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SYNTACTIC PROOFS FOR YABLO'S PARADOXES IN TEMPORAL LOGIC

Abstract. Temporal logic is of importance in theoretical computer science for its application in formal verification, to state requirements of hardware or software systems. Linear temporal logic is an appropriate logical environment to formalize Yablo's paradox which is seemingly non-self-referential and basically has a sequential structure. We give a brief review of Yablo's paradox and its various versions. Formalization of these paradoxes yields some theorems in Linear Temporal Logic (LTL) for which we give syntactic proofs using an appropriate axiomatization of LTL.

Keywords: non-self-referential paradox; Yablo's paradox; linear temporal logic; syntactic proofs

Introduction

Yablo's paradox that seemingly avoids self-reference was published by Stephen Yablo in [1993]. There have been a lot of discussions on the issue of self-referentiality or circularity of this paradox [see e.g., Beall, 2001; Bringsjord, 2003; Bueno and Colyvan, 2003a,b; Ketland, 2004, 2005; Priest, 1997; Sorensen, 1998; Yablo, 2004]. Unlike the liar paradox, which uses a single sentence, Yablo's paradox applies an infinite sentences, each of which refers only to later ones in the sequence. There is no consistent way to assign truth values to all the statements, although no statement directly refers to itself. Yablo considers the following sequence of sentences $\{\mathcal{Y}_i\}$:

$$\mathcal{Y}_1 : \forall k > 1;\ \mathcal{Y}_k \ \text{ is untrue,}$$
$$\mathcal{Y}_2 : \forall k > 2;\ \mathcal{Y}_k \ \text{ is untrue,}$$

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$$\mathcal{Y}_3 : \forall k > 3;\ \mathcal{Y}_k \text{ is untrue,}$$
$$\vdots$$

The paradox follows from the following deductions. Suppose $\mathcal{Y}_1$ is true. Then for any $k > 1$, $\mathcal{Y}_k$ is not true. In particular, $\mathcal{Y}_2$ is not true. Also, $\mathcal{Y}_k$ is not true for any $k > 2$. But this is exactly what $\mathcal{Y}_2$ says, hence $\mathcal{Y}_2$ is true after all. Contradiction! Suppose then that $\mathcal{Y}_1$ is false. This means that there is a $k > 1$ such that $\mathcal{Y}_k$ is true. But we can repeat the reasoning, this time with respect to $\mathcal{Y}_k$ and reach a contradiction again. No matter whether we assume $\mathcal{Y}_1$ to be true or false, we reach a contradiction. Hence the paradox.

Yablo's paradox can be viewed as a non-self-referential liar's paradox; it has been used to give alternative proof for Gödel's first incompleteness theorem [Cieśliński and Urbaniak, 2013; Leach-Krouse, 2014]. Recently in [Karimi and Salehi, 2014, 2017], a formalization of Yablo's paradox is proposed in linear temporal logic. For the semantical interpretation, appropriate class of Kripke models in linear temporal logic is used. However, in [Karimi, 2017] a non-self-reference version of Brandenburger-Keisler paradox [Brandenburger, 2006] is presented in epistemic game theory based on Yablo's strategy in his paradox. In this paper, after a short discussion on Yablo's paradox and ω-inconsistency, we present a way to have formal versions of Yablo's paradoxes in Linear Temporal Logic (LTL). An axiomatization for LTL and syntactic proofs for these formal versions of Yablo's paradox will be presented.

The rest of paper is organized as follows: Section 1 shows the existence of Yablo formulas using diagonal techniques. Section 2 discusses on Yablo's paradox and ω-inconsistency. Section 3 gives a brief review on Linear Temporal Logic, and will focus on the formalization of Yablo's paradoxes in LTL. Finally, Section 4 will propose an axiom system for LTL as well as syntactic proofs for various versions of Yablo's paradox.

1. Yablo's paradox

Let S be a theory formulated in the language L_T, a language of first order arithmetic extended with a one place predicate $T(x)$. By $\forall x T(\ulcorner \varphi(\dot{x}) \urcorner)$ we mean: for all natural numbers x, the result of substituting a numeral denoting x for a variable free in φ is true.

DEFINITION 1.1 (Cieśliński, 2013). Let S be a theory in the language L_T. Formula $Y(x)$ is called a Yablo formula in S if it satisfies the Yablo condition, i.e., if $S \vdash \forall x\big[Y(x) \equiv \forall z > x \, \neg T(\ulcorner Y(\dot{z})\urcorner)\big]$. Yablo sentences are obtained by substituting numerals for x in $Y(x)$.

It is easy to prove the existence of Yablo formulas for all theories extending Robinson's arithmetic [Ketland, 2005; Priest, 1997]. For this, we apply the generalized diagonal lemma:

THEOREM 1.1 (Generalized Diagonal Lemma). *Let S be a theory in L_T extending Robinson's arithmetic. Then for any formula $\mathcal{G}(x, y)$ in L_T, there is a formula $\varphi(x)$ such that*

$$S \vdash \varphi(x) \equiv \mathcal{G}\big(x, \ulcorner\varphi(x)\urcorner\big).$$

To construct Yablo formula, consider an arithmetized formula of Yablo's sequence as follows:

$$\mathcal{G}(x, y) := \forall z > x \, \neg T\big(Sub(y, \dot{z})\big),$$

where $Sub(y, \dot{z})$ is the substitution function. Applying diagonal lemma for $\mathcal{G}(x, y)$, there exists a Yablo formula $Y(x)$ for which

$$S \vdash Y(x) \equiv \forall z > x \, \neg T\big(Sub(\ulcorner Y(x)\urcorner, \dot{z})\big).$$

Yablo's paradox appears in several varieties [Yablo, 2004]:

$$\mathcal{Y}_n \iff \forall i > n \ (\mathcal{Y}_i \text{ is not true}) \qquad \text{(always Yablo's paradox)}$$
$$\mathcal{Y}_n \iff \exists i > n \ (\mathcal{Y}_i \text{ is not true}) \qquad \text{(sometimes Yablo's paradox)}$$
$$\mathcal{Y}_n \iff \exists i > n \, \forall j \geqslant i \ (\mathcal{Y}_i \text{ is not true}) \qquad \text{(almost always Yablo's paradox)}$$
$$\mathcal{Y}_n \iff \forall i > n \, \exists j \geqslant i \ (\mathcal{Y}_i \text{ is not true}) \qquad \text{(infinitely often Yablo's paradox)}$$

Sometimes Yablo's paradox is the dual version of the *Always* one. Indeed, using $\neg\mathcal{Y}_n$ instead of $\mathcal{Y}_n$ in Always Yablo's paradox, one can easily derive the Sometimes Yablo's paradox.

To see that the *Almost Always* version is paradoxical, let $\mathcal{Y}_0$, $\mathcal{Y}_1$, $\mathcal{Y}_2, \ldots$ be a sequence of sentences that each sentence says "all sentences, except finitely many, after this sentence are false":

$$\mathcal{Y}_0 \quad : \quad \exists i > 0 \, \forall j \geqslant i \ (\mathcal{Y}_j \text{ is not true}),$$
$$\mathcal{Y}_1 \quad : \quad \exists i > 1 \, \forall j \geqslant i \ (\mathcal{Y}_j \text{ is not true}),$$

$$\mathcal{Y}_2 \quad : \quad \exists i > 2\, \forall j \geqslant i\, (\mathcal{Y}_j \text{ is not true}),$$
$$\vdots$$

Assigning truth values in a consistent way to all sentences $\mathcal{Y}_i$'s leads us to a paradox. To this end, assume for a moment that there is a sentence $\mathcal{Y}_n$ which is true; so there exists $i > n$ for which all $\mathcal{Y}_j$ with $j \geqslant i$ are untrue. In particular, $\mathcal{Y}_i$ is untrue. Since all the sentences $\mathcal{Y}_{i+1}, \mathcal{Y}_{i+2}$, ... are untrue, so $\mathcal{Y}_i$ has to be true. Therefore, $\mathcal{Y}_i$ is true and false the same time, which is a contradiction. Whence, all $\mathcal{Y}_n$'s are untrue, so $\mathcal{Y}_0$ is true: a contradiction!

Note that, applying $\neg\mathcal{Y}_n$ instead of $\mathcal{Y}_n$ in Almost Always Yablo's paradox, we see that the *Infinitely Often* version of Yablo's paradox is the dual version of this paradox.

2. Yablo's paradox and ω-inconsistency

Ketland [2005] shows that Yablo's sentences have a non-standard model. He argues that the list of Yablo sentences is ω-paradoxical, in the sense that it is unsatisfiable on the standard model $\mathbb{N}$ of arithmetic. Ketland has translated Yablo's paradox into first-order logic which is call the Uniform Homogeneous Yablo Scheme (UHYS) in [Ketland, 2005]:

$$\forall x(\varphi(x) \leftrightarrow \forall y[xRy \to \neg\varphi(y)]), \tag{UHYS}$$

where R is a binary relation symbol, with the auxiliary axioms stating that R is serial and transitive:

$$\forall x \exists y(xRy), \tag{SER}$$
$$\forall x, y, z(xRyRz \to xRz). \tag{TRANS}$$

A Yablo-like argument can show that the formula $\neg(\texttt{UHYS} \wedge \texttt{SER} \wedge \texttt{TRANS})$ is a first-order tautology [Karimi and Salehi, 2014], i.e. (UHYS) is inconsistent, together with (SER) and (TRANS). Note that the inconsistency of (UHYS) arises irrespective of what φ means, provided that R is serial and transitive. However, Ketland [2005] shows that the associated set of numerical instances of (UHYS) is consistent as it has a non-standard model.

Barrio [2010] argues that Yablo's sequences yield new boundaries to the expressive capabilities of certain axiomatic theories of truth. He shows that one can produces an ω-inconsistent, but consistent theory of truth by adding Yablo's sentences and the Local Yablo Disquotational Scheme to first-order arithmetic. This is the reason why Ketland states Yablo's Paradox is not strictly a paradox but actually an ω-paradox [Ketland, 2005].

Barrio [2010] shows that Yablo's sentences have no model in second order PA, but even in this case, the sequence is consistent. Barrio and Picollo [2013] show that adopting ω-inconsistent truth theories for arithmetic in the second-order case leads to unsatisfiability.

Yatabe [2011] reviews Yablo's paradox to analyze the computational content of ω-inconsistent theories and explains the correspondence between co-induction and ω-inconsistent theories of truth. Yatabe argues in favour of ω-inconsistent first-order theories of truth as he believes that they are intrinsically equipped with a machinery, co-induction, which is useful for proving properties of infinite structures, e.g. infinite streams, infinite trees, infinite process or infinite data structures. Co-induction is of importance as it enables us to represent infinite process and it is widely used in computer science. Against Yatabe, Barrio and Da Ré [2018] present some undesirable philosophical features of ω-inconsistent theories. They focus on the classical theory of symmetric truth FS and the non-classical theory of naïve truth based on Łukasiewicz infinitely-valued logic PAŁT in which they identify five conceptual problems as results of ω-inconsistency.

3. Formalization of Yablo's paradoxes in LTL

Paradoxes are interesting in philosophy and mathematics as they can be turned into concrete theorems by the way of formalizing them in some appropriate logics. For example, Liar's paradox when translated into the propositional logic is a sentence L such that $L \leftrightarrow \neg L$, which is inconsistent. Here, the inconsistency is equivalent to the fact that the formula $\neg(\varphi \leftrightarrow \neg\varphi)$ is a tautology in propositional logic. As another example, Forster and Goré [2016] demystify Yablo's paradox by showing that the formula $\Box(p \leftrightarrow \Box\neg p)$ is unsatisfiable in the modal logic KD4 characterized by frames that are strict partial orders without maximal elements. For others, see [Karimi and Salehi, 2014].

Luna [2009] analyses the structure of Yablo's paradox to show that beginning-less step-by-step determination processes can be used to provoke paradoxes. Applying temporal version of Yablo's setup, he shows that the ungroundedness, under the form of a beginningless time or time-like process, leads to incompatible intuitions. To give an outline, Luna imagines a time without a beginning inhabited only by an infinite row of temporally successive thinkers with no first thinker, each of which is in the absolute past of all the following thinkers and thinks or states only this: "*nobody has ever been right*". He calls such thinkers "*Yabloesque thinkers*". Yabloesque thinkers are exactly the same as Yablo's sentences but in the past-time temporal logic with a beginningless time. Luna [2009] argues that if the chain of the Yabloesque thinkers existed, each thinker in it would succeed in asserting a definite state of affairs to which Excluded Middle would apply; therefore, each thinker would succeed in making a statement with a definite truth-value which is impossible and leads us to a paradox.

Here, we show that there is another way to have a formal version of Yablo's paradox, and that is in Linear Temporal Logic (LTL) [Karimi and Salehi, 2014, 2017]. The (propositional) linear temporal logic is a logical formalism that can refer to time; in LTL one can encode formulas about the future, e.g., a condition will eventually be true, a condition will be true until another fact becomes true, etc.

3.1. Linear Temporal Logic

Linear Temporal Logic (LTL) was first proposed for the formal verification of computer programs by Pnueli [1977]. We assume the reader is familiar with the general framework of LTL, but for the sake of accessibility, we list the main notations and definitions which will be referred to later on. For more details [see Kröger and Merz, 2008].

Let $\mathbf{V}$ be a set of *propositional constants*. The alphabet of a basic language $\mathcal{L}_{\mathrm{LTL}}(\mathbf{V})$ (also shortly: $\mathcal{L}_{\mathrm{LTL}}$) of propositional linear temporal logic LTL is given by all the propositional constants of $\mathbf{V}$ and the symbols $\{\mathbf{false}, \rightarrow, \bigcirc, \Box, (,)\}$. The Backus-Naur form for well-formed formulas of linear temporal logic is as follows:

$$\varphi ::= \mathbf{false} \mid v \mid \varphi \rightarrow \varphi \mid \bigcirc\varphi \mid \Box\varphi,$$

where $v \in \mathbf{V}$ is a propositional constant. Connectives $\neg, \vee, \wedge, \leftrightarrow, \mathbf{true}$ are defined as in classical logic, and the operator $\Diamond$ as $\Diamond\varphi \equiv \neg\Box\neg\varphi$. Formulas $\bigcirc\varphi$, $\Box\varphi$, and $\Diamond\varphi$ are read as "next φ", "always φ", and "sometime φ".

For the semantical interpretations of LTL we use Kripke structures. Let $\mathbf{V}$ be a set of propositional constants. A *temporal (or Kripke)* *structure* for $\mathbf{V}$ is an infinite sequence $\mathcal{K} = (\eta_0, \eta_1, \eta_2, \ldots)$ of mappings $\eta_i \colon \mathbf{V} \to \{\mathrm{ff}, \mathrm{tt}\}$ called *states*, and η_0 is called the *initial state* of $\mathcal{K}$. For $\mathcal{K}$ and $i \in \mathbb{N}$, we define $\mathcal{K}_i(F) \in \{\mathrm{ff}, \mathrm{tt}\}$ for every formula F inductively as follows:

1. $\mathcal{K}_i(v) = \eta_i(v) \quad$ for $v \in \mathbf{V}$.
2. $\mathcal{K}_i(\mathbf{false}) = \mathrm{ff}$.
3. $\mathcal{K}_i(\varphi \to \psi) = \mathrm{tt} \iff \mathcal{K}_i(\varphi) = \mathrm{ff}$ or $\mathcal{K}_i(\psi) = \mathrm{tt}$.
4. $\mathcal{K}_i(\bigcirc\varphi) = \mathcal{K}_{i+1}(\varphi)$.
5. $\mathcal{K}_i(\Box\varphi) = \mathrm{tt} \iff \mathcal{K}_j(\varphi) = \mathrm{tt}$ for every $j \geqslant i$.

The formula $\bigcirc\varphi$ informally means "φ holds in temporal state" and $\Box\varphi$ means "φ holds in all forthcoming states including the present one". Truth values for other formula are defined as follows:

6. $\mathcal{K}_i(\neg\varphi) = \mathrm{tt} \iff \mathcal{K}_i(\varphi) = \mathrm{ff}$.
7. $\mathcal{K}_i(\varphi \vee \psi) = \mathrm{tt} \iff \mathcal{K}_i(\varphi) = \mathrm{tt}$ or $\mathcal{K}_i(\psi) = \mathrm{tt}$.
8. $\mathcal{K}_i(\varphi \wedge \psi) = \mathrm{tt} \iff \mathcal{K}_i(\varphi) = \mathrm{tt}$ and $\mathcal{K}_i(\psi) = \mathrm{tt}$.
9. $\mathcal{K}_i(\varphi \leftrightarrow \psi) = \mathrm{tt} \iff \mathcal{K}_i(\varphi) = \mathcal{K}_i(\psi)$.
10. $\mathcal{K}_i(\mathbf{true}) = \mathrm{tt}$.
11. $\mathcal{K}_i(\Diamond\varphi) = \mathrm{tt} \iff \mathcal{K}_j(\varphi) = \mathrm{tt}$ for some $j \geqslant i$.

Definition 3.1 (Kröger and Merz, 2008). A formula φ of $\mathcal{L}_{LTL}(\mathbf{V})$ is called *valid* in the temporal structure $\mathcal{K}$ for $\mathbf{V}$ (or $\mathcal{K}$ satisfies φ), denoted by $\models_{\mathcal{K}} \varphi$, if $\mathcal{K}_i(\varphi) = \mathrm{tt}$ for every $i \in \mathbb{N}$. The formula φ is called a consequence of a set $\mathcal{F}$ of formulas ($\mathcal{F} \models \varphi$) if $\models_{\mathcal{K}} \varphi$ holds for every $\mathcal{K}$ such that $\models_{\mathcal{K}} \psi$ for all $\psi \in \mathcal{F}$. The formula φ is called *(universally) valid* ($\models \varphi$) if $\emptyset \models \varphi$. A formula φ is called (locally) satisfiable if there is a temporal structure $\mathcal{K}$ and $i \in \mathbb{N}$ such that $\mathcal{K}(\varphi) = \mathrm{tt}$.

The following theorem will be used in our arguments.

Theorem 3.1 (Kröger and Merz, 2008). LTL $\models \varphi$ *if and only if* $\neg\varphi$ *is not satisfiable.*

3.2. Yablo's paradox in linear temporal logic

In order to formalize Yablo's sentences, we consider a version of (*Always*) Yablo's paradox in the form:

$$\forall n(\mathcal{Y}_n \leftrightarrow \forall\, i > n\, \neg\mathcal{Y}_i)$$

where the variables range over $\mathbb{N}$. We go further by thinking of the $\mathcal{Y}_n$ in the statement of Yablo's paradox not as an infinite family of atomic propositions but as a single proposition evaluated in lots of worlds in a (*temporal*) Kripke model. So, an interpretation of each sentence of (*Always*) Yablo's paradox in linear temporal logic is a sentence φ that satisfies the equivalence $\varphi \leftrightarrow \bigcirc\Box\neg\varphi$ [Karimi and Salehi, 2014].

The temporal counterpart of Yablo's paradox is the temporal formula

$$\Box(\varphi \leftrightarrow \bigcirc\Box\neg\varphi)$$

which says: always (in the temporal states) is it the case that φ holds iff from the next step onward φ is not true. Therefore, the inconsistency here is equivalent to the fact that the formula $\neg\Box(\varphi \leftrightarrow \bigcirc\Box\neg\varphi)$ is a tautology in propositional linear temporal logic. The proof is exactly the same as Yablo's argument but this time in linear temporal logic [Karimi and Salehi, 2014]:

THEOREM 3.2 (Always Yablo's Paradox). LTL $\models \neg\Box(\varphi \leftrightarrow \bigcirc\Box\neg\varphi)$.

PROOF. To show this formula is valid we will follow exactly the line of Yablo's reasoning to obtain his paradox, this time in LTL. By Theorem 3.1, to prove the formula $\neg\Box(\varphi \leftrightarrow \bigcirc\Box\neg\varphi)$ is valid in LTL, we need to show the formula $\Box(\varphi \leftrightarrow \bigcirc\Box\neg\varphi)$ is not satisfiable. For a moment assume that there is a Kripke structure $\mathcal{K}$ and $n \in \mathbb{N}$ for which $\mathcal{K}_n\big(\Box(\varphi \leftrightarrow \bigcirc\Box\neg\varphi)\big) = \mathtt{tt}$. Then $\forall i \geqslant n\ \mathcal{K}_i(\varphi \leftrightarrow \bigcirc\Box\neg\varphi) = \mathtt{tt}$ which implies that $\forall i \geqslant n\ \mathcal{K}_i(\varphi) = \mathcal{K}_i(\bigcirc\Box\neg\varphi) = \mathcal{K}_{i+1}(\Box\neg\varphi)$. We distinguish two cases:

(1) For some $j \geqslant n$ we have $\mathcal{K}_j(\varphi) = \mathtt{tt}$. Then $\mathcal{K}_{j+1}(\Box\neg\varphi) = \mathtt{tt}$ so $\mathcal{K}_{j+l}(\varphi) = \mathtt{ff}$ for all $l \geqslant 1$. In particular $\mathcal{K}_{j+1}(\varphi) = \mathtt{ff}$ whence $\mathcal{K}_{j+2}(\Box\neg\varphi) = \mathtt{ff}$ which is in contradiction with $\mathcal{K}_{j+1}(\Box\neg\varphi) = \mathtt{tt}$.

(2) For all $j \geqslant n$ we have $\mathcal{K}_j(\varphi) = \mathtt{ff}$. So $\mathtt{ff} = \mathcal{K}_n(\varphi) = \mathcal{K}_{n+1}(\Box\neg\varphi)$ hence there must exist some $i > n$ with $\mathcal{K}_i(\varphi) = \mathtt{tt}$ which contradicts (1) above.

Thus, the formula $\Box(\varphi \leftrightarrow \bigcirc\Box\neg\varphi)$ cannot be satisfiable in LTL. ⊣

Each version of Yablo's paradox can be seen as a theorem in linear temporal logic for which we will give syntactic proofs based on an appropriate axiomatization of LTL.

THEOREM 3.3 (Karimi and Salehi, 2014).

$$\begin{array}{ll}
\text{LTL} \models \neg\Box(\varphi \leftrightarrow \bigcirc\Diamond\neg\varphi) & \text{(Sometimes Yablo's Paradox)}\\
\text{LTL} \models \neg\Box(\varphi \leftrightarrow \bigcirc\Diamond\Box\neg\varphi) & \text{(Almost Always Yablo's Paradox)}\\
\text{LTL} \models \neg\Box(\varphi \leftrightarrow \bigcirc\Box\Diamond\neg\varphi) & \text{(Infinitely Often Yablo's Paradox)}
\end{array}$$

4. Syntactic proofs for Yablo's paradoxes

Now, we present a syntactic argument. Consider axioms and rules for LTL as follows:

$$\bigcirc\neg\varphi \leftrightarrow \neg\bigcirc\varphi \tag{$\mathtt{Ax}_1$}$$

$$\bigcirc(\varphi \to \psi) \to (\bigcirc\varphi \to \bigcirc\psi) \tag{$\mathtt{Ax}_2$}$$

$$\Box\varphi \to (\varphi \wedge \bigcirc\Box\varphi) \tag{$\mathtt{Ax}_3$}$$

$$\frac{\varphi, \quad \varphi \to \psi}{\psi} \tag{$\mathtt{mp}$}$$

$$\frac{\varphi \to \psi, \quad \varphi \to \bigcirc\varphi}{\varphi \to \Box\psi} \tag{$\mathtt{ind}$}$$

$$\frac{\varphi}{\bigcirc\varphi} \tag{$\mathtt{nex}$}$$

Theorem 4.1 (Deduction Theorem for LTL [Kröger and Merz, 2008]). *Let φ, ψ be formulas in* LTL *and $\mathcal{F}$ be a set of formulas. If $\mathcal{F} \cup \{\varphi\} \vdash \psi$ then $\mathcal{F} \vdash \Box\varphi \to \psi$. In particular, if $\varphi \vdash \psi$ then $\vdash \Box\varphi \to \psi$.*

Theorem 4.2. *Using axioms and rules of* LTL, *following rules and formulas are derivable in* LTL:

1. $\dfrac{\varphi \to \psi}{\bigcirc\varphi \to \bigcirc\psi}$
2. $\Box\varphi \to \Box\Box\varphi$
3. $\bigcirc\Box\varphi \to \Box\bigcirc\varphi$
4. $\Box\bigcirc\varphi \to \bigcirc\Box\varphi$
5. *if* $\dfrac{\varphi}{\psi}$, *then* $\dfrac{\bigcirc\varphi}{\bigcirc\psi}$
6. $(\bigcirc\varphi \to \bigcirc\psi) \to \bigcirc(\varphi \to \psi)$
7. $\varphi \wedge \bigcirc\Box\varphi \to \Box\varphi$
8. $\dfrac{\varphi}{\Box\varphi}$
9. $\dfrac{\varphi \to \psi}{\Box\varphi \to \Box\psi}$

Using the axioms and rules, we can directly show:

Theorem 4.3. LTL $\vdash \neg\Box(\varphi \leftrightarrow \bigcirc\Box\neg\varphi)$.

PROOF. For a moment, assume that LTL $\vdash \Box(\varphi \leftrightarrow \bigcirc\Box\neg\varphi)$, then we have LTL $\vdash \Box(\varphi \leftrightarrow \Box\bigcirc\neg\varphi)$. Thus,

1. $\Box(\varphi \leftrightarrow \Box\bigcirc\neg\varphi)$
2. $\Box(\varphi \leftrightarrow \Box\bigcirc\neg\varphi) \rightarrow (\varphi \leftrightarrow \Box\bigcirc\neg\varphi) \wedge \bigcirc\Box(\varphi \leftrightarrow \Box\bigcirc\neg\varphi)$ (Ax$_3$)
3. $\varphi \leftrightarrow \Box\bigcirc\neg\varphi$ (1), (2), (mp), (prop)
4. $\bigcirc\varphi \leftrightarrow \bigcirc\Box\bigcirc\neg\varphi$ (nex), (Ax$_1$), (mp)
5. $\varphi \leftrightarrow \bigcirc\neg\varphi \wedge \bigcirc\Box\bigcirc\neg\varphi$ (3), (Ax$_3$)
6. $\varphi \leftrightarrow \bigcirc\neg\varphi \wedge \bigcirc\varphi$ (4), (5)
7. $\varphi \leftrightarrow$ **false** (6), (prop)
8. $\neg\varphi$ (7)
9. $\bigcirc\neg\varphi$ (8), (nex)
10. $\Box\bigcirc\neg\varphi$ (9)
11. φ (3), (10), (mp)
12. $\perp$ (8), (11)

Therefore, LTL $\vdash \neg\Box(\varphi \leftrightarrow \bigcirc\Box\neg\varphi)$. ⊣

THEOREM 4.4. LTL $\vdash \neg\Box(\varphi \leftrightarrow \bigcirc\Diamond\neg\varphi)$.

PROOF. Applying the proof of Theorem 4.3 for the formula $\neg\varphi$ instead of φ yields the assertion. ⊣

THEOREM 4.5. LTL $\vdash \neg\Box(\varphi \leftrightarrow \bigcirc\Box\Diamond\neg\varphi)$.

PROOF. For a moment, assume that LTL $\vdash \Box(\varphi \leftrightarrow \bigcirc\Box\Diamond\neg\varphi)$, then

1. $\Box(\varphi \leftrightarrow \bigcirc\Box\Diamond\neg\varphi)$
2. $(\varphi \leftrightarrow \bigcirc\Box\Diamond\neg\varphi) \wedge \bigcirc\Box(\varphi \leftrightarrow \bigcirc\Box\Diamond\neg\varphi)$ (Ax$_3$) , (mp)
3. $\varphi \leftrightarrow \bigcirc\Box\Diamond\neg\varphi$ (2), (prop)
4. $\varphi \leftrightarrow \bigcirc(\Diamond\neg\varphi \wedge \bigcirc\Box\Diamond\neg\varphi)$ (Ax$_3$), Theorem 4.2(7)
5. $\Box\varphi \leftrightarrow \Box\bigcirc\Box\Diamond\neg\varphi$ (1), Theorem 4.2(9)
6. $\Box\varphi \leftrightarrow \bigcirc\Box\Box\Diamond\neg\varphi$ (5), Theorem 4.2(3,4)
7. $\Box\varphi \leftrightarrow \bigcirc\Box\Diamond\neg\varphi$ (6), Theorem 4.2(2)
8. $\varphi \leftrightarrow \bigcirc(\Diamond\neg\varphi \wedge \Box\varphi)$ (4), (7)
9. $\varphi \leftrightarrow \bigcirc(\neg\Box\varphi \wedge \Box\varphi)$ (8)
10. $\varphi \leftrightarrow \perp$ (9), (prop)
11. $\neg\varphi$ (10), (prop)
12. $\neg\varphi \leftrightarrow \neg\bigcirc\Box\Diamond\neg\varphi$ (3), (prop)
13. $\neg\varphi \leftrightarrow \bigcirc\Diamond\Box\neg\neg\varphi$ (12), Theorem 4.2(3,4)
14. $\neg\varphi \leftrightarrow \bigcirc\Diamond\Box\varphi$ (13), (prop)
15. $\neg\varphi \leftrightarrow \bigcirc\Diamond(\Box\varphi \wedge \bigcirc\Box\varphi)$ (14), (Ax$_3$), Theorem 4.2(7)

16.	$\neg\varphi \leftrightarrow \bigcirc\Diamond(\bigcirc\Box\Diamond\neg\varphi \wedge \bigcirc\Box\varphi)$	(7), (15)
17.	$\neg\varphi \leftrightarrow \bigcirc\Diamond\bigcirc(\Box\Diamond\neg\varphi \wedge \Box\varphi)$	(16)
18.	$\neg\varphi \leftrightarrow \bigcirc\bigcirc\Diamond(\Box\Diamond\neg\varphi \wedge \Box\Box\varphi)$	(17), ($\mathtt{Ax}_1$), (definition of $\Diamond$)
19.	$\neg\varphi \leftrightarrow \bigcirc\bigcirc\Diamond\Box(\Diamond\neg\varphi \wedge \Box\varphi)$	(18)
20.	$\neg\varphi \leftrightarrow \bigcirc\bigcirc\Diamond\Box(\neg\Box\varphi \wedge \Box\varphi)$	(19)
21.	$\neg\varphi \leftrightarrow \bot$	(20), (prop)
22.	$\neg\varphi$	(21), (prop)
23.	$\varphi \wedge \neg\varphi$	(11), (22)
24.	$\bot$	(23), (prop)

Thus, $\mathrm{LTL} \vdash \neg\Box(\varphi \leftrightarrow \bigcirc\Box\Diamond\neg\varphi)$. ⊣

THEOREM 4.6. $\mathrm{LTL} \vdash \neg\Box(\varphi \leftrightarrow \bigcirc\Diamond\Box\neg\varphi)$.

PROOF. Applying the proof of Theorem 4.5 for the formula $\neg\varphi$ instead of φ yields the assertion. ⊣

5. Conclusions

Paradoxes are of great importance in philosophy and mathematics as they can be turned into concrete theorems by means of formalization in appropriate logics. Linear temporal logic is a logical environment in which Yablo's paradox can be formalized as basically it has a sequential structure. In this paper, we have turned Yablo's paradox into a theorem in Linear Temporal Logic (LTL). We have also use this approach to other versions of Yablo's paradox, including the original version and its dual (Sometimes Yablo's paradox), as well as Almost Always Yablo's paradox and its dual (Infinitely Often Yablo's paradox). Therefore, each version of Yablo's paradox is appeared as a theorem in linear temporal logic for which we give syntactic proofs based on an appropriate axiomatization of LTL.

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