Logic and Logical Philosophy
Volume 28 (2019), 277–330
DOI: 10.12775/LLP.2019.014

Damian E. Szmuc

AN EPISTEMIC INTERPRETATION OF PARACONSISTENT WEAK KLEENE LOGIC

Abstract. This paper extends Fitting's epistemic interpretation of some Kleene logics to also account for Paraconsistent Weak Kleene logic. To achieve this goal, a dualization of Fitting's "cut-down" operator is discussed, leading to the definition of a "track-down" operator later used to represent the idea that no consistent opinion can arise from a set including an inconsistent opinion. It is shown that, if some reasonable assumptions are made, the truth-functions of Paraconsistent Weak Kleene coincide with certain operations defined in this track-down fashion. Finally, further reflections on conjunction and disjunction in the weak Kleene logics accompany this paper, particularly concerning their relation with containment logics. These considerations motivate a special approach to defining sound and complete Gentzen-style sequent calculi for some of their four-valued generalizations.

Keywords: weak Kleene logic; infectious logic; containment logic; sequent calculus

1. Introduction

Paraconsistent Weak Kleene logic (PWK, for short) is the three-valued logic that arises from the *weak* truth-tables due to Kleene [36], when the intermediate value (which we will, provisionally, call $\mathbf{u}$) is taken to be a *designated* value. These truth-tables — represented in Figure 1 below — are referred to as weak, because they exhibit a sort of "infectious" behavior of the intermediate value: in fact, this value is assigned to a complex formula whenever one of its components is assigned so.

Moreover, notice that the Paracomplete Weak Kleene logic ($\mathsf{K}_3^{\mathrm{w}}$, for short) is another three-valued logic that arises from these truth-tables,

Received November 13, 2017. *Revised* July 5, 2018. *Published online* May 24, 2019

when the intermediate value is not taken to be designated. Naturally, it is because these logics are defined making essential use of the weak Kleene truth-tables that, e.g., in [30], Fitting refers to them (and by extension to their eventual four-valued generalizations, on which more below) as *weak Kleene logics*.[1]

	$\neg$
t	**f**
u	**u**
f	**t**

$\wedge$	**t**	**u**	**f**
t	**t**	**u**	**f**
u	**u**	**u**	**u**
f	**f**	**u**	**f**

$\vee$	**t**	**u**	**f**
t	**t**	**u**	**t**
u	**u**	**u**	**u**
f	**t**	**u**	**f**

Figure 1. The weak Kleene truth-tables

These systems can be compared, primarily, with the three-valued logics defined in terms of the otherwise *strong* Kleene truth-tables — represented in Figure 2 below. Thus, when the intermediate value featured in this strong truth-tables is taken to be non-designated the induced system is usually referred to as Strong Kleene logic ($\mathsf{K_3}$, for short), while the system induced by taking the intermediate value to be designated is usually referred to as Priest's Logic of Paradox (LP, for short). Analogously to the previous remarks, then, it is because these logics are defined making essential use of the strong Kleene truth-tables that, e.g. in [29], Fitting refers to them (and by extension to their eventual four-valued generalizations, on which more below) as *strong Kleene logics*.

	$\neg$
t	**f**
u	**u**
f	**t**

$\wedge$	**t**	**u**	**f**
t	**t**	**u**	**f**
u	**u**	**u**	**f**
f	**f**	**f**	**f**

$\vee$	**t**	**u**	**f**
t	**t**	**t**	**t**
u	**t**	**u**	**u**
f	**t**	**u**	**f**

Figure 2. The strong Kleene truth-tables

Among the weak Kleene logics, an increasing amount of recent work has been focused on Paraconsistent Weak Kleene [see, e.g., 9, 11, 12, 13, 39, 40, 44, 51]. However, what is salient among these works is the absence of a cogent philosophical interpretation for it. The aim of this paper is to try to overcome this deficit, by offering an *epistemic* interpretation for PWK; that is, an epistemic understanding of its truth-values, its consequence relation and its characteristic truth-tables.

[1] It should be noted that PWK and $\mathsf{K_3^w}$ are usually identified as the classical,"internal" or $\{\neg, \wedge, \vee\}$-fragments of Halldén's and Bochvar's logics of nonsense presented, respectively, in [7] and [32].

To achieve this goal, we will benefit from Fitting's epistemic interpretation of some of the Kleene logics in works such as [28, 29, 30]. Interestingly, Fitting himself showed how his epistemic interpretation is flexible enough not only to endow the *strong* Kleene logic K_3 and one of its four-valued generalizations, the logic FDE [studied in 6, 19]) with an epistemic interpretation, but also to provide such a reading for the paracomplete *weak* Kleene logic $\mathsf{K}_3^{\mathrm{w}}$ and one of its four-valued generalizations, the logic $\mathsf{S}_{\mathtt{fde}}$ (on which more below). Quite surprisingly, an attempt to broaden the range of application of this reading to cover the case of Paraconsistent Weak Kleene has not been proposed so far. This is, specifically, what we intend to do in this paper: to show how Fitting's epistemic interpretation of the Kleene logics can also account for the case of PWK. We are, so to speak, after the missing piece of the puzzle.

To this extent, the paper is structured as follows. In Section 2 we give a few formal preliminaries and discuss a little bit more rigorously some aspects of the weak Kleene logics and their four-valued generalizations. Section 3 is the main section of the paper, where Fitting's epistemic interpretation of the Kleene logics is reviewed, and our novel reading of Paraconsistent Weak Kleene is presented in full detail. Section 4 discusses a number components of the epistemic interpretation of many-valued logics, in light of some alternative philosophical and technical approaches appearing in the literature. Section 5 has two parts, and in each of them additional formal results are presented. First, some new results are provided, concerning the relation between subsystems of the weak Kleene logics and containment logics — i.e., systems whose valid inference comply with certain set-theoretic containment principles relating the set of propositional variables appearing in the premises and the set of propositional variables appearing in the conclusion. Secondly, making essential use of these new results, sound and complete Gentzen-style sequent calcui for some four-valued generalization of the weak Kleene logics are introduced. Finally, Section 6 outlines some concluding remarks.

2. Preliminaries

Let $\mathcal{L}$ be a propositional language and let Var be a set of propositional variables, assumed to be countably infinite. By $\mathbf{FOR}(\mathcal{L})$ we denote the absolutely free algebra (of formulae), freely generated by Var, with universe $\mathrm{FOR}(\mathcal{L})$. In all of the cases considered in this paper the propositional language will be limited to the set $\{\neg, \wedge, \vee\}$. In what follows, capital Greek letters Γ, Δ, etc. will denote sets of formulae, and lower-case Greek letters φ, ψ, etc. will denote arbitrary formulae. As usual, a logic L is a pair $\langle \mathrm{FOR}(\mathcal{L}), \vdash_{\mathsf{L}}\rangle$, where $\vdash_{\mathsf{L}} \subseteq \wp(\mathrm{FOR}(\mathcal{L})) \times \mathrm{FOR}(\mathcal{L})$ is a substitution-invariant consequence relation.

For $\mathcal{L}$ a propositional language, an $\mathcal{L}$-matrix (a matrix, for short) is a structure $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$, where $\langle \mathcal{V}, \mathcal{O}\rangle$ is an algebra of the same similarity type as $\mathcal{L}$, with universe $\mathcal{V}$ and a set of operations $\mathcal{O}$, and $\mathcal{D} \subset \mathcal{V}$.[2] Given a matrix, a valuation v is an homomorphism from $\mathrm{FOR}(\mathcal{L})$ to $\mathcal{V}$, for which we denote by $v[\Gamma]$ the set $\{v(\gamma) \mid \gamma \in \Gamma\}$, i.e., the image of v under Γ. Finally, by a matrix logic L we understand a pair $\langle \mathrm{FOR}(\mathcal{L}), \vDash_{\mathcal{M}}\rangle$ where $\vDash_{\mathcal{M}} \subseteq \wp(\mathrm{FOR}(\mathcal{L})) \times \mathrm{FOR}(\mathcal{L})$ is a substitution-invariant consequence relation defined by letting

$$\Gamma \vDash_{\mathcal{M}} \varphi \iff \text{for every valuation } v, \text{ if } v[\Gamma] \subseteq \mathcal{D}, \text{ then } v(\varphi) \in \mathcal{D}$$

Moreover, when $\mathsf{L} = \langle \mathrm{FOR}(\mathcal{L}), \vDash_{\mathcal{M}}\rangle$ we may alternatively denote $\vDash_{\mathcal{M}}$ as $\vDash_{\mathsf{L}}$.

Moving on to some specifics of our investigation, when analyzing logical systems below, it will be useful to consider *infectious* matrix logics, as defined below.[3]

Definition 2.1. A matrix logic $\mathsf{L} = \langle \mathrm{FOR}(\mathcal{L}), \vDash_{\mathcal{M}}\rangle$ is *infectious* if and only if there is an element $\mathbf{x} \in \mathcal{V}$ such that for every n-ary $f^{\diamond}_{\mathcal{M}} \in \mathcal{O}$ it holds that

$$\text{if } \mathbf{x} \in \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}, \text{ then } f^{\diamond}_{\mathcal{M}}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \mathbf{x}$$

As is easy to see, both Paraconsistent Weak Kleene and Paracomplete Weak Kleene can be faithfully described as three-valued *infectious* logics, with the subtle difference that in the former the infectious value is taken to be designated, whereas in the latter it is not. Furthermore, of much interest to us and of much use in finding an epistemic interpretation for PWK, is to look at infectious *subsystems* — most particularly *four-valued* subsystems — of these three-valued matrix logics. Without loss of generality, in what follows we will be assuming that semantics for these

[2] Notice that $\mathcal{O}$ is a set that includes for every n-ary operator $\diamond$ in the language $\mathcal{L}$, a corresponding n-ary truth-function $f^{\diamond}_{\mathcal{M}} \colon \mathcal{V}^n \longrightarrow \mathcal{V}$.

[3] The following definition is inspired by [34], although a similar definition might be found in [22]. For a generalization of this notion that also applies to non-deterministic matrices (as defined in [3]) see [51].

systems count with the classical truth-values $\mathbf{t}$ and $\mathbf{f}$, and two additional truth-values: $\top$ and $\bot$.

Four-valued subsystems of the Kleene logics in general, and some subsystems of PWK in particular, have been studied in recent works [e.g., 46, 51, 53]. Along this line, *normal* four-valued generalizations of the Kleene logics are taken to be those in which the truth-functions for the connectives coincide with those of Classical Logic, when restricted to the set $\{\mathbf{t}, \mathbf{f}\}$. Among the normal generalizations, however, there are two further salient families which have caught the attention of scholars: the family of regular and the family of monotonic generalizations.

Regular systems are defined in [53, p. 226] to be such that all of their truth-functions comply with the criterion — quoted from [46, p. 4] with notation adjusted to fit ours — that a given column (row) of the truth-table contains $\mathbf{t}$ in the $\top$ or $\bot$ row (column), only if the column (row) has $\mathbf{t}$ in all of its cells; and likewise for $\mathbf{f}$. *Monotonic* systems are presented, as is usual, in terms of all of the truth-functions of the underlying matrix preserving some previously-defined order over the truth-values. As detailed in [46, p. 4], the four truth-values of Kleene's four-valued generalizations are ordered in [53] by letting: $\bot \leq \mathbf{f}$, $\bot \leq \mathbf{t}$, $\mathbf{f} \leq \top$, $\mathbf{t} \leq \top$, allowing $\mathbf{t}$ and $\mathbf{f}$ to be incompatible — this is, precisely, the "information" order of the lattice $\mathsf{A4}$ detailed in [6].

As reported in [46], it was proved in [53] that there are 81 four-valued monotonic logics of which only 6 are regular. To this extent, it must be highlighted that the four-valued generalizations of the weak Kleene logics that we are going to discuss next are not regular, although they are indeed monotonic — as is routine to check.

In this vein, it is worth looking at the four-valued logic $\mathsf{S}_{\mathtt{fde}}$ presented below, in Definition 2.2. This system, introduced by Harry Deutsch in [15], has since then been discussed several times in the literature, with different purposes, e.g., in [25, 29, 38, 45, 48, 51].

DEFINITION 2.2. $\mathsf{S}_{\mathtt{fde}}$ is the four-valued logic induced by the matrix $\langle \mathcal{V}_{\mathsf{S}_{\mathtt{fde}}}, \mathcal{D}_{\mathsf{S}_{\mathtt{fde}}}, \mathcal{O}_{\mathsf{S}_{\mathtt{fde}}} \rangle$, where $\mathcal{V}_{\mathsf{S}_{\mathtt{fde}}} = \{\mathbf{t}, \top, \bot, \mathbf{f}\}$, $\mathcal{D}_{\mathsf{S}_{\mathtt{fde}}} = \{\mathbf{t}, \top\}$, $\mathcal{O}_{\mathsf{S}_{\mathtt{fde}}} = \{f^{\neg}_{\mathsf{S}_{\mathtt{fde}}}, f^{\wedge}_{\mathsf{S}_{\mathtt{fde}}}, f^{\vee}_{\mathsf{S}_{\mathtt{fde}}}\}$ and these truth-functions are defined by the truth-tables in Figure 3.

The most important thing about $\mathsf{S}_{\mathtt{fde}}$ is that Fitting, in [30], has taken this logic to be *a four-valued generalization* of $\mathsf{K}^{\mathrm{w}}_{3}$. Why so? On the one hand, it is legitimate to say it is a *generalization* of Paracomplete Weak Kleene first, because its semantics include an undesignated infec-

	$f^{\neg}_{\mathsf{S_{fde}}}$
$\mathbf{t}$	$\mathbf{f}$
$\top$	$\top$
$\bot$	$\bot$
$\mathbf{f}$	$\mathbf{t}$

$f^{\wedge}_{\mathsf{S_{fde}}}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\top$	$\top$	$\top$	$\bot$	$\mathbf{f}$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\bot$	$\mathbf{f}$

$f^{\vee}_{\mathsf{S_{fde}}}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\bot$	$\mathbf{t}$
$\top$	$\mathbf{t}$	$\top$	$\bot$	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
$\mathbf{f}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$

Figure 3. Truth-tables for $\mathsf{S_{fde}}$ (the four-valued generalization of $\mathsf{K}^{\mathrm{w}}_3$)

tious value, just like the semantics for $\mathsf{K}^{\mathrm{w}}_3$; secondly, because this fact — together with the information that it is normal, in the above technical sense — secures that when valuations are restricted to the set $\{\mathbf{t}, \bot, \mathbf{f}\}$ we obtain nothing more than the semantics for $\mathsf{K}^{\mathrm{w}}_3$. On the other hand, it is a *four-valued* generalization, because its semantics include an additional non-classical truth-value, namely $\top$, such that both it and its negation are designated — hence e.g. Explosion, the inference $\varphi \wedge \neg\varphi \vDash \psi$, is invalid in $\mathsf{S_{fde}}$. This informs us, moreover, that we are in front of a *paraconsistent* subsystem of $\mathsf{K}^{\mathrm{w}}_3$.

The importance of considering four-valued generalizations of Kleene logics does not rely, however, just on their technical interest. In fact, as we will see later in Section 3, it is only after offering an epistemic interpretation for Belnap-Dunn logic FDE — whose truth-functions are discussed in Section 3.2 below — and looking at Strong Kleene logic K_3 through the eyes of such a reading, that Fitting provided an epistemic interpretation of the latter. Similarly, it was only after offering an epistemic interpretation for $\mathsf{S_{fde}}$ and looking at Paracomplete Weak Kleene with the tools provided by such a reading, that Fitting arrived at an *epistemic interpretation* for $\mathsf{K}^{\mathrm{w}}_3$.

Thus, it will be through a similar path that we will arrive at an epistemic interpretation for PWK. We will, then, find a suitable four-valued generalization of Paraconsistent Weak Kleene, which we will later endow with an epistemic interpretation. Hence, it will be only after looking at PWK through this interpretation, that we will be able to provide an epistemic understanding for it. Our target four-valued system, which we will call $\mathsf{dS_{fde}}$, is presented below in Definition 2.3. This system was first introduced in [51], although with a different name.[4] Here, we prefer to call it $\mathsf{dS_{fde}}$ for it clearly is the dual of Deutsch's $\mathsf{S_{fde}}$.[5]

[4] In the context of [51], the logic $\mathsf{dS_{fde}}$ is referred to as the system $L_{\mathbf{nb}'}$.

[5] Meaning that, letting $\Sigma^{\neg} = \{\neg\sigma \mid \sigma \in \Sigma\}$ for every $\Sigma \subseteq \mathrm{FOR}(\mathcal{L})$, we can prove: $\Gamma \vDash_{\mathsf{S_{fde}}} \Delta \Longleftrightarrow \Delta^{\neg} \vDash_{\mathsf{dS_{fde}}} \Gamma^{\neg}$, which we leave to the reader as an exercise. Moreover, for

DEFINITION 2.3. The logic $\mathsf{dS}_{\mathtt{fde}}$ is induced by the matrix $\langle \mathcal{V}_{\mathsf{dS}_{\mathtt{fde}}}, \mathcal{D}_{\mathsf{dS}_{\mathtt{fde}}}, \mathcal{O}_{\mathsf{dS}_{\mathtt{fde}}}\rangle$, where $\mathcal{V}_{\mathsf{dS}_{\mathtt{fde}}} = \{\mathbf{t}, \top, \bot, \mathbf{f}\}$, $\mathcal{D}_{\mathsf{dS}_{\mathtt{fde}}} = \{\mathbf{t}, \top\}$, $\mathcal{O}_{\mathsf{dS}_{\mathtt{fde}}} = \{f^{\neg}_{\mathsf{dS}_{\mathtt{fde}}}, f^{\wedge}_{\mathsf{dS}_{\mathtt{fde}}}, f^{\vee}_{\mathsf{dS}_{\mathtt{fde}}}\}$ and these truth-functions are defined by the truth-tables in Figure 4.

	$f^{\neg}_{\mathsf{dS}_{\mathtt{fde}}}$
$\mathbf{t}$	$\mathbf{f}$
$\top$	$\top$
$\bot$	$\bot$
$\mathbf{f}$	$\mathbf{t}$

$f^{\wedge}_{\mathsf{dS}_{\mathtt{fde}}}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\top$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\bot$	$\top$	$\bot$	$\mathbf{f}$
$\mathbf{f}$	$\mathbf{f}$	$\top$	$\mathbf{f}$	$\mathbf{f}$

$f^{\vee}_{\mathsf{dS}_{\mathtt{fde}}}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\top$	$\mathbf{t}$	$\mathbf{t}$
$\top$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\mathbf{t}$	$\top$	$\bot$	$\bot$
$\mathbf{f}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$

Figure 4. Truth-tables for $\mathsf{dS}_{\mathtt{fde}}$ (the four-valued generalization of PWK)

The most important thing to say about $\mathsf{dS}_{\mathtt{fde}}$ is that this logic can, indeed, be regarded as *a four-valued generalization* of PWK. Why so? On the one hand, it is a *generalization* of Paraconsistent Weak Kleene, first, because its semantics include a designated infectious value, just like the semantics for PWK. Secondly, because this fact — together with the information that it is normal, in the above technical sense — secures that when valuations are restricted to the set $\{\mathbf{t}, \top, \mathbf{f}\}$ we obtain nothing more than the semantics for PWK. On the other hand, it is a *four-valued* generalization, because its semantics include an additional non-classical truth-value, namely $\bot$, such that both it and its negation are undesignated — hence e.g. Implosion, the inference $\psi \vDash \varphi \vee \neg\varphi$, is invalid in $\mathsf{dS}_{\mathtt{fde}}$. This informs us, moreover, that we are in front of a *paracomplete* subsystem of PWK.

Before moving on to discuss the target epistemic interpretations of these logics, let us take a brief pause to notice that the weak Kleene logics and their four-valued generalizations are equipped with conjunctions and disjunctions that are utterly peculiar, least to say. In the case of Paracomplete Weak Kleene, a remarkable feature of it is that it invalidates the inference called *Addition*, i.e. $\varphi \vDash \varphi \vee \psi$. Whereas, in the case of Paraconsistent Weak Kleene, a remarkable feature of it is that

---

the purpose of giving a deeper meaning to referring to this four-valued generalization of PWK with the name $\mathsf{dS}_{\mathtt{fde}}$, it would be interesting to present an intensional system dS which dualizes Deutsch's S and, then, prove that $\mathsf{dS}_{\mathtt{fde}}$ is in fact its first-degree fragment. We conjecture this intensional system can be designed by substituting the content-inclusion clause $\gamma_{w'}(B) \leq_{w'} \gamma_{w'}(A)$ featured in Ferguson's semantics for conditionals in the system S (detailed in [23, pp. 77–79]), with the alternative clause $\gamma_{w'}(A) \leq_{w'} \gamma_{w'}(B)$.

it invalidates the inference called *Simplification*, i.e. $\varphi \wedge \psi \vDash \psi$. It is reasonable to ask, then, how the failures of inferences so basic can be made sense of.

Firstly, regarding the failure of Addition, various explanations have been given, which appeal to the meaninglessness [7] or the off-topic character [4] of the issues represented by the newly added disjuncts, to free-choices in deontic logic [56], computational failures [21, 24], analytic connections between premises and conclusions [26] and many other things. Of all these, we will come back to the explanation of the failure of Addition in terms of the absence of analytic connections, when we discuss analytic entailments as modeled by containment logics, in Section 5. We will, obviously, also come back the epistemic explanation of the failure of Addition when we spell out Fitting's epistemic interpretation of Paracomplete Weak Kleene, in Section 3.3.

Notwithstanding the importance of each of these particular explanations, we can refer to a unifying account of all these treatments of disjunctions which do not satisfy Addition, proposed by Thomas Ferguson in the recent paper [21]. There, Ferguson follows Zimmerman's reflections in [56], noting that in many of these accounts the failure of Addition is explained by disjunction having a *conjunctive* flavor to it — or otherwise being nothing more than a conjunction in disguise. The conjunction in question is formed by the explicitly stated disjunction and the implicit requirement that *both* disjuncts satisfy a certain enabling condition, to be further specified in each interpretation. The failure of Addition is accounted for, in this way, by noticing that the fact that one of the disjuncts holds does not guarantee the satisfaction of all the required constraints. In fact, were some of the disjuncts not to satisfy the required enabling condition, then the (apparent) disjunction would not be satisfied — for more on this [see 21, pp. 344–349].

Secondly, regarding the failure of Simplification, various explanations have also been given, which appeal to the meaninglessness of one of the conjuncts [32], to a causal, explanatory or otherwise grounding-like connection between premises and conclusions [41], to regressive analytic connections between premises and conclusions [41], and many other things. Of all these, we will come back to the explanation of the failure of Simplification in terms of the absence of regressive analytic connections, when we discuss regressive analytic entailments as modeled by containment logics, in Section 5. We will, again, also come back the epistemic explanation of the failure of Simplification, when we give a our novel epistemic interpretation of Paraconsistent Weak Kleene, in Section 3.4

Until now no unifying account of all these treatments of conjunctions which do not satisfy Simplification has been proposed. Whether or not it is actually possible to do so, is something we do not know and, in fact, an issue whose discussion will take us probably too far afield. Nevertheless, we can still point out that our epistemic explanation of the failure of Simplification will share some features with the account — implicitly — proposed by Ciuni in [11] to understand the failure of Simplification in Paraconsistent Weak Kleene, when this system is understood as a logic devised to handle paradoxes and other semantic pathologies.

There, Ciuni proposed to explain the failure of Simplification in Paraconsistent Weak Kleene by pointing out that, when it is employed as a logic to handle paradoxes, conjunction has a *disjunctive* flavor to it — or otherwise is nothing more than a disjunction in disguise. The disjunction in question is formed by the explicitly stated conjunction and the possibility that *either* of the conjuncts satisfies a certain overriding condition: in the particular case he is discussing, that of being a pathological proposition. In this way, were some of the conjuncts to satisfy this overriding condition, the (apparent) conjunction would be satisfied. We shall highlight that, as the reader will notice in the sequel, our own epistemic interpretation of PWK and therefore of the failure of Simplification in it, will exhibit a similar pattern — although the ingredients will be completely different. We will come back to this similarity below, at the end of Section 3.4.

Having said this, let us now turn to the epistemic interpretation of the Kleene logics.

3. The Epistemic Interpretation of Kleene logics

In this section we will, first, review Fitting's epistemic interpretation of the strong Kleene logic K_3 and its four-valued generalization FDE. After that, we will look at Fitting's epistemic interpretation of the Paracomplete Weak Kleene logic $\mathsf{K}_3^{\mathrm{w}}$ and its four-valued generalization $\mathsf{S}_{\mathtt{fde}}$. We will, finally, advance an epistemic interpretation of Paraconsistent Weak Kleene logic and its four-valued generalization $\mathsf{dS}_{\mathtt{fde}}$, which is novel to this work.

3.1. What is an Epistemic Interpretation?

Before jumping to the interpretations themselves, we should explain what are we trying to do in providing an epistemic interpretation for the Kleene logics, i.e., what Fitting did and what we will, consequently, try to do. To briefly answer this question we shall say our aim is to provide an epistemic reading of the truth-values featured in the corresponding Kleene logics, their notion of logical consequence, and the truth-functions characteristic of these systems.

We will devote Sections 3.2 to 3.4 below to a discussion of the epistemic interpretation of the distinctive truth-functions of each of the strong and weak Kleene logics. Here, we shall talk about what is shared by the epistemic interpretations of each of these systems, namely, the reading of the truth-values and the consequence relation at play.

In relation to these, in e.g. [29], Fitting suggests we consider the following situation. Suppose we have a certain group of experts $\mathcal{E}$ whose opinion we value and whom we are consulting on certain matters, in the form of a series of yes/no questions. When asking these experts about a certain proposition φ some will say it is true, some will say it is false, some may be willing to decline expressing and opinion and some may have reasons for calling it both true and false. Fitting suggests that in these cases we, correspondingly, assign φ a sort of *generalized truth-value*

$$v(\varphi) = \langle P, N\rangle$$

where P is the set of experts who say φ is true, and N is the set of experts who say φ is false [30, p. 57]. Thus, it is possible that $P \cup N \neq \mathcal{E}$ and it is also possible that $P \cap N \neq \emptyset$.[6]

Given this picture, let us now focus on the epistemic reading of the four truth-values $\mathbf{t}$, $\top$, $\bot$, $\mathbf{f}$. These values are usually interpreted—that is, outside of the epistemic interpretation, e.g. in [47]—as, respectively, true only, both-true-and-false, neither-true-nor-false and false only. However, in the context of the epistemic reading we are currently discussing, the assignment of the non-classical values $\bot$ and $\top$ to a certain formula corresponds, respectively, to the case where no expert

[6] As Fitting highlights in many places, considerations along these lines already suggest we are going to end up, down the road, with a lattice-theoretic structure (called a bilattice) which can be put to very good use for logical investigations. But we will try not to go into the formal details of the connection between this investigations and those concerning bilattices, leaving this discussion for another time.

expresses an opinion towards the formula in question, i.e., to the generalized truth-value $\langle\emptyset,\emptyset\rangle$, and the case where all experts say that the formula in question is true, and at the same time they say it is false, i.e., to the generalized truth-value $\langle\mathcal{E},\mathcal{E}\rangle$. In the former case, we might say they have an *indeterminate* opinion, and in the latter case we might say they have an *inconsistent* opinion.

Similarly, the assignment of the classical values $\mathbf{t}$ and $\mathbf{f}$ corresponds, respectively, to the case where all experts say the formula in question is true and no expert says it is false, i.e., to the generalized truth-value $\langle\mathcal{E},\emptyset\rangle$, and the case where no expert says that the formula is true and all experts say that the formula is false, i.e., to the generalized truth-value $\langle\emptyset,\mathcal{E}\rangle$. Moreover, with regard to the full set of the four generalized truth-values, we will take the terminological liberty — in alignment with the previous remarks — of calling $\mathbf{t}, \top$ and $\mathbf{f}$ the *determinate* values, while calling $\mathbf{t}, \bot$ and $\mathbf{f}$ the *consistent* values. Thus, to account for the strong and weak Kleene logics presented above, it is necessary to think that every time the experts are consulted on a certain proposition φ the resulting general opinion can be represented by one of the four truth-values $\mathbf{t}, \top, \bot, \mathbf{f}$.

This epistemic interpretation of the truth-values $\mathbf{t}, \top, \bot, \mathbf{f}$ certainly puts things under a different light, but it still does not account for an epistemic understanding of a logic. For that purpose we need to give an epistemic understanding of the underlying truth-functions of the given logic (whether it be one of the strong or the weak Kleene ones) and of the accompanying definition of *logical consequence.* The latter issue is easier to settle. Being relatively conservative, in what follows we will always be taking logical consequence to be somehow related to *truth-preservation.* More specifically, by this we mean that an argument will be valid if and only if whenever the premises are taken to be true by all experts, so is the conclusion.

The task of giving an *epistemic understanding of the strong and the weak Kleene truth-tables* and of their four-valued generalizations will demand a little bit more work. Fitting achieved this by taking these truth-functions to embody different approaches to determine what experts think of certain complex formulae such as $\varphi \wedge \psi$ and $\varphi \vee \psi$ — i.e., whether they think they are true or false — given how these experts stand concerning their components. In a nutshell, we can say that Fitting took the truth-functions characteristic of each of the Kleene logics discussed by him, to embody different policies applicable when pooling the opinions of the consulted experts. Thus, in what remains of this section we will be discussing what these difference policies are in the case of the strong Kleene logics and of the Paracomplete Weak Kleene logic, showing at last how it is possible to extend this account to provide an epistemic interpretation of the truth-functions of Paraconsistent Weak Kleene.

3.2. The Epistemic Interpretation of strong Kleene logics

Given the above remarks, the last thing required to provide an epistemic interpretation of the strong Kleene logics K_3 and its four-valued generalization FDE is to clarify which policies for pooling the opinions of the consulted experts are characteristic of these logics. The unsurprising answer will be: the most intuitive and straightforward ones.

In fact, concerning a conjunction $\varphi \wedge \psi$, Fitting says that we should calculate its generalized truth-value — the pair comprising, first, the set of experts which think it is true and, second, the set of experts which think it is false — as follows. On the one hand, it seems intuitive to say that the experts who believe $\varphi \wedge \psi$ is true are those who believe both φ *and* ψ are true. That is to say, the set of experts who believe $\varphi \wedge \psi$ is true can be calculated by taking the *intersection* of two sets: the set of experts who think φ is true, and the set of experts who think ψ is true. On the other hand, it also seems intuitive to say that the experts who believe $\varphi \wedge \psi$ is false are those who believe either φ *or* ψ is false. That is to say, the set of experts who believe $\varphi \wedge \psi$ is false can be calculated by taking the *union* of two sets: the set of experts who think φ is false, and the set of experts who think ψ is false. Fitting proposes to formally represent this, given two propositions φ and ψ whose generalized truth-values are $v(\varphi) = \langle P_1, N_1\rangle$ and $v(\psi) = \langle P_2, N_2\rangle$, by defining an operation $\sqcap$ between them as

$$\varphi \sqcap \psi = \langle P_1 \cap P_2, N_1 \cup N_2\rangle$$

Analogous reasoning establishes that for the case of a disjunction $\varphi \vee \psi$, its generalized truth-value should be calculated as follows. On the one hand, it seems intuitive to say that the experts who believe $\varphi \vee \psi$ is true are those who believe φ *or* ψ are true. That is to say, the set of experts who believe $\varphi \vee \psi$ is true can be calculated by taking the *union* of two sets: the set of experts who think φ is true, and the set of experts who think ψ is true. On the other hand, it also seems intuitive to say that the experts who believe $\varphi \vee \psi$ is false are those and only those who believe both φ *and* ψ are false. That is to say, the set of experts who believe $\varphi \vee \psi$ is false can be calculated by taking the *intersection* of two sets: the set of experts who think φ is false, and the set of experts who think ψ is false. Fitting proposes to formally represent this, given two propositions φ and ψ whose generalized truth-values are $v(\varphi) = \langle P_1, N_1 \rangle$ and $v(\psi) = \langle P_2, N_2 \rangle$, by defining an operation $\sqcup$ between them as[7]

$$\varphi \sqcup \psi = \langle P_1 \cup P_2, N_1 \cap N_2 \rangle$$

Finally, for the case of a negation $\neg\varphi$, its generalized truth-value should be calculated by switching the role of the set of experts saying φ is true and the set of experts saying φ is false. That is, those experts who say that φ is true, should be counted as saying that $\neg\varphi$ is false, and those experts saying that φ is false should be counted as saying that $\neg\varphi$ is true. Formally, for a given proposition φ whose generalized truth-value is $v(\varphi) = \langle P_1, N_1 \rangle$, Fitting defines the operation $\neg$ as

$$\neg\varphi = \langle N_1, P_1 \rangle$$

Let us now consider the case where every time the experts are consulted on a certain proposition φ, the resulting general opinion can be represented by one of the four truth-values $\mathbf{t}, \top, \bot, \mathbf{f}$. If, in this context, we were to graphically summarize the outcome of the previous pooling directives concerning negation, conjunction and disjunction, we will arrive at the following "truth-tables"

	$\neg$
$\mathbf{t}$	$\mathbf{f}$
$\top$	$\top$
$\bot$	$\bot$
$\mathbf{f}$	$\mathbf{t}$

$\sqcap$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\top$	$\top$	$\top$	$\mathbf{f}$	$\mathbf{f}$
$\bot$	$\bot$	$\mathbf{f}$	$\bot$	$\mathbf{f}$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$

$\sqcup$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$
$\top$	$\mathbf{t}$	$\top$	$\mathbf{t}$	$\top$
$\bot$	$\mathbf{t}$	$\mathbf{t}$	$\bot$	$\bot$
$\mathbf{f}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$

which are, respectively, those of the truth-functions $f^{\neg}_{\mathsf{FDE}}$, $f^{\wedge}_{\mathsf{FDE}}$ and $f^{\vee}_{\mathsf{FDE}}$ of Belnap-Dunn four-valued logic FDE, as discussed e.g. in [6, 19]. This suggests that the above remarks amount to an epistemic interpretation of Belnap-Dunn four-valued logic.

[7] Fitting actually denotes the operations $\sqcap$ and $\sqcup$ with the symbols $\wedge$ and $\vee$, respectively. However, in an effort to minimize confusion as much as possible, we decided to change them in order to differentiate them from the connectives usually employed to represent conjunction and disjunction. However, this is only a stylistic choice, and none of the formal results depends on this.

The question remains, however, of how to use the above considerations to provide an epistemic interpretation for Strong Kleene logic K_3. If we imagine a situation in which all experts are consulted and, for no proposition φ all experts express an inconsistent opinion, this will amount to restricting the FDE valuations to the "consistent" values: namely, $\mathbf{t}, \bot, \mathbf{f}$. The three-valued logic induced by this restriction is K_3 and it is, thus, through this reflections that Fitting arrived at an epistemic interpretation for Strong Kleene logic.

Interestingly, this can be further applied to provide an epistemic interpretation for Priest's Logic of Paradox LP. In fact, if we imagine a situation where, for all propositions on which the experts are asked about, no expert refrains from expressing an opinion, this will amount to restricting the FDE valuations to the "determinate" values: namely, $\mathbf{t}, \top, \mathbf{f}$. The three-valued logic rendered by this constraints is, thus, LP — which, additionally, provides an epistemic interpretation for this interesting Kleene logic. Let us, then, see how a modification of these remarks may lead to an epistemic interpretation of the *weak* Kleene logics.

3.3. The Epistemic Interpretation of Paracomplete Weak Kleene

How does Fitting arrive at the desired interpretation of Paracomplete Weak Kleene? It is only after taking the weak Kleene truth-tables to summarize a quite distinctive approach to pooling the opinions of the consulted experts. An approach, that is, which must — in a very sensible way — diverge from that of the strong Kleene logics reviewed in the previous subsection.

In fact, in [30] Fitting is quite clear about this, noting that sometimes we may want to collect and ponder on the opinions of the consulted experts in special ways. Recall that the framework allows experts to be silent about certain matters when they are asked about their opinions. Thus, e.g., when evaluating a conjunction $\varphi \wedge \psi$ or a disjunction $\varphi \vee \psi$ we may

> want to 'cut this down' by considering people who have actually expressed an opinion on both propositions [φ] and [ψ]. [30, pp. 66–67]

Hence, we shall call the resulting alternative conjunctions and disjunctions — following Ferguson in [23] and [25] — the "cut-down variants" of these logical operations. This has as a result that the "cut-down" way in which we calculate e.g. the set of experts who believe the conjunction

$\varphi \wedge \psi$ is true (alternatively, false) requires taking the *intersection* of set of experts *that we previously classified* as saying $\varphi \wedge \psi$ is true (false), with the set of experts who have actually expressed an opinion towards both φ and ψ. Similarly for a disjunction $\varphi \vee \psi$ and a negation $\neg\varphi$. In these cases, we may say these variants operate within Fitting's epistemic interpretation of the Kleene logics, following the motto

> NO DETERMINATE OPINION CAN ARISE FROM A SET THAT INCLUDES AN INDETERMINATE OPINION.

Let us now have a closer look at Fitting's proposal to formally model this cut-down approach.

The first step is to define a unary operator — eloquently called a "cut-down operator", by Ferguson — which, for a given proposition φ outputs the set of experts who have expressed any determinate opinion whatsoever (either positive or negative) towards φ. This is done by taking the *union* of the experts who said it is true and the set of experts who said it is false.

This can be rigorously represented with the help of a further operation on generalized truth-values, called the gullibility or "accept anything" operation $\oplus$ [see, e.g., 30, p. 56]. Given two propositions φ and ψ, the gullibility operation between them gives as a result a generalized truth-value where, on the one hand, all who think φ or ψ are true are brought together and, on the other hand, all who think φ or ψ are false are also brought together.

That is to say, the result of calculating the gullibility operation between φ and ψ is a pair, obtained as follows. As the first coordinate, we have the *union* of the set of experts who think φ is true with the set of experts who think ψ is true. As the second coordinate, we have the *union* of the set of experts who think φ is false with the set of experts who think ψ is false. Speaking more formally, consider two propositions φ and ψ such that their generalized truth-values are, respectively, $v(\varphi) = \langle P_1, N_1 \rangle$ and $v(\psi) = \langle P_2, N_2 \rangle$. The gullibility operation applied to them is defined such that

$$\varphi \oplus \psi = \langle P_1 \cup P_2, N_1 \cup N_2 \rangle$$

It is, then, with the aid of this operation that the cut-down $[\![\varphi]\!]$ of a proposition φ, whose generalized truth-value is $v(\varphi) = \langle P_1, N_1 \rangle$, can be defined as

$$[\![\varphi]\!] = \varphi \oplus \neg\varphi = \langle P_1 \cup N_1, P_1 \cup N_1 \rangle$$

noting, furthermore, that the only case where $[\![\varphi]\!] = \langle\emptyset, \emptyset\rangle$ is the case where no expert expressed a determinate opinion towards φ — corresponding to the assignment of the truth-value $\perp$ to φ.

The second step to formally model Paracomplete Weak Kleene's operations in this epistemic setting, is to design, with the help of these tools, e.g. "cut-down" conjunctions and disjunctions. To accomplish this we can use the help of yet another operation on generalized truth-values, called the consensus or "agreement" operation $\otimes$ [see, e.g., 30, p. 56]. Given two propositions φ and ψ, the consensus operation between them gives as a result a generalized truth-value where, on the one hand, all those who agree that φ and ψ are true are brought together and, on the other hand, all those who agree that φ and ψ are false are also brought together.

That is to say, the result of calculating the consensus between φ and ψ is a pair, obtained as follows. As the first coordinate, we have the *intersection* of the set of experts who think φ is true with the set of experts who think ψ is true. As the second coordinate, we have the *intersection* of the set of experts who think φ is false with the set of experts who think ψ is false. Speaking more formally, consider two propositions φ and ψ such that their generalized truth-values are, respectively, $v(\varphi) = \langle P_1, N_1\rangle$ and $v(\psi) = \langle P_2, N_2\rangle$. The consensus operation applied to them is defined such that

$$\varphi \otimes \psi = \langle P_1 \cap P_2, N_1 \cap N_2\rangle$$

It is, then, with the aid of these formal instruments that we are able to define the target cut-down variants of conjunction and disjunction. Let us focus, for instance, in the case of conjunction. Fitting indicates that we ought to take the generalized truth-value of $\varphi \wedge \psi$ and cut it down to the set of people who have actually expressed an opinion towards both φ and ψ. In other words, the generalized truth-value of this cut-down conjunction should be obtained as follows.

On the one hand, we should cut down the set of experts who believe both φ and ψ are true. This can be done by taking the *intersection* of set of experts who think both φ and ψ are true, with the set of experts who have actually expressed a determinate opinion towards both propositions. On the other hand, we should cut down the set of experts who believe either φ or ψ are false. This can be done by taking the *intersection* of set of experts who think either φ or ψ are false, with the set of experts who have actually expressed a determinate opinion towards both propositions.

Both these moves, together, amount to nothing other than taking the consensus $\otimes$ between the set of experts *that we previously classified* as saying $\varphi \wedge \psi$ is true (or false) — i.e. $\varphi \sqcap \psi$ — and the set of experts who have actually expressed an opinion towards both φ and ψ — i.e., the cut-downs of φ and ψ, namely $[\![\varphi]\!]$ and $[\![\psi]\!]$. Similar reasoning leads to similar results for disjunction and negation. More formally, Fitting defines a cut-down conjunction $\triangle$ and a cut-down disjunction $\triangledown$ as follows, noting that negation is not altered by these modifications.[8]

$$\varphi \triangle \psi = (\varphi \sqcap \psi) \otimes [\![\varphi]\!] \otimes [\![\psi]\!] \qquad \varphi \triangledown \psi = (\varphi \sqcup \psi) \otimes [\![\varphi]\!] \otimes [\![\psi]\!]$$

Finally, if the four values $\mathbf{t}, \top, \bot, \mathbf{f}$ are taken into account, the "truth-tables" for the operations of conjunction, disjunction and negation — understood in this "cut-down" fashion — would be the following, as is easy to check.

	$\neg$
$\mathbf{t}$	$\mathbf{f}$
$\top$	$\top$
$\bot$	$\bot$
$\mathbf{f}$	$\mathbf{t}$

$\triangle$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\top$	$\top$	$\top$	$\bot$	$\mathbf{f}$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\bot$	$\mathbf{f}$

$\triangledown$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\bot$	$\mathbf{t}$
$\top$	$\mathbf{t}$	$\top$	$\bot$	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
$\mathbf{f}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$

These are, respectively, the truth-functions $f^{\neg}_{\mathsf{S}_{\mathtt{fde}}}$, $f^{\wedge}_{\mathsf{S}_{\mathtt{fde}}}$ and $f^{\vee}_{\mathsf{S}_{\mathtt{fde}}}$ of the four-valued logic $\mathsf{S}_{\mathtt{fde}}$, depicted in Figure 3 above. Hence, the previous account can be taken to represent nothing more than an epistemic interpretation of this four-valued logic.

Of most importance to us, though, is that our discussion of $\mathsf{S}_{\mathtt{fde}}$ as a four-valued generalization of Paracomplete Weak Kleene in Section 2 already suggests how we are going to provide an epistemic interpretation for $\mathsf{K}^{\mathrm{w}}_{3}$. Indeed, if we imagine a situation in which experts are consulted, and for no proposition φ all the experts think φ is true and all experts think φ is also false, i.e., if all of them have consistent opinions on absolutely all propositions, this will formally amount to restricting the valuations of $\mathsf{S}_{\mathtt{fde}}$ to the "consistent" values: namely, the truth-values $\mathbf{t}, \bot, \mathbf{f}$. The three-valued logic induced by this restriction will be no other than $\mathsf{K}^{\mathrm{w}}_{3}$ and it is, therefore, through these remarks that Fitting arrived at an epistemic interpretation for Paracomplete Weak Kleene.

[8] See Fitting [29, p. 67] and Ferguson [23, p. 24], [25, p. 3].

Let us now illustrate how the *failure of Addition* is, thus, properly understood in this framework. Consider, for example, the case where all experts think φ is true and no expert thinks it is false and, simultaneously, all experts think ψ is neither true nor false — i.e. $v(\varphi) = \langle \mathcal{E}, \emptyset \rangle$ and $v(\psi) = \langle \emptyset, \emptyset \rangle$. Given our previous considerations, we can equivalently say that φ is assigned the truth-value $\mathbf{t}$, whereas ψ is assigned the truth-value $\perp$. As a result of this, the value of the 'cut-down' disjunction $\varphi \triangledown \psi$ will be $\langle \emptyset, \emptyset \rangle$ — that is, the epistemic counterpart of $\perp$. In other words, the failure of Addition is understood in this interpretation by taking disjunction to be a *cut-down disjunction*. Under such a reading, it is clear how from e.g. the fact that all experts think φ is true it does *not* follow that all experts think $\varphi \vee \psi$ is true — the reason being that all experts may have no opinion whatsoever with regard to ψ. Furthermore, allowing us to establish that in the context of this epistemic interpretation the fact that all experts have an opinion on the given disjuncts works as the aforementioned "enabling condition" (cf. Section 2) for a disjunction to be true.

An anonymous referee wondered whether or not under this interpretation we are claiming that $\varphi \vee \psi$ may be unknown, even if φ is true. That is not what we are claiming. What we have just said is that, if we want to *report* which propositions are supported by all experts, and — additionally — we want to adopt a cut-down policy in providing these reports, making sure that in such cases all experts have an opinion on all of the component propositions thereof, then even though all experts may support φ, then we may not regard all experts as supporting $\varphi \vee \psi$. The reason is that all of them may have no opinion towards ψ.[9]

In what follows we will be proceeding similarly to arrive at our desired epistemic interpretation for Paraconsistent Weak Kleene.

3.4. The Epistemic Interpretation of Paraconsistent Weak Kleene

Our aim, now is to provide a novel understanding of Paraconsistent Weak Kleene's truth-functions by taking them to summarize a distinctive

[9] This touches upon the difference between a proposition φ being assigned the truth-value $\mathbf{t}$, which under the current interpretation is to be read as reporting that "all experts said that φ is true, and no expert said that φ is false", and a proposition φ being true *simpliciter*, i.e., ontologically speaking. Not only are these two properties different, but additionally we do not even have the means to say that a proposition *is* true — ontologically speaking — in the current epistemic interpretation of these semantics. We elaborate on these issues, in the discussion provided in Section 4.1.

approach to pooling the opinion of the experts being consulted. This approach will have to, yet again, diverge quite sensibly not only from the pooling policies embodied by the strong Kleene logics — but also from the one embodied by Paracomplete Weak Kleene, discussed in the previous section.

We will motivate our approach noting that there might be further special ways in which we might want to collect and ponder on the opinions of the experts. Let us recall, for example, that the general framework outlined by Fitting allows experts to have inconsistent opinions about certain matters, i.e., some experts may have reasons for calling a proposition φ both true and false.

Thus, e.g. when considering a conjunction $\varphi \wedge \psi$ or a disjunction $\varphi \vee \psi$ we may want — in a way that is perfectly dual to Fitting's suggestions above — to "track down" people who have expressed an inconsistent opinion towards *either* φ or ψ. Hence, we shall call the resulting alternative conjunctions and disjunctions, the "track-down" variants of these famous logical operations. In these cases, we may say these variant operate within Fitting's epistemic interpretation, following the motto

> NO CONSISTENT OPINION CAN ARISE FROM A SET THAT INCLUDES AN INCONSISTENT OPINION

Let us now have a closer look at our proposal to formally model this track-down approach.

The first step to technically represent these track-down variants, is to define a unary operator — to be called a "track-down" operator — which, for a given proposition φ outputs the set of experts who have expressed an inconsistent opinion towards φ. This is done by taking the *intersection* of the set of experts who said it is true and the set of experts who said it is false.

This can be formally represented with the help of the consensus operation, letting the track-down $\|\varphi\|$ of a proposition φ, whose generalized truth-value is $v(\varphi) = \langle P_1, N_1 \rangle$, be defined as

$$\|\varphi\| = \varphi \otimes \neg\varphi = \langle P_1 \cap N_1, P_1 \cap N_1 \rangle$$

noting, furthermore, that the only case where $\|\varphi\| = \langle \mathcal{E}, \mathcal{E} \rangle$ is the case where all experts expressed an inconsistent opinion towards φ — corresponding to the assignment of the truth-value $\top$ to φ.

The second step to formally model Paraconsistent Weak Kleene's operations in this epistemic setting, is to design, with the help of these tools, e.g. "track-down" conjunctions and disjunctions. Let us focus, for instance, in the case of conjunction. Our previous motivations indicate we ought to take the generalized truth-value of $\varphi \wedge \psi$ and "track down" the set of people who have expressed an inconsistent opinion towards either φ or ψ. In other words, the generalized truth-value of this track-down conjunction should be obtained as follows.

On the one hand, we should collect the set of experts who believe either φ or ψ is false, together with the set of people who have expressed an inconsistent opinion towards either φ or ψ. This can be done by taking the *union* of set of experts who think both φ and ψ are true, with the set of experts who have expressed an inconsistent opinion towards either propositions. On the other hand, we should collect the set of experts who believe either φ or ψ is false, together with the set of people who have expressed an inconsistent opinion towards either φ or ψ. This can be done by taking the *union* of set of experts who think either φ or ψ is false, with the set of experts who have expressed an inconsistent opinion towards either propositions.

Both these moves amount to nothing other than taking the result of the $\oplus$ operation between the set of experts *that we previously classified* as saying $\varphi \wedge \psi$ is true (or false) — i.e. $\varphi \sqcap \psi$ — and the set of experts who have expressed an inconsistent opinion towards either φ or ψ — i.e., the track-downs of φ and ψ, namely $\|\varphi\|$ and $\|\psi\|$. Similar reasoning leads to similar results for disjunction and negation. More formally, we can define a track-down conjunction $\blacktriangle$ and a track-down disjunction $\blacktriangledown$ as follows, noticing that negation is not altered by these modifications.

$$\varphi \blacktriangle \psi = (\varphi \sqcap \psi) \oplus \|\varphi\| \oplus \|\psi\| \qquad \varphi \blacktriangledown \psi = (\varphi \sqcup \psi) \otimes \|\varphi\| \otimes \|\psi\|$$

Thus, just as in the cut-down case where the set of experts being considered is *shrunk* to cover all those experts expressing a determinate opinion on *all* the relevant propositions, in the track-down case something similar happens. In fact, in the track-down case the set of experts being considered is *enlarged* to cover all those experts expressing an inconsistent opinion on *some* of the relevant propositions, hence it is asked of the pooling procedure not to forget that some people do not have a consistent opinion towards the issues in question.

Finally, if the four values $\mathbf{t}, \top, \bot, \mathbf{f}$ are taken into account, the "truth-tables" for the operations of conjunction, disjunction and negation — understood in this "track-down" fashion — would be the following, as is easy to check.

	$\neg$
$\mathbf{t}$	$\mathbf{f}$
$\top$	$\top$
$\bot$	$\bot$
$\mathbf{f}$	$\mathbf{t}$

$\blacktriangle$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\top$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\bot$	$\top$	$\bot$	$\mathbf{f}$
$\mathbf{f}$	$\mathbf{f}$	$\top$	$\mathbf{f}$	$\mathbf{f}$

$\blacktriangledown$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\top$	$\mathbf{t}$	$\mathbf{t}$
$\top$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\mathbf{t}$	$\top$	$\bot$	$\bot$
$\mathbf{f}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$

As advertised, these are, respectively, the truth-functions $f^{\neg}_{\mathsf{dS_{fde}}}$, $f^{\wedge}_{\mathsf{dS_{fde}}}$ and $f^{\vee}_{\mathsf{dS_{fde}}}$ of the four-valued logic $\mathsf{dS_{fde}}$, whose truth tables we depicted in Figure 4 above. Hence, the previous account can be taken to constitute an epistemic interpretation of this four-valued logic.

Of most importance to us, however, is that our discussion of $\mathsf{dS_{fde}}$ as a four-valued generalization of Paraconsistent Weak Kleene in Section 2 allows us to transition from the above remarks to an epistemic interpretation for PWK. In fact, if we consider a situation in which experts are consulted, and for no proposition φ all experts refrain from expressing an opinion about it, i.e., if all of them have determinate opinions on absolutely all propositions, this will formally amount to restricting the valuations of $\mathsf{dS_{fde}}$ to the "determinate" values: namely $\mathbf{t}, \top, \mathbf{f}$. The three-valued logic induced by this restriction is our target logic PWK. It is, therefore, through these considerations that we arrive at our desired *epistemic interpretation for Paraconsistent Weak Kleene*—thus fulfilling the main goal of this paper.[10]

Let us now illustrate, as expected, how the *failure of Simplification* is, thus, properly understood in this framework. Consider, for example, the case where all experts think φ is true and at the same time all experts think it is false, while also all experts think ψ is false and no expert thinks it is true—i.e. $v(\varphi) = \langle \mathcal{E}, \mathcal{E} \rangle$ and $v(\psi) = \langle \emptyset, \mathcal{E} \rangle$. Given our previous considerations, we can equivalently say that φ is assigned the truth-value $\top$, whereas ψ is assigned the truth-value $\mathbf{f}$. As a result of this, the value of the "track-down" conjunction $\varphi \blacktriangle \psi$ will be $\langle \mathcal{E}, \mathcal{E} \rangle$—that is, the epistemic counterpart of $\top$. In a nutshell, the failure of Simplification is understood in this interpretation by taking conjunction to be a *track-down conjunction*. Under such a reading, it is clear how from e.g. the fact that all experts think $\varphi \wedge \psi$ is true it does *not* follow that all experts think that ψ is true—the reason being that we may

[10] These ideas were formally developed in full extent in [52], were it is shown that $\mathsf{dS_{fde}}$ is the logic of track-down operations on logical bilattices.

regard all experts as thinking that the conjunction is true because all of them have an inconsistent opinion concerning φ.

We can, furthermore, connect this to our purported understanding of the failure of Simplification in terms of conjunction being a "disguised disjunction", formed by the conjuncts in question and two more disjuncts representing the possibility that either of the conjuncted propositions triggers a certain overriding condition. It is clear from the above that the epistemic interpretation of Paraconsistent Weak Kleene outlined allows for this reading. In fact, in the context of such an epistemic interpretation, the fact that all experts have an inconsistent opinion on one of the given conjuncts works as the aforementioned "overriding condition" (cf. Section 2) for a conjunction to be true.

The previous reflections put PWK and its four-valued generalization under a new light with regard to philosophical and logical investigations. But, speaking of a formal logic, we believe some philosophical achievements can be made of a broader significance if we can supplement them with attractive formal results. This is why, below, we devote ourselves to offer some of these. We present sound and complete two-sided sequent calculi for the four-valued generalizations of the weak Kleene logics. We motivate the particular calculi below, by pointing out some results on the relation between weak Kleene logics and containment logics.

However, before doing this we shall provide a discussion and a comparison of the account presented in this paper, with works in the literature which discuss the relation between many-valued logics and epistemic concepts. To this we now turn.

4. Related Work

In this section we offer a comparison of some work, present in the literature about many-valued logics, concerning their relation with epistemic concepts. These discussions are not meant to be exhaustive, as there is a great deal of related works which we cannot — for considerations of space — address with here. What follows should, then, be regarded as a rather brief overview and comparison of what we have done in the previous sections, with some works that form a part of the constantly growing literature on these topics.[11]

[11] We would like to thank an anonymous reviewer for urging us to comment on these issues.

4.1. Epistemic Interpretations and Truth-Functionality

In a number of papers coauthored by Didier Dubois and colleagues, some deeply interesting observations concerning giving epistemic interpretations to many-valued logics are thoroughly considered. Prime among them is the worry that there is some inherent tension between giving an epistemic interpretation to a many-valued logic, and the logic in question being truth-functional, as discussed e.g. in [17, 18]. This concern is further explained by Dubois' claim that — when involved in such a project — the very assumption of truth-functionality is debatable, precisely because "belief is never truth-functional" [18, p. 195].

Our response to these observations will follow closely the replies formulated by Heinrich Wansing and Nuel Belnap in their recent paper [55], which addresses the previous concerns, as directed to the "Told Interpretation" (an epistemic interpretation, indeed) of the logic FDE. This reading of Belnap-Dunn's four-valued logic interprets this system in terms of how a computer should reason when receiving information from different reliable sources about the semantic status of propositional variables. This reading takes the values $\mathbf{t}, \top, \bot, \mathbf{f}$ to represent, respectively, the situation in which the computer has been told that the given proposition is true, but has not been told that it is false; the situation in which the computer has been told that the given proposition is true, and it has also been told that it is false; the situation in which the computer has not been told that the given proposition is true, and has not been told that it is false; and, finally, the situation in which the computer has not been told that the given proposition is true, but has been told that it is false. We will see that more or less the majority of the components of their defense apply to Fitting's epistemic interpretation, if appropriate care is taken. As such, their arguments can be turned into a defense of our interpretation of the weak Kleene logics and, in particular, of Paraconsistent Weak Kleene.

Dubois' argument for the non-truth-functionality of belief appears in many forms in various of his works, and can be reasonably summarized as follows. As long as φ can only be either true or false, even if we have no information concerning its truth-value or if we have conflicting information concerning its truth-value, the proposition $\varphi \wedge \neg\varphi$ can be unmistakably at any time predicted as being false and the proposition $\varphi \vee \neg\varphi$ can be unmistakably at any time predicted as being true [cf. 18, p. 195]. Regarding all the Kleene logics (the strong and the weak, the three- and the four-valued), the upshot of this line of argumentation appears to be that — as long as they are given an epistemic interpretation — they will be highly inadequate formalisms. In all of these systems, the previously referred formulae are not evaluated as Dubois' remarks require them to be.

Dubois and his colleagues conclude, then, that the lack of expressive power of epistemically interpreted truth-functional many-valued logics suggests the need to look for more appropriate frameworks. In this vein, epistemic logics appear to provide a more suitable environment. In fact, with these ideas in mind, Dubois developed epistemic logics designed to recast the intuitions behind the epistemic interpretation of some of the Kleene logics. For instance, in [18] the systems MEL and MELC are presented as epistemic logics "to model reasoning about incomplete information", and to model reasoning about incomplete as well as "conflicting information" — respectively.[12] These are intended to recast, as epistemic logics, the systems K_3 and FDE.

To provide an example of the difference in expressive power between many-valued logics interpreted epistemically and proper epistemic logics, let us focus on the case of the three-valued strong Kleene logic. In the epistemic logic MEL, $\Box p$ is interpreted as "the source claims to know that p is true" [18, p. 201], and correspondingly $\Diamond p$ as "the source does not claim to know that p is false". Given this, it is possible to provide a translation Φ of a formula in Kleene logic in terms of a formula in MEL involving only possibility and necessity as applied to literals. This is done along the following lines, granted p is a propositional variable [cf. 18, p. 202].

$$\Phi(p) = \Box p \qquad\qquad \Phi(\neg p) = \Box\neg p$$
$$\Phi(\varphi \wedge \psi) = \Phi(\varphi) \wedge \Phi(\psi) \qquad\qquad \Phi(\varphi \vee \psi) = \Phi(\varphi) \vee \Phi(\psi)$$

Interestingly, even if both K_3 and MEL can represent the fact that the truth-value of e.g. φ is unknown — by $\nvDash_{\mathsf{K}_3} \varphi \vee \neg\varphi$ and $\nvDash_{\mathsf{MEL}} \Box\varphi \vee \Box\neg\varphi$ — some things are expressible in MEL that are not expressible in K_3. In particular, in the latter it is possible to express e.g. that it is known that φ is either true or false — by $\vDash_{\mathsf{MEL}} \Box(\varphi \vee \neg\varphi)$ — whereas there is no corresponding expression in K_3. This, according to Dubois,

[12] While the former can be described as the subjective fragment of the modal logic KD without nesting, the latter can be understood as the subjective fragment of the bimodal logic KD_2 without nesting. See [18] for more on this.

witnesses K_3's "lack of expressiveness and inferential power compared to the proposed epistemic logic" [18, p. 195].

Putting aside the technical comparison with epistemic logics, let us go back to the conceptual part of Dubois' argument against the truth-functionality of belief. It is clear that a particular instantiation of his argument could be used to criticize the logics considered in the present paper. However, notice that, as Wansing and Belnap explain in [55, p. 926], his objection could not be read as requiring that e.g. even if no expert did express her opinion towards φ, it still should be the case that $\varphi \vee \neg\varphi$ is true *simpliciter* — i.e., ontologically speaking. Or, similarly, that e.g. even if all experts did express an inconsistent opinion towards φ, it still should be the case that $\varphi \vee \neg\varphi$ is false — in the ontological sense. As explained by these authors, this cannot be required precisely because their epistemic interpretation — and, we may add, also Fitting's interpretation — of the Kleene logics does not have a mix of truth-values, some of which are epistemically interpreted and some of which are ontologically interpreted. All truth-values are epistemically interpreted.

Thus, in the "Told Interpretation" of FDE assigning a proposition the truth-value $\mathbf{t}$ means that the computer has been told that the formula is true, or that it has arrived at the conclusion that the formula is true, given what it has been told about the component formulae. Through analogous reasoning we may conclude that in Fitting's epistemic interpretation of the Kleene logics, assigning a proposition the truth-value $\mathbf{t}$ means that all of the experts have reported thinking that the formula is true and none has reported to think it is false, or that it is possible to arrive at such a conclusion, given what the experts have reported to think about the component formulae.[13] With Wansing and Belnap, then, we may argue that we cannot be required to say that a formula is true (or false) ontologically speaking, because in the context of these epistemic interpretations these truth-values "are just not available" [55, p. 926].

Therefore, Dubois' objection against the truth-functionality of the epistemic interpretation of the Kleene logics should be posed against these accounts in a different way instead. For example, by stating that

[13] As discussed in a footnote to Section 3.3, it is in this sense that — for instance — the failure of Addition in e.g. $\mathsf{K}_3^{\mathrm{w}}$ should not be interpreted in terms of having a true disjunct without a true disjunction. This is mainly due to the fact that the assignment of the truth-value $\mathbf{t}$ cannot be read in this ontological way, in the context of the outlined epistemic interpretation.

e.g. even if no expert did express her opinion towards φ, it is still true that all experts *believe* in the truth of $\varphi \vee \neg\varphi$. Similarly, by saying that e.g. even if all experts did express an inconsistent opinion towards φ, it is still true that all experts *believe* in the falsity of $\varphi \vee \neg\varphi$. However, since this is not what happens in the truth-functional semantics outlined above, there is something wrong.

But, then again, we must reply that this cannot be required, as the truth-values of the Kleene logics are not meant to be interpreted in terms of belief states — but merely in terms of information states, in the sense of data records [55, p. 924]. Thus, if e.g. the computer has *been told* that a given proposition is true and has also been told that it is false then this does not imply that either the computer or we as users are bound to *believe* that the proposition *is* in fact both true and false. In the same way, if e.g. all experts *say* a given proposition is true and that, at the same time, all experts say the given proposition is false then this does not imply that either these experts or we are bound to *believe* the proposition is in fact both true and false. All that this means is that we have received the information, provided by the corresponding sources, that the proposition is true and false. Put it differently, our assignment of the corresponding truth-value to the proposition corresponds to a faithful labeling of the recorded data. Thus, truth-values are not to be interpreted in terms of belief states in the context of the epistemic interpretations of the Kleene logics — be it the one due to Belnap, or in the one due to Fitting. Realizing this seems to dissolve the criticism.

But then — an objector might insist — we could still ask the semantics to evaluate formulae in such a way that, regardless of the information received about φ, it is always the case that e.g. the computer has not been told that $\varphi \wedge \neg\varphi$ it is true and that it has been told it is false, and similarly that the computer has been told that $\varphi \vee \neg\varphi$ is true and that it has not been told that it is false. And, analogously, that regardless of the opinion expressed by the consulted experts towards φ, we should always *report* that no expert thinks $\varphi \wedge \neg\varphi$ is true and that all of them think it is false, and at the same time that we should always *report* that all experts think $\varphi \vee \neg\varphi$ is true and none think it is false.

However, as Wansing and Belnap state in their reply to this constraints, in [55, p. 926], these evaluation requirements seem to be inaccurate. For what is worth, such an interpretation would not model reasoning based on the received, recorded, or stored information. This applies both to the epistemic interpretation due to Belnap, as well as to the interpretation due to Fitting. In answering its questions, the computer is to reply strictly in terms of what it has been told. Thus, it seems inappropriate for the computer to report that it has been told that e.g. $\varphi \vee \neg\varphi$ is true and that it has not been told that it is false, when it has not been told anything about φ. In the same way, in reporting what the experts think, Fitting's interpretation requires us to gather the information strictly in terms of what the experts have reported thinking. Thus, it seems off limits for us to report that e.g. all experts think $\varphi \vee \neg\varphi$ is true and none thinks it is false, if all experts refrained from expressing an opinion towards φ. Therefore, just as the computer is set to answer in terms of what it has been told and not in terms of what it could be programmed to answer, we are bound to report the experts' opinion in terms of the opinions actually expressed by the experts and not in terms of what we could be required to answer. In Dubois' words, the role of the semantics is "not to interpret the information provided by the sources, but just to store it." [18, p. 205].

These reflections, taken together, explain why the allegations against the truth-functionality of the Kleene logics do not — after all — strike a final blow against giving them an epistemic interpretation.

4.2. Orthopairs

Throughout many papers coauthored by Jonathan Lawry and collaborators, an interesting semantics using mathematical structures referred to as orthopairs, is presented. Interestingly, much like the previously discussed semantics due to Fitting, orthopair semantics also work with pairs of sets to account for three- and four-valued logics. In what follows, we will compare some formal aspects of orthopairs with the previous technical approaches to the semantics of Kleene logics.

By an orthopair we mean a pair $\langle P, N\rangle$ of disjoint subsets of propositional variables $P, N \subseteq \mathsf{Var}$ such that $P \cap N = \emptyset$. Thus, orthopairs can be put into a one-to-one correspondence with three-valued valuations. By this we mean that each three-valued valuation v induces an orthopair $\langle P_v, N_v\rangle$ and each orthopair $\langle P, N\rangle$ induces a three-valued valuation $v_{\langle P,N\rangle}$, as follows [cf. 10, p. 1867].

$$P_v = \{p \in \mathsf{Var} \mid v(p) = \mathbf{t}\} \qquad v_{\langle P,N\rangle}(p) = \begin{cases} \mathbf{t} & \text{if } p \in P \\ \mathbf{f} & \text{if } p \in N \\ \bot & \text{if } p \in \mathsf{Var} \setminus (P \cup N) \end{cases}$$
$$N_v = \{p \in \mathsf{Var} \mid v(p) = \mathbf{f}\}$$

Additionally, Lawry and his collaborators also suggest understanding an orthopair $\langle P, N\rangle$ in terms of an epistemic state, associating the propositional variables in P and N to those propositions, respectively, known to be true, and known to be false — later associating the propositional variables in $\mathsf{Var} \setminus (P \cup N)$ to those propositions for which the agent has no knowledge [10, p. 1870].

In a similar vein, if we lift the restriction that P and N are disjoint from the definition of an orthopair, we get the more general notion of a paraconsistent orthopair which can, in turn, be put into a bijection with four-valued valuations.[14] In fact, each four-valued valuation v induces an orthopair $\langle P_v, N_v\rangle$ and each paraconsistent orthopair $\langle P, N\rangle$ induces a four-valued valuation $v_{\langle P,N\rangle}$, as follows [10, p. 1877]:

$$P_v = \{p \in \mathsf{Var} \mid v(p) = \mathbf{t} \text{ or } v(p) = \top\}$$
$$N_v = \{p \in \mathsf{Var} \mid v(p) = \mathbf{f} \text{ or } v(p) = \top\}$$

$$v_{\langle P,N\rangle}(p) = \begin{cases} \mathbf{t} & \text{if } p \in P \text{ and } p \notin N \\ \mathbf{f} & \text{if } p \in N \text{ and } p \notin P \\ \top & \text{if } p \in P \text{ and } p \in N \\ \bot & \text{if } p \in \mathsf{Var} \setminus (P \cup N) \end{cases}$$

Analogously, paraconsistent orthopairs $\langle P, N\rangle$ can be seen in terms of a fusion of the consistent epistemic states of two conflicting agents, letting one of the agents have an epistemic state represented by the orthopair $\langle P, N \backslash P\rangle$ and the other agent have an epistemic state represented by the orthopair $\langle P \backslash N, N\rangle$ [10, p. 1877].

In Lawry's words, given orthopairs allow for a representation in terms of epistemic states "it is natural to compare them in terms of their respective amount of information" [10, p. 1880]. In this respect, the authors consider a number of different orderings which — they show — have a close relation with many of the Kleene logics. First among them, is the so-called truth-ordering $\leq_t$ intended to express that an orthopair makes propositional variables "more true" than another. This relation, defined as

$$\langle P_1, N_1\rangle \leq_t \langle P_2, N_2\rangle \Longleftrightarrow P_1 \subseteq P_2 \ \text{ and } \ N_2 \subseteq N_1$$

[14] These authors differentiate orthopairs and paraconsistent orthopairs also in notation, usually denoting the former by $\langle P, N\rangle$ and the latter by $\langle F, G\rangle$. However, for the sake of making the presentation more readable, we will not be adopting this disambiguation here.

It has its corresponding meet and join operations $\sqcap_t$ and $\sqcup_t$ which — when understood as operations on $\{\mathbf{t}, \top, \bot, \mathbf{f}\}$ — coincide with Fitting's operations $\sqcap$ and $\sqcup$ defined above. As such, they provide yet another interpretation of the truth-functions of the strong Kleene logics.

Another interesting relation on orthopairs considered by these authors is the information ordering $\leq_I$, defined as

$$\langle P_1, N_1\rangle \leq_I \langle P_2, N_2\rangle \iff P_1 \subseteq P_2 \ \text{ and } \ N_1 \subseteq N_2$$

It also has its corresponding meet meet and join operations $\sqcap_I$ and $\sqcup_I$ which, yet again — when understood as operations on $\{\mathbf{t}, \top, \bot, \mathbf{f}\}$ — coincide with Fitting's operation $\otimes$ and $\oplus$ defined above.

In this regard, it is instructive to compare Fitting's reading of these operations in terms of consensus and gullibility operations, with Lawry's reading of these operations as pessimistic and optimistic combinations. He considers that the former represents a *pessimistic combination* "as it only retains what both orthopairs hold as being true or false", resulting in an epistemic state that is less informative than the epistemic states being combined [10, p. 1880]; whereas he considers that the latter represents an *optimistic combination* "as it retains what at least one orthopair holds as being true or false", resulting in an epistemic state that is more informative than the epistemic states being combined [10, p. 1881].

Lawry's work is also attractive because it puts forward a new consensus operation — entirely different from the one discussed by Fitting. This new operation can be intuitively described as a consistent fusion of the epistemic states of two conflicting agents. To this extent, this way of getting a consensus requires a subsidiary operation that allows to — so to say — prepare the epistemic states of the conflicting agents, so that they can be consistently fused. This auxiliary notion, called the *difference operation* in [37] is denoted by $\ominus$ and is defined as follows

$$\langle P_1, N_1\rangle \ominus \langle P_2, N_2\rangle = \langle P_1 \backslash N_2, N_1 \backslash P_2\rangle$$

It can be conceptually described as a contraction operation, used to eliminate the discrepancies between the conflicting orthopairs or epistemic states, prior to merging — by eliminating conflicting propositional variables. Given this, the *new consensus operation* $\odot$ is defined as follows

$$\langle P_1, N_1\rangle \odot \langle P_2, N_2\rangle = \langle (P_1 \backslash N_2) \cup (P_2 \backslash N_1), (N_1 \backslash P_2) \cup (N_2 \backslash P_1)\rangle$$

This suggests two further projects that can be carried out, drawing inspiration from the previous remarks. First, it would be interesting to investigate the alternative cut- and track-down operations defined out of switching Fitting's consensus operation — in all of the definitions featured in Section 3 — by Lawry's new consistent consensus operation. Secondly, in [10, p. 1884] Lawry and others remark that their new consensus operations eliminates the difference between the conflicting orthopairs before merging them, in the sense of merging the information contained in them — by taking their information-theoretic disjunction, i.e. $\langle P_1, N_1\rangle \odot \langle P_2, N_2\rangle = (\langle P_1, N_1\rangle \ominus \langle P_2, N_2\rangle) \sqcup_I (\langle P_2, N_2\rangle \ominus \langle P_1, N_1\rangle)$. However, it would be intriguing to explore what different consensus operations arise from eliminating the differences between the conflicting orthopairs and then merging them, in the sense of taking the truth-theoretic disjunction.

Finally, one of the most salient aspects of a comparison between Fitting's and Lawry's formalism to provide semantics for three- and four-valued logics relies in the way the weak Kleene operations are modeled. In particular, the three-valued Paracomplete weak Kleene logic was interpreted, in Section 3.4, according to Fitting's cut-down operations. Differing from this, Lawry shows that these three-valued operations can be understood in an alternative way, if we focus on certain operations defined out of the two new orderings on orthopairs $\leq_P$ and $\leq_N$, defined below

$$\langle P_1, N_1\rangle \leq_P \langle P_2, N_2\rangle \iff P_1 \subseteq P_2 \text{ and } P_1 \cup N_1 \subseteq P_2 \cup N_2$$
$$\langle P_1, N_1\rangle \leq_N \langle P_2, N_2\rangle \iff N_1 \subseteq N_2 \text{ and } P_1 \cup N_1 \subseteq P_2 \cup N_2$$

As explained by the authors, "these relations are one-sided in the sense that the negative and positive literals do not play the same role. In one case, we keep the inclusion condition on the negative part of the orthopair and on the positive part in the other case" [10, p. 1881]. While the relation $\leq_P$ is intended to represent the idea of "at least as positive, and no less informative than", the relation $\leq_N$ is intended to represent the idea of "at least as negative, and no less informative than".

One of the most salient results is, then, that the meet operations $\sqcap_P$ and $\sqcap_N$ coincide — when understood as operations on $\{\mathbf{t}, \bot, \mathbf{f}\}$ — with Fitting's cut-down operations $\vartriangle$ and $\triangledown$, respectively (for more on how to translate these operations on orthopairs into operations on truth-values see [10, p. 1882]). This, already, means that Lawry's work allows for a different, alternative, interpretation of the weak Kleene operations in terms of one-sided orderings. But, furthermore, this means that it is also possible to ask if this result extends to the four-valued case — that is, understanding $\sqcap_P$ and $\sqcap_N$ as operations on $\{\mathbf{t}, \top, \bot, \mathbf{f}\}$. Luckily, simple calculations show they do, still, coincide with Fitting's cut-down opeartions, leading us to conclude that they can be legitimately regarded as at least as plausible formalizations of the weak Kleene operations as Fitting's.

To conclude this section, and given these positive results, the question imposes itself regarding whether or not it is possible to define new different orderings on orthopairs whose meet or joint operations coincide with our track-down operations $\blacktriangle$ and $\blacktriangledown$ — when understood as operations on $\{\mathbf{t}, \top, \bot, \mathbf{f}\}$. If the answer is positive, then once more Lawry's work will have shown its fruitfulness, for it will allow us to provide a different alternative interpretation of Paraconsistent Weak Kleene's truth-functions. We leave this hugely intriguing open problem here, hoping to investigate it in future work.

5. Sequent Calculi

In what follows we will endow the previously discussed four-valued generalizations of the weak Kleene logics with suitable Gentzen-style sequent calculi.

Before moving on, let us notice that *natural deduction* calculi have been recently offered for the weak Kleene logics and some of their four-valued generalizations. Indeed, these were introduced for $\mathsf{K}_3^{\mathrm{w}}$ and PWK in [44], for $\mathsf{S}_{\mathtt{fde}}$ in [45], and for some subsystems thereof in [46].[15] Although a similar presentation can be carried out for $\mathsf{dS}_{\mathtt{fde}}$, for matters of space we focus here on Gentzen-style sequent calculi for this logic, leaving the investigation of a natural deduction calculus for it for another occasion.

To arrive at our desired sequent calculi, we will draw inspiration from the techniques introduced by Coniglio and Corbalán in [13] to provide calculi of this kind for the systems PWK and $\mathsf{K}_3^{\mathrm{w}}$. The main feature of such proof systems is that they are obtained by taking an appropriate sequent calculus for Classical Logic, and applying different restrictions to the operational rules featured in it. More particularly, these restrictions pertain to some inclusion requirements between the set of propositional

[15] In particular, [46] presents natural deduction calculi for the logics $\mathsf{K}_{\mathsf{4b}}^{w}$ and $\mathsf{K}_{\mathsf{4n}}^{w}$ referred in [51], respectively, as $\mathsf{L}_{\mathbf{eb'}}$ and $\mathsf{L}_{\mathbf{b'e}}$.

variables of the active formulae of the corresponding rules, and the set of propositional variables appearing in some of the side formulae.

The main reason for requiring the rules to comply with these provisos is that weak Kleene logics happen to be closely connected with a family of systems whose valid inferences enjoy certain variable inclusion features. This family of systems, denominated *containment logics* e.g. in [49], gathers logics where an inference holds *only if* certain set-theoretic containment principle holds between the set of propositional variables appearing in the premises and the set of propositional variables appearing in the conclusion.[16]

Thus, Coniglio and Corbalán arrived at sequent calculi for PWK and $\mathsf{K}_3^{\mathrm{w}}$ by noticing that these weak Kleene logics can be described as being pretty close — in a sense to be made precise below — to containment subsystems of Classical Logic. By following a similar path, we will arrive at sequent calculi for $\mathsf{dS}_{\mathtt{fde}}$ and $\mathsf{S}_{\mathtt{fde}}$, by noticing that these logics can be rightfully described as proper containment subsystems of some other Kleene logics. To accomplish this, we will benefit from connecting weak Kleene logics, in general, with containment logics, later looking at these four-valued systems as an instance of a more general phenomenon.

5.1. Connecting weak Kleene logics and Containment logics

Let $\mathrm{var}(\Gamma)$ represent the set of propositional variables appearing in the set of formulae Γ, allowing us to refer e.g. to $\mathrm{var}(\{\varphi\})$ and $\mathrm{var}(\{\varphi,\psi\})$ as $\mathrm{var}(\varphi)$ and $\mathrm{var}(\varphi,\psi)$, respectively.

A very well-known family of containment logics — which we will refer to as Parry logics — are such that all its valid inferences enjoy a property we may call the $\vDash$-Parry Principle, i.e., the property that

$$\Gamma \vDash \varphi \quad \text{only if} \quad \mathrm{var}(\varphi) \subseteq \mathrm{var}(\Gamma)$$

Indeed, a species of Parry's Proscriptive Principle discussed in [43], applied to entailment.[17] Thus, logics satisfying the $\vDash$-Parry Principle clearly inavalidate Addition — i.e. $\varphi \vDash \varphi \vee \psi$ — for it may well happen that the propositional variables appearing in ψ are *not* included among those appearing in φ.

---

[16] Some important works revolving around these systems are e.g. [1, 12, 22, 27, 41, 42, 43, 50], among others.

[17] In fact, [22] calls it $\vDash$-Proscriptive Principle. There is nothing substantial in the choice of denomination here, though.

Systems of this sort have been studied, discussed and advanced by logicians such as Angell [1], Fine [27], Paoli [41], Epstein [20], Correia [14] and Ferguson [24] — alongside, of course, Parry [43] himself — with the specific aim of modelling *analytic* entailments. An analytic connection between premises and conclusion holds, these authors claim, when the content of the conclusion is included in the content of the premises. Indeed, e.g. [1] and [33] understand a logical behavior of this sort as extending Kant's notion of *analyticity*, according to which the predicate is included in the subject, to also apply to arguments. When the content of a given complex formula is obtained by collecting the content of the propositional variables appearing in it, it is straightforward to see how the requirement that an entailment is analytic directly implies the failure of Addition — for, in this sense, the content of $\varphi \vee \psi$ is usually not taken to be included in the content of φ.

However, notwithstanding the fact that $\vDash$-Parry logics invalidate Addition, it is *not* true that all systems where Addition is invalid are $\vDash$-Parry logics. In fact, Paracomplete Weak Kleene logic attests to this. For Explosion — i.e., the inference $\varphi, \neg\varphi \vDash \psi$ — is valid in it, but does not enjoy the $\vDash$-Parry property. This can be generalized, as the following characterization of logical consequence in $\mathsf{K}_3^{\mathrm{w}}$ shows.

Observation 5.1 ([54]). *For all sets of formulae $\Gamma \cup \{\varphi\}$,*

$$\Gamma \vDash_{\mathsf{K}_3^{\mathrm{w}}} \varphi \iff \begin{cases} \Gamma \vDash_{\mathsf{CL}} \varphi \text{ and } \mathrm{var}(\varphi) \subseteq \mathrm{var}(\Gamma), \text{ or} \\ \Gamma \vDash_{\mathsf{CL}} \emptyset \end{cases}$$

Thus, we can say that $\mathsf{K}_3^{\mathrm{w}}$ is close to a containment subsystem of Classical Logic. In fact, letting the $\vDash$-Parry fragment of a logic L, denoted $\mathsf{L}_{\mathsf{PP}^{\vDash}}$, be defined such that

$$\Gamma \vDash_{\mathsf{L}_{\mathsf{PP}^{\vDash}}} \varphi \iff \Gamma \vDash_{\mathsf{L}} \varphi \text{ and } \mathrm{var}(\varphi) \subseteq \mathrm{var}(\Gamma)$$

it can be observed that the only thing standing between the $\vDash$-Parry fragment of Classical Logic (i.e. $\mathsf{CL}_{\mathsf{PP}^{\vDash}}$) and Paracomplete Weak Kleene are the $\mathsf{K}_3^{\mathrm{w}}$-valid inferences involving inconsistent premises.[18]

Interestingly, in [22] it is pointed out how to obtain $\vDash$-Parry logics, taking $\mathsf{K}_3^{\mathrm{w}}$ as a starting point. Let us say that a logic $\mathsf{L} = \langle \mathrm{FOR}(\mathcal{L}), \vDash_{\mathsf{L}} \rangle$ has *anti-theorems* if there is some $\psi \in \mathrm{FOR}(\mathcal{L})$ such that $\psi \vDash_{\mathsf{L}} \emptyset$. Then,

[18] For an extensive discussion of $\mathsf{CL}_{\mathsf{PP}^{\vDash}}$, its relation to Parry logics and weak Kleene logics, see [22].

a connection between weak Kleene logics and $\vDash$-Parry logics can be established by the following general result.

PROPOSITION 5.2 ([22]). *Let $\mathcal{L}$ be a language and let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an $\mathcal{L}$-matrix such that $\mathcal{V} \setminus \mathcal{D}$ contains an infectious value. If $\mathsf{L} = \langle \mathrm{FOR}(\mathcal{L}), \vDash_{\mathcal{M}} \rangle$ has no anti-theorems, then L is a $\vDash$-Parry logic.*

Notice first that this explicitly appeals to subsystems of Paracomplete Weak Kleene, as every logic induced by a matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ such that $\mathcal{V} \setminus \mathcal{D}$ contains an infectious value is a *subsystem* of $\mathsf{K}_3^{\mathrm{w}}$. Note, moreover, that in light of this observation it is *sufficient* to consider certain paraconsistent subsystems of Paracomplete Weak Kleene to arrive at a $\vDash$-Parry logic. It is in this sense that, in [22] and [21] the four-valued logic $\mathsf{S}_{\mathtt{fde}}$ — which is both a paraconsistent logic and a subsystem of $\mathsf{K}_3^{\mathrm{w}}$ — is described as the $\vDash$-Parry fragment of Priest's Logic of Paradox.

OBSERVATION 5.3 ([22]). *For all sets of formulae $\Gamma \cup \{\varphi\}$,*

$$\Gamma \vDash_{\mathsf{S}_{\mathtt{fde}}} \varphi \Longleftrightarrow \Gamma \vDash_{\mathsf{LP}} \varphi \textit{ and } \mathrm{var}(\varphi) \subseteq \mathrm{var}(\Gamma)$$

Now, moving on to the relation that Paraconsistent Weak Kleene and subsystems thereof have with containment logics, we will highlight that just as we connected containment logics to systems where Addition fails, we can do the same with systems where Simplification fails. Let us consider another family of containment logics — to which we will refer as *Dual* Parry logics — such that all its valid inferences enjoy a property we may call the $\vDash$-Dual Parry Principle, i.e., the property that

$$\Gamma \vDash_{\mathsf{L}} \varphi \quad \text{only if} \quad \exists \Gamma' \subseteq \Gamma, \Gamma' \neq \emptyset, \mathrm{var}(\Gamma') \subseteq \mathrm{var}(\varphi)$$

which is a clear dualization of the $\vDash$-Parry Principle, arrived at by reversing the direction of the famous containment principle discussed before.[19] Thus, logics satisfying the $\vDash$-Dual Parry Principle saliently invalidate Simplification — i.e. $\varphi \wedge \psi \vDash \psi$ — for it may well happen that the propositional variables appearing in φ are *not* included among those appearing in ψ.

Systems of this sort have been considered by e.g. Epstein [20] and Paoli [41] with the specific aim of modeling what the latter calls *regressive*

[19] In [16], [20] and [41], similar properties have been called, respectively, Converse Parry Property, Dual Dependence and Regressive Analiticity. Yet again, there is nothing substantial going on in the choice of denomination here — but for a criticism of the terminology employed in [16], see [35, p. 176, fn. 20].

analytic entailments. A regressive analytic connection between premises and conclusions is holds when we

> proceed from simple ingredients (simple ideas as primitive concepts, simple propositions as axioms), down to more complex ones; by analyzing a derived concept or a theorem, we can overturn the procedure and regress to the basic components. [41, p. 2]

Hence, this seemingly gives regressive analytic entailments a sort of *explanatory* flavor, the symptom of which appears to be the complexity increase (or stability) from premises to conclusions [41, p. 2]. Yet again, if we apply these ideas to the *content* of premises and conclusions, and then obtain the content of complex expressions by collecting those of the propositional variables appearing in it, it is straightforward to see how the requirement that an inference is regressive analytic directly implies the failure of Simplification — for, in this sense, the content of $\varphi \wedge \psi$ is usually not taken to be included in the content of ψ.[20]

Again, notwithstanding the fact that $\vDash$-Dual Parry systems invalidate Simplification, it is *not* true that all logics that invalidate Simplification are $\vDash$-Dual Parry systems. Indeed, Paraconsistent Weak Kleene logic attests to this. For Implosion — i.e., the inference $\psi \vDash \varphi \vee \neg\varphi$ — is valid in it, although this inference does not enjoy the $\vDash$-Dual Parry property. This can be generalized, as the following characterization of logical consequence in PWK shows.

OBSERVATION 5.4 ([12]). *For all sets of formulae $\Gamma \cup \{\varphi\}$,*

$$\Gamma \vDash_{\mathsf{PWK}} \varphi \Longleftrightarrow \begin{cases} \Gamma \vDash_{\mathsf{CL}} \varphi \textit{ and } \exists \Gamma' \subseteq \Gamma, \Gamma' \neq \emptyset, \mathrm{var}(\Gamma') \subseteq \mathrm{var}(\varphi), \textit{or} \\ \emptyset \vDash_{\mathsf{CL}} \varphi \end{cases}$$

Thus, we can say that PWK is close enough to a containment subsystem of Classical Logic. In fact, letting the $\vDash$-Dual Parry fragment of a logic L, denoted $\mathsf{L}_{\mathsf{DPP}^{\vDash}}$, be defined such that

$$\Gamma \vDash_{\mathsf{L}_{\mathsf{DPP}^{\vDash}}} \varphi \Longleftrightarrow \Gamma \vDash_{\mathsf{L}} \varphi \text{ and } \exists \Gamma', \emptyset \neq \Gamma' \subseteq \Gamma, \mathrm{var}(\Gamma') \subseteq \mathrm{var}(\varphi)$$

[20] Moreover — as is also argued in [41] — consequence relations enjoying this feature have been motivated by those who favor the idea that the entailments have some causal or grounding flavor to it, as e.g. in [8], which would explain that simple constituents entail some of the compounds they constitute, but not the other way around.

it can be observed that the only thing standing between the $\vDash$-Dual Parry fragment of Classical Logic (i.e. $\mathsf{CL}_{\mathsf{DPP}^{\vDash}}$) and Paraconsistent Weak Kleene are the PWK-valid inferences involving tautlogical conclusions.

Interestingly, by a dualization of [22, Observation 1] advanced in [51, p. 297], it can be pointed out how to obtain $\vDash$-Dual Parry logics, taking PWK as a starting point. Let us say that a logic $\mathsf{L} = \langle \mathrm{FOR}(\mathcal{L}), \vDash_{\mathsf{L}} \rangle$ has *theorems* if there is some $\psi \in \mathrm{FOR}(\mathcal{L})$ such that $\emptyset \vDash_{\mathsf{L}} \psi$. Then, a connection between weak Kleene logics and $\vDash$-Dual Parry logics can be established by the following general observation.

PROPOSITION 5.5. *Let $\mathcal{L}$ be a language and let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an $\mathcal{L}$-matrix such that $\mathcal{D}$ contains an infectious value. If $\mathsf{L} = \langle \mathrm{FOR}(\mathcal{L}), \vDash_{\mathcal{M}} \rangle$ has no theorems, then L is a $\vDash$-Dual Parry logic.*

PROOF. Assume all of the antecedent conditions hold and suppose, for *reductio*, that L is not a $\vDash$-Dual Parry logic. This implies there is an inference $\Gamma \vDash_{\mathcal{M}} \varphi$ such that it is *not true* that $\exists \Gamma' \subseteq \Gamma, \Gamma' \neq \emptyset, \mathrm{var}(\Gamma') \subseteq \mathrm{var}(\varphi)$. This implies that for all $\gamma \in \Gamma$, $\mathrm{var}(\gamma) \nsubseteq \mathrm{var}(\varphi)$.

Let $\Sigma \setminus \Delta$ be the result of subtracting from Σ all the elements that are in Δ. Since L has no theorems, moreover, we can assume that there is a valuation v such that $v(\varphi) \notin \mathcal{D}$. Let us refer to the infectious value contained in $\mathcal{D}$ as $\mathbf{x}$. We can construct a valuation v^* such that

$$v^*(p) = \begin{cases} \mathbf{x} & \text{if } p \in \mathrm{var}(\Gamma) \setminus \mathrm{var}(\varphi) \\ v(p) & \text{otherwise} \end{cases}$$

Since, by the above, we are justified to assume that for all $\gamma \in \Gamma$, $\mathrm{var}(\gamma) \setminus \mathrm{var}(\varphi) \neq \emptyset$, we know that for all $\gamma \in \Gamma$, there is a $q \in \mathrm{var}(\gamma) \setminus \mathrm{var}(\varphi)$ such that $v^*(q) = \mathbf{x}$. Hence, for all $\gamma \in \Gamma$, $v^*(\gamma) = \mathbf{x} \in \mathcal{D}$, further implying that $v^*[\Gamma] \subseteq \mathcal{D}$, while at the same time $v^*(\varphi) \notin \mathcal{D}$. Then, v^* witnesses that $\Gamma \nvDash_{\mathcal{M}} \varphi$, which contradicts our initial assumption. Therefore, L is a $\vDash$-Dual Parry logic. $\square$

Notice first that this explicitly appeals to subsystems of Paraconsistent Weak Kleene, as every matrix logic induced by a matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ such that $\mathcal{D}$ contains an infectious value is a subsystem of PWK. Note, moreover, that in light of this observation it is *sufficient* to consider certain paracomplete subsystems of Paraconsistent Weak Kleene to arrive at a $\vDash$-Dual Parry logic. Thus, as a consequence of these remarks and those made in [51], the four-valued logic $\mathsf{dS}_{\mathtt{fde}}$ — which is both a paracomplete logic and a subsystem of PWK — can be regarded as the $\vDash$-Dual Parry fragment of Strong Kleene logic.

OBSERVATION 5.6. *For all sets of formulae $\Gamma \cup \{\varphi\}$,*

$$\Gamma \vDash_{\mathsf{dS_{fde}}} \varphi \Longleftrightarrow \Gamma \vDash_{\mathsf{K_3}} \varphi \text{ and } \exists \Gamma', \emptyset \neq \Gamma' \subseteq \Gamma, \mathrm{var}(\Gamma') \subseteq \mathrm{var}(\varphi)$$

PROOF. That $\Gamma \vDash_{\mathsf{dS_{fde}}} \varphi$ implies $\Gamma \vDash_{\mathsf{K_3}} \varphi$ is established by the fact that $\mathsf{dS_{fde}}$ is a subsystem of $\mathsf{K_3}$, established in [51], it being easy to check this by looking at their matrices. That $\Gamma \vDash_{\mathsf{dS_{fde}}} \varphi$ implies $\exists \Gamma', \emptyset \neq \Gamma' \subseteq \Gamma, \mathrm{var}(\Gamma') \subseteq \mathrm{var}(\varphi)$ follows from Proposition 5.5 above. □

Finally, having looked at the systems $\mathsf{dS_{fde}}$ and $\mathsf{S_{fde}}$ as *containment subsystems* of Strong Kleene logic and Priest's Logic of Paradox, respectively, we will now move on to present their corresponding sequent calculi. As advertised, these will be obtained by imposing certain appropriate *containment provisos* to the operational rules of appropriate Gentzen-style sequent calculi for $\mathsf{K_3}$ and LP.

5.2. Definitions

DEFINITION 5.7. By a sequent $\Gamma \succ \Delta$ we mean an ordered pair $\langle \Gamma, \Delta \rangle$ of (non-simultaneously empty) finite sets of formulae of $\mathrm{FOR}(\mathcal{L})$.[21]

DEFINITION 5.8. Let L be a matrix logic $\mathsf{L} = \langle \mathrm{FOR}(\mathcal{L}), \vDash_{\mathcal{M}} \rangle$ such that $\mathcal{M} = \langle \mathcal{D}, \mathcal{V}, \mathcal{O} \rangle$. An $\mathcal{M}$ valuation v satisfies a sequent $\Gamma \succ \Delta$ (symbolized $v \vDash_{\mathcal{M}} \Gamma \succ \Delta$) if and only if $v(\gamma) \in \mathcal{D}$ for all $\gamma \in \Gamma$, then $v(\delta) \in \mathcal{D}$ for some $\delta \in \Delta$. A sequent $\Gamma \succ \Delta$ is valid (symbolized $\vDash_{\mathcal{M}} \Gamma \succ \Delta$) if for every $\mathcal{M}$ valuation v, $v \vDash \Gamma \succ \Delta$

Thus, we may interchangeably refer to an inference or sequent $\Gamma \succ \Delta$ which is valid in the logic $\mathsf{L} = \langle \mathrm{FOR}(\mathcal{L}), \vDash_{\mathcal{M}} \rangle$ as $\Gamma \vDash_{\mathcal{M}} A$ or $\vDash_{\mathcal{M}} \Gamma \succ \Delta$. Recall, also, that in such cases we may alternatively denote $\vDash_{\mathcal{M}}$ as $\vDash_{\mathsf{L}}$.[22]

[21] Note that, since we are working with sequents built from *sets*, the Contraction and Exchange rules are going to be built into the system, and no explicit mention of them is going to be necessary.

[22] Notice, that in dealing with sequent calculi we are moving from consequence relations relating sets of premises with a single conclusion, to consequence relations relating sets of premises with multiple conclusions. Our discussion was carried out in the former setting, but can be understood in terms of the latter, and so there is nothing worrisome in this.

DEFINITION 5.9. A sequent rule $\mathfrak{R}$ preserves validity in $\mathcal{M}$ if for every instance $\dfrac{\mathfrak{r}}{\Gamma \succ \Delta}$ of $\mathfrak{R}$ and for every $\mathcal{M}$ valuation v, if $v \vDash_{\mathcal{M}} \Sigma \succ \Pi$ for every $\Sigma \succ \Pi \in \mathfrak{r}$, then $v \vDash_{\mathcal{M}} \Gamma \succ \Delta$

DEFINITION 5.10 ([13]). The sequent calculus $\mathcal{G}\mathsf{CL}$ contains the following rules.[23]

$$\frac{}{\varphi \succ \varphi}\ [Id]$$

$$\frac{\Gamma \succ \Delta}{\Gamma, \varphi \succ \Delta}\ [WL] \qquad \frac{\Gamma \succ \Delta}{\Gamma \succ \varphi, \Delta}\ [WR] \qquad \frac{\Gamma, \varphi \succ \Delta \quad \Gamma \succ \varphi, \Delta}{\Gamma \succ \Delta}\ [Cut]$$

$$\frac{\Gamma \succ \varphi, \Delta}{\Gamma, \neg\varphi \succ \Delta}\ [\neg L] \qquad \frac{\Gamma, \varphi \succ \Delta}{\Gamma \succ \neg\varphi, \Delta}\ [\neg R]$$

$$\frac{\Gamma, \varphi, \psi \succ \Delta}{\Gamma, \varphi \wedge \psi \succ \Delta}\ [\wedge L] \qquad \frac{\Gamma \succ \varphi, \Delta \quad \Gamma \succ \psi, \Delta}{\Gamma \succ \varphi \wedge \psi, \Delta}\ [\wedge R]$$

$$\frac{\Gamma, \varphi \succ \Delta \quad \Gamma, \psi \succ \Delta}{\Gamma, \varphi \vee \psi \succ \Delta}\ [\vee L] \qquad \frac{\Gamma \succ \varphi, \psi, \Delta}{\Gamma \succ \varphi \vee \psi, \Delta}\ [\vee R]$$

PROPOSITION 5.11 ([13]). *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{CL}$ if and only if $\vDash_{\mathsf{CL}} \Gamma \succ \Delta$.*

DEFINITION 5.12. Let us refer to the rules below as the De Morgan rules.

$$\frac{\Gamma, \varphi \succ \Delta}{\Gamma, \neg\neg\varphi \succ \Delta}\ [\neg\neg L] \qquad \frac{\Gamma \succ \varphi, \Delta}{\Gamma \succ \neg\neg\varphi, \Delta}\ [\neg\neg R]$$

$$\frac{\Gamma, \neg\varphi \succ \Delta \quad \Gamma, \neg\psi \succ \Delta}{\Gamma, \neg(\varphi \wedge \psi) \succ \Delta}\ [\neg\wedge L] \qquad \frac{\Gamma \succ \neg\varphi, \neg\psi, \Delta}{\Gamma \succ \neg(\varphi \wedge \psi), \Delta}\ [\neg\wedge R]$$

$$\frac{\Gamma, \neg\varphi, \neg\psi \succ \Delta}{\Gamma, \neg(\varphi \vee \psi) \succ \Delta}\ [\neg\vee L] \qquad \frac{\Gamma \succ \neg\varphi, \Delta \quad \Gamma \succ \neg\psi, \Delta}{\Gamma \succ \neg(\varphi \vee \psi), \Delta}\ [\neg\vee R]$$

OBSERVATION 5.13. *The rules $[\neg\neg L], [\neg\wedge L], [\neg\vee L], [\neg\neg R], [\neg\wedge R], [\neg\vee R]$ are admissible in $\mathcal{G}\mathsf{CL}$.*

DEFINITION 5.14 ([2]). Let the calculus $\mathcal{G}\mathsf{K}_3$ be the result of subtracting the rule $[\neg R]$ and adding the De Morgan rules to $\mathcal{G}\mathsf{CL}$. Let the calculus

[23] Coniglio and Corbalán call this system C, but for matters of uniformity we will adopt the name $\mathcal{G}\mathsf{CL}$, since it gives a more suggestive idea that we are working with a Gentzen-style sequent calculus for CL.

$\mathcal{G}\mathsf{LP}$ bet he result of subtracting the rule $[\neg L]$ and adding the De Morgan rules to $\mathcal{G}\mathsf{CL}$.[24]

THEOREM 5.15 ([2]). *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{K}_3$ if and only if $\vDash_{\mathsf{K}_3} \Gamma \succ \Delta$.*

THEOREM 5.16 ([2]). *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{LP}$ if and only if $\vDash_{\mathsf{LP}} \Gamma \succ \Delta$.*

THEOREM 5.17 ([2]). *Let $\Gamma \cup \Delta$ be finite non-empty set of formulae of $\mathcal{L}$. The sequent $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{K}_3$, then there is a Cut-free derivation of it. Similarly for $\mathcal{G}\mathsf{LP}$.*

DEFINITION 5.18 ([13]). Let the calculus $\mathcal{G}\mathsf{PWK}$ result from $\mathcal{G}\mathsf{CL}$ minus the rules $[\wedge R]$ and $[\wedge L]$, and the additional restriction that the rule $[\neg L]$

---

[24] Let us clarify a number of things. First, the sequent calculus for LP is presented by [5] without a proper name, hence we call it $\mathcal{G}\mathsf{LP}$ here. The sequent calculus for K_3 is not presented in [5], but it is pointed out that it should be constructed this way, which is done in e.g. [31] — although in [31] the axioms only feature literals, i.e., propositional variables or their negations, which is again inessential given the rest of the rules. Secondly, in [5] and [31], these calculi are presented with both the left and right Weakening rules being *absorbed* into the axioms. There is nothing substantial to this, given in the context of a calculus satisfying $[Cut]$, both sets of rules are interderivable. Thirdly, both in the context of [5] and [31], the calculi for LP and K_3 are taken as the result of subtracting from $\mathcal{G}\mathsf{CL}$ both negation rules $[\neg L]$ and $[\neg R]$ and adding, alongside with the De Morgan rules, the axioms (which can be traced back to [2]) we call $[Exhaustion]$ and $[Exclusion]$, respectively

$$\frac{}{\Gamma \succ \varphi, \neg\varphi, \Delta}\,[Exhaustion] \qquad\qquad \frac{}{\Gamma, \varphi, \neg\varphi \succ \Delta}\,[Exclusion]$$

However, this difference in presentation is inessential. For, in the context of a calculus satisfying $[Id]$, $[WL]$, $[WR]$ and $[Cut]$, on the one hand $[Exhaustion]$ and $[\neg R]$ are interderivable and, on the other, $[Exclusion]$ and $[\neg L]$ are interderivable. This can be seen by the following derivations

$$\dfrac{\Gamma, \varphi \succ \Delta \qquad \dfrac{}{\Gamma \succ \varphi, \neg\varphi, \Delta}\,[Exhaustion]}{\Gamma \succ \neg\varphi, \Delta}\,[Cut} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \succ \varphi}\,[Id]}{\Gamma, \varphi \succ \varphi}\,[WL]}{\Gamma, \varphi \succ \varphi, \Delta}\,[WR]}{\Gamma \succ \varphi, \neg\varphi, \Delta}\,[\neg R]$$

$$\dfrac{\Gamma \succ \varphi, \Delta \qquad \dfrac{}{\Gamma, \varphi, \neg\varphi \succ \Delta}\,[Exclusion]}{\Gamma, \neg\varphi \succ \Delta}\,[Cut] \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \succ \varphi}\,[Id]}{\Gamma, \varphi \succ \varphi}\,[WL]}{\Gamma, \varphi \succ \varphi, \Delta}\,[WR]}{\Gamma, \varphi, \neg\varphi \succ \Delta}\,[\neg L]$$

must comply with the proviso that $\mathrm{var}(\varphi) \subseteq \mathrm{var}(\Delta)$ — in which case, we will call this rule $[\neg^H L]$.[25]

Definition 5.19 ([13]). Let the calculus $\mathcal{G}\mathsf{K}_3^{\mathrm{w}}$ result from $\mathcal{G}\mathsf{CL}$ minus the rules $[\vee R]$ and $[\vee L]$, and the additional restriction that the rule $[\neg R]$ must comply with the proviso that $\mathrm{var}(\varphi) \subseteq \mathrm{var}(\Gamma)$ — in which case, we will call this rule $[\neg^B R]$.[26]

Theorem 5.20 ([13]). *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{PWK}$ if and only if $\vDash_{\mathsf{PWK}} \Gamma \succ \Delta$.*

Theorem 5.21 ([13]). *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{K}_3^{\mathrm{w}}$ if and only if $\vDash_{\mathsf{K}_3^{\mathrm{w}}} \Gamma \succ \Delta$.*

Theorem 5.22 ([13]). *Let $\Gamma \cup \Delta$ be finite non-empty set of formulae of $\mathcal{L}$. If the sequent $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{PWK}$, then there is a Cut-free derivation of it. Similarly for $\mathcal{G}\mathsf{K}_3^{\mathrm{w}}$.*

Let us now turn to the calculi $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$ and $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$ for the four-valued generalizations of PWK and $\mathsf{K}_3^{\mathrm{w}}$, i.e. $\mathsf{dS}_{\mathtt{fde}}$ and $\mathsf{S}_{\mathtt{fde}}$. Their presentation is heavily inspired in the above discussed calculi presented in [13] for PWK and $\mathsf{K}_3^{\mathrm{w}}$ — where they are properly discussed as the $\{\neg, \wedge, \vee\}$-fragment of Halldén's and Bochvar's logics of nonsense, respectively.

Definition 5.23. Let the calculus $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$ result from $\mathcal{G}\mathsf{K}_3$, adding the restrictions that the rule $[\neg L]$ must comply with the proviso that $\mathrm{var}(\varphi) \subseteq \mathrm{var}(\Delta)$, and the rules $[\wedge L]$ and $[\neg\vee L]$ must comply with the proviso that $\mathrm{var}(\varphi, \psi) \subseteq \mathrm{var}(\Delta)$ — in which case, we will call these rules $[\neg^H L]$, $[\wedge^H L]$ and $[\neg\vee^H L]$.

Definition 5.24. Let the calculus $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$ result from $\mathcal{G}\mathsf{LP}$, adding the restrictions that the rule $[\neg R]$ must comply with the proviso that $\mathrm{var}(\varphi) \subseteq \mathrm{var}(\Gamma)$, and the rules $[\vee R]$ and $[\neg\wedge R]$ must comply with the proviso that $\mathrm{var}(\varphi, \psi) \subseteq \mathrm{var}(\Gamma)$ — in which case, we will call these rules $[\neg^B R]$, $[\vee^B R]$ and $[\neg\wedge^B R]$.

[25] Coniglio and Corbalán call this system H_3, but for matters of uniformity we will adopt the name $\mathcal{G}\mathsf{PWK}$, since it gives a more suggestive idea that we are working with a Gentzen-style sequent calculus for PWK.

[26] Coniglio and Corbalán call this system B_3 but, yet again, for matters of uniformity we will adopt the name $\mathcal{G}\mathsf{K}_3^{\mathrm{w}}$, since it gives a more suggestive idea that we are working with a Gentzen-style sequent calculus for $\mathsf{K}_3^{\mathrm{w}}$.

5.3. Soundness and Completeness for $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$

In what follows we prove the soundness and completeness results for the sequent calculus $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$. For soundness, the proof is standard, by the usual means.

LEMMA 5.25. *Every sequent rule of the calculus $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$ preserves $\mathsf{dS}_{\mathtt{fde}}$-validity.*

PROOF. Obviously the axiom and the structural rules preserve validity. We prove the case for the restricted operational rules and leave the rest as an exercise for the reader.

Ad $[\neg^H L]$ Let v be a $\mathsf{dS}_{\mathtt{fde}}$ valuation such that $v \vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma \succ \varphi, \Delta$ and assume $\mathrm{var}(\varphi) \subseteq \mathrm{var}(\Delta)$. Suppose $v(\gamma) \in \{\mathbf{t}, \top\}$ for all $\gamma \in \Gamma \cup \{\neg\varphi\}$. Then, by hypothesis, $v(\delta) \in \{\mathbf{t}, \top\}$ for some $\delta \in \Delta$, or $v(\varphi) \in \{\mathbf{t}, \top\}$. Since $v(\neg\varphi) \in \{\mathbf{t}, \top\}$, then $v(\varphi) \in \{\mathbf{f}, \top\}$. If $v(\varphi) = \mathbf{f}$, then $v(\delta) \in \{\mathbf{t}, \top\}$ for some $\delta \in \Delta$. If $v(\varphi) = \top$, then $v(p) = \top$ for some $p \in \mathrm{var}(\varphi)$. Since $\mathrm{var}(\varphi) \subseteq \mathrm{var}(\Delta)$, there is a $\delta \in \Delta$ such that $q \in \mathrm{var}(\delta)$ and $v(q) = \top$, hence $v(\delta) = \top$ for some $\delta \in \Delta$. In both cases it follows that $v(\delta) \in \{\mathbf{t}, \top\}$ for some $\delta \in \Delta$, establishing that $v \vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma, \neg\varphi \succ \Delta$.

Ad $[\wedge^H L]$ Let v be a $\mathsf{dS}_{\mathtt{fde}}$ valuation such that $v \vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma, \varphi, \psi \succ \Delta$ and assume $\mathrm{var}(\varphi, \psi) \subseteq \mathrm{var}(\Delta)$. Suppose $v(\gamma) \in \{\mathbf{t}, \top\}$ for all $\gamma \in \Gamma \cup \{\varphi \wedge \psi\}$. If $v(\varphi \wedge \psi) = \mathbf{t}$, then given the $\mathsf{dS}_{\mathtt{fde}}$ truth-function for conjunction, we can establish that $v(\varphi) = \mathbf{t}$ and $v(\psi) = \mathbf{t}$, hence by hypothesis $v(\delta) \in \{\mathbf{t}, \top\}$ for some $\delta \in \Delta$. If $v(\varphi \wedge \psi) = \top$, then either there is a $p \in \mathrm{var}(\varphi)$ such that $v(p) = \top$, or there is a $q \in \mathrm{var}(\psi)$ such that $v(q) = \top$. Either way, since $\mathrm{var}(\varphi, \psi) \subseteq \mathrm{var}(\Delta)$ we know that there is a $\delta \in \Delta$ such that there is an $r \in \mathrm{var}(\delta)$ for which $v(r) = \top$, hence $v(\delta) \in \{\mathbf{t}, \top\}$ for some $\delta \in \Delta$. In both cases it follows that $v(\delta) \in \{\mathbf{t}, \top\}$ for some $\delta \in \Delta$, establishing $v \vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma, \varphi \wedge \psi \succ \Delta$.

Ad $[\neg\vee^H L]$ Let v be a $\mathsf{dS}_{\mathtt{fde}}$ valuation such that $v \vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma, \neg\varphi, \neg\psi \succ \Delta$ and assume $\mathrm{var}(\varphi, \psi) \subseteq \mathrm{var}(\Delta)$. Suppose $v(\gamma) \in \{\mathbf{t}, \top\}$ for all $\gamma \in \Gamma \cup \{\neg(\varphi \vee \psi)\}$. If $v(\neg(\varphi \vee \psi)) = \mathbf{t}$, then given the $\mathsf{dS}_{\mathtt{fde}}$ truth-functions for negation and disjunction, we can establish that $v(\neg\varphi) = \mathbf{t}$ and $v(\neg\psi) = \mathbf{t}$, and hence by hypothesis $v(\delta) \in \{\mathbf{t}, \top\}$ for some $\delta \in \Delta$. If $v(\neg(\varphi \vee \psi)) = \top$, then either there is a $p \in \mathrm{var}(\varphi)$ such that $v(p) = \top$, or there is a $q \in \mathrm{var}(\psi)$ such that $v(q) = \top$. Either way, since $\mathrm{var}(\varphi, \psi) \subseteq \mathrm{var}(\Delta)$ we know that there is a $\delta \in \Delta$ such that there is an $r \in \mathrm{var}(\delta)$ for which $v(r) = \top$, hence $v(\delta) \in \{\mathbf{t}, \top\}$ for some $\delta \in \Delta$.

In both cases it follows that $v(\delta) \in \{\mathbf{t}, \top\}$ for some $\delta \in \Delta$, establishing $v \vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma, \neg(\varphi \vee \psi) \succ \Delta$. $\square$

THEOREM 5.26 (Soundness of $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$). *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. If $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$, then $\vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma \succ \Delta$.*

PROOF. If $\Gamma \succ \Delta$ is an axiom, then it is valid in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$. By induction on the depth of a derivation of $\Gamma \succ \Delta$ in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$ it follows, by the above Lemma 5.25, that $\Gamma \succ \Delta$ is valid in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$. $\square$

PROPOSITION 5.27 (Non-triviality of $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$). *Let Γ be a finite non-empty set of formulae of $\mathcal{L}$. The sequent $\Gamma \succ \emptyset$ is not provable in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$.*

PROOF. Let v be a $\mathsf{dS}_{\mathtt{fde}}$-valuation such that $v(p) = \top$ for every $p \in \mathrm{var}(\Gamma)$. It follows that $v \nvDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma \succ \emptyset$ and thus $\nvDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma \succ \emptyset$. By contraposition of Soundness, we can conclude that the sequent $\Gamma \succ \emptyset$ is not provable in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$. $\square$

We now turn to completeness.

PROPOSITION 5.28. *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. If $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$, then it is provable in $\mathcal{G}\mathsf{K}_3$.*

PROOF. Straightforward, since $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$ is a restriction of $\mathcal{G}\mathsf{K}_3$. $\square$

LEMMA 5.29. *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. If $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{K}_3$ and $\mathrm{var}(\Gamma) \subseteq \mathrm{var}(\Delta)$, then $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$ without using the Cut rule.*

PROOF. Remember that proofs in sequent calculi are rooted binary trees such that the root is the sequent being proved and the leaves of the tree are instances of $[Id]$ in other words, sequents of the form $\varphi \succ \varphi$.

Now assume that Π is a Cut-free derivation of $\Gamma \succ \Delta$ in $\mathcal{G}\mathsf{K}_3$ such that $\mathrm{var}(\Gamma) \subseteq \mathrm{var}(\Delta)$. If Π is a Cut-free derivation in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$, then the result is established. If Π is not a Cut-free derivation in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$, then there must be in Π applications of the rules $[\neg L]$, $[\wedge L]$ and $[\neg\vee L]$ where the required provisos are not satisfied

$$\frac{\Gamma^* \succ \Delta^*, \varphi}{\Gamma^*, \neg\varphi \succ \Delta^*}\ [\neg L] \qquad \frac{\Gamma^*, \varphi, \psi \succ \Delta^*}{\Gamma^*, \varphi \wedge \psi \succ \Delta^*}\ [\wedge L] \quad \frac{\Gamma^*, \neg\varphi, \neg\psi \succ \Delta^*}{\Gamma^*, \neg(\varphi \vee \psi) \succ \Delta^*}\ [\neg\vee L]$$

Now, since Π is a Cut-free proof, we are guaranteed that the root sequent $\Gamma \succ \Delta$ contains *all* the propositional variables appearing in Π. Since, by hypothesis, we know that $\mathrm{var}(\varGamma) \subseteq \mathrm{var}(\varDelta)$, we can affirm that $\mathrm{var}(\Pi) = \mathrm{var}(\varDelta)$.

What is left is, then, to design a procedure to transform Π into a Cut-free proof of $\varGamma \succ \varDelta$ in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$. We do this in two steps. First, we enlarge every node of Π by adding $\varDelta$ to its right-hand side. By doing this, we obtain a rooted binary tree Π', whose leaves are sequents of the form $\varphi \succ \varphi, \varDelta$. Second, we extend each leaf with a branch starting in an instance of $[Id]$, that is, a sequent of the form $\varphi \succ \varphi$, followed by any number of necessary iterated applications of the right Weakening rule $[WR]$, so that the sequent $\varphi \succ \varphi, \varDelta$ is obtained.

From this procedure, we get a rooted binary tree Π'' which is undoubtedly a Cut-free derivation in $\mathcal{G}\mathsf{K}_3$ of the sequent $\varGamma \succ \varDelta$, such that the critical instances of the rules $[\neg L]$, $[\vee L]$ and $[\neg\wedge R]$ have in Π'' the form

$$\frac{\varGamma^* \succ \varphi, \varDelta^*, \varDelta}{\varGamma^*, \neg\varphi \succ \varDelta^*, \varDelta}\ [\neg L] \qquad \frac{\varGamma^*, \varphi, \psi \succ \varDelta^*, \varDelta}{\varGamma^*, \varphi \wedge \psi \succ \varDelta^*, \varDelta}\ [\wedge L]$$

$$\frac{\varGamma^*, \neg\varphi, \neg\psi \succ \varDelta^*, \varDelta}{\varGamma^*, \neg(\varphi \vee \psi) \succ \varDelta^*, \varDelta}\ [\neg\vee L]$$

and are, thus, admissible in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$. Finally, from this we infer that Π'' is a Cut-free derivation in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$ of the sequent $\varGamma \succ \varDelta$. □

Corollary 5.30. *Let $\varGamma \cup \varDelta$ be a finite non-empty set of formulae of $\mathcal{L}$. If $\models_{\mathsf{dS}_{\mathtt{fde}}} \varGamma \succ \varDelta$ but $\mathrm{var}(\varGamma) \subsetneq \mathrm{var}(\varDelta)$, then there is a $\varGamma' \subseteq \varGamma$ such that $\models_{\mathsf{dS}_{\mathtt{fde}}} \varGamma' \succ \varDelta$, where $\mathrm{var}(\varGamma') \subseteq \mathrm{var}(\varDelta)$.*

Proof. First, notice that if $\models_{\mathsf{dS}_{\mathtt{fde}}} \varGamma \succ \varDelta$, then $\mathrm{var}(\varGamma) \neq \emptyset \neq \mathrm{var}(\varDelta)$. Now assume $\models_{\mathsf{dS}_{\mathtt{fde}}} \varGamma \succ \varDelta$ but $\mathrm{var}(\varGamma) \nsubseteq \mathrm{var}(\varDelta)$. Hence, define $\varGamma' = \varGamma \setminus \{\gamma \in \varGamma \mid \mathrm{var}(\gamma) \nsubseteq \mathrm{var}(\varDelta)\}$, hence $\varGamma' \subset \varGamma$ and $\mathrm{var}(\varGamma') \subseteq \mathrm{var}(\varDelta)$. Suppose there is a $\mathsf{dS}_{\mathtt{fde}}$ valuation v such that $v(\gamma) \in \{\mathbf{t}, \top\}$ for all $\gamma \in \varGamma'$. If $v(\gamma) = \top$ for some $\gamma \in \varGamma'$, then $v(p) = \top$ for some $p \in \mathrm{var}(\gamma)$ and, therefore, $v(p) = \top$ for some $p \in \mathrm{var}(\varGamma')$. Since $\mathrm{var}(\varGamma') \subseteq \mathrm{var}(\varDelta)$, then $v(q) = \top$ for some $q \in \mathrm{var}(\varDelta)$, hence there is a $\delta \in \varDelta$ such that $v(\delta) = \top$. This establishes $\models_{\mathsf{dS}_{\mathtt{fde}}} \varGamma' \succ \varDelta$. If $v(\gamma) = \mathbf{t}$ for all $\gamma \in \varGamma'$, then suppose for *reductio* that $v(\delta) \in \{\bot, \mathbf{f}\}$ for all $\delta \in \varDelta$, which implies that $\not\models_{\mathsf{dS}_{\mathtt{fde}}} \varGamma' \succ \varDelta$. But then, $v(p) \in \{\mathbf{t}, \bot, \mathbf{f}\}$ for all $p \in \mathrm{var}(\varDelta)$. And since $\mathrm{var}(\varGamma') \subseteq \mathrm{var}(\varDelta)$, this will also require that $v(q) \in \{\mathbf{t}, \bot, \mathbf{f}\}$ for all $q \in \mathrm{var}(\varGamma')$. Consider, now, a $\mathsf{dS}_{\mathtt{fde}}$ valuation v^* such that

$$v^*(p) = \begin{cases} \top & \text{if } p \in \mathrm{var}(\varGamma) \setminus \mathrm{var}(\varDelta) \\ v(p) & \text{if } p \in \mathrm{var}(\varDelta) \end{cases}$$

Then, by the above this will imply $v^*(\gamma) \in \{\mathbf{t}, \top\}$ for all $\gamma \in \Gamma$, but $v^*(\delta) \in \{\bot, \mathbf{f}\}$ for all $\delta \in \Delta$, hence v^* witnesses $\nvDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma \succ \Delta$, contrdaciting our initial assumption. Therefore, if $\vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma \succ \Delta$ but $\mathrm{var}(\Gamma) \subsetneq \mathrm{var}(\Delta)$, then there is a $\Gamma' \subseteq \Gamma$ such that $\vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma' \succ \Delta$, where $\mathrm{var}(\Gamma') \subseteq \mathrm{var}(\Delta)$. □

THEOREM 5.31 (Completeness of $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$). *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. If $\vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma \succ \Delta$, then $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$ without using the Cut rule.*

PROOF. Assume $\vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma \succ \Delta$. By Observation 5.6, we know that $\vDash_{\mathsf{K}_3} \Gamma \succ \Delta$, and also by Theorem 5.15 we are granted that $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{K}_3$. To finally establish that $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$ without using the Cut rule, we consider two cases. First, if $\mathrm{var}(\Gamma) \subseteq \mathrm{var}(\Delta)$, we know by Lemma 5.29 that this is the case. Second, if $\mathrm{var}(\Gamma) \subsetneq \mathrm{var}(\Delta)$, we know by Corollary 5.30 that there is a $\Gamma' \subseteq \Gamma$ such that $\vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma \succ \Delta'$, where $\mathrm{var}(\Gamma') \subseteq \mathrm{var}(\Delta)$. Now, by Lemma 5.29 we know that $\Gamma' \succ \Delta$ is provable in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$ without using the Cut rule, by means of a proof Π_1 (i.e., a rooted binary tree) whose root is $\Gamma' \succ \Delta$ and whose leaves are instances of $[Id]$, of the form $\varphi \succ \varphi$. Finally, we transform Π_1 into a proof Π_1', by extending down the node $\Gamma' \succ \Delta$ by means of the required iterated applications of the left Weakening rule $[WL]$, until we arrive at the sequent $\Gamma \succ \Delta$. But this rooted binary tree Π_1' is now a proof in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$ of the sequent $\Gamma \succ \Delta$, without using the Cut rule. □

COROLLARY 5.32 (Cut-elimination for $\mathsf{dS}_{\mathtt{fde}}$). *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae in $\mathcal{L}$. If the sequent $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$, then there is a Cut-free derivation of $\Gamma \succ \Delta$ in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$.*

PROOF. Assume that $\Gamma \succ \Delta'$ is provable in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$. By Theorem 5.34, that is, because the system is sound, we know that $\vDash_{\mathcal{G}\mathsf{dS}_{\mathtt{fde}}} \Gamma \succ \Delta$. But then by Theorem 5.39, that is, because the system is complete, we know that $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$ without using the Cut rule. □

5.4. Soundness and Completeness for $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$

LEMMA 5.33. *Every sequent rule of the calculus $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$ preserves $\mathsf{S}_{\mathtt{fde}}$-validity.*

PROOF. Obviously the axiom and the structural rules preserve validity. We prove the case for the restricted operational rules and leave the rest as an exercise for the reader.

Ad $[\neg^B R]$ Let v be a $\mathsf{S}_{\mathtt{fde}}$ valuation such that $v \vDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma, \varphi \succ \Delta$ and assume that $\mathrm{var}(\varphi) \subseteq \mathrm{var}(\Gamma)$. Suppose $v(\gamma) \in \{\mathbf{t}, \top\}$ for all $\gamma \in \Gamma$. Thus, $v(p) \in \{\mathbf{t}, \top, \mathbf{f}\}$ for all $p \in \mathrm{var}(\Gamma)$. Since $\mathrm{var}(\varphi) \subseteq \mathrm{var}(\Gamma)$, we also know that $v(q) \in \{\mathbf{t}, \top, \mathbf{f}\}$ for all $q \in \mathrm{var}(\varphi)$. Hence, $v(\varphi) \in \{\mathbf{t}, \top, \mathbf{f}\}$. If $v(\varphi) \in \{\top, \mathbf{f}\}$, then $v(\neg\varphi) \in \{\mathbf{t}, \top\}$, hence $v(\delta) \in \{\mathbf{t}, \top\}$ for some $\delta \in \Delta \cup \{\neg\varphi\}$. If $v(\varphi) = \mathbf{t}$, then by hypothesis there is a $\delta \in \Delta$ such that $v(\delta) \in \{\mathbf{t}, \top\}$, hence $v(\delta) \in \{\mathbf{t}, \top\}$ for some $\delta \in \Delta \cup \{\neg\varphi\}$. Either way, this establishes $v \vDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \neg\varphi, \Delta$.

Ad $[\vee^B R]$ Let v be a $\mathsf{S}_{\mathtt{fde}}$ valuation such that $v \vDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \varphi, \psi, \Delta$ and assume that $\mathrm{var}(\varphi, \psi) \subseteq \mathrm{var}(\Gamma)$. Suppose $v(\gamma) \in \{\mathbf{t}, \top\}$ for all $\gamma \in \Gamma$. Hence, $v(p) \in \{\mathbf{t}, \top, \mathbf{f}\}$, for all $p \in \mathrm{var}(\Gamma)$. Since $\mathrm{var}(\varphi, \psi) \subseteq \mathrm{var}(\Gamma)$ we know that $v(q) \in \{\mathbf{t}, \top, \mathbf{f}\}$, for all $q \in \mathrm{var}(\varphi, \psi)$ and, moreover, $v(\varphi) \in \{\mathbf{t}, \top, \mathbf{f}\}$ and $v(\psi) \in \{\mathbf{t}, \top, \mathbf{f}\}$. By hypothesis, there is a $\delta \in \Delta \cup \{\varphi, \psi\}$ such that $v(\delta) \in \{\mathbf{t}, \top\}$. Thus, either there is a $\delta \in \Delta$ such that $v(\delta) \in \{\mathbf{t}, \top\}$, or $v(\varphi) \in \{\mathbf{t}, \top\}$, or $v(\psi) \in \{\mathbf{t}, \top\}$. Finally, given $v(\varphi) \in \{\mathbf{t}, \top, \mathbf{f}\}$ and $v(\psi) \in \{\mathbf{t}, \top, \mathbf{f}\}$, and given the $\mathsf{S}_{\mathtt{fde}}$ truth-function for disjunction, we can establish that in all these cases it follows that there is a $\delta \in \Delta \cup \{\varphi \vee \psi\}$ such that $v(\delta) \in \{\mathbf{t}, \top\}$. Therefore, $v \vDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \varphi \vee \psi, \Delta$.

Ad $[\neg\wedge^B R]$ Let v be a $\mathsf{S}_{\mathtt{fde}}$ valuation such that $v \vDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \neg\varphi, \neg\psi, \Delta$ and assume that $\mathrm{var}(\varphi, \psi) \subseteq \mathrm{var}(\Gamma)$. Suppose $v(\gamma) \in \{\mathbf{t}, \top\}$ for all $\gamma \in \Gamma$. Thus, $v(p) \in \{\mathbf{t}, \top, \mathbf{f}\}$, for all $p \in \mathrm{var}(\varphi, \psi)$. Since $\mathrm{var}(\varphi, \psi) \subseteq \mathrm{var}(\Gamma)$ we know that $v(q) \in \{\mathbf{t}, \top, \mathbf{f}\}$, for all $q \in \mathrm{var}(\varphi, \psi)$ and, moreover, that $v(\neg\varphi) \in \{\mathbf{t}, \top, \mathbf{f}\}$ and $v(\neg\psi) \in \{\mathbf{t}, \top, \mathbf{f}\}$. By hypothesis, there is a $\delta \in \Delta \cup \{\neg\varphi, \neg\psi\}$ such that $v(\delta) \in \{\mathbf{t}, \top\}$. Thus, either there is a $\delta \in \Delta$ such that $v(\delta) \in \{\mathbf{t}, \top\}$, or $v(\neg\varphi) \in \{\mathbf{t}, \top\}$, or $v(\neg\psi) \in \{\mathbf{t}, \top\}$. Finally, given $v(\neg\varphi) \in \{\mathbf{t}, \top, \mathbf{f}\}$ and $v(\neg\psi) \in \{\mathbf{t}, \top, \mathbf{f}\}$, and given the $\mathsf{S}_{\mathtt{fde}}$ truth-functions for negation and conjunction, we can establish that in all these cases it follows that there is $\delta \in \Delta \cup \{\neg(\varphi \wedge \psi)\}$ such that $v(\delta) \in \{\mathbf{t}, \top\}$. Therefore, $v \vDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \neg(\varphi \wedge \psi), \Delta$. □

THEOREM 5.34 (Soundness of $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$). *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. If $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$, then $\vDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \Delta$.*

PROOF. If $\Gamma \succ \Delta$ is an axiom, then it is valid in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$. By induction on the depth of a derivation of $\Gamma \succ \Delta$ in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$ it follows, by the above Lemma 5.33, that $\Gamma \succ \Delta$ is valid in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$. □

PROPOSITION 5.35 (Non-triviality of $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$). *Let Γ be a finite non-empty set of formulae of $\mathcal{L}$. The sequent $\Gamma \succ \emptyset$ is not provable in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$.*

PROOF. Let v be a $\mathsf{S}_{\mathtt{fde}}$-valuation such that $v(p) = \top$ for every $p \in \mathrm{var}(\Gamma)$. It follows that $v \nvDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \emptyset$ and thus $\nvDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \emptyset$. By contraposition of Soundness, we can conclude that the sequent $\Gamma \succ \emptyset$ is not provable in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$. □

We now turn to completeness.

PROPOSITION 5.36. *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. If $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$, then it is provable in $\mathcal{G}\mathsf{LP}$.*

PROOF. Straightforward, since $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$ is a restriction of $\mathcal{G}\mathsf{LP}$. □

LEMMA 5.37. *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. If $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{LP}$ and $\mathrm{var}(\Delta) \subseteq \mathrm{var}(\Gamma)$, then $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$ without using the Cut rule.*

PROOF. Remember that proofs in sequent calculi are rooted binary trees such that the root is the sequent being proved and the leaves of the tree are instances of $[Id]$ in other words, sequents of the form $\varphi \succ \varphi$.

Now assume that Π is a Cut-free derivation of $\Gamma \succ \Delta$ in $\mathcal{G}\mathsf{LP}$ such that $\mathrm{var}(\Delta) \subseteq \mathrm{var}(\Gamma)$. If Π is a Cut-free derivation in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$, then the result is established. If Π is not a Cut-free derivation in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$, then there must be in Π applications of the rules $[\neg R]$, $[\vee R]$ or $[\neg\wedge R]$ where the required provisos are not satisfied

$$\frac{\Gamma^*, \varphi \succ \Delta^*}{\Gamma^* \succ \neg\varphi, \Delta^*}\ [\neg R] \qquad\qquad \frac{\Gamma^* \succ \varphi, \psi, \Delta^*}{\Gamma^* \succ \varphi \vee \psi, \Delta^*}\ [\vee R]$$

$$\frac{\Gamma^* \succ \neg(\varphi \wedge \psi), \Delta^*}{\Gamma^* \succ \neg\varphi, \neg\psi, \Delta^*}\ [\neg\wedge R]$$

Now, since Π is a Cut-free proof, we are guaranteed that the root sequent $\Gamma \succ \Delta$ contains *all* the propositional variables appearing in Π. Since, by hypothesis, we know that $\mathrm{var}(\Delta) \subseteq \mathrm{var}(\Gamma)$, we can affirm that $\mathrm{var}(\Pi) = \mathrm{var}(\Gamma)$.

What is left is, then, to design an algorithmic procedure to transform Π into a Cut-free proof of $\Gamma \succ \Delta$ in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$. We do this in two steps. First, we enlarge every node of Π by adding Γ to its left-hand side. By doing this, we obtain a rooted binary tree Π', whose leaves are sequents of the form $\Gamma, \varphi \succ \varphi$. Second, we extend each leaf with a branch starting in an instance of $[Id]$, that is, a sequent of the form $\varphi \succ \varphi$, followed by any number of necessary iterated applications of the left Weakening rule $[WL]$, so that the sequent $\Gamma, \varphi \succ \varphi$ is obtained.

From this procedure, we get a rooted binary tree Π'' which is undoubtedly a Cut-free derivation in $\mathcal{G}\mathsf{LP}$ of the sequent $\Gamma \succ \Delta$, such that the critical instances of the rules $[\neg R]$, $[\vee L]$ and $[\neg\wedge R]$ have in Π'' the form

$$\frac{\Gamma, \Gamma^*, \varphi \succ \Delta^*}{\Gamma, \Gamma^* \succ \neg\varphi, \Delta^*}\ [\neg R] \qquad \frac{\Gamma, \Gamma^* \succ \varphi, \psi, \Delta^*}{\Gamma, \Gamma^* \succ \varphi \vee \psi, \Delta^*}\ [\vee R]$$

$$\frac{\Gamma, \Gamma^* \succ \neg(\varphi \wedge \psi), \Delta^*}{\Gamma, \Gamma^* \succ \neg\varphi, \neg\psi, \Delta^*}\ [\neg\wedge R]$$

and are, thus, admissible in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$. Finally, from this we infer that Π'' is a Cut-free derivation in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$ of the sequent $\Gamma \succ \Delta$. $\square$

COROLLARY 5.38. *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. If $\vDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \Delta$ but $\mathrm{var}(\Delta) \subsetneq \mathrm{var}(\Gamma)$, then there is a $\Delta' \subseteq \Delta$ such that $\vDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \Delta'$, where $\mathrm{var}(\Delta') \subseteq \mathrm{var}(\Gamma)$.*

PROOF. First, notice that if $\vDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \Delta$, then $\mathrm{var}(\Gamma) \neq \emptyset \neq \mathrm{var}(\Delta)$. Now assume $\vDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \Delta$ but $\mathrm{var}(\Delta) \nsubseteq \mathrm{var}(\Gamma)$. Hence, define $\Delta' = \Delta \setminus \{\delta \in \Delta \mid \mathrm{var}(\delta) \nsubseteq \mathrm{var}(\Gamma)\}$, hence $\Delta' \subset \Delta$ and $\mathrm{var}(\Delta') \subseteq \mathrm{var}(\Gamma)$. Suppose additionally, for *reductio*, that there is a $\mathsf{S}_{\mathtt{fde}}$ valuation v such that $v(\gamma) \in \{\mathbf{t}, \top\}$ for all $\gamma \in \Gamma$, but $v(\delta) \in \{\bot, \mathbf{f}\}$ for all $\delta \in \Delta'$, thus implying $\nvDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \Delta'$. Construct now a $\mathsf{S}_{\mathtt{fde}}$ valuation v^* such that

$$v^*(p) = \begin{cases} \bot & \text{if } p \in \mathrm{var}(\Delta) \setminus \mathrm{var}(\Gamma) \\ v(p) & \text{if } p \in \mathrm{var}(\Gamma) \end{cases}$$

Hence, v^* is such that $v^*(\gamma) \in \{\mathbf{t}, \top\}$ for all $\gamma \in \Gamma$, but $v^*(\delta) \in \{\bot, \mathbf{f}\}$ for all $\delta \in \Delta$, hence v^* witnesses $\nvDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \Delta$, contradicting our initial assumption. Thus, there is a $\Delta' \subset \Delta$ such that $\vDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \Delta'$, where $\mathrm{var}(\Delta') \subseteq \mathrm{var}(\Gamma)$. Therefore, if $\vDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \Delta$ but $\mathrm{var}(\Delta) \subsetneq \mathrm{var}(\Gamma)$, then there is a $\Delta' \subseteq \Delta$ such that $\vDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \Delta'$, where $\mathrm{var}(\Delta') \subseteq \mathrm{var}(\Gamma)$. $\square$

THEOREM 5.39 (Completeness of $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$). *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. If $\vDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \Delta$, then $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$ without using the Cut rule.*

PROOF. Assume $\vDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \Delta$. By Observation 5.3, we know that $\vDash_{\mathsf{LP}} \Gamma \succ \Delta$, and also by Theorem 5.16 we are granted that $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{LP}$. To finally establish that $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$ without using the Cut rule, we consider two cases. First, if $\mathrm{var}(\Delta) \subseteq \mathrm{var}(\Gamma)$, we know by Lemma 5.37 that this is the case. Second, if $\mathrm{var}(\Delta) \subsetneq \mathrm{var}(\Gamma)$, we know by Corollary 5.38 that there is a $\Delta' \subseteq \Delta$ such that $\vDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \Delta'$, where $\mathrm{var}(\Delta') \subseteq \mathrm{var}(\Gamma)$. Now, by Lemma 5.37 we know that $\Gamma \succ \Delta'$ is provable in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$ without using the Cut rule, by means of a proof Π_1 (i.e., a rooted binary tree) whose root is $\Gamma \succ \Delta'$ and whose leaves are instances of $[Id]$, of the form $\varphi \succ \varphi$. Finally, we transform Π_1 into a proof Π_1', by extending down the node $\Gamma \succ \Delta'$ by means of the required iterated applications of the right Weakening rule $[WR]$, until we arrive at the sequent $\Gamma \succ \Delta$. But this rooted binary tree Π_1' is now a proof in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$ of the sequent $\Gamma \succ \Delta$, without using the Cut rule. □

COROLLARY 5.40 (Cut-elimination for $\mathsf{S}_{\mathtt{fde}}$). *Let $\Gamma \cup \Delta$ be a finite nonempty set of formulae in $\mathcal{L}$. If the sequent $\Gamma \succ \Delta$ is provable in $\Gamma \succ \Delta$ in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$, then there is a Cut-free derivation of $\Gamma \succ \Delta$ in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$.*

PROOF. Assume that $\Gamma \succ \Delta'$ is provable in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$. By Theorem 5.34, that is, because the system is sound, we know that $\vDash_{\mathcal{G}\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \Delta$. But then by Theorem 5.39, that is, because the system is complete, we know that $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$ without using the Cut rule. □

6. Conclusion

In this paper we have shown that, by following Fitting's epistemic interpretation of the strong Kleene logics K_3 and FDE, and the Paracomplete Weak Kleene logic $\mathsf{K}_3^{\mathrm{w}}$, an up-to-now unnoticed epistemic interpretation of Paraconsistent Weak Kleene logic PWK is available. This interpretation is revealed by focusing on a four-valued generalization of PWK, namely the logic $\mathsf{dS}_{\mathtt{fde}}$, and showing that its truth-functions can be interpreted in terms of what we called track-down operations. These operations, built inspired by the idea that no consistent opinion can arise from a set that includes an inconsistent opinion, coincide with the truth-functions of Paraconsistent Weak Kleene when certain reasonable constraints are assumed.

In addition to providing this novel interpretation of Paraconsistent Weak Kleene, the failure of Conjunctive Simplification in such a system and its sublogics is discussed in terms of track-down conjunctions and also in connection with containment logics. Concerning this latter relation, Paraconsistent Weak Kleene is shown to be closely related, and its theoremless subsystems are shown to belong, to a family of systems that respect a containment principle dual to Parry's Proscriptive Principle for entailment. These considerations mirror the previous remarks made in the literature concerning the other three-valued weak Kleene logic, namely $\mathsf{K}_3^{\mathrm{w}}$, whose subsystems containing no anti-theorems were shown by Ferguson to respect Parry's Proscriptive Principle for entailment.

These observations allowed us to design sound and complete Gentzen-style sequent calculi for this four-valued generalizations of PWK and $\mathsf{K}_3^{\mathrm{w}}$, i.e., the systems we referred to as $\mathsf{dS}_{\mathtt{fde}}$ and $\mathsf{S}_{\mathtt{fde}}$, drawing inspiration from the techniques recently applied by Coniglio and Corbalán to provide calculi of these sort for the three-valued weak Kleene logics. The main feature of these calculi, both for logics of the three- and four-valued kinds, was the presence of linguistic (i.e., variable inclusion) provisos in some of the operational rules for the calculi, pertaining to the set of propositional variables of the active formulae of the corresponding rules, and the set of propositional variables appearing in some of the side formulae of such rules.

Acknowledgments. I would like to thank Eduardo Barrio, Federico Pailos, Lucas Rosenblatt, the members of the Buenos Aires Logic Group, and an anonymous reviewer for their comments and suggestions. I am very thankful to Thomas Macaulay Ferguson for discussing a previous version of this material in great detail, which led to substantial improvements in the paper. The ideas contained here were presented at the 5th Workshop on Philosophical Logic in Buenos Aires, and at a CAPE Lecture in Kyoto University, whose audiences I would like to thank for their feedback. This work was done while enjoying a PhD scholarship from the National Scientific and Technical Research Council of Argentina (CONICET).

References

[1] Angell, R. B., "Deducibility, entailment and analytic containment", pages 119–143 in J. Norman and R. Sylvan (eds.), *Directions in Relevant Logic*, Springer, Dordrecht, 1989. DOI: 10.1007/978-94-009-1005-8_8

[2] Avron, A., "Natural 3-valued logics — characterization and proof theory", *The Journal of Symbolic Logic* 56, 1 (1991): 276–294. DOI: 10.2307/2274919

[3] Avron, A., and A. Zamansky, "Non-deterministic semantics for logical systems", pages 227–304 in D. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic*, volume 16, Springer, Dodrecht, 2011. DOI: 10.1007/978-94-007-0479-4_4

[4] Beall, J., "Off-topic: A new interpretation of weak-Kleene logic", *The Australasian Journal of Logic* 13, 6 (2016): 136–142.

[5] Beall, J. C., "Multiple-conclusion LP and default classicality", *The Review of Symbolic Logic* 4, 2 (2011): 326–336. DOI: 10.1017/S1755020311000074

[6] Belnap, N., "A useful four-valued logic", pages 5–37 in M. Dunn and G. Epstein (eds.), *Modern uses of Multiple-Valued Logic*, volume 2, Springer, Dodrecht, 1977. DOI: 10.1007/978-94-010-1161-7_2

[7] Bochvar, D. A., "On a three-valued logical calculus and its application to the analysis of the paradoxes of the classical extended functional calculus", *History and Philosophy of Logic* 2, 1–2 (1981): 87–112. DOI: 10.1080/01445348108837023

[8] Bolzano, B., *Wissenschaftslehre*, Seidel, Sulzbach, 1837.

[9] Bonzio, S., J. Gil-Ferez, F. Paoli and L. Peruzzi, "On paraconsistent weak Kleene logic: Axiomatization and algebraic analysis", *Studia Logica* 105, 2 (2017): 253–297. DOI: 10.1007/s11225-016-9689-5

[10] Ciucci, D., D. Dubois and J. Lawry, "Borderline vs. unknown: Comparing three-valued representations of imperfect information", *International Journal of Approximate Reasoning* 55, 9 (2014): 1866–1889. DOI: 10.1016/j.ijar.2014.07.004

[11] Ciuni, R., 'Conjunction in paraconsistent weak Kleene Logic", pages 61–76 in P. Arazim and M. Dančák (eds.), *The Logica Yearbook 2014*, College Publications, London, 2015.

[12] Ciuni, R., and M. Carrara, "Characterizing logical consequence in paraconsistent weak Kleene", pages 165–176 in L. Felline, A. Ledda, F. Paoli and E. Rossanese (eds.), *New Developments in Logic and the Philosophy of Science*, College Publications, London, 2016.

[13] Coniglio, M., and M. I. Corbalán, "Sequent calculi for the classical fragment of Bochvar and Halldén's nonsense logics", pages 125–136 in D. Kesner and P. Viana (eds.), *Proceedings of the Seventh Workshop on Logical and Semantic Frameworks with Applications (LSFA 2012)*, volume 113 of *EPTCS*, 2012.

[14] Correia, F., "Semantics for analytic containment", *Studia Logica* 77, 1 (2004): 87–104. DOI: 10.1023/B:STUD.0000034187.37935.24

[15] Deutsch, H., "Relevant analytic entailment", *The Relevance Logic Newsletter* 2, 1 (1977): 26–44.

[16] Došen. K., "A note on the law of identity and the converse Parry property", *Notre Dame Journal of Formal Logic* 19, 1 (1978): 174–176. DOI: 10.1305/ndjfl/1093888223

[17] Dubois, D., "On ignorance and contradiction considered as truth-values", *Logic Journal of the IGPL* 16, 2 (2008): 195–216. DOI: 10.1093/jigpal/jzn003

[18] Dubois, D., "Reasoning about ignorance and contradiction: Many-valued logics versus epistemic logic", *Soft Computing* 16, 11 (2012): 1817–1831. DOI: 10.1007/s00500-012-0833-5

[19] Dunn, M., "Intuitive semantics for first-degree entailments and 'coupled trees'", *Philosophical Studies* 29, 3 (1976): 149–168. DOI: 10.1007/BF00373152

[20] Epstein, R. L., *The Semantic Foundations of Logic*, volume I: Propositional Logics, Oxford University Press, New York, 2nd edition, 1995.

[21] Ferguson. T. M., "A computational interpretation of conceptivism", *Journal of Applied Non-Classical Logics* 24, 4 (2014): 333–367. DOI: 10.1080/11663081.2014.980116

[22] Ferguson, T. M., "Logics of nonsense and Parry systems", *Journal of Philosophical Logic* 44, 1 (2015): 65–80. DOI: 10.1007/s10992-014-9321-y

[23] Ferguson, T. M., "Cut-down operations on bilattices", pages 24–29 in *Proceedings of the 45th International Symposium on Multiple-Valued Logic (ISMVL 2015)*, IEEE Computer Society, 2015. DOI: 10.1109/ISMVL.2015.14

[24] Ferguson, T. M., "Faulty Belnap computers and subsystems of FDE", *Journal of Logic and Computation* 26, 5 (2016): 1617–1636. DOI: 10.1093/logcom/exu048

[25] Ferguson, T. M., "Rivals to Belnap-Dunn logic on interlaced trilattices", *Studia Logica* 105, 6 (2017): 1123–1148. DOI: 10.1007/s11225-016-9695-7

[26] Ferguson, T. M. "The proscriptive principle and logics of analytic implication", PhD thesis, City University of New York, 2017.

[27] Fine, K., "Angellic content", *Journal of Philosophical Logic* 45, 2 (2016): 199–226. DOI: 10.1007/s10992-015-9371-9

[28] Fitting, M., "Bilattices and the semantics of logic programming", *The Journal of Logic Programming* 11, 2 (1991): 91–116. DOI: 10.1016/0743-1066(91)90014-G

[29] Fitting, M., "Kleene's three valued logics and their children", *Fundamenta Informaticae* 20, 1–3 (1994): 113–131. DOI: 10.3233/FI-1994-201234

[30] Fitting, M., "Bilattices are nice things", pages 53–78 in T. Bolander, V. Hendricks and S. A. Pedersen (eds.), *Self-Reference*, CSLI Publications, 2006.

[31] French, R., and S. Standefer, "Non-triviality done proof-theoretically", pages 438–450 in A. Baltag, J. Seligman, and T. Yamada (eds.), *Logic, Rationality, and Interaction: 6th International Workshop (LORI 2017)*, Springer, Berlin, 2017. DOI: 10.1007/978-3-662-55665-8_30

[32] Halldén, S., *The Logic of Nonsense*, Uppsala Universitets Arsskrift, Uppsala, 1949.

[33] Hintikka, J., *Logic, Language-Games and Information: Kantian Themes in the Philosophy of Logic*, Clarendon Press, Oxford, 1973.

[34] Humberstone, L., "Power matrices and Dunn-Belnap semantics: Reflections on a remark of Graham Priest", *The Australasian Journal of Logic* 11, 1 (2014): 14–45.

[35] Humberstone, L., and R. Meyer, "The relevant equivalence property", *Logic Journal of the IGPL* 15, 2 (2007): 165–181.

[36] Kleene, S. C., *Introduction to Metamathematics*, North-Holland, Amsterdam, 1952.

[37] Lawry, J., and D. Dubois, "A bipolar framework for combining beliefs about vague propositions", pages 530–540 in G. Brewka, T. Eiter and S. McIlraith (eds.), *Principles of Knowledge Representation and Reasoning: Proceedings of the Thirteenth International Conference*, AAAI Press, 2012.

[38] Oller, C. A., "Paraconsistency and analyticity", *Logic and Logical Philosophy* 7 (1999): 91–99. DOI: 10.12775/LLP.1999.008

[39] Omori, H., "Halldén's logic of nonsense and its expansions in view of logics of formal inconsistency", pages 129–133 in *Proceedings of the 27th International Workshop on Database and Expert Systems Applications (DEXA 2016)*, IEEE Computer Society, 2016. DOI: 10.1109/DEXA.2016.039

[40] Omori, H., and D. Szmuc, "Conjunction and disjunction in infectious logics", pages 268–283 in A. Baltag, J. Seligman, and T. Yamada (eds.), *Logic, Rationality, and Interaction: 6th International Workshop (LORI 2017)*, Springer, Berlin, 2017. DOI: 10.1007/978-3-662-55665-8_19

[41] Paoli, F., "Regressive analytical entailments", Technical Report number 33, *Konstanzer Berichte zur Logik und Wissenschaftstheorie*, 1992.

[42] Paoli. F., "Tautological entailments and their rivals", pages 153–175 in J.-Y. Béziau, W. A. Carnielli and D. M. Gabbay (eds.), *Handbook of Paraconsistency*, College Publications, 2007.

[43] Parry, W. T., "Ein axiomensystem für eine neue art von implikation (analytische implikation)", *Ergebnisse eines mathematischen Kolloquiums* 4 (1933): 5–6.

[44] Petrukhin, Y., "Natural deduction for three-valued regular logics", *Logic and Logical Philosophy* 26, 2 (2017): 197–206. DOI: 10.12775/LLP.2016.025

[45] Petrukhin, Y., "Natural deduction for Fitting's four-valued generalizations of Kleene's logics", *Logica Universalis* 11, 4 (2017): 525–532. DOI: 10.1007/s11787-017-0169-0

[46] Petrukhin, Y., "Natural deduction for four-valued both regular and monotonic logics", *Logic and Logical Philosophy* 27, 1 (2018): 53–66. DOI: 10.12775/LLP.2017.001

[47] Priest, G., *An Introduction to Non-Classical Logic: From If to Is*, Cambridge University Press, 2nd edition, 2008. DOI: 10.1017/CBO9780511801174

[48] Priest, G.. "Plurivalent logics", *The Australasian Journal of Logic* 11, 1 (2014): 2–13.

[49] Routley, R., V. Plumwood, R. Meyer and R. Brady, *Relevant Logics and Their Rivals: The Basic Philosophical and Semantical Theory*, Ridgeview, Atascadero, CA, 1982.

[50] Sylvan, R., "Relevant containment logics and certain frame problems of AI", *Logique et Analyse* 31 (1988): 11–25.

[51] Szmuc, D., "Defining LFIs and LFUs in extensions of infectious logics", *Journal of Applied Non-Classical Logics* 26, 4 (2017): 286–314. DOI: 10.1080/11663081.2017.1290488

[52] Szmuc, D., "Track-down operations on bilattices", in *Proceedings of the 48th International Symposium on Multiple-Valued Logic (ISMVL 2018)*, IEEE Computer Society, 2018. Forthcoming.

[53] Tomova, N., "About four-valued regular logics", *Logical Investigations* 15 (2009): 223–228.

[54] Urquhart, A., "Basic many-valued logic", pages 249–295 in D. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic*, volume 2, Springer, Dodrecht, 2001. DOI: 10.1007/978-94-017-0452-6_4

[55] Wansing, H., and N. Belnap, "Generalized truth values. A reply to Dubois", *Logic Journal of the IGPL* 18, 6 (2010): 921–935. DOI: 10.1093/jigpal/jzp068

[56] Zimmermann, T. E., "Free choice disjunction and epistemic possibility", *Natural Language Semantics* 8, 4 (2000): 255–290. DOI: 10.1023/A:1011255819284

Damian E. Szmuc
Department of Philosophy
University of Buenos Aires
and
IIF-SADAF
CONICET
Buenos Aires, Argentina
szmucdamian@gmail.com