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CONNEXIVE LOGICS
An overview and current trends

Abstract. In this introduction, we offer an overview of main systems developed in the growing literature on connexive logic, and also point to a few topics that seem to be collecting attention of many of those interested in connexive logic. We will also make clear the context to which the papers in this special issue belong and contribute.

Keywords: connexive logic; principle of conjunctive contrariety; consequential implication; cancellation account of negation; contra-classical logics

1. Introduction

This second special issue on connexive logic, after [35], consists of papers presented at *the third workshop on connexive logic* held at Kyoto University, Japan, in September 2017, as well as papers submitted in response to an open call for papers.[1] Connexive logics are traditionally characterized as systems validating the theses of Aristotle and Boethius, namely the following theses:
Aristotle $\sim(\sim A \to A)$, $\sim(A \to \sim A)$;
Boethius $(A \to B) \to \sim(A \to \sim B)$, $(A \to \sim B) \to \sim(A \to B)$.
In this introduction, we present a survey of systems developed in the growing literature on connexive logic, and also point to a few topics that seem to be collecting attention of many of those interested in connexive

[1] See https://sites.google.com/site/connexivelogic/events/kncl3 for the details of the workshop.

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logic. We will also make clear the context to which the papers in this special issue belong and contribute.

2. Systems of connexive logic: an overview

In this section, we offer an overview of systems of connexive logic from a semantic perspective.[2] Note that this overview is not meant to be complete by any means, but still covers the main systems that contributed substantially in the development of connexive logic.[3] The systems are presented in chronological order, with the semantics, as well as some key features.

2.1. Preliminaries

Our propositional language consists of a finite set C of propositional connectives and a countable set Prop of propositional variables which we refer to as $\mathcal{L}_{\mathsf{C}}$. Furthermore, we denote by $\mathsf{Form}_{\mathsf{C}}$ the set of formulas defined as usual in $\mathcal{L}_{\mathsf{C}}$. In this paper, we always assume that $\{\sim, \rightarrow\} \subseteq \mathsf{C}$ and just include the symbols for the propositional connective(s) not from $\{\sim, \rightarrow\}$ in the subscript of '$\mathcal{L}_{\mathsf{C}}$'. For example, we write $\mathcal{L}_{\wedge}$ and $\mathsf{Form}_{\wedge}$ to mean $\mathcal{L}_{\{\sim,\rightarrow,\wedge\}}$ and $\mathsf{Form}_{\{\sim,\rightarrow,\wedge\}}$ respectively. Moreover, we denote a formula of $\mathcal{L}_{\mathsf{C}}$ by A, B, C, etc. and a set of formulas of $\mathcal{L}_{\mathsf{C}}$ by Γ, Δ, Σ, etc.

2.2. Angell-McCall: a many-valued approach

The founders of modern connexive logic are Richard Angell and Storrs McCall. The motivation for Angell, in [1], was to devise a formal system that realizes what he calls the *principle of subjunctive contrariety*, i.e., the principle that 'If p were true then q would be true' and 'If p were true then q would be false' are incompatible. In order to show the consistency of his formal system, Angell made use of the following four-valued matrix.

[2] There is, for example, also the early, purely axiomatic approach presented by Everett Nelson in [14], and there exist different kinds of proof system for various connexive logics that one could consider in an attempt to systematize the field.

[3] One of the contributions to connexive logic that is not mentioned in the following overview is the work by Shahid Rahman and Helge Rückert, who introduced dialogical games and dialogical tableaux rules for a connexive conditional in [27].

DEFINITION 1. An *Angell-McCall-interpretation* for $\mathcal{L}_\wedge$ is a function $I\colon \mathsf{Prop} \longrightarrow \{1,2,3,4\}$. I is then extended to $v\colon \mathsf{Form}_\wedge \longrightarrow \{1,2,3,4\}$ by the following truth tables:

A	$\sim A$
1	4
2	3
3	2
4	1

$A \wedge B$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	3	4
4	4	3	4	3

$A \to B$	1	2	3	4
1	1	4	3	4
2	4	1	4	3
3	1	4	1	4
4	4	1	4	1

Finally, $\Gamma \models A$ iff for all Angell-McCall-interpretations I, $v(A) \in \{1,2\}$ if $v(B) \in \{1,2\}$ for all $B \in \Gamma$.

REMARK 2. McCall then later axiomatized the above matrix in [10], and also discussed its relation to the syllogism in [11]. More recently, in [12], McCall adds **FDE**-conjunction and the classical conditional to his system with the following tables (notation modified):

$A \wedge_{\mathbf{FDE}} B$	1	2	3	4
1	1	2	3	4
2	2	2	4	4
3	3	4	3	4
4	4	4	4	4

$A \supset B$	1	2	3	4
1	1	1	3	4
2	1	1	3	3
3	1	1	1	1
4	1	1	1	1

This motivates to represent Angell's values 1, 2, 3, and 4 in analogy to Michael Dunn's representation of the four truth values of **FDE** (i.e., **t**, **b**, **n**, and **f**) as the four subsets $\{1\}$, $\{0,1\}$, $\varnothing$, and $\{0\}$, respectively, of the set $\{0,1\}$ (seen as the set of classical truth values).[4] Under this reading, from an algebraic perspective, McCall's negation emerges as classical negation, [see 4, p. 828]. Moreover, what we then obtain is the following truth condition for the non-classical conditional, by applying the mechanical procedure described in [19] for turning truth tables employing the four truth values of **FDE** into Dunn conditions (i.e., pairs of positive and negative conditions in terms of containing or not containing the classical values 0 or 1):

$$1 \in I(A \to B) \text{ iff } ((1 \notin I(A) \text{ or } 1 \in I(B)) \;\&\; (0 \in I(A) \text{ iff } 0 \in I(B)).$$

This shows that the connexive arrow of Angell-McCall is obtained by adding a condition to the *truth condition* for the classical material conditional.

[4] For a recent overview of **FDE** and related systems, see [20].

2.3. Routley, Mortensen and Brady: an approach based on relevant logics

The system of Angell and McCall based on the four-valued matrix received some immediate criticisms by Richard Routley and Hugh Montgomery in [29], as well as by John Woods in [see 36]. Routley later came up with a formal system based on one of his specialist fields namely relevant logics (also called ‘relevance logics’). The following semantics was presented in [28] for a language containing fusion (intensional, multiplicative) conjunction, $\bullet$.

Definition 3. A *Routley* interpretation for $\mathcal{L}_\bullet$ is an 8-tuple $\langle T, O, K, R, S, *, G, I\rangle$, where O, K are nonempty sets with $O \subseteq K$, $T \in O$ is the base world, R, S are ternary relations on K, $*\colon K \longrightarrow K$, G is a relation on $\mathsf{Form} \times K$ and $I\colon \mathsf{Form} \times K \to \{0, 1\}$, such that

- $a \leq a$, where $b \leq c =_{\text{def.}}$ for some $x \in O$, $Rxbc$;
- if $a \leq d$ and $Rdbc$ then $Rabc$;
- $a = a^{**}$;
- if $a \leq b$ then $b^* \leq a^*$;
- if $a \leq b$ and $Scda$ then $Scdb$;
- $I(p, a) = 1$ and $a \leq b$ then $I(p, b) = 1$;
- $I({\sim}A, a) = 1$ iff $I(A, a^*) \neq 1$;
- $I(B \to C, a) = 1$ iff for all $b, c \in W$: if $Rabc$ and $I(B, b) = 1$ then $I(C, c) = 1$;
- $I(B \bullet C, a) = 1$ iff for some $b, c \in W$: $Sbca$, $I(B, b) = 1$ and $I(C, c) = 1$;
- If AGb then $I(A, b) = 1$.

Finally, $\Gamma \models A$ iff for every Routley interpretation I, $I(T, A) = 1$ if $I(T, B) = 1$ for all $B \in \Gamma$.

Remark 4. There are two deviations from the standard semantics for relevant logics. First, the relation S is used to interpret fusion instead of using R because of the ‘distance’ of $\bullet$ and $\to$. Second, another additional element, namely G, is introduced, and ‘AGb’ is read as: proposition A generates situation b, by which is meant that everything that holds in situation b is implied by A. This is needed for capturing connexive theses.

Remark 5. Further developments following Routley’s system include the contributions by Chris Mortensen and Ross Brady. More specifically, Mortensen [13] models Aristotle’s theses by using slightly simpler models (still with 6-tuples!), motivated by Routley’s 8-tuple models. This was followed by Brady [2], who presented yet another kind of models which aims at working with less *ad hoc* models.

A note on the papers in the special issue.[5] Nissim Francez's contribution is related to the combination of relevant logics and connexive principles. More specifically, Francez constructs a system of natural deduction that combines the relevant conditional of **R** with the connexive theses inspired by Wansing's approach (see §2.6 below on this approach).

2.4. Pizzi: consequential implication

Claudio Pizzi is yet another scholar who was motivated by the works of Angell and McCall, and has been continuing to work on connexive logics since [23]. Pizzi's works builds on the themes of counterfactual logics as well as relevance logics, but here we focus on the more recent work related to the notion of relevance for conditionals. One of Pizzi's key ideas on relevance is called *consequential relevance*, and requires the following: "the antecedent and the consequent of a true conditional cannot have incompatible modal status." [24, p. 127] Pizzi, who considers $\mathcal{L}_{\wedge,\vee,\supset}$ then suggests to realize his idea by requiring the following three conditions:

- A strictly implies B;[6]
- It is false that A and B have incompatible modal status, or in other words we have:
  1. $\sim(\Box A \wedge \sim\Box B)$, i.e., $\Box A \supset \Box B$,
  2. $\sim(\Diamond A \wedge \sim\Diamond B)$, i.e., $\Diamond A \supset \Diamond B$,
  3. $\sim(\sim\Box A \wedge \Box B)$, i.e., $\Box B \supset \Box A$,
  4. $\sim(\sim\Diamond A \wedge \Diamond B)$, i.e., $\Diamond B \supset \Diamond A$.
- Finally, $\Box A$ and $\Box\sim A$ are contraries, which means that a basic thesis for the logic of such operators is $\sim(\Box A \wedge \Box\sim A)$, i.e. $\Box A \supset \Diamond A$ which is the well-known deontic axiom D.

Based on these considerations, we may introduce the following semantics which is an alternative presentation of the semantics for the system **CI** introduced in [25].[7]

[5] We will use this way of highlighting contributions to this special issue with a bar on the left.

[6] In the sense of strict implication: $(A \to B)$ only if $\Box(A \supset B)$.

[7] Pizzi's original presentation deploys translation into modal language.

DEFINITION 6. A *Pizzi-Williamson interpretation* for $\mathcal{L}_{\wedge,\vee,\supset}$: a triple $\langle W, R, I\rangle$, where W is a non-empty set, R is a serial binary relation on W, and $I\colon W \times \mathsf{Prop} \longrightarrow \{0, 1\}$. The function I is extended to $V\colon W \times \mathsf{Form}_{\wedge,\vee,\supset} \longrightarrow \{0, 1\}$ as follows:

- $V(w, p) = I(w, p)$,
- $V(w, {\sim}A) = 1$ iff $V(w, A) \neq 1$,
- $V(w, A \vee B) = 1$ iff $V(w, A) = 1$ or $V(w, B) = 1$,
- $V(w, A \wedge B) = 1$ iff $V(w, A) = 1$ and $V(w, B) = 1$,
- $V(w, A \supset B) = 1$ iff $V(w, A) \neq 1$ or $V(w, B) = 1$,
- $V(w, A \to B) = 1$ iff for all $x \in W$ s.t. wRx: both $V(x, A) \neq 1$ or $V(x, B) = 1$, and for every modal status Θ: $V(w, \Theta A \equiv \Theta B) = 1$.

Finally, $\Sigma \models A$ iff for every Pizzi-Williamson interpretation $\langle W, R, I\rangle$ and every $w \in W$, $V(w, A) = 1$ if $V(w, B) = 1$ for all $B \in \Sigma$.

REMARK 7. First, the above system **CI** is definitionally equivalent to the modal logic **KD** [see 25]. Second, assume the following in addition to **CI**:

- $(A \to B) \supset (A \supset B)$,
- $(A \to B) \to {\sim}(A \to {\sim}B)$, i.e., one of Boethius' theses.

Then, we obtain the following: $(A \to B) \equiv (B \to A)$. This can be seen as a tension between the above thesis of Boethius and consequential implication. It is worth noting here in relation to the above result that Pizzi lays emphasis on the following slightly different version of Boethius' thesis since the very first paper on consequential implication in [23]:

$$(A \to B) \supset {\sim}(A \to {\sim}B).$$

This thesis is called *weak Boethius' thesis*, for example, in [25]. Finally, note that Aristotle's theses and the weak Boethius thesis are valid in **CI**.

2.5. Priest: a cancellation account of negation and connexivity

One of the later developments in connexive logics that deserves special attention is the contribution made by Graham Priest in [26]. In brief, Priest gave a formalization of the cancellation account of negation, and then observed some connections between the cancellation account of negation and connexive logics. According to the cancellation account of negation as discussed, for example, in Richard and Valerie Routley's [30], "${\sim}A$ deletes, neutralizes, erases, cancels A (and similarly, since the relation is symmetrical, A erases $\sim A$), so that $\sim A$ together with A leaves nothing, no content." In the Routleys' opinion, the cancellation view of negation not only validates Aristotle's and Boethius' theses but also invalidates some "degenerate" cases of conjunction elimination, namely $(A \wedge \sim A) \vdash A$ and $(A \wedge \sim A) \vdash \sim A$, because they subscribe to a variable containment condition for valid entailments as a relevance criterion [see also 28]. Since they believe that connexive logic is arrived at by endorsing the notion of negation as cancellation, they also come to believe that the failure of conjunction elimination is characteristic of connexive logic. However, there are other roads to connexivity, and the cancellation account of negation has been criticized in [34].

Definition 8. A *Priest interpretation* for $\mathcal{L}_{\wedge,\vee}$ is a triple $\langle W, g, I\rangle$, where W is a non-empty set, $g \in W$ and $I : W \times \mathsf{Prop} \to \{0,1\}$. The function I is extended to $V\colon W \times \mathsf{Form}_{\wedge,\vee} \longrightarrow \{0,1\}$ as follows:

- $V(w,p) = I(w,p)$,
- $V(w, \sim A) = 1$ iff $V(w,A) \neq 1$,
- $V(w, A \vee B) = 1$ iff $V(w,A) = 1$ or $V(w,B) = 1$,
- $V(w, A \wedge B) = 1$ iff $V(w,A) = 1$ and $V(w,B) = 1$,
- $V(w, A \to B) = 1$ iff $\begin{cases} \text{for some } x_0 \in W\colon\ V(x_0, A) = 1, \text{ and} \\ \text{for all } x \in W\colon\ V(x,A) \neq 1 \text{ or } V(x,B) = 1. \end{cases}$

Finally, $\Sigma \models A$ iff (i) for some $\langle W, g, I\rangle$, $V(g,B) = 1$ for all $B \in \Sigma$, and (ii) for every $\langle W, g, I\rangle$, $V(g,A) = 1$ if $V(g,B) = 1$ for all $B \in \Sigma$.

Remark 9. Note that the following holds for $\models$:that

- $\models \sim(p \to \sim p)$
- $\models \sim(\sim p \to p)$
- $\models \sim((p \to q) \wedge (p \to \sim q)$
- $\not\models (p \to q) \to \sim(p \to \sim q)$
- $p \to q \models \sim(p \to \sim q)$

Therefore, as in Pizzi's system **CI**, Boethius' thesis is not valid, but a weaker version, here in the rule form, holds in Priest's formalization. Finally, $\models$ is not monotonic or closed under uniform substitution, but if we define the semantic consequence relation as follows, then these properties will be recovered: $\Sigma \models A$ iff for every $\langle W, g, I\rangle$, $V(g,A) = 1$ if $V(g,B) = 1$ for all $B \in \Sigma$.

2.6. Systems adjusting falsity conditions

The systems so far can be seen as achieving connexivity by searching for additional conditions on top of the *truth* conditions for material, strict, or relevant conditionals. However, the family of systems in this last group will have a different perspective on connexivity. More specifically, connexivity is achieved by looking at the *falsity* condition for the conditional. We will start by considering the first system that was based on this idea in [31].

DEFINITION 10. A **C**-model for the language $\mathcal{L}_{\wedge,\vee}$ is a triple $\langle W, \leq, I\rangle$, where W is a non-empty set (of states), $\leq$ is a partial order on W, and $I\colon W \times \mathsf{Prop} \longrightarrow \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$ is an assignment of truth values to state-variable pairs with the condition that $i \in I(w_1, p)$ and $w_1 \leq w_2$ only if $i \in I(w_2, p)$ for all $p \in \mathsf{Prop}$, all $w_1, w_2 \in W$ and $i \in \{0,1\}$. Interpretations I are then extended to valuations V assigning truth values to state-formula pairs by the following conditions:

- $V(w, p) = I(w, p)$,
- $1 \in V(w, {\sim}A)$ iff $0 \in V(w, A)$,
- $0 \in V(w, {\sim}A)$ iff $1 \in V(w, A)$,
- $1 \in V(w, A \wedge B)$ iff $1 \in V(w, A)$ and $1 \in V(w, B)$,
- $0 \in V(w, A \wedge B)$ iff $0 \in V(w, A)$ or $0 \in V(w, B)$,
- $1 \in V(w, A \vee B)$ iff $1 \in V(w, A)$ or $1 \in V(w, B)$,
- $0 \in V(w, A \vee B)$ iff $0 \in V(w, A)$ and $0 \in V(w, B)$,
- $1 \in V(w, A \to B)$ iff for all $x \in W$: if $w \leq x$ and $1 \in V(x, A)$ then $1 \in V(x, B)$,
- $0 \in V(w, A \to B)$ iff for all $x \in W$: if $w \leq x$ and $1 \in V(x, A)$ then $0 \in V(x, B)$.

Finally, the semantic consequence relation is now defined as follows: $\Sigma \models_C A$ iff for all **C**-models $\langle W, \leq, I\rangle$, and for all $w \in W$: $1 \in V(w, A)$ if $1 \in V(w, B)$ for all $B \in \Sigma$.

REMARK 11. Like David Nelson's logic **N4**, the system **C** is a paraconsistent and constructive logic.[8] Nelson's **N4** is obtained from **C** by replacing the falsity condition for the conditional by the following condition:

$$0 \in V(w, A \to B) \text{ iff } 1 \in V(w, A) \text{ and } 0 \in V(w, B).$$

[8] For the details of **N4** [see, e.g., 8, 15].

The system **C** validates Aristotle's and Boethius' theses. One of the striking features of **C**, as well as other logics following the idea developed in [31], is that this system is inconsistent (but nevertheless non-trivial). Indeed, both $(A \wedge {\sim}A) \rightarrow A$ and ${\sim}((A \wedge {\sim}A) \rightarrow A)$ are valid. Yet another feature of **C** that deserves to be highlighted is that it enjoys nice proof-theoretical properties. Note that this was not the case with systems before the birth of **C**.

A note on the papers in the special issue. Thomas Ferguson's contribution is concerned with Peano arithmetic based on the above system **C**. In particular, Ferguson establishes that the theory is Post consistent, among other things.

Then, by considering the classical extension of **C**, namely taking a singleton for the set of states W in **C**-models, we obtain the following four-valued logic, introduced in [33].

DEFINITION 12. An ***MC**-interpretation* for $\mathcal{L}_{\wedge,\vee}$ is a function $I\colon \mathsf{Prop} \longrightarrow \{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$. I is then extended to $v\colon \mathsf{Form}_{\wedge,\vee} \longrightarrow \{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$ by the following truth tables.

A	${\sim}A$
t	**f**
b	**b**
n	**n**
f	**t**

$A \wedge B$	**t**	**b**	**n**	**f**
t	**t**	**b**	**n**	**f**
b	**b**	**b**	**f**	**f**
n	**n**	**f**	**n**	**f**
f	**f**	**f**	**f**	**f**

$A \vee B$	**t**	**b**	**n**	**f**
t	**t**	**t**	**t**	**t**
b	**t**	**b**	**t**	**b**
n	**t**	**t**	**n**	**n**
f	**t**	**b**	**n**	**f**

$A \rightarrow B$	**t**	**b**	**n**	**f**
t	**t**	**b**	**n**	**f**
b	**t**	**b**	**n**	**f**
n	**b**	**b**	**b**	**b**
f	**b**	**b**	**b**	**b**

REMARK 13. Note that by considering a further extension of **MC** by the law of excluded middle, namely $A \vee {\sim}A$, we obtain the system **CN** introduced by John Cantwell in [3]. Moreover, a further expansion of **CN**, which is functionally complete and thus Post complete, is discussed in [16] as well as in [18].

A note on the papers in the special issue. Norihiro Kamide's contribution considers an expansion of **MC** by co-implication, introduces a sequent calculus, and establishes results such as cut-elimination and completeness. Moreover, a modal expansion is also discussed. Hitoshi Omori's contribution can be seen as introducing a generalization of **MC** by considering a variant of Wansing's falsity condition by making use of modal vocabularies.

REMARK 14. Instead of strengthening the conditional in **C**-models, we may also apply the idea of a connexive reading of negated conditionals to a weaker conditional, which may also lead to further applications of connexive logic, such as introducing a negation connective to the Lambek calculus in Categorial Grammar [32]. For example, as shown in [17], we may require Wansing's falsity condition on top of the weak relevant logic **BD**.

> *A note on the papers in the special issue.* Heinrich Wansing and Matthias Unterhuber's contribution, among other things, combines the idea leading from **N4** to **C** with conditional logics, making use of Chellas-Segerberg semantics, thereby working with a very weak implication. Wansing and Unterhuber also introduce an expansion by a constructive conditional and compare their system with other systems discussed in the literature.

Remark 15. Here are two more remarks related to the approach via the falsity condition. First, we emphasize the flexibility of this approach. We have already seen that a wide range of truth conditions for the conditional is compatible with the connexive falsity condition. This is in sharp contrast, for example, with the relevant logic approach based on the star interpretation of negation. Indeed, the relevant logic **R** is *not* compatible with Aristotle's thesis [cf. 13, Theorem 2]. Second, the approach based on the falsity conditions also suggests a systematic study of contra-classical logics [cf. 7], which is a larger family of nonclassical logics that connexive logics belong to. The idea, in brief, is to consider less explored falsity conditions for other connectives than the conditional. For a first step towards this direction see [21].

2.7. Summary so far

Table 1 summarizes the review of systems in the literature. In the form of a slogan, from a semantic perspective, connexive logics can be classified into two groups: one group consists of systems obtained by tweaking the *truth* conditions, and the other group consists of systems obtained by tweaking the *falsity* conditions of conditionals.[9]

[9] Needless to say, the classification here is not the only way. For example, we may classify according to a list of theses and check which ones are validated and which ones are not. For such an attempt [see 6].

	conditional	negation	consequence relation
Angell-McCall	material + tweak	classical	standard
Routley	relevant	star	standard
Pizzi	strict + tweak	classical	standard
Priest	strict +tweak	classical	nonstandard
Wansing	various kinds	de Morgan	standard

Table 1. Summary of systems in literature

A note on the papers in the special issue. Tomasz Jarmużek and Jacek Malinowski's contribution suggests yet another way to tweak the truth condition of the material conditional. More specifically, they use the framework of relating semantics to require that the antecedent and the consequent of the material conditional are related.

3. Current trends in connexive logic

After four workshops on connexive logic[10], we are realizing that there are some topics that are collecting special attention. We will here briefly discuss three of them.

3.1. On the very notion of connexivity

Recall that connexive logics are traditionally characterized as systems validating the theses of Aristotle and Boethius:
Aristotle $\sim(\sim A \to A)$, $\sim(A \to \sim A)$;
Boethius $(A \to B) \to \sim(A \to \sim B)$, $(A \to \sim B) \to \sim(A \to B)$.
Given that characterization, the central concern of connexive logic consists of developing connexive systems that are naturally motivated conceptually or in terms of applications, that admit of a simple and plausible semantics, and that can be equipped with proof systems possessing nice proof-theoretical properties, such as the eliminability of the cut-rule enjoyed by the system **C**.

This established characterization of connexive logic was reconsidered with an interesting alternative by Andreas Kapsner in [9], where the following requirements are given:

[10] The fourth workshop on connexive logic already took place at the Ruhr University Bochum, Germany, in October, 2018. See `https://sites.google.com/site/connexivelogic/events/logic-in-bochum-4` for the details of the workshop.

Unsat1 In no model, $A \rightarrow {\sim}A$ is satisfiable (for any A).
Unsat2 In no model $(A \rightarrow B)$ and $(A \rightarrow {\sim}B)$ are simultaneously satisfiable (for any A and B).

Based on these conditions, Kapsner coined the term *strong connexivity* for the property that consists of all four conditions above. Correspondingly, Kapsner called logics that only satisfy the earlier two conditions *weakly connexive*.[11]

In view of these notions, there are a lot of weakly connexive logics with intuitive semantics such as **C**, Cantwell's **CN**, and other systems obtained by adjusting falsity conditions. However, the only strongly connexive logic, to the best of our knowledge, is the heavily criticized system of Angell-McCall, and it remains to be seen if there are strongly connexive systems with more intuitive semantics.

> *A note on the papers in the special issue.* Andreas Kapsner's contribution suggests yet another kind of connexivity, called *humble connexivity*. The main idea is to restrict the connexive principles with the condition that antecedents are possible. He then explores different ways of formalizing this idea, and observes some implications on conditional logics and paraconsistent logics.

It should also be noted that some experiments on Aristotle's theses and/or Boethius' theses have been carried out, e.g., in [5, 22], and some results seem to speak in favor of Aristotle's and Boethius' theses being endorsed with no restriction. What we may say in view of empirical investigations seems to be an interesting topic for future research.

3.2. Historical considerations on connexivity

As we can see from the names of the theses, the most prominent connexive principles are named after historically very important philosophers: Aristotle and Boethius. Other ancient, respectively medieval, philosophers who have been said to endorse connexive principles include Chrysippus, Peter Abelard, Peter of Spain, and Robert Kilwardby. It is then unsurprising that there are some historical considerations related to the notion of connexivity, pursued with both exegetical as well as systematical concerns. In this special issue, we have two contributions.

[11] Logics that satisfy **Unsat1** and **Unsat2** are called *Kapsner connexive* in [6].

A note on the papers in the special issue. First, Wolfgang Lenzen's contribution is dealing with Leibniz's algebra of concepts, and suggests that Leibniz came to think that the connexive theses must be restricted to self-consistent concepts. Second, Spencer Johnstone's contribution discusses Robert Kilwardby's understanding of the modal syllogism, his notion of *per se necessity*, and his distinction between on the one hand the notion of 'accidental' implication (entailment) and on the other hand the paraconsistent 'natural' implication relationship that validates Aristotle's thesis $\sim(\sim A \to A)$ and Boethius' thesis $(A \to B) \to \sim(A \to \sim B)$.

Note that the early modern development of connexive logics was also related to Aristotle's syllogistic [cf. 11], and it seems to be interesting to return to this topic after the development of more recent systems of connexive logic.

3.3. Connexive logic and conditional logic

Since the very beginning of the modern development of connexive logics, the theme of conditional logics has formed important part of the motivations. Indeed, Angell and Pizzi's motivations were closely related to conditional logic, and this focus continues to be present to date. In this special issue, we have three contributions to conditional logic.

A note on the papers in the special issue. Two papers have already been mentioned, namely the contribution of Kapsner, and the contribution of Wansing and Unterhuber. The third one is Yale Weiss' contribution in which Weiss explores connexive extensions of the basic regular conditional logic **CR**. His observations are led by proof-theoretic considerations, but sound and complete algebraic semantics are also presented.

Conditional logic is a branch of philosophical logic with a number of recent developments both technically and philosophically. How the findings in conditional logic will give new insights into connexivity is yet another interesting topic which seems to be still in progress.

4. Concluding remarks

We hope that this introduction gives a useful systematic perspective of how the research on connexive logic has been developing since the 1960s, and also hope that it has become clear that there are some interesting ongoing discussions. We have identified three current trends in work on connexive logic, and we have indicated how the contributions to the present special issue fit into the systematic picture and the recent debates.

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