following are some V-theorems derived by the application of (Def $>_+$) and (Def $>_-$). In some of them both operators occurr in the same formula:

1. $A \supset A >_+ A$
2. $A \supset A >_- A$

3. $A >_+ B \supset \neg(A >_- B)$
4. $A >_- B \supset \neg(A >_+ B)$
5. $A >_+ A \vee A >_- A$
6. $\neg(A >_- A) \equiv (A >_+ A)$
7. $A >_+ B \supset A > B$
8. $A >_- B \supset A > B$

The following are examples of non-theorems of V+(Def$>_+$)+(Def$>_-$):

9. $A >_+ A$
10. $A >_- A$
11. $(A \wedge B) >_+ A$
12. $(A \wedge B) >_- A$
13. $\neg(A >_+ \neg A)$
14. $\neg(A >_- \neg A)$
15. $A >_+ B \supset \neg B >_+ \neg A$
16. $A >_+ B \supset \neg(A >_+ \neg B)$
17. $A >_- B \supset \neg(A >_- \neg B)$

The proof of the non-theoremhood of the preceding formulas is given by showing that, if they were theorems, an inconsistency or a known non-theorem would follow. As an example, $A >_+ B \supset \neg B >_+ \neg A$ would yield by instantiation $\top >_+ \top \supset \neg\top >_+ \neg\top$. But the antecedent $\top >_+ \top$ is a theorem by (Def$>_+$) and $\vdash \top$, so by Modus Ponens and (Def$>_+$) $\neg\top$ would also be such, which is impossible. $\neg(A >_+ \neg A)$ would yield $\neg(\top >_+ \neg\top)$, so $\Diamond\top$, which does not belong to K, so does not belong to V.

Remark 3. The wff 16 has the same pattern of what is called *Strawson's Thesis* or *Weak Boethius' Thesis*. An instance of 16 would be $\top >_+ \top \supset \neg(\top >_+ \neg\top)$, in which the antecedent is a theorem while the consequent is equivalent to the non-theorem $\Diamond\top$. As for its negative mirror image $A >_- B \supset \neg(A >_- \neg B)$, it is a non-theorem of V since an instantiation would be $(\neg\bot \wedge (\bot > \top)) \supset (\neg\bot \supset (\top > \neg\top))$, so $\bot$. See however in §4 what follows from adding 16 as an axiom to the stronger system VW.

For other non-theorems, such as the transitivity of $>_+$, i.e.

(Trans$_+$) $\quad (A >_+ B \wedge B >_+ C) \supset A >_+ C$

one needs to define a semantics for the background system V. A class of models for V based on so-called set-selection functions is defined as follows.[5]

[5] For a discussion of set-selection functions see [Lewis, 1973, pp. 58–59]. The ax-

If X is a sentence expressible in the language of V and the truth values of any statement is represented as 1 or 0, we will use the notation $[X]$ to denote the set of all X-worlds, i.e. the set of all worlds j s.t. the sentence X has value 1 at j (the evaluation function v being defined at steps (v) and (vi)).

A V-model is a 4-ple $\langle W, F, R, v\rangle$, where

(i) W is a non-empty set of possible worlds;
(ii) R is a binary relation on W;
(iii) F is a function from pairs of sentences and worlds to sets of worlds such that, for every sentence A and world i, $F(A,i)$ has the following properties:
  (a) $F(A,i) \subseteq [A]$,
  (b) if $F(A,i) \subseteq [B]$ and $F(B,i) \subseteq [A]$, then $F(A,i) = F(B,i)$,
  (c) either $F(A \vee B, i) \subseteq [A]$ or $F(A \vee B, i) \subseteq [B]$ or $F(A \vee B, i) = F(A,i) \cup F(B,i)$;
(iv) if $F(A,i) \neq \emptyset$ and $j \in F(A,i)$ then iRj;
(v) v is an evaluation function defined in a standard way as far as truth-functional operators are concerned, and for the corner operator satisfies the following clause:
(vi) $v(A{>}B, i) = 1$ iff either or no k exists such that iRk or $v(B,j) = 1$ for every j in $F(A,i)$.

It is easy to see that from the given conditions it follows that $v(\Box A, i) = 1$ iff $v(A,j)$ at every j such that iRj. A wff A is V-*valid* iff $v(A,i) = 1$ at every i of every V-model.

It is trivial to show that all the theorems of V are V-valid, so that V is sound and consistent w.r.t. the given semantics. The completeness proof for this and other systems may be derived from the one exposed in [Lewis, 1973, pp. 124 ff.].

The underivability of Trans$_+$ in V is proved by constructing a refuting V-model M with the following properties: $W = \{i,j,k\}$, $[B] = \{i,j\}$, $[A] = \{i,k\}$ $[C] = \{j,k\}$, R is arbitrary, $F(A,i) = \{i\}$ and $F(B,i) = \{j\}$. Clearly $A{>_+}B$ and $B{>_+}C$ have value 1 at the world i of the model, while $A >_+ C$ has value 0 at i because $F(A,i)$, i.e. $\{i\}$, is not included in $[C]$.

Note that the underivability of (Trans$_+$) in V implies the underivability in V of (Trans) $(A > B \wedge B > C) \supset A > C$.

---

ioms introduced here for selection functions in V-models are reproduced from [Lewis, 1971, p. 75].

If Trans were a V-theorem, in fact, by *Theorema Praeclarum*, i.e. $((A \supset B) \wedge (C \supset D)) \supset ((A \wedge C) \supset (B \wedge D))$, Trans might be conjoined with the PC-thesis $(A \wedge B) \supset A$ so to yield the V-theorem $((A > B \wedge B > C) \wedge (A \wedge B)) \supset (A > C \wedge A)$, which, by (Def $>_+$), is equivalent to (Trans$_+$).

§3. We now turn our attention to a system whose language has $>_+$ and $>_-$ as two primitive conditional operators. The formation rules are as before but include the two following clauses:

(i) If A and B are wffs, $A >_+ B$ is a wff
(ii) If A and B are wffs, $A >_- B$ is a wff

Auxiliary Symbols:

(Def $>$)	$A > B := A >_+ B \vee A >_- B$
(Def $\Supset_+$)	$A \Supset_+ B := A \wedge \neg(A >_+ \neg B) \wedge \neg(A >_- \neg B)$
(Def $\Supset_-$)	$A \Supset_- B := \neg A \wedge \neg(A >_+ \neg B) \wedge \neg(A >_- \neg B)$
(Def $\Box$)	$\Box A := \neg A >_+ \bot \vee \neg A >_- \bot$
(Def $\boxdot$)	$\boxdot A := \top >_+ A$

REMARK 4. Note that the disjunction $A \asymp_+ B \vee A \asymp_- B$ implies $A \asymp B$, i.e. $(A >_+ B \vee A >_- B) \wedge (B >_+ A \vee B >_- A)$ thanks to the PC-law $((A \wedge B) \vee (C \wedge D)) \supset ((A \vee C) \wedge (B \vee D))$, but the converse implication does not hold. The translation of $A \asymp B$ in 2-conditional language is $(A >_+ B \vee A >_- B) \wedge (B >_+ A \vee B >_- A)$.

The minimal axiom system written in this language will be called V^2 and consists of the following axioms:

1_+	$A \supset (A >_+ A)$
1_-	$\neg A \supset (A >_- A)$
2_+	$\Box A \supset (B \supset (B >_+ A))$
2_-	$\Box A \supset (\neg B \supset (B >_- A))$
3_+	$((A >_+ B \vee A >_- B) \wedge (B >_+ A \vee B >_- A)) \supset (A >_+ C \supset (B >_+ C \vee B >_- C))$
3_-	$((A >_+ B \vee A >_- B) \wedge (B >_+ A \vee B >_- A)) \supset (A >_- C \supset (B >_+ C \vee B >_- C))$
4_+	$(A >_+ B \wedge (A \Supset_+ C \vee A \Supset_- C)) \supset (A \wedge C) >_+ B$
4_-	$(A >_- B \wedge (A \Supset_+ C \vee A \Supset_- C)) \supset (A \wedge C) >_- B$
5_+	$A >_+ B \supset A$
5_-	$A >_- B \supset \neg A$

6_+ $A >_+ B \supset \neg(A >_- B)$
6_- $A >_- B \supset \neg(A >_+ B)$
7_+ $(A \wedge \Box(A \supset B)) \supset A >_+ B$
7_- $(\neg A \wedge \Box(A \supset B)) \supset A >_- B$

and the following rules:

MP: Modus Ponens for $\supset$
Eq: Replacement of proved material equivalents
RCK_+ From $\vdash (A_1 \wedge \cdots \wedge A_n) \supset B$ infer $\vdash (C >_+ A_1 \wedge \cdots \wedge C >_+ A_n) \supset C >_+ B$, for any $n > 0$
RCK_- From $\vdash (A_1 \wedge \cdots \wedge A_n) \supset B$ infer $\vdash (C >_- A_1 \wedge \cdots \wedge C >_- A_n) \supset C >_- B$, for any $n > 0$

Semantics. A V^2-model is a 4-ple $\langle W, F, R, v\rangle$ as the one before defined for V-model, but with the difference that the truth conditions for $>_+$ and $>_-$ are now as follows:

(i) $v(A >_+ B, i) = 1$ iff $v(A, i) = 1$ and no k exists such that iRk or $v(B, j) = 1$ for every j in $F(A, i)$,
(ii) $v(A >_- B, i) = 1$ iff $v(A, i) = 0$ and no k exists such that iRk or $v(B, j) = 1$ for every j in F(A,i).

The notion of validity is the same as for V.

Remark 5. a. The definition $\boxdot A := \top >_+ A$ is equivalent to $\boxdot A := \top >_+ A \vee \top >_- A$, since $\top >_- A$ in V is equivalent to $\bot$; so $\boxdot A := \top >_+ A$ is equivalent to the known definition $\boxdot A := \top > A$.

b. The rule RCE is translated via (Def$>$) into "From $\vdash A \supset B$ infer $\vdash A >_+ B \vee A >_- B$" and follows by each of the two rules RCK_+ and RCK_- by taking A as value of C.

c. $A \supset (A >_+ B)$ and $\neg A \supset (A >_- B)$ jointly imply $A >_+ B \vee A >_- B$, so $A > B$.

Now we define in V a function f from the language of V to the language of V^2 which preserves the properties of truth-functional operators (i.e. $fA = A$ for every A atomic, $f\bot = \bot$ and $f(A \supset B) = (fA \supset fB)$) and furthermore has the following property:

(f) $f(A > B) := fA >_+ fB \vee fA >_- fB.$

In V^2 we define a function g from the language of V^2 to the language of V which behaves as f with respect to the truth functional connectives but for the two conditional operators behaves as follows:

(g1) $g(A >_+ B) := gA \wedge (gA > gB)$,
(g2) $g(A >_- B) := g\neg A \wedge (gA > gB)$.

It is lengthy but *routine* to prove that the two systems V and V^2 are definitionally equivalent or, otherwise said, that they translate each other, in the sense that f and g provide a one-one translation among the theorems of the two systems.

This amounts to proving the following metatheorem (the details of the proof are given in the Appendix):

PROPOSITION 1. V *and* V^2 *are definitionally equivalent systems, i.e. for every* A, A *is a* V*-thesis iff* fA *is a* V^2*-thesis and* A *is a* V^2*-thesis iff* gA *is a* V*-thesis.*

As a consequence of Proposition 1, some properties of V such as consistency and decidability turn out to be properties also of V^2. The completeness of V w.r.t. the class of V-models may be proved by applying the Henkin method along the same lines of [Lewis, 1973, pp. 124–127]. Since the definitions of the modal operators $\Box$ and $\boxdot$ in the two systems have equivalent *definientia* (see Remark 5), the modal fragments of V and V^2 are also coincident.

§4. Let us now look at a system, named VW in the literature, which is V extended with the so-called *Conditional Modus Ponens*:

CMP $A > B \supset (A \supset B)$

or with the equivalent axiom:

CMP′ $(A \wedge B) \supset A \Supset B$

CMP yields a special form of contraposition which is

WC $(A > B \wedge \neg B) \supset \neg A$

It is straightforward to derive from CMP the wff $\neg A > A \supset (\neg A \supset A)$, so also $\Box A \supset A, A \supset \Diamond A$ and *a fortiori* the deontic axiom $\Diamond\top$. Thus VW is a proper extension of V and its modal fragment is KT, as proved in [Lewis, 1973].

At page 166 it was stated that WBT_+, i.e. $A >_+ B \supset \neg(A >_+ \neg B)$, is not a V-theorem. Another proof of the same fact is a consequence of the following two more interesting propositions.

PROPOSITION 2. *In* VW+(Def $>_+$), WBT_+ *is logically equivalent to* T: $\Box A \supset A$.

PROOF. From the modal law $A \supset \Diamond A$ (equivalent to $\Box A \supset A$), the schema $A >_{+} B \supset A$ and the transitivity of $\supset$, (Def $>$) and PC one obtains $(*)$ $A >_{+} B \supset (\Diamond A \wedge A > B)$. Suppose now by a contradiction that $A > B \wedge A > \neg B$. Then we have $A > (B \wedge \neg B)$, so $A > \bot$, which means $\neg\Diamond A$; thus, by PC, $(**)$ $(\Diamond A \wedge A > B) \supset \neg(A > \neg B)$. Then by the transitivity of $\supset$ applied to $(*)$ and $(**)$ we have $A >_{+} B \supset \neg(A > \neg B)$ and, by PC and (Def $>_{+}$), $A >_{+} B \supset \neg(A >_{+} \neg B)$.

In the other direction:

1.	$(A \wedge A > B) \supset (\neg A \vee \neg(A > \neg B))$	WBT_{+}, (Def $>_{+}$)
2.	$(A \wedge A > B) \supset (A \supset \neg(A > \neg B))$	1, PC
3.	$(A \wedge A > B) \supset \neg(A > \neg B)$	2, PC
4.	$(\neg A \wedge \neg A > A) \supset \neg(\neg A > \neg A)$	3, $\neg A$ for A, A for B
5.	$(\neg A \wedge \Box A) \supset \bot$	4, (Def $\Box$), PC
6.	$\Box A \supset A$	5, PC ⊣

PROPOSITION 3. WBT_{+} *is underivable in* V^2+(Def $>$) *and in any of its extensions not containing* $\Box A \supset A$.

PROOF. Obvious consequence of Proposition 2 and of the fact that the modal fragment of V^2 does not contain $\Box A \supset A$ since it is the minimal normal system K. ⊣

Let us now consider the following 2-conditional axiom which aims to be a mirror-image of CMP in 2-conditional language:

CMP° $\quad A >_{+} B \supset (A \supset B)$

Given that $(A >_{+} B) \supset A$ is a V^2-axiom and that B implies $A \supset B$, two axioms which are equivalent to CMP° are

CMP°° $\quad A >_{+} B \supset B$

and

CMP_{+} $\quad A >_{+} B \supset (A \wedge B)$

CMP° in V+(Def $>_{+}$) is equivalent to $(A \wedge A > B) \supset (A \supset B)$, so by PC to $A \supset (A > B \supset (A \supset B))$ and, again by PC, to $A > B \supset (A \supset B)$.

An useful remark is that in the light of CMP_{+} we might introduce a new operator ${}_{+}{>}_{+}$

(Def ${}_{+}{>}_{+}$) $\quad A \;{}_{+}{>}_{+}\; B := A \wedge B \wedge A > B$

and realize that in any system containing CMP_{+}, $A \;{}_{+}{>}_{+}\; B$ is equivalent to $A >_{+} B$.

Furthermore, in any system containing CMP_+ we have the following two theorems which surrogate contraposition:

(ci) $(A >_+ B \wedge \neg B) \supset \neg A$
(cii) $(A\ {}_+{>}_+\ B \wedge \neg B) \supset \neg A$

which are clearly both easily seen to be equivalent to CMP_+ via the contrapositivity of $\supset$.

The conjunction $A \wedge B \wedge A > B$ could be qualified as a *fully explicit* factual conditional (for a reading of it in natural language see Remark 2). In a parallel way, we could define in V a fully explicit counterfactual conditional as follows:

(Def ${}_{\_}{>}_{\_}$) $A\ {}_{\_}{>}_{\_}\ B := \neg A \wedge \neg B \wedge A > B$

Obviously, $A\ {}_{\_}{>}_{\_}\ B$ entails $A >_{\_} B$, but the converse is unprovable in all non trivial conditional systems, due to the failure of unrestricted contraposition in all them.

Let us now call VW^2 the system $\mathrm{V}^2{+}\mathrm{CMP}_+$. Note that we do not need to introduce a mirror image of CMP_+ for $>_{\_}$, since $A >_{\_} B \supset (A \supset B)$ is an obvious theorem following from axiom 5_-.

Now we may prove what follows:

PROPOSITION 4. VW^2 *and* VW *are definitionally equivalent systems.*

PROOF. We already know that V and V^2 are definitionally equivalent systems. So we have simply to prove that CMP and CMP_+ are intertranslatable axioms. Given CMP_+, it is straightforward to see that in VW^2 both $A >_+ B$ and $A >_{\_} B$ imply $A \supset B$, so $(A >_+ B \vee A >_{\_} B) \supset (A \supset B)$ is a VW^2-thesis. Suppose A and B are atomic, so $fA = A$ and $fB = B$. $(fA >_+ fB \vee fA >_{\_} fB) \supset (fA \supset fB)$ reduces to $f(A > B \supset (A \supset B))$, which means that the f-image of CMP is a VW^2-theorem. The extension to the case of non-atomic wffs is trivial.

In the other direction: the g-image of CMP_+ is $(gA > gB \wedge gA) \supset (gA \wedge gB)$, which follows from $(gA > gB) \supset (gA \supset gB)$, so from CMP ($A$ and B atomic). Then the g-image of CMP_+ is a VW-theorem.

Thus, all and only the f-images of VW-theses are VW^2-theses and all and only the g-images of VW^2-theses are VW-theses. $\dashv$

In what follows we prove that there is formula involving both $>_+$ and $>_{\_}$ which is equivalent to CMP_+ in VW^2. This wff is

(BA) $(A >_+ B \wedge \neg B) \supset A >_{\_} B$

What is the intuitive meaning of $(\neg B \wedge A >_{+} B) \supset A >_{-} B$? Suppose you say $A >_{+} B$, i.e. "Since A, B", but that it turns out that B is false ($\neg B$): then, since A is also false by contraposition, you have to correct your factual conditional into a counterfactual one. This conversion is possible only thanks to the special contrapositivity formula (ci) above. As a matter of fact, BA establishes a simple but non-trivial relation between factual and counterfactual conditionals.

In the following proof of the equivalence between BA and CMP_{+} no use is made of the translation between the $>_{-}/>_{+}$-language and the $>_{-}$-language.

Proposition 5. V^2+BA *and* V^2+CMP_{+} *are equivalent systems.*

Proof. ($\Rightarrow$) Deriving BA from CMP_{+} in V^2 is trivial since, given CMP_{+}, the two antecedents $\neg B$ and $A >_{+} B$ are inconsistent. The steps are:

1. $A >_{+} B \supset (A \wedge B)$	CMP_{+}
2. $(A >_{+} B \wedge \neg B) \equiv \perp$	1, (ci), Ax 5_{+}, PC
3. $(A >_{+} B \wedge \neg B) \supset A >_{-} B$	$\vdash \perp \supset (A >_{-} B)$, 2, Eq

($\Leftarrow$) The derivation from left to right is as follows:

1. $(\neg B \wedge A >_{+} B) \supset A >_{-} B$	BA
2. $\neg B \supset (A >_{+} B \supset A >_{-} B)$	1, PC
3. $B \vee (A >_{+} B \supset (A \wedge A >_{-} B))$	2, Ax 5_{+}
4. $B \vee (A >_{+} B \supset \perp)$	3, Ax5_{-} $\vdash (A \wedge A >_{-} B) \supset \perp$
5. $B \vee \neg(A >_{+} B)$	4, PC
6. $A >_{+} B \supset B$	5, PC
7. $A >_{+} B \supset (A \wedge B)$	6, Ax 5_{+} $\dashv$

The semantics for VW^2 is a simple extension of the semantics for V^2: VW^2-models are defined as V^2-models satisfying the additional condition

(vii) For every $i \in W$, if $v(A, i) = 1$ then $i \in F(A, i)$

The proof that VW^2 is sound and complete w.r.t. this class of models is a standard extension of the proof already known for V^2.

§5. It is a well-known fact that transitivity of the corner operator "$>$" is not a law of classical conditional logic. Counterexamples based on counterfactual conditionals are well-known in the literature. But also conditionals which are "neutral-to-fact" do not support transitivity. For instance:

(a) If Smith were to win the elections against Allen, Allen would reform his party.
(b) If Allen were to die before the elections, Smith would win the elections.
(c) If Allen were to die before the elections, Allen would reform his party.

On the other hand, if CMP is accepted as a principle, transitivity is intuitive for factual conditionals. Suppose in fact that A is true. Then "since A, B" implies B by CMP and "since B, C" implies C by CMP. So, given such premises, the conclusion "since A, C" is intuitively sound. However, we may prove the following theorem.

Proposition 6. Trans$_+$, *i.e.* $(A >_+ B \wedge B >_+ C) \supset A >_+ C$, *is underivable in* VW^2.

Proof. Writing for sake of simplicity $z \in [X]$ instead of $v(X, z) = 1$, a countermodel to Trans$_+$ is provided by a VW^2-model $\langle W, F, R, v\rangle$, where $W = \{i, j, k, l, m\}$, $[A] = \{i, l, m\}$, $[B] = \{i, j, l\}$, $[C] = \{i, k\}$, $F(A, i) = \{i, l\}$, $F(B, i) = \{i\}$, R is reflexive. Thus, A, B and C are true at i and $F(A, i)$ and $F(B, i)$ satisfy the conditions required for VW^2-models. Now $F(A, i)$ and $F(B, i)$ are not empty, $F(A, i) \subseteq [B]$, $F(B, i) \subseteq [C]$, so, being A, B, C all true at i, $A >_+ B$ and $B >_+ C$ have value 1 at i. But $F(A, i)$ (i.e. $\{i, l\}$) is not included in $[C]$ (i.e. in $\{i, k\}$). Thus $A >_+ C$ has value 0 at i. By the completeness of VW^2 Trans$_+$ then is not a VW^2-theorem. ⊣

There are plausible reasons then to propose the introduction of Trans$_+$ as an axiom schema which can be usefully subjoined to VW^2. In the $>_-$-language a parallel axiom schema, which we will name Factual Transitivity, would be simply formulated as

FT $\quad (A \wedge A > B \wedge B > C) \supset A > C$

The system VW^2+Trans$_+$ will be named VWTr^2, while VW+FT will be named VWTr. It is easy then to prove what follows:

Proposition 7. VWTr^2 (*i.e.* VW^2+Trans$_+$) *and* VWTr (*i.e.* VW+FT) *are definitionally equivalent systems.*

Proof. It is enough to prove that $g((A >_+ B \wedge B >_+ C) \supset A >_+ C)$ is a VWTr^2-theorem and that $f((A \wedge A > B \wedge B > C) \supset A > C)$ is a VWTr-theorem. Now $g((A >_+ B \wedge B >_+ C) \supset A >_+ C)$ reduces, for A, B, C atomic, to $(A \wedge B \wedge A > B \wedge B > C) \supset (A \wedge A > C)$, which in VWTr follows from FT by PC. On the other hand $f((A \wedge A > B \wedge B > C) \supset A > C)$ reduces, by CMP, to (F^*) $(A \wedge (A >_+ B \vee A >_- B) \wedge B \wedge (B >_+ C \vee B >_- C)) \supset (A \wedge (A >_+ C \vee A >_- C))$. Let us here observe that $\vdash (A \wedge A >_- B) \equiv \bot$, while $\vdash (A \wedge A >_+ B) \equiv A >_+ B$, by axiom 5_+: so $A \wedge (A >_+ B \vee A >_- B)$ is actually equivalent to $A >_+$ B. By an analogous argument the remaining conjuncts of (F^*) containing conditionals turn out to be equivalent to $B >_+ C$ and $A >_+ C$, respectively. So $f((A \wedge A > B \wedge B > C) \supset A > C)$ is equivalent to Trans_+. ⊣

The semantic counterpart of the new axiom FT which is subjoined to VW is provided by the following condition:

(viii) if $v(A, i) = 1$ and if, for every j in $F(A, i)$, $v(B, j) = 1$, $F(A, i) \subseteq F(B, i)$.

A VWTr-model is a VW-model with the further property (viii). Moreover, a VWTr^2-model is a VW^2-model with the further property (viii).

Remark 6. The new axiom FT added to VW does not allow deriving Trans in VWTr. Unrestricted transitivity of the $>$-operator is invalid in all systems of classical conditional logic. We have simply to remark that in order to validate Trans, clause (viii) above should omit the condition that $v(A, i) = 1$. See also the following Remark 8.

The definition of VWTr-models and VWTr^2-models allows proving the following results.

Proposition 8. *All* VWTr^2-*theorems are* VWTr^2-*valid.*

Proof. Clearly all VW^2-theorems are VWTr^2-valid. We need only show that Trans_+ has the required property. Suppose there is a world i at which $v(A >_+ B, i) = 1$ and $v(B >_+ C, i) = 1$. This means that B is true at all worlds in $F(A, i)$, C is true at all worlds in $F(B, i)$ and that $v(A, i) = 1$. But if $v(A, i) = 1$, $F(A, i) \subseteq F(B, i)$, and this implies that C is true at all worlds in $F(A, i)$. On the other hand, since $v(A, i) = 1$, $v(A >_+ C, i) = 1$. ⊣

By a parallel argument one could easily prove:

Proposition 9. *All* VWTr *theorems are* VWTr-*valid.*

The consistency of VWTr^2 and VWTr trivially follows from Propositions 8 and 9. Two other propositions of interest are the following:

Proposition 10. VW^2 *and* VWTr^2 *are distinct systems.*

Proof. At page 11 it was proved that Trans_{+} is underivable in VW^2, so that VWTr^2 is a proper extension of VW^2. ⊣

An obvious consequence of Proposition 10 is

Proposition 11. VW *and* VWTr *are distinct systems.*

To conclude we need a proof that the conditional operator $>$ in VWTr does not collapse on $\supset$ or on $\prec\!\!3$ (non triviality), and more generally that VWTr^2 and WWTr are non-trivial system. Such result follows from a theorem which will be proved in the following Proposition 12 by observing that VWTr is a proper subsystem of the well-known system VC, which is known to be non-trivial.

Remark 7. It is well-known that various variants of transitivity may surrogate transitivity for the standard conditional operator [see Lewis, 1973, p. 33]. One of them is the so-called "Limited Transitivity", which belongs to V and to all its extensions:[6]

LT $\quad (A > B \wedge (A \wedge B) > C) \supset A > C$

Now by LT, via the PC law $(\neg A \wedge \neg(A \wedge B)) \supset \neg A$ and *Theorema Praeclarum*, we obtain $(\neg A \wedge \neg(A \wedge B) \wedge (A > B \wedge (A \wedge B) > C)) \supset (\neg A \wedge A > C)$, so by (Def $>_{\_}$) and Exportation

$\mathrm{CLT}_{-} \quad A >_{\_} B \supset ((A \wedge B) >_{\_} C \supset A >_{\_} C)$

But from LT it is also straightforward to derive

CLT $\quad (A \wedge B \wedge A > B \wedge (A \wedge B) > C) \supset (A \wedge A > C)$

so also

$\mathrm{CLT}_{+} \quad A >_{+} B \supset ((A \wedge B) >_{+} C \supset A >_{+} C)$

which is the factual mirror-image of CLT_{-}.

However, $A >_{\_} B \supset (B >_{\_} C \supset A >_{\_} C)$ is underivable in VWTr^2 for the reasons which will be exposed in Remark 8, p. 178. This establishes

[6] The proof of LT is based on axiom CSO of p. 2: $A \asymp B \supset (A > C \supset B > C)$. Suppose in fact $A > B$, so $A > (A \wedge B)$. Since $(A \wedge B) > A$ is a theorem, the supposition is equivalent to $A \asymp (A \wedge B)$. Then from the law $(A \wedge B) > C \supset (A \wedge B) > C$ we obtain by CSO the conclusion $(A \wedge B) > C \supset A > C$ and, by Importation, $(A > B \wedge (A \wedge B) > C) \supset A > C$.

an important feature of system VWTr^2. In fact, its characteristic axiom Trans_+ identifies properties of $>_+$ which are not mirrored by properties of $>_-$.

§6. We recall that the characteristic axiom of VW^2 is CMP_+, i.e. $A >_+ B \supset (A \wedge B)$, and is validated by the semantic clause (vii): For every $i \in W$, if $v(A, i) = 1$ then $i \in F(A, i)$ (Weak Centering). We are now interested in seeing what happens in extending VW^2 with an axiom which is the converse of CMP_+, i.e.

$\mathrm{CS}_+ \quad (A \wedge B) \supset A >_+ B$

This system will be called VC^2. In a parallel way, one may extend VW with the axiom

$\mathrm{CS} \quad (A \wedge B) \supset A > B$

obtaining the system called VC by Lewis in [1971]. The proof of the definitional equivalence of VC and VC^2 is trivial and is left to the reader.

We have simply to remark that the counterfactual mirror image of CS_+, i.e. $(A \wedge B) \supset A >_- B$, entails the non-theorem $(A \wedge B) \supset \neg A$, so it is inconsistent with every classical conditional logic.

As is well-known, the semantic counterpart of CS is provided by the so-called *Centering Condition*:

(CC) if $v(A, i) = 1$ then $F(A, i) = \{i\}$.

A VC-model is then a VW-model with the property (CC), and a VC^2-model is a VW^2-model with the same property. An acquired result is that VC is sound and complete w.r.t. to the class of VC-models and then that VC^2 is sound and complete w.r.t. the class of VC^2-models. Now a straightforward consequence of CS_+ in VC^2 is the equivalence

SemiColl $\quad A >_+ B \equiv (A \wedge B)$

This amounts to a collapse of every explicit factual conditional on the conjunction of its clauses: we could call it *semicollapse* of the operator $>_+$. The intuitive meaning of Semicoll is that "Since A, B" and "A and B" make the same assertion, which appears to be counterintuitive or simply false. Another consequence of Semicoll, given that B is equivalent to $(A \wedge B) \vee (\neg A \wedge B)$, is that replacing each of the disjuncts with the equivalents occurring in Semicoll one obtains

ET $\quad (A >_+ B \vee \neg A >_+ B) \equiv B$

ET states that "it rains", for instance, is equivalent to "it rains since the Loch Ness monster exists or it rains since the Loch Ness monster does not exist", two disjuncts which intuitively appear both to be false.

Another consequence of Semicoll concerns transitivity. Given that a PC-theorem is

$(*)\qquad ((A \wedge B) \wedge (B \wedge C)) \supset (A \wedge C)$

by replacement in $(*)$ of the proved equivalents occurring in Semicoll we obtain

$(**)\qquad (A >_+ B \wedge B >_+ C) \supset (A >_+ C)$

i.e. the transitivity of factual conditionals, which is the characteristic axiom of VWTr^2. A consequence of this result are the two following propositions:

PROPOSITION 12. VWTr *is a proper subsystem of* VC.

PROPOSITION 13. VWTr^2 *is a proper subsystem of* VC^2.

REMARK 8. $(**)$ of course does not imply that transitivity is a universal property of conditional operators both in VWTr and VWTr^2. In particular, transitivity of explicit counterfactuals $(A >_- B \wedge B >_- C) \supset A >_- C$ does not hold in VC nor in weaker systems. A countermodel to it is simply provided by a VC-model in which $W = \{i, j, k\}$, iRj, iRk, $F(A, i) = \{k\}$, $F(B, i) = \{j\}$. The assignment are $[A] = \{k\}$, $[B] = \{j, k\}$, $[C] = \{j\}$. In this model $\neg A \wedge A > B$ and $\neg B \wedge B > C$ are true at i, but $\neg C \wedge A > C$ is false at i.

The derivation of Semicoll suggests that VC^2 is not a good logic for explicit factual conditionals. A further negative consideration is suggested by the following VC^2-theorem:

$(***)\qquad A \equiv \Diamond A >_+ A$

$(***)$ follows from the instance of Semicoll $A \wedge \Diamond A \equiv (\Diamond A >_+ A)$ and from the equivalence $A \equiv (\Diamond A \wedge A)$, which is an obvious KT-theorem. Furthermore, given that $A \supset \Diamond A$ and $A > \Diamond A$ are both VC-theorems, we have also the theorem:

$(***')\qquad A \equiv \Diamond A \asymp A$

The formula $(***)$ states that asserting that A is true is equivalent to asserting that A is true *since* $\Diamond A$ is true. This may be read as a clear assertion of the fact that the truth of A depends from the truth of $\Diamond A$

(and also vice versa, as stated in ($***'$)). Furthermore, given that an instance of ($***'$) is

$$(***'') \qquad \neg A \equiv \Diamond\neg A >\!\!<_+ \neg A,$$

a simple PC-argument leads to the proof of

$$(***\vee) \qquad \Diamond A >\!\!<_+ A \vee \Diamond\neg A >\!\!<_+ \neg A$$

i.e. to the strange assertion that either A or $\neg A$ are true and conditionally equivalent to their own possibility.[7]

Other surprising properties of VC and VC[2] result not from their theorems but from their non-theorems. The formula

$$(\Diamond>) \qquad \Diamond(A > B) \supset \Diamond(\neg A > \neg B)$$

is incompatible with VC and also with VW. This is proved by the fact that its instance $\Diamond(\bot>\top) \supset \Diamond(\top>\bot)$ leads by Modus Ponens (given that $\Diamond(\bot > \top)$ is a thesis) to the contradiction $\Diamond\bot$. But we ask whether an inconsistency also follows from the following variant of ($\Diamond>$) for explicit conditionals:

$$(\Diamond\pm) \qquad \Diamond(A >_+ B) \supset \Diamond(\neg A >_- \neg B)$$

Via CMP we may straightly derive from ($\Diamond\pm$) its variant containing what we called at page 10 "fully explicit conditionals".

$$(\Diamond\pm\text{fe}) \qquad \Diamond(A >_+ B \wedge B) \supset \Diamond(\neg A >_- \neg B \wedge B)$$

From ($\Diamond\pm$fe), via the KT-thesis $A \supset \Diamond A$, we obtain

$$(\pm) \qquad (A >_+ B \wedge B) \supset \Diamond(\neg A >_- \neg B \wedge B)$$

The last formula is not devoid of philosophical interest. In fact $A>_+ B\wedge B$ should be read as "B since A", and according to the counterfactual theory of causation, $\neg A >_- \neg B \wedge B$ should be read "A is a causal factor of B". The implication represented in ($\pm$) is strongly intuitive, since it asserts that, if an explanation of B in terms of A exists, it may be that a causal explanation of B in terms of A also exists. A slightly different

[7] Lewis was well aware of the problematic consequences of the Centering Condition and suggested as a possible way out [see Lewis, 1973, p. 29] the notion of "weakly centered" systems of spheres. According to this idea the innermost non-empty sphere around i is made of worlds which are similar to i as i itself but distinct from i: an idea which appears to be incompatible with the so-called Identity of Indiscernables. This proposal enhances systems weaker than VC but at the price of making unreliable the very idea of similarity among possible worlds. For semantic analyses which reject CS see [Gundersen, 2004; von Kutschera, 1974].

interpretation says that if A is a *ceteris paribus* sufficient condition for B, it might be a *ceteris paribus* necessary condition for it. Note that also the converse of ($\Diamond\pm$), i.e. $\Diamond(\neg A >_{-} \neg B) \supset \Diamond(A >_{+} B)$, is *prima facie* plausible; however, if A is $\top$ and B is $\bot$, the antecedent $\Diamond(\top \wedge \bot > \top)$ is a thesis while the consequent $\Diamond(\bot \wedge \bot > \bot)$ is a contradiction. In the light of this, we have to check that ($\Diamond\pm$) is consistent with all conditional systems considered up to now.

We now may prove what follows:

Proposition 14. ($\Diamond\pm$) *is consistent with all systems included in* VC^2.

Proof. It is enough to prove that the wff ($\Diamond\pm$) is consistent with VC^2. This property is proved via a standard kind of argument used in modal logic, which is simply sketched here. Let tr be a function applied to VC^2-wffs which commutes with truth-functional operators and such that $\mathrm{tr}(A >_{+} B) = \mathrm{tr}(A) \wedge \mathrm{tr}(B)$ and $\mathrm{tr}(A >_{-} B) = \neg(\mathrm{tr}A)$. It follows, by means of (Def$\Box$), that $\mathrm{tr}(\Box A) = \mathrm{tr}(\Diamond A) = \mathrm{tr}(A)$. The reader can check that applying the tr-reduction to the axioms of VC^2 yields PC-valid wffs and that the rules of inference preserve such property. The tr-reduction applied to ($\Diamond\pm$) yields the PC-tautology $(A \wedge B \wedge B) \supset A \wedge B$, and this is enough to prove that VC^2+($\Diamond\pm$) is a consistent system. ⊣

However, ($\Diamond\pm$) is not derivable in VC^2 and in any of its subsystems.

Proposition 15. *The wff* ($\Diamond\pm$), *i.e.* $\Diamond(A >_{+} B) \supset \Diamond(\neg A >_{-} \neg B)$, *is not a* VC^2*-theorem.*

Proof. Let us consider a VC^2-model where $W = \{i, j\}$, iRi, iRj and for A, B atomic wffs, $\{i\} = F(A, i)$, $\{j\} = F(\neg A, i)$, $v(A, i) = v(B, i) = 1$, $v(A, j) = 0$, $v(B, j) = 1$. It is easy to check that $v(A >_{+} B \wedge B, i) = 1$, so also $\Diamond(A >_{+} B)$, by KT. But $v(\neg A >_{-} \neg B, i) = 0$: in fact, A is true at i and false at j, $F(\neg A, i)$ is $\{j\}$ and $v(\neg B, j) = 0$, since $v(B, j) = 1$. Also, $F(\neg A, j)$ is $\{j\}$, by Centering, and $v(\neg B, j) = 0$. For this reason $\neg A >_{-} \neg B$ is false at both i and j. So $\Diamond(\neg A >_{-} \neg B)$ is false at both i and j, while $\Diamond(A >_{+} B)$ is true at i. So ($\Diamond\pm$) cannot be a VC^2-theorem. ⊣

In the light of Propositions 14 and 15, we are allowed to build two systems of different strength: one which is VW^2+($\Diamond\pm$), and will be called VW$\Diamond\pm^2$, and another which is VWTr^2+($\Diamond\pm$) and will be called VWTr$\Diamond\pm^2$. The semantics for such system is provided by defining a new condition on models:

(SfNec) if iRj, $v(A, j) = 1$ and $F(A, j) \subseteq [B]$, then there is a k such that iRk and $F(\neg A, k) \subseteq [\neg B]$.

VW^2-models and VWTr^2-models endowed with the additional property (SfNec) will be called VW$\Diamond\pm^2$-models and VWTr$\Diamond\pm^2$-models, respectively. It is then a trivial exercise to prove what follows:

Proposition 16. VW$\Diamond\pm^2$ *is sound w.r.t. the class of* VW$\Diamond\pm^2$*-models.*

Proposition 17. VWTr$\Diamond\pm^2$ *is sound w.r.t. the class of* VWTr$\Diamond\pm^2$*-models.*

Proposition 18. VWTr$\Diamond\pm^2$ *and* VW$\Diamond\pm^2$ *are distinct systems.*

The reader can check that extending VWTr$\Diamond\pm^2$ with $\Box A \equiv A$ yields an inconsistency. This amounts to a proof of the following proposition.

Proposition 19. VWTr$\Diamond\pm^2$ *is a non-trivial system.*

The interrelation among the 2-conditional systems examined in this paper is then visualized as follows, where the lines form top to bottom symbolize proper inclusion:

$$\begin{tikzcd}
 & & \mathrm{V}^2 \arrow[d, dash] & & \\
 & & \mathrm{VW}^2 \arrow[dll, dash] \arrow[drr, dash] & & \\
\mathrm{VWTr}^2 \arrow[d, dash] \arrow[drr, dash] & & & & \mathrm{VW}\Diamond\pm^2 \arrow[dll, dash] \\
\mathrm{VC}^2 & & \mathrm{VWTr}\Diamond\pm^2 & &
\end{tikzcd}$$

§7. The directions of inquiry which can be developed in the framework of conditional systems written in a 2-conditional language are various. A hint to one of the most natural perspectives will be sufficient.

The logics having $>_+$ and $>_-$ as primitives which have been examined up to now are intertranslatable with logics written in the $>_-$-language. But the translation functions proposed in the preceding pages are not the only possible translations which have some plausibility. Surely it is difficult to imagine that $>_+$ and $>_-$ may be defined in terms of $>$ in some different way as the one represented at p. 4. But we may conceive translations of the $>$-operator in terms of $>_+$ and $>_-$ which are different from the one which has been introduced in (Def$>$).

An analogy with tense-logic gives some useful suggestion in this concern. As is well-known, tense-logic is a bimodal logic whose language has two primitive operators P and F for past and future respectively, or alternatively their dual H and G. But in terms of H and G various definitions of the notion of temporal necessity have been introduced by ancient and modern philosophers[8], and the same we may propose to do with the notion of a conditional operator.

As an example, a plausible alternative definition of the corner operator (now symbolized by $>^*$) to be introduced in some 2-conditional systems is the following:

$$(\text{Def}\,>^*) \qquad A >^* B \;:=\; A >_+ B \vee A >_- B \vee (A \wedge B)$$

In the light of the old definition (Def >) the preceding definition could also be written as

$$(\text{Def}\,>^*) \qquad A >^* B \;:=\; A > B \vee (A \wedge B)$$

The dual operator $A \Supset^* B$ turns out to be equivalent to $A \Supset B \wedge (A \supset B)$. A straightforward consequence of (Def $>^*$) is that in any 2-conditional system extended with (Def $>^*$) we have the three theorems $A >_+ B \supset A >^* B$, $A >_- B \supset A >^* B$ and $(A \wedge B) \supset A >^* B$.

Let us suppose to take the weak system VW^2 (equivalent to VW) as background system. The question is: which is the fragment of VW^2+ (Def $>^*$) containing only theorems whose language has only truthfunctional operators and $>^*$? We already know that (Def >) allows deriving the system VW, so all the axioms ID, MOD, CSO, CV, CMP and the rules MP, RCK, Eq.

The theorem $(A \wedge B) \supset A >^* B$ is the $>^*$-version of CS (we will call it CS*). Now it is not difficult to realize that not only CS but that all the theorems of the system VC may be reobtained for $>^*$. An useful premise is provided by the fact that, obviously, $A > B$ implies $A >^* B$ and $A \asymp B$ and $(A \wedge B)$ both imply $A \asymp^* B$.

We have then in VW2+(Def $>^*$) the following theorems:

ID*	$A >^* A$
MOD*	$\Box A \supset (B >^* A)$
CSO*	$A \asymp^* B \supset (A >^* C \supset B >^* C)$

[8] The three following definitions are the most well-known: $\Box A := A \wedge \mathrm{G}A$ (Diodorean); $\Box A := A \wedge \mathrm{H}A \wedge \mathrm{G}A$ (Megaric); $\Box A := \mathrm{HG}A$ (Smirnov).

(Notice that $A \mathrel{>\!\!<}^* B$ is equivalent to $A \mathrel{>\!\!<} B \vee (A \wedge B)$. In VW $A \wedge B$, coinjoined with $A > C \vee (A \wedge C)$, implies $B > C \vee (B \wedge C)$: in fact $A \wedge B$ and $A > C$ by CMP has as a consequence $B \wedge C$, so also $B > C \vee (B \wedge C)$. Furthermore, $A \wedge B$ coinjoined with $A \wedge C$ implies $B \wedge C$, so again $B > C \vee (B \wedge C)$. On the other hand, by CSO, the first $A \mathrel{>\!\!<} B$ implies $A > C \supset B >^* C$. And $A \mathrel{>\!\!<} B$ coinjoined with $A \wedge C$ by CMP implies $B \wedge C$, so $B >^* C$. Then $A \mathrel{>\!\!<} B$ implies $(A > C \vee A \wedge C) \supset B >^* C$, so $A >^* C \supset B > *C$.)

CV* $\quad (A >^* B \wedge A \Supset^* C) \supset (A \wedge C) >^* B$

(Notice that $(A > B \wedge A \Supset C) \supset (A \wedge C) > B$ is the axiom CV. So $(A > B \wedge A \Supset C) \supset (A \wedge C) >^* B$. A PC-theorem is $(A \wedge B \wedge (A \supset C)) \supset (A \wedge B \wedge C)$. $(A \wedge B \wedge C)$ implies $(A \wedge C) >^* B$, so $(A > B \wedge A \Supset C) \vee (A \wedge B \wedge (A \supset C))$ implies $(A \wedge C) >^* B$. But $(A >^* B \wedge A \Supset^* C)$ implies $(A > B \wedge A \Supset C) \vee (A \wedge B \wedge (A \supset C))$, so by transitivity $(A \wedge C) >^* B$.)

Two basic derived rules are the following:

RCK* $\quad$ From $\vdash (A_1 \wedge \cdots \wedge A_n) \supset B$ infer $\vdash (C >^* A_1 \wedge \cdots \wedge C >^* A_n) \supset C >^* B$, for any $n > 0$.

(Notice that from the hypothesis $\vdash (A_1 \wedge \cdots \wedge A_n) \supset B$ we have (∘) $\vdash ((A_1 \wedge \cdots \wedge A_n) \wedge C) \supset (B \wedge C)$. Now a PC-rule is (∘∘): $\vdash (K \supset Z)$ and $\vdash (K' \supset Z')$ jointly imply $\vdash (K \vee K') \supset (Z \vee Z')$. By RCK $\vdash (A_1 \wedge \cdots \wedge A_n) \supset B$ implies $\vdash (C > A_1 \wedge \cdots \wedge C > A_n) \supset C > B$, and this by (∘) and (∘∘) implies $\vdash ((C > A_1 \wedge \cdots \wedge C > A_n) \vee (A_1 \wedge \cdots \wedge A_n) \wedge C) \supset (C > B \vee (B \wedge C))$. So from $\vdash (A_1 \wedge \cdots \wedge A_n) \supset B$ we may infer $\vdash (C >^* A_1 \wedge \cdots \wedge C >^* A_n) \supset C >^* B$.)

RCE* $\quad$ From $\vdash A \supset B$ infer $\vdash A >^* B$.

(Obvious consequence of RCE: From $\vdash A \supset B$ infer $\vdash A > B$.)

A conjecture (that will not be proved here), is that VC is exactly the fragment containing all and only the theorems of VW^2+(Def $>^*$) whose symbols are the ones for truth-functional operators and also $>^*$.

As a conclusive suggestion, we list two other possible alternative definitions of the corner operator which would yield, inside VW^2, mono-conditional fragments stronger than VC:

(Def $>^{**}$) $\quad A >^{**} B \ := \ A >_+ B \vee A >_- B \vee A \Supset B$

(Def $>^{***}$) $\quad A >^{***} B \ := \ A >_+ B \vee A >_- B \vee A \Supset_+ B$

Appendix

LEMMA 1. *For every A, if A is a* V^2*-thesis, gA is a* V*-thesis.*

PROOF. By induction on the length of the proofs. It is left to reader to check that the g-translations of the axioms of V^2 may be proved to be V-theorems. The inference rules MP and Eq hold in both systems. From the inference rule RCK for $>$ one derives the following two rules:

$g\mathrm{RCK}_+$ From $\vdash (A_1 \wedge \dots \wedge A_n) \supset B$ infer $\vdash ((C \wedge C > A_1) \wedge \dots \wedge (C \wedge C > A_n)) \supset (C \wedge C > B)$ for any $n > 0$

$g\mathrm{RCK}_-$ From $\vdash (A_1 \wedge \dots \wedge A_n) \supset B$ infer $\vdash ((\neg C \wedge C > A_1) \wedge \dots \wedge (\neg C \wedge C > A_n)) \supset (\neg C \wedge C > B)$, for any $n > 0$

It is easy to see that for any A: $C > A \supset C > B$ follows from $((C \wedge C > A) \supset (C \wedge C > B)) \wedge ((\neg C \wedge C > A) \supset (\neg C \wedge C > B))$. So the conjunction of $g\mathrm{RCK}_+$and $g\mathrm{RCK}_-$ implies RCK. But $g\mathrm{RCK}_+$ and $g\mathrm{RCK}_-$ are the g-translations of RCK_+ and RCK_-,and they are both derived in V *via* the PC-rule $A \supset B/(C \wedge A) \supset (C \wedge B)$ ⊣

LEMMA 2. *For every A, if A is a* V*-thesis, fA is a* V^2*-thesis.*

PROOF. By induction on the length of the proofs. The f-images of the axioms of V are derived from the ones of V^2. The proof runs as follows for each one of the V-axioms:

ID: $f(A > A)$, i.e. $fA >_+ fA \vee fA >_- fA$, follows by simple steps from axiom 1_-: $\neg A \supset (A >_- A)$ and from axiom 1_+: $A \supset (A >_+ A)$ and the clause $fA = A$ for A atomic.

MOD: $f(\neg A > A \supset B > A)$, follows by simple steps and (Def$\Box$) from axiom 7_+, i.e. $\Box A \supset (B \supset (B >_+ A)$ and axiom 7: $\Box A \supset (\neg B \supset B >_- A)$.

CSO: we have to prove in V^2 $f(A \asymp B \supset (A > C \supset B > C))$, i.e. $((fA >_+ fB \vee fA >_- fB) \wedge (fB >_+ fA \vee fB >_- fA)) \supset ((fA >_+ fC \vee fA >_- fC) \supset (fB >_+ fC \vee fB >_- fC))$.

For A, B, C atomic this means that we have to prove (#): $((A >_+ B \vee A >_- B) \wedge (B >_+ A \vee B >_- A)) \supset ((A >_+ C \vee A >_- C) \supset (B >_+ C \vee B >_- C))$. From axiom 3_+, i.e. $((A >_+ B \vee A >_- B) \wedge (B >_+ A \vee B >_- A)) \supset (A >_+ C \supset (B >_+ C \vee B >_- C))$ and from axiom 3_-, i.e. $((A >_+ B \vee A >_- B) \wedge (B >_+ A \vee B >_- A)) \supset (A >_- C \supset (B >_+ C \vee B >_- C))$, by composition of the consequents and by the PC law $((C \supset A) \wedge (D \supset A)) \supset ((C \vee D) \supset A)$ it is straightforward to derive (#).

CV: $f((A > B \wedge A \Supset C) \supset (A \wedge C > B))$ is provable from the two axioms 4_+: $(A >_+ B \wedge (A \Supset_+ C \vee A \Supset_- C)) \supset (A \wedge C) >_+ B$ and 4_-: $(A >_- B \wedge (A \Supset_+ C \vee A \Supset_- C)) \supset (A \wedge C) >_- B$.

Note that $A \Supset_+ C \vee A \Supset_- C$ may be exported from 4_+ and 4_-, and this disjunction is $f(A \Supset C)$. Now (i): $f(A \Supset C) \supset (A >_+ B \supset (A \wedge C) >_+ B)$ and (ii): $f(A \Supset C) \supset (A >_- B \supset (A \wedge C) >_- B)$ in PC respectively entail (i*): $f(A \Supset C) \supset (A >_+ B \supset ((A \wedge C) >_+ B \vee (A \wedge C) >_- B)$ and (ii*): $f(A \Supset C) \supset (A >_- B \supset ((A \wedge C) >_- B \vee (A \wedge C) >_- B)$. Now a valid PC-rule is $Z \supset (K \supset R)$, $Z \supset (K' \supset R)/Z \supset ((K \vee K') \supset R)$. By applying this rule to (i*) and (ii*) and reimporting $f(A \Supset C)$ we reach the required $f((A > B \wedge A \Supset C) \supset (A \wedge C) > B)$. ⊣

LEMMA 3. *For every A of the* V^2*-language,* $\vdash_{\mathrm{V}^2} A \equiv fgA$.

PROOF. By induction on the complexity of the wffs. Suppose that the theorem holds for arbitrary wffs A and B. Then we have to prove in V^2 the two equivalences

$(_+1)$ $\quad A >_+ B \equiv fg(A >_+ B)$
$(_-1)$ $\quad A >_- B \equiv fg(A >_- B)$

As for $(_+1)$: what we have to prove is equivalent, by the clause $g1$ of the definition of g (see p. 170), to $A >_+ B \equiv f(gA \wedge (gA > gB))$ and by the definition (f) (see p. 169) to $A >_+ B \equiv fgA \wedge ((fgA >_+ fgB) \vee (fgA >_- fgB))$. Hence, given that by hypothesis, $\vdash A \equiv fgA$ and $\vdash B \equiv fgB$, for every A and B, we have to prove in V^2 $A >_+ B \equiv (A \wedge (A >_+ B \vee A >_- B))$.

From left to right the argument is as follows. By virtue of axiom 5_+ of V^2, i.e. $(A >_+ B) \supset A$, by the PC-law $A >_+ B \supset (A >_+ B \vee A >_- B)$ and by composition of the consequents we have $A >_+ B \supset (A \wedge (A >_+ B \vee A >_- B))$, as required.

From right to left: $A \wedge (A >_+ B \vee A >_- B)$ implies by PC $(A \wedge A >_+ B) \vee (A \wedge A >_- B)$. But the second disjunct is equivalent to $\perp$, by axiom 5_-, so the right hand side of the equivalence boils down to $A \wedge A >_+ B$ and implies $A >_+ B$.

The argument to prove $(_-1)$ is obviously specular to the preceding one. ⊣

LEMMA 4. *For every A of the* V*-language,* $\vdash_{\mathrm{V}} A \equiv gfA$.

PROOF. Supposing that the theorem holds for arbitrary A and B, the only interesting case is to prove in V the equivalence $A > B \equiv gf(A > B)$. The equivalence is derived from the first formula (i) of the following list, which in its turn is derived via the induction hypothesis from the equivalence (iv) (a PC-law), and the equivalences (iii) and (ii):

(i) $A > B \equiv g(fA >_{+} fB \vee fA >_{-} fB)$
(ii) $A > B \equiv g(fA >_{+} fB) \vee g(fA >_{-} fB)$
(iii) $A > B \equiv (gfA \wedge gfA > gfB) \vee (gf\neg A \wedge gfA > gfB)$
(iv) $A > B \equiv ((A \wedge A > B) \vee (\neg A \wedge A > B))$ $\dashv$

From Lemmas 1–4 we conclude with the statement of Proposition 1 (see p. 170): V *and* V^2 *are definitionally equivalent systems.*

Acknowledgements. The author is grateful to Walter Carnielli, John Corcoran, Andrea Jacona and Mauro Nasti for useful comments to a preceding draft of the paper. An anonymous referee was also helpful in suggesting the developments outlined in the last section.

References

Arlò-Costa, I., 2007, "The logic of conditionals", in *Stanford Encyclopedia of Philosophy*, https://plato.stanford.edu/entries/logic-conditionals/.

Corcoran, John, 1993, "Meanings of implication", *Dialogos* 9, 1973: 59–76. Reprinted in R. Hughes (ed.), *Philosophical Companion to First Order Logic*, Indianapolis: Hackett, 1993.

Edgington, D., 2014, "Indicative conditionals" in *Stanford Encyclopedia of Philosophy*, https://plato.stanford.edu/entries/conditionals/.

Goodman, N., 1947, "The problem of counterfactual conditionals", *Journal of Philosophy* 44: 113–128.

Gundersen, L., 2004, "Outline of a new semantics for counterfactuals", *Pacific Philosophical Quarterly* 85 (1): 1–20. DOI: 10.1111/j.1468-0114.2004.00184.x

Lewis, D. K., 1971, "Completeness and decidability of three logics of counterfactual conditionals", *Theoria* 37: 74–85

Lewis, D. K., 1973, *Counterfactuals*, Oxford: Blackwell.

Nute, D., 1980, *Topics in Conditional Logic*, Dordrecht: Reidel. DOI: 10.1007/978-94-009-8966-5

Pizzi, C., 1980, "'Since', 'Even if', 'As if'", pages 73–87, chapter 6, in M. L. Dalla Chiara (ed.), *Italian Studies in the Philosophy of Science*, Studies in the Philosophy of Science, vol. 47, Reidel, Dordrecht. DOI: 10.1007/978-94-009-8937-5_6

Pizzi, C., 2013, "Counterfactuals and Modus Tollens in abductive arguments", *Logic Journal of the IGPL* 21 (6): 962–979. DOI: 10.1093/jigpal/jzt014

Pollock, J. L., 1975, "Four kinds of conditionals", *American Philosophical Quarterly* 1: 51–59.

von Kutschera, F., 1974, "Indicative conditionals", *Theoretical Linguistics* 1 (1–3): 257–269. DOI: 10.1515/thli.1974.1.1-3.257

Claudio E. A. Pizzi
University of Siena
Italy
bonafous@outlook.it