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A “DISTRIBUTIVE” OR A “COLLECTIVE” APPROACH TO SENTENCES?

Abstract. It is a well-known fact that Russell’s antinomy arises within distributive set theory whereas it does not do so within collective set theory. In this paper, I shall propose what I shall call a “collective” understanding of a sentence as opposed to the standard, truth-functional approach which I shall term a “distributive” approach. Similar to the case with sets, the liar antinomy appears when the liar sentence is treated distributively. If, however, the sentence is understood collectively, then the liar antinomy does not appear.

Keywords: content implication; pluripropositionalism; Suszko’s identity; liar sentence; liar antinomy; Buridan; Prior; Grice’s implicature; definition of truth

Introduction

Alfred Tarski’s contribution to contemporary logic, metamathematics, model theory, set theory, analytic philosophy, and many others is one of the most impressive achievements of human thought. He is someone who is justly renowned for his knowledge of and results in mathematics, logic and philosophy but also for the accuracy of his logical and philosophical intuitions. Of central concern in this paper is his semantic theory of truth containing a theorem on the indefinability of truth in the formal universal languages, definition of truth and a postulate of the stratification of the language. They are all fundamental elements of contemporary logic. It is Tarski that established the result that seems so obvious to us today that a statement that some sentence of the language \mathcal{L} is true or false cannot be expressed in \mathcal{L} . Such a statement has to be formulated in the

metalanguage of \mathcal{L} , which consequently leads to the idea of stratification of the language. Tarski's postulate was a result of the well-known problem of the logical value of the liar sentence. The definition of truth collapses in the face of the liar sentence — simple reasoning shows that the liar sentence is true if and only if it is false. Of course, Tarski's postulate of the stratification of the language makes the liar sentence semantically incorrect — no sentence can state its own truthfulness or falsity as no sentence can simultaneously belong to a given language and its metalanguage. Thus, every identification of a sentence p of level n of the language \mathcal{L} with a sentence “ p is true” or “ p is false” belonging to the next level $n + 1$ is incorrect and so forbidden, although only the second case, i.e., $p =$ “ p is false”, leads to contradiction.

However, it seems that that one can fully respect Tarski's understanding of the liar problem by proposing another, parallel, Lesniewski-like approach to the liar sentence. The word “parallel” means that Lesniewski's approach does not invalidate Tarski's approach. For if a language is equipped with appropriate connectives, then both approaches to the liar sentence can be used in the same language. Although the aforementioned identification of the sentence p and the sentence “ p is true” or “ p is false” is illegal, it should always be possible to express the relation between the contents of the sentences p and “ p is true” or “ p is false” — contents understood as thoughts expressed by sentences. It seems that this not only solves some semantic problems but primarily introduces a new perspective of understanding of the sentences.

1. Some fundamental facts about the sense of a sentence

1.1. Buridan's thesis: *Every sentence says of itself that it is true.*

More than fifty years ago, Arthur Prior developed an idea of the calculus with proposition-forming ‘functors’ of propositions. Using Łukasiewicz's notation, he enriched the classical propositional calculus with the proposition-forming functor ‘ d ’ whose intended meaning is “It is said by a Cretan that”, and universal U and existential E quantifiers binding variables of any categories [see Prior, 1961, p. 16]. To avoid misunderstanding, Prior explains this meaning more precisely:

‘It is said by a Cretan that p ’ is not a sentence about the sentence ‘ p ’ but a new sentence which, like ‘Not p ’, is about whatever ‘ p ’ is about;

e.g., ‘?t is said by a Cretan that Socrates is ill’ is not about the sentence ‘Socrates is ill’ but is another sentence which, like that one, is about Socrates. [Prior, 1961, p. 17]

This calculus, which displays a sensitivity to the senses of sentences, was constructed by Prior to deepen the discussion of the liar problem. In addition, the passage above implies explicitly that the sense (content) of a sentence is a thought (proposition) expressed by this sentence. As Prior stated, his logic is the ordinary propositional calculus where d is not truth-functional. Only an axiomatic extension of this calculus by $CdpCdNpdq$, $CQpqCdpdq$ ¹ and $CddUppdp$ results in the truth-functional trivialization of d . Using his calculus, Prior proves the thesis $C(dUpCdpNp)K(EpKdpp)(EpKdpNp)$ according to Church’s remark recalled by Prior at the start of his article. Prior comments that, what this thesis asserts, with an assumed illustrative value for d , is that if it is said by a Cretan that whatever is said by a Cretan is not the case, then something said by a Cretan is the case, and something said by a Cretan is not the case [see Prior, 1961, p. 17]. Later, Prior proves that statement “There are at least two statements said by a Cretan (or Cretans)” is a logical consequence of this thesis [see Prior, 1961, pp. 17–20]. Thus, if it is said by a Cretan that whatever is said by a Cretan is not the case, then at least two things are said-by-a-Cretan.

Prior’s solution of the liar antinomy belongs to the class of those approaches which assume that every uttered and well-understood sentence is regarded as true. Otherwise their senses cannot be properly recognized. Of course, this idea is not new. In the medieval period it was a quite popular opinion that, once ‘mental contact’ had been made with the sentence, it had to be considered as true in order to be correctly understood. The philosophers most associated with his view are probably Jean Buridan (c. 1295–1363), Thomas Bradwardine (c. 1300–1349) and Albert of Saxony (c. 1320–1390) [see Parsons, 2008, p. 132].

It seems that Buridan was the philosopher of Middle Ages who explicitly made this remark. Almost all the essays in the book *Unity, Truth and the Liar: The Modern Relevance of Medieval Solutions to the Liar Paradox* [Rahman et al., 2008] present various ways of understanding Buridan’s famous statement. Eugene Mills’s chapter probably delivers the best interpretation. His solution is based on the same assumptions as Read’s [see Mills, 2008, p. 125]. However, in opposition to Read’s inter-

¹ “ Q ” is the connective of material equivalence.

pretation [see [Read, 2008](#)], Mills does not understand Buridan's thesis as the statement "Every sentence is true". In other words, Buridan's thesis cannot be identified with the claim "Sentence p is true if and only if p ". Otherwise, all false sentences would be simultaneously true and false, which is not true. It means that Buridan's thesis does not establish any theory of truth. Rather it is connected with the meaning of the sentence:

I argue that every proposition attributes truth to itself, in the sense that (for example) the proposition that the grass is green is strictly identical with the proposition that it is true that the grass is green. [...] I assert that every proposition says of itself that it's true. This assertion simply doesn't entail that every sentence says of itself that it's true. [[Mills, 2008](#), p. 127]

This reformulation of Buridan's thesis means that in Mills's opinion, Buridan's thesis concerns the meanings of sentences than of sentences. Although this explanation is not clearly expressed by Mills, it seems that the closest interpretation might be the following:

every proposition first needs to be treated as true for it to be accurately understood and only afterwards can it be recognised as true or false.

Here it is necessary to emphasize the difference between the meanings of "to be treated as true" and "to assume truthfulness". In the case of Buridan's thesis, no assumption is made as to the logical value of the sentence when the sentence is treated as true. If we do not treat the sentence as if it were true, we will get the complementary meaning of this sentence. The sentence treated as true has a certain meaning A , and the sentence treated as a false complementary meaning A^- . Treating the sentence as true helps determine the meaning of the sentence, and not its logical value.

More precisely, correct recognition of the logical value of the sentence is based on the comparison of two situations: the first being the one expressed by the sentence as if it were true; and the second being the actual state of affairs. If both situations agree with each other, we consider the sentence to be truthful, and therefore true. If both situations do not agree with one another, we reject the veracity of the sentence and consider it to be false.

As an example, let us consider the sentence "The Sun orbits the Earth". Obviously, this is a false sentence. However, we are only able

to recognise it as false if we are able to correctly grasp its sense. But accurate recognition of the meaning is possible only if this sentence is understood as true. Otherwise, the sense of this sentence would be quite different as in, for example “The Earth orbits the Sun” or “The Sun travels through the Universe along a straight line”. Thus, we must first grasp the meaning of this sentence by treating as it if it were true. Only then are we able to consider the sentence in relation to the facts and ultimately determine that the sentence is false.

Almost 200 years after Buridan, a similar approach to the problem of the significance of sentences, especially in the context of the liar antinomy, was proposed by the Belgian theologian Jodocus Clichtoveus (1472–1543) and presented by the French theologian Jacques Le Fèvre d’Estaples (c. 1450–1537) in his *Jacobi Fabri Stapulensis artificiales nonnullae introductiones per Iodocum Clichtoveum in unum diligenter collecte* (Parisius 1520) [see [Tworak, 2004](#), p. 63]. Jodokus’s idea is based on the following assumptions he made: (a) *insolubilia* are determined by the meanings of sentences equivalent to them; (b) every sentence implies itself; (c) every sentence implies a statement of its own truth; (d) if a given sentence implies some other sentences (each one separately), it also implies their conjunction; (e) each sentence is equivalent to a conjunction, the factors of which are this sentence and a sentence stating its truthfulness; (f) equivalent sentences imply each other and have the same logical value. Using these assumptions, it is not difficult to prove that the liar sentence is equivalent to a conjunction of which one conjunct is the negation of the other. Therefore, the liar sentence is false, and its negation true [see [Tworak, 2004](#), p. 64].

Surprisingly, modern research has confirmed the interpretation of Buridan’s thesis presented here. In a series of behavioral experiments [Gilbert et al.](#) [see [1990](#), [1993](#)] showed that interfering with the process of truth-verification makes humans prone to making only one type of mistake when judging the truthfulness of sentences. People mistake false sentences for true sentences, but not the other way around. In other words, if the process of reasoning about a sentence is interrupted, then people stop at the initial judgment that the sentence is true and are unable to modify that belief. This automatic and (in some sense) unconscious process of accepting sentences shows that at the very beginning of ‘contact’ with the sentences, i.e., just before they start to think through them, people treat sentences as true.

1.2. Woleński's thesis: *The liar sentence is antinomial only in the logic of truth, because in the logic of falsehood it does not lead to contradiction; the truth-teller sentence is antinomial only in the logic of falsehood, because in the logic of truth it does not lead to contradiction* [Woleński, 1992, 1995].

In other words, the liar problem cannot be considered without taking into account which logical value is the designed value of our thinking — the paradoxicality of the liar as well as of the truth-teller is connected with the dualisation of logic.

Traditionally, by a “logic of truth” is understood a logic for which the truth is a designated value of its adequate semantics. Similarly, a logic of falsehood is a logic for which falsehood is the designated value of its adequate semantics. A good test of what logic we are dealing with is its way of understanding logical connectives. For example, in the logic of truth disjunction has a designated value if and only if at least one disjunct has a designated value, while conjunction has a designated value if and only if both conjuncts have a designated value. By contrast, in the logic of falsehood, disjunction has a designated value if and only if both disjuncts have a designated value, while conjunction has a designated value if and only if at least one conjunct has a designated value. Similarly with the other connectives. A good example of both the logic of truth and the logic of falsehood is the pair of intuitionistic sentential calculi that are the Heyting logic and the Brouwerian logic respectively. In Heyting logic, the Modus Ponens rule uses a well-known intuitionistic implication. In Brouwerian logic, the role of implication is played by intuitionistic co-implication². It is not difficult to notice that the traditional and standard senses of all the connectives — conjunction, disjunction, co-implication and others — is kept only if falsehood is a designated value of models for Brouwerian logic.

In order to prove Woleński's thesis let us firstly recall the fundamental fact that falsehood converts the sense of the sentence on its opposite. By way of illustration, if, in the logic of truth, sentence $A = “2 + 2 = 4”$ says that 2 plus 2 is equal to 4, then in the logic of falsehood, the opposite sense that that sentence has would be 2 plus 2 is not equal to 4. Therefore, in the logic of truth, this sentence represents exactly

² Co-implication is a connective that is the dual to implication considered by Rauszer [1974, 1980] in the Heyting-Brouwer logic she constructs. See also [Łukowski, 1996].

one precise piece of information (e.g. $2 + 2 = 4$), whereas in the logic of falsehood, a huge number of them (e.g., $2 + 2 = 2$, $2 + 2 = 0$, $2 + 2 = 131$, ...).³ As has been mentioned earlier, in the logic of truth, the falsity of a sentence changes its meaning to the opposite of what it would mean if it were true. In what follows, we will say that the falsity of a sentence *reverses* the meaning of a given sentence in the logic of truth.⁴

Let us consider two general names “*a*” and “*b*” for two logical values such that $a = \text{not-}b$ and $b = \text{not-}a$.⁵ Moreover, let us assume that *a* keeps the sense of every sentence, and *b* reverses this sense on the opposite.

Let us now consider two self-referential sentences in their general forms:⁶

$S_1 = \text{“}S_1 \text{ has the logical value } a\text{”}$ and $S_2 = \text{“}S_2 \text{ has the logical value } b\text{”}$.

For the logic with a as a designated value (the logic of truth).

1.1. Let assume that S_1 has *a*. Since, it is a logic of the value *a* which keeps the senses of sentences, the sense of this assumption, i.e., the sense of “ S_1 has *a*” is kept, and so S_1 has *a*. Again, since *a* keeps

³ On March 30, 2017, at the University of Łódź, Jan Woleński gave a lecture entitled “Is the generality and universality of logic the same?”. In the course of it, he referred to the question of the possibility of thinking in the logic of falsehood. He said that for some reason, in the process of human evolution, the logic of truth has beaten logic of falsehood, and today we all think in the logic of truth. He then added that it is probably the result of the fact that the truth concentrates information, while falsehood disperses it.

⁴ Something quite different from the senses of sentences is the sense of the connectives of a given logic and the logic that is the dual of that logic. In both cases, the senses of the connectives are the same. For example, a characteristic of the connective of conjunction in the logic of truth is given by the equivalence: $v(\alpha \wedge \beta) \in D$ iff $v(\alpha) \in D$ and $v(\beta) \in D$; whereas in the logic of falsehood it is: $v(\alpha \wedge \beta) \in D$ iff $v(\alpha) \in D$ or $v(\beta) \in D$. In the first case, *D* is the set of all values designated in the logic of truth, so it is a set of the semantic correlates of true sentences. In the second case, *D*, as a set of all values designated in the logic of falsehood, is a set of the semantic correlates of false sentences. It is easy to see that both characteristics are equivalent. Thus, a given logic and its dual are one and the same logic, with the difference that one determines inferences from true sentences to true sentences, and the other from false to false [cf. Łukowski, 2002]. Duality is here defined by Wójcicki’s operator *d*, such that *C* and C^d are mutually dual, and moreover, $C^{dd} = C$ [see Wójcicki, 1973].

⁵ To make this easier, the general values *a* and *b* should be understood as truth and falsehood, respectively.

⁶ Of course, S_1 should be connoted with the truth-teller sentence, while S_2 with the liar sentence.

senses, the sense of $S_1 = "S_1 \text{ has the logical value } a"$ is kept, so S_1 has the logical value a .

1.2. Let assume that S_1 has b . Since, it is a logic of the value a which keeps the senses of sentences, the sense of this assumption, i.e., the sense of " S_1 has b " is kept, and so S_1 has b . Then, since b reverses senses, the sense of $S_1 = "S_1 \text{ has the logical value } a"$ is reversed, so S_1 has the logical value b .

Thus, there is no contradiction in case 1: S_1 has a if and only if S_1 has a .

2.1. Let assume that S_2 has a . Since, it is a logic of the value a which keeps the senses of sentences, i.e., the sense of " S_2 has a " is kept, and so S_2 has a . Again, since a keeps senses, the sense of $S_2 = "S_2 \text{ has the logical value } b"$ is kept, so S_2 has the logical value b .

2.2. Let assume that S_2 has b . Since, it is a logic of the value a which keeps the senses of sentences, i.e., the sense of " S_2 has b " is kept, and so S_2 has b . Then, since b reverses senses, the sense of $S_2 = "S_2 \text{ has the logical value } b"$ is reversed, so S_2 has the logical value a .

Thus, there is a contradiction in case 2: S_2 has a if and only if S_2 has b .

For the logic with b as a designated value (the logic of falsehood).

3.1. Let assume that S_1 has a . Since, it is a logic of the value b which reverses the senses of sentences, the sense of this assumption, i.e., the sense of " S_1 has a " is reversed, and so S_1 has b . Again, since b reverses senses, the sense of $S_1 = "S_1 \text{ has the logical value } a"$ is reversed, so S_1 has the logical value b .

3.2. Let assume that S_1 has b . Since, it is a logic of the value b which reverses the senses of sentences, the sense of this assumption, i.e., the sense of " S_1 has b " is reversed, and so S_1 has a . Then, since a keeps senses, the sense of $S_1 = "S_1 \text{ has the logical value } a"$ is kept, so S_1 has the logical value a .

Thus, there is a contradiction in case 3: S_1 has a if and only if S_1 has b .

4.1. Let assume that S_2 has a . Since, it is a logic of the value b which reverses the senses of sentences, the sense of this assumption, i.e., the sense of " S_2 has a " is reversed, and so S_2 has b . Again, since b reverses senses, the sense of $S_2 = "S_2 \text{ has the logical value } b"$ is reversed, so S_2 has the logical value a .

4.2. Let assume that S_2 has b . Since, it is a logic of the value b which reverses the senses of sentences, the sense of this assumption, i.e., the sense of “ S_2 has b ” is reversed, and so S_2 has a . Then, since a keeps senses, the sense of $S_2 =$ “ S_2 has the logical value b ” is kept, so S_2 has the logical value b .

Thus, there is no contradiction in case 4: S_2 has a if and only if S_2 has a .

Therefore, the conclusion coming from the above cases 1–4 is simple and proves Woleński’s thesis which is crucial for our approach: the liar sentence is antinomial only in the logic of truth, since in the logic of falsehood it does not lead to a contradiction, unlike the truth-teller sentence which is antinomial only in the logic of falsehood, whereas in the logic of truth it does not lead to a contradiction.

1.3. The hypothesis: *The existence of two types of set theory — distributive and collective — means that there are two corresponding ways of understanding sentences. Understood in a distributive sense, the liar sentence generates an antimony whereas understood in the collective sense, it does not.*

As already mentioned an aim of the paper is to compare two possible ways of understanding of sentences — distributive and collective — which derive from two ways of understanding the concept of a set. As with sets, when understood as distributive rather than collective, certain sentences will lead to different results. It is well-known that Russell’s set $\{x : x \notin x\}$ is antinomial if understood distributively whereas it is not antinomial in Leśniewski’s theory as normal sets⁷ do not exist in it. This difference can be illustrated more vividly. Almost anything can be considered distributively or collectively, so let us take the example of a finger. Understood distributively, a finger is not a part of a hand but an element of the set of all fingers. The set of all fingers and the set of both hands have nothing in common — they are essentially disjoint. Understood collectively, a finger is a part of some hand and has nothing in common with the fingers of all other hands.

In a similar way, one can distinguish two ways of understanding a sentence. On the distributive approach the liar sentence leads to the well-known liar antinomy, while on the collective approach the same sentence does not lead to any contradiction. In this paper, I will explain

⁷ A set x is normal iff $x \notin x$.

how these different reactions to the liar sentence arise. The solution is to adapt the idea of the set-theoretical distinction between distributive and collective sets. It proposes a perspective from which the main connective of the complex sentence decides if this sentence should be understood in a distributive or a collective way. We have no choice in how to understand complex sentences. To put it bluntly, we have no choice in how to understand complex sentences. Some connectives require a distributive understanding whilst others require a collective understanding. It is different with atomic sentences that can be understood both distributively and collectively. To look ahead a little, it will be shown that the expression of the sense of the liar sentence L in terms of logical values of truth and falsehood inevitably leads to the contradictory identification of L with its negation $\neg L$. This is the result of a distributive understanding of the liar sentence. However, a quite different interpretation is possible in which the sense of a sentence consists in some parts being senses of other sentences. In the collective understanding of sentences, a sense of the sentence is understood as a thought expressed by this sentence and so, called the content of sentence. More precisely, the content of a given sentence is composed out of the contents of other sentences. So, the content of a sentence can be an ingredient (in Leśniewski's sense) of a content of another sentence — a content of one sentence can be the whole or a proper part of the content of other sentence. That is why this approach is called “collective”. The relation between the contents of p and q becomes expressible thanks to the new connective of content implication, and is expressed by the sentence $p : q$. The meaning of the liar sentence refers to this fact.

2. The “distributive” approach to sentences

Classical logic, like many other formal logics, is determined exclusively by extensional connectives, i.e., by connectives whose characteristics depend on the logical values of their arguments. The concept of extensionality, however, should be extended to those intentional connectives whose characteristics are expressed using the logical values of their arguments. Instead of extensionality, we will talk about truth-functionality. Thus, in a logic with only truth-functional connectives, the logical value is a semantic correlate of the sentence. Let us add that this the only possible sense of the sentence. Therefore, the use of such logic to express

thinking in natural language is a serious mistake. After all, the content of a sentence is the essence of a natural language, of course, the content understood as the thought expressed by the sentence. This mistake leads to the so-called paradoxes of material implication. Forgetting about the truth-functional nature of the connectives of implication and disjunction, we argue that the classical tautology $(\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha)$ is paradoxical. Indeed, it is not true that from any two chosen sentences, at least one follows from the other. However, this interpretation of this formula is unacceptable. Its real sense is simple: $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$, for any $x, y \in \{1, 0\}$. Moreover, it is easy to see that the sense of the sentence understood in this way is an indivisible whole — it has no parts. Neither 1 nor 0 has components. Even if 1 is the sense of the complex sentence $\neg p$, no one can say that 0 is a part of it. The same applies to the other complex sentences, like $\neg p$, $p \wedge q$, $p \vee q$, $p \rightarrow q$, $p \leftrightarrow q$.

It is a well-known fact that, on the basis of Tarski's definition of truth, the supposition that the liar sentence possesses a logical value leads to contradiction. The argument can be repeated for almost any formal logic including classical logic. It is a result of the fact that the semantic correlate of the liar sentence is understood as a logical value, and so as a whole without parts. One sense (this being one logical value of the liar sentence) implies the other, i.e., the other logical value. Such a treatment of the senses of sentences as indivisible wholes will be here called “distributive” as opposed to the “collective” understanding, according to which the sense of the sentence can be a part or the whole of the sense of other sentence. The essence of the distributive approach is expressed by the fact that connectives of the language reduce all relations between parts of sentences to substitutions, e.g., the sentence $p \rightarrow q$ can be substituted for p in $p \vee s$ without any relation to the actual thoughts expressed by p , q , s , as if they were blocks in a puzzle. Just as a completely abstract sets in the distributive sense do not exist in time and space, the meanings of sentences in the distributive approach are related neither to reality nor to the language expressing contents communicable with its help.

COMMENT. Usually, such a distributive approach is associated with the wordless assumption that the correct characterization of the liar sentence L can be given only by some equivalence: $L \leftrightarrow \neg L$, $L \equiv \neg L$, $L \approx \neg L$, etc. It is a direct result of the fact that every sentence, and so the liar, is treated and understood as an indivisible whole. Thus, a standard

and usually undisputed approach to the liar sentence identifies L with $\neg L$. However, a simple understanding of the sense of L leads us to the opposite conclusion. The liar sentence says of itself that it is false (or not true). That is all, and nothing more. The well-known, commonly accepted equivalence $L \leftrightarrow \neg L$ is an inferential consequence of the sense of L , but this equivalence is not the essence of L . It seems that a sentence whose sense could be expressed by such an equivalence should have the form

$$C = \text{“}C \text{ is true iff } C \text{ is false”}$$

or

$$C = \text{“truthfulness of } C \text{ is its falsehood, and vice versa”}.$$

3. The “collective” approach to sentences

A situation rather different to the one discussed in the previous section is one in which the language of a logic contains at least one non-truth-functional connective, i.e., one whose characteristics are not determined by the logical values of its arguments. A good example of such a connective, presented below, is Suszko’s sentential identity and our connective of content implication. The existence of such a connective forces the semantics to have models with sets of semantic correlates which have more than two elements. Such a multitude of semantic correlates makes it impossible to reduce them to two logical values—the understanding of the sense of sentence must be different. Here, we have accepted that the sense of a sentence is understood as a thought expressed by this sentence. Therefore, the sense of the sentence is here identified with the content of the sentence. As will be shown below, this approach meets the idea expressed in *Tractatus Logico-Philosophicus* by Wittgenstein.

According to the intended understanding of content implication, $\alpha : \beta$ is a true sentence if the content of the sentence (i.e., the thought expressed by the sentence, or truth conditions of the sentence) β is included in the content of the sentence α . Thus, the sentence $\alpha : \beta$ is true if and only if the content of β is a part, not necessarily proper, of the content of α . In other words, $\alpha : \beta$ is true, iff the sentence α says what is said by β . Of course, α can say something more than what is said by β . The simultaneous truthfulness of $\alpha : \beta$ and $\beta : \alpha$ means that the content of α is equal to the content of β , and so α says what is said by β and β says what is said by α . It means that α and β are two different sentences with

the same sense. Thus, the triviality of the content implication connective means that the only tautologies with it as the main functor can be one of only two forms:

$$\alpha : \alpha$$

or

$$(\alpha_1 \wedge \cdots \wedge \alpha_n) : \alpha_i,$$

for $i \in \{1, \dots, n\}$. This fact agrees with Wittgenstein’s theses of the *Tractatus Logico-Philosophicus*:

5.122 If p follows from q , the sense of “ p ” is contained in that of “ q ” [Wittgenstein, 2015, p. 310].

5.14 If a proposition follows from another, then the latter says more than the former, the former less than the latter [Wittgenstein, 2015, p. 314].

5.141 If p follows from q and q from p , then they are one and the same proposition [Wittgenstein, 2015, p. 314].

The intended application for the new connective is simple: to express the fact that the content of one sentence is a part or the whole of the content of another sentence. If the part is not proper, then both sentences have the same content. This means that the understanding of the sentence $\alpha : \beta$ means that the content of β is an *ingredient* of the content of α in precisely Leśniewski’s sense.⁸

Thanks to its triviality, the content implication connective is useful for expressing various contentual relations between sentences such as those typical for metaphors or implicatures. In his famous and influential paper “Logic and conversation” Grice [1977] notices that every utterance says more than what it literally says.⁹ Every utterance implies, suggests, means, . . . , etc. something above and beyond its literal content. In a similar way, every sentence expresses more than it is said by the words that compose this sentence, i.e., it goes far beyond its conventional meaning. Since every sentence is notoriously imprecise, the context of utterance of a sentence adds additional possible senses. Moreover, for

⁸ “Definition I. The expression ‘ingredient of object A ’ is used to denote A , and every part of A . That is: Object B is an ingredient of object A if and only if either B is A or B is part of A . Nowadays the term ‘ingredient’ is often simply rendered as ‘part’ and Leśniewski’s ‘part’ is called ‘proper part’” [Simons, 2015].

⁹ “In the sense in which I am using the word *say*, I intend what someone has said to be closely related to the conventional meaning of the words (the sentence) he has uttered” [Grice, 1977, p. 21].

every sentence there are conditions expressed by other sentences which must be satisfied for this sentence to be true. In this way, Grice defines two kinds of *implicature*: *conversational* and *conventional*. The first one is stricter and in some sense, inferential. The second is less precise, more intuitive and arbitrary. Conversational implicature is explained by Grice in the following way:

A man who, by (in, when) saying (or making as if to say) that p has implicated that q , may be said to have conversationally implicated that q , PROVIDED THAT (1) he is to be presumed to be observing the conversational maxims¹⁰, or at least the cooperative principle (abbr. CP); (2) the supposition that he is aware that, or thinks that, q is required in order to make his saying or making as if to say p (or doing so in THOSE terms) consistent with this presumption; and (3) the speaker thinks (and would expect the hearer to think that the speaker thinks) that it is within the competence of the hearer to work out, or grasp intuitively, that the supposition mentioned in (2) IS required.

[Grice, 1977, p. 30]

The second kind of implicature is presented by Grice in opposition to the first one:

The presence of a conversational implicature must be capable of being worked out; for even if it can in fact be intuitively grasped, unless the intuition is replaceable by an argument, the implicature (if present at all) will not count as a CONVERSATIONAL implicature; it will be a CONVENTIONAL implicature.

[Grice, 1977, p. 31]

Grice explains also how to understand the working out of a conversational implicature:

A general pattern for the working out of a conversational implicature might be given as follows: 'He has said that p ; there is no reason to suppose that he is not observing the maxims, or at least the CP; he could not be doing this unless he thought that q ; he knows (and knows that I know that he knows) that I can see that the supposition that he thinks that q IS required; he has done nothing to stop me thinking that q ; he intends me to think, or is at least willing to allow me to think, that q ; and so he has implicated that q '.

[Grice, 1977, p. 31]

According to Grice, both implicatures are not precisely determined by the words uttered. The context of the utterance is essential for its

¹⁰ Maxims of *Quantity, Quality, Relation, and Manner*.

accurate, current and unique understanding, and as a consequence, for successful communication. This explains why it was not Grice’s intention to provide a formal definition of an implicature. He did not accept that there was a single and unique implicature for a given utterance, be it conversational or conventional. It therefore follows that to propose a single formal structure shared by all implicatures would be to completely misunderstand Grice’s thinking. It would be also a mistake to suppose that each utterance has a unique formal structure of implicature. One possible way of accommodating all possible understandings of the implicature of a given sentence p is to consider them as a maximally broad class of sentences q , where the sentence “ p says q ” would be accepted by anyone who hears an utterance of p . Thus, every formalization of implicatures should define as general an inference as possible. That is why no formal structure of implicature can be accepted. However, it seems that a construction based on the content implication connective proposed below satisfies that postulate, because acceptance of the sentence $p : q$ neither requires any prior acceptance of any connection between the forms of sentences p and q nor the simultaneous presence of shared words in these both sentences. Sometimes we are willing to accept the sentence “the inscription ‘Beware!’¹¹ says that you should not pass the border” (i.e., $p : \neg q$) and sometimes “the inscription ‘Beware!’ says that you should smile and just pass the border” (i.e., $p : r \wedge q$) — everything depends on the context of utterance.

The examples below should shed some light on the nature of content implication disclosing advantages and limitations of the new connective.

Example 1. “(This suit is elegant) : (You can go to the opera in this suit)” — $p : q$.

Example 2. “(This suit is elegant) : (You can go to the theatre in this suit)” — $p : s$.

Example 3. “((This suit is elegant) : (You can go to the opera in this suit)) : (You should be smartly dressed going to the opera)” — $(p : q) : r$.

The difference between examples 1 and 2 seems to be analogous to that between conversational and conventional implicature. In the first example, the relation between the contents of sentences p and q seems to be more objective than in the second example. Indeed, it is still a

¹¹ Or ‘you are entering at your own risk’.

commonly held view that people attending the opera should be elegantly dressed. In the case of the theatre, convenience may be more important than elegance. This is why the sentence $p : q$ from the Example 1 illustrates conversational implicature, whereas $p : s$ from the second example illustrates conventional implicature. The third example shows that a sentence of the form $\alpha : \beta$ can be an argument of other sentence of the same form. In brief, all content relations should be expressible in a language with the content-implication connective.

The new connective extends the expressiveness of the language not only in the case of implicatures. It can help us express relations between the contents of sentences that hitherto it has not been possible to express. For example, it is possible to capture relations involved in metaphors. For this purpose, in order to avoid any misunderstanding, it is necessary to extend the set of propositional variables $\{p, s, q, \dots\}$ by p', s', q', \dots , which will be reserved for the metaphorical understandings of p, s, q, \dots .

Example 4. “(It costs an arm and a leg) : (It is extremely expensive)” — $t' : r$

although the first sentence in this example can be understood literally in the following way:

Example 5. “([A moment of inattention] cost him an arm and a leg) : (He lost an arm and a leg)” — $t : u$

Examples 4 and 5 show why the correct understanding of the content implication $\alpha : \beta$ cannot depend on any relations between the forms of α and β , i.e., it should be independent of the structures of both sentences and the words appearing in them.

Of course, content implication also has some pretty strong limitations. It might seem that the new connective results in universality of the language extended by it. The expression “ $p : q$ ” looks like a sentence saying what p says about q . However, it is not true. The fact that every pair of sentences can be connected by the new connective whose intended speaking is “... says, that ...” might suggest that a construction of sentences of successive metalanguages should be possible. This is why it is necessary to underline that relations between a sentence’s contents are the only ones which can be expressed by “:”. It means, for example, that it is impossible to express the fact that some sentence possess a property, e.g., the property of being or not being a sentence

of some language. The only possible way is to use atomic sentences on both sides of the connective:

Example 6. “(These boots are smart): (‘These boots are smart’ is an English sentence)”.

It means that no sentence cannot be named by using the content implication connective. Thus, a formal language extended with the connective of content implication remains a subjective language only. Relations between sentences belonging to various levels of the language can be expressed in a trivial way only by using atomic sentences, i.e., sentences without connectives including the connective of content implication.

4. Classical propositional logic with the content implication

Classical propositional logic with content implication had been already presented in [Łukowski, 1997, 2006, 2011], but under another and not well-chosen name. Let $\mathcal{L}_{\text{CCL}} = \langle \text{For}_{\text{CCL}}, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, : \rangle$, the Contentual Propositional Language, be a language for classical propositional logic extended by the content implication connective “:”. Contentual Classical Logic (CCL) is given by an axiom set for classical propositional logic and the following formulas:

$$((\alpha : \beta) \wedge (\beta : \delta)) \rightarrow (\alpha : \delta), \quad (\text{A1})$$

$$(\alpha \wedge \beta) : \alpha, \quad (\text{A2})$$

$$(\alpha \wedge \beta) : (\beta \wedge \alpha), \quad (\text{A3})$$

$$\alpha : (\alpha \wedge \alpha), \quad (\text{A4})$$

$$((\alpha : \beta) \wedge (\beta : \alpha)) \rightarrow ((\neg\alpha : \neg\beta) \wedge (\neg\beta : \neg\alpha)), \quad (\text{A5})$$

$$((\alpha : \beta) \wedge (\beta : \alpha) \wedge (\delta : \gamma) \wedge (\gamma : \delta)) \rightarrow \\ ((\alpha \S \delta) : (\beta \S \gamma)) \wedge ((\beta \S \gamma) : (\alpha \S \delta)), \quad (\text{A6})$$

$$((\alpha : \beta) \wedge (\delta : \gamma)) \rightarrow ((\alpha \circ \delta) : (\beta \circ \gamma)), \quad (\text{A7})$$

$$(\alpha : \beta) \rightarrow (\alpha \rightarrow \beta), \quad (\text{A8})$$

for $\S \in \{\rightarrow, \leftrightarrow, :\}$ and $\circ \in \{\wedge, \vee\}$. Moreover, *Modus Ponens* (MP):

$$\text{if } \alpha \rightarrow \beta \text{ and } \alpha \text{ are theses, then } \beta \text{ is thesis} \quad (\text{MP})$$

is the only inference rule of CCL. One of the most important CCL-theses is $\alpha : \alpha$, a trivial formula easily inferred by (A1), (A2) and (A4). Despite

its trivial form, this formula is not trivial because it expresses Buridan's famous thesis. It prevents us from forgetting that we "breathe" the logic of truth and not falsehood.

A semantics adequate for CCL is the class of all so-called CCL-models, i.e., matrices $\mathfrak{M} = \langle \mathfrak{A}, D \rangle$ such that $\mathfrak{A} = \langle A, -, \cap, \cup, \Rightarrow, \Leftrightarrow, \supset \rangle$ is an algebra similar to \mathcal{L}_{CCL} , D is a nonempty subset of A and for all $a, b \in A$,

1. $a = a \cap a$,
2. $a \cap b = b \cap a$,
3. $a \cap (b \cap c) = (a \cap b) \cap c$,
4. $\neg a \in D$ iff $a \notin D$,
5. $a \cap b \in D$ iff $a \in D$ and $b \in D$,
6. $a \cup b \in D$ iff $a \in D$ or $b \in D$,
7. $a \Rightarrow b \in D$ iff $a \notin D$ or $b \in D$,
8. $a \supset b \in D$ iff $a = b \cap c$, for some $c \in A$.

Semantic inference is defined in a standard way:

$$\begin{aligned} \Pi \models_{\text{CCL}} \alpha \text{ iff for arbitrary CCL-model } \mathfrak{M} = \langle A, D \rangle \text{ and} \\ v \in \text{Hom}(\mathcal{L}_{\text{CCL}}, \mathfrak{A}), v(\alpha) \in D, \text{ if for any } \beta \in \Pi, v(\beta) \in D. \end{aligned}$$

A proof of the completeness theorem is presented in [Łukowski, 1997].

As was previously assumed, the meaning of content implication refers directly to the connective of conjunction. The truthfulness of the sentence $p : q$ means that p is a conjunction, in which q is one of its conjuncts, and so the content of the sentence q is a part of the content of p . That is why, as had been assumed, a content implication is a CCL-tautology only if it is either in the form " $\alpha : \alpha$ " or " $(\alpha_1 \wedge \dots \wedge \alpha_n) : \alpha_i$ ", for $i \in \{1, \dots, n\}$.

Moreover, the logical value of the sentence $p : q$ is independent from logical values of p and q with one understandable exception — $p : q$ cannot be true if p is true and q is false. With this exception noted, relations between the contents of sentences are independent of the logical values of sentences. For example, the content of the sentence "Dwarfs exist" is included in the content of the sentence "Last night, two dwarfs played with a butterfly before the sushi restaurant in Łódź". Thus, although both sentences are obviously false, the sentence "(Last night, two dwarfs played with the butterfly before the sushi restaurant in Łódź) : (Dwarfs exist)" is true.

5. A solution of the liar problem

A solution of the liar paradox, first presented in [Łukowski, 1997], is quite simple in a propositional language extended with the content implication connective “:”.¹² It is likely that such an extension frees any propositional language and logic from the well-known problem with the liar sentence in a way consistent with the approach by Jean Buridan as well as Jodocus Clichtoveus.

Let us assume that L is the liar sentence. In a language with the content implication connective, its sense is expressed by the formula:

$$L : \neg L$$

The sentence L says no more and no less than that L is false. Thus, contrary to the distributive approach, in CCL the sense of the liar sentence is not identified with “the sentence L is equivalent to its own negation”. Formally, the sense of L is not identified with the formula $L \leftrightarrow \neg L$.

Before giving the proper solution of the liar antinomy let us first make a simple remark. Since $L : \neg L$ and $L : L$, thus $L : (L \wedge \neg L)$, by (A1), (A2) and (A4), and so, L is such a conjunction

$$L = (L) \wedge (\neg L) \wedge (L \wedge \neg L) \wedge \dots$$

where one of its conjuncts will assuredly be false. Unfortunately, no infinite formula belongs to the CCL-language. Fortunately, truthfulness of

$$L : (L) \wedge (\neg L) \wedge (L \wedge \neg L)$$

is enough to recognize the falseness of the sentence L .

The solution of the liar antinomy now proceeds quickly. Let $\mathfrak{M} = \langle \mathfrak{A}, D \rangle$ be a CCL-model, $v \in \text{Hom}(\mathcal{L}_{\text{CCL}}, \mathfrak{A})$ be a homomorphism such, that $v(L) = a_0 \in A$. Since L is a liar sentence, the formula $L : \neg L$ is satisfied in \mathfrak{M} by v . Thus,

$$a_0 \supset -a_0 \in D.$$

By condition 8 of the CCL-model, $a_0 = -a_0 \cap c$, for some $c \in A$. There are two cases: either $a_0 \in D$ or $a_0 \notin D$.

Let $a_0 \in D$. Then, $-a_0 \cap c \in D$. By condition 5, $-a_0 \in D$, and so by 4, $a_0 \notin D$ — a contradiction. Thus, sentence L cannot be satisfied in \mathfrak{M} by v .

¹² This solution first appeared in [Łukowski, 1997, 2006, 2011].

Let $a_0 \notin D$. By condition 4, $-a_0 \in D$, and so $a_0 = -a_0 \cap c \notin D$, for some $c \in A$. Thus, $a_0 \notin D$ or $c \notin D$. Since $a_0 \in D$, so $c \notin D$. The proof is completed by the discovery of a false sentence z , that $L : z$. But the existence of such a sentence has already been proven: $L \wedge \neg L$. Thus, let $c = v(z) = v(L \wedge \neg L) \notin D$ – no contradiction.

Summarizing, L is a false sentence as it is not true. Although L says that L is false, L is not true because it says much more, namely, that L is true and false at the same time.

6. Suszko's sentential calculus with identity

The formal “collective” approach to sentences has been a feature of the literature for many years. As long ago as 1972, Bloom and Suszko published their famous “Investigations into the sentential calculus with identity” [Bloom and Suszko, 1972]. Let us recall that the connective of propositional identity is defined by the following axioms:

$$\alpha \equiv \alpha, \quad (\text{A1}_{\equiv})$$

$$(\alpha \equiv \beta) \rightarrow (\neg\alpha \equiv \neg\beta), \quad (\text{A2}_{\equiv})$$

$$((\alpha \equiv \beta) \wedge (\gamma \equiv \delta)) \rightarrow ((\alpha \S \gamma) \equiv (\beta \S \delta)), \quad (\text{A3}_{\equiv})$$

$$(\alpha \equiv \beta) \rightarrow (\alpha \rightarrow \beta), \quad (\text{A4}_{\equiv})$$

for $\S \in \{\wedge, \vee, \rightarrow, \leftrightarrow, \equiv\}$.

An axiomatic extension of the Classical Propositional Calculus by formulas (A1_≡)–(A4_≡) is known as the *Sentential Calculus with Identity* (SCI) [see, e.g., Suszko, 1975]. Within SCI the liar sentence L should be understood as a sentence such that $L \equiv \neg L$ is true. However, by (A1_≡), $L \equiv L$, and so, by (A3_≡), $L \wedge L \equiv L \wedge \neg L$. Since $L \equiv L \wedge L$ is not a SCI-tautology, we cannot receive $L \equiv L \wedge \neg L$. However, an understanding the semantic correlates of sentences as their senses justifies such an extension of SCI, in which the formula $L \equiv L \wedge L$ would be a tautology. It is especially justified if we accept an appropriate strengthening of Suszko's identity. Indeed, extending Suszko's connective by three axioms:

$$(\alpha \wedge \alpha) \equiv \alpha,$$

$$(\alpha \wedge \beta) \equiv (\beta \wedge \alpha),$$

$$(\alpha \wedge \beta) \wedge \gamma \equiv \alpha \wedge (\beta \wedge \gamma),$$

gives an identity “ \equiv_c ” such that

$$((\alpha : \beta) \wedge (\beta : \alpha)) \leftrightarrow (\alpha \equiv_c \beta)$$

This means that the liar sentence can be understood as $L \equiv_c L \wedge \neg L$.

7. A “collective” definition of truth

The absence of the liar antinomy means that a definition of the should be possible for the collective approach to sentences, i.e., for the language with the connective of content implication. Let us extend the CCL-language by the logical constant “ $\mathbf{1}$ ”. Let \mathbf{I} be the semantic interpretation of $\mathbf{1}$, such that \mathbf{I} is an intersection of all designated elements of the CCL^1 -model $\mathfrak{M}^{\mathbf{I}} = \langle \mathfrak{A}^{\mathbf{I}}, D \rangle$, with the algebra $\mathfrak{A}^{\mathbf{I}} = \langle A^{\mathbf{I}}, -, \cap, \cup, \Rightarrow, \Leftrightarrow, \supset, \mathbf{I} \rangle$ similar to the CCL^1 -language $\mathcal{L}^1 = \langle \text{For}_{\text{CCL}^1}, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, :, \mathbf{1} \rangle$. Thus,

$$\mathbf{I} := \Pi\{a : a \in D\}.$$

Therefore, $\mathbf{1}$ can be understood as a conjunction of all true sentences of the given theory including all logical truths. Now, a definition of the truthfulness of the sentence α is following,

Definition of a sentence:

$$\alpha \leftrightarrow (\mathbf{1} : \alpha)$$

This definition can be presented in more traditional way, i.e., by using an operator T :

*Definition of a sentence**

$$T\alpha \leftrightarrow (\mathbf{1} : \alpha).$$

It is not unexpected that “ $\mathbf{1}$ ” represents truth—but not only logical truth. The last equivalence explains when a sentence α is by us treated as true. Our mental picture of reality, here represented by some CCL^1 -model, explains which beliefs for us are true and which are false. All the beliefs we accept as true, expressed by appropriate sentences, create the picture of the world we believe in. Thus, the conjunction of all the sentences satisfied in the CCL^1 -model represents the truth we believe in. In such a way, our CCL^1 -model defines the truth we believe in. It means that a sentence is true for us, if it is a part of this truth which is presented by our CCL^1 -model. Fortunately, strategies of acceptance of sentences composing the truth is not determined by our construction, and so, the collective definition of truth does not depend on the experience.

The definition of truth presented above ought to be compared with other, well-known philosophical concepts of truth. However, for the moment, let us just notice that this understanding of truth matches our everyday thinking, including scientific practice. Even belief in a single and changeless Truth does not contradict this definition, which expresses our unceasing search for the Truth. Our permanently improved, and so changing understanding of the world is represented by respectively changing truth written with small “t”, in the CCL¹-model represented by “1”. It is our understanding of the truth, written with a lowercase letter, that constantly follows the one and unchanging Truth, written with a capital letter.

8. Conclusion

There is no tradition of combining distributive and collective set theories into one theory. Indeed, it may not be possible. It seems that a set theory must be either distributive or collective. In other words, sets must be understood either as normal or not normal. A quite different situation is probable in the case of language, be it a formal language or a natural language. Sentences which have to be understood in a distributive manner as well as sentences from treated in a collective way can consistently belong to the one and the same language. In the paper this fact has been shown with the example of two formal languages: our CCL-language and Suszko’s SCI-language. This is even more obvious when one looks at natural language. It seems that such hybrid, i.e., distributive–collective, languages are closer to natural language.

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