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ON DEFINABILITY OF CONNECTIVES AND MODAL LOGICS OVER FDE

Abstract. The present paper studies two approaches to the expressiveness of propositional modal logics based on first-degree entailment logic, **FDE**. We first consider the basic **FDE**-based modal logic **BK** and certain systems in its vicinity, and then turn to some **FDE**-based modal logics in a richer vocabulary, including modal bilattice logic, **MBL**. On the one hand, model-theoretic proofs of the definability of connectives along the lines of [7] and [17] are given for various **FDE**-based modal logics. On the other hand, building on [10], expressibility is considered in terms of mutual faithful embeddability of one logic into another logic. A distinction is drawn between definitional equivalence, which is defined with respect to a pair of structural translations between two languages, and weak definitional equivalence, which is defined with respect to a weaker notion of translations. Moreover, the definitional equivalence of some **FDE**-based modal logics is proven, especially the definitional equivalence of **MBL** and a conservative extension of the logic $\mathbf{BK}^{\Box} \times \mathbf{BK}^{\Box}$, which underlines the central role played by **BK** among **FDE**-based modal logics.

Keywords: definability of connectives; first-degree entailment logic; modal logic; modal bilattice logic; functional completeness; translations between logics; weak definitional equivalence; definitional equivalence

Introduction

The present paper studies two approaches to the expressiveness of propositional modal logics based on first-degree entailment logic, **FDE**. To begin with, we consider the basic **FDE**-based modal logic **BK** and certain fragments of it that make use of different implications. Then we turn

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to some recently studied **FDE**-based modal logics in a richer vocabulary, including modal bilattice logic, **MBL**, [12].

In the first approach, expressiveness is understood in terms of the definability of logical connectives. A set of connectives is expressively complete if from that inventory of connectives every semantically possible connective can be explicitly defined. For many-valued logics, including two-valued classical logic, expressive completeness means functional completeness. Given a many-valued logic and a finite set of n-ary truth-functions, if every possible n-ary truth-function of the logic in question can be obtained by finite compositions of members of the given set of truth-functions, this set is called functionally complete. In particular, the search space for the functional completeness result is well defined, namely the class of all possible n-ary truth-functions. For a relational semantics as for **FDE**-based modal logics, however, it is not clear what the search space should be, since on the one hand the verification (support of truth) conditions and the falsification (support of falsity) conditions are given in terms of metalogical verification and falsification clauses. On the other hand, the definitions of the modal operators rely on an accessibility relation between worlds, which can not be expressed in terms of many-valued truth-functions. Our strategy for obtaining results about the definable connectives is therefore somewhat different from that for many-valued logics. Nevertheless, in conformance with [7], we will speak of functional completeness in our consideration of the class of definable connectives for various **FDE**-based modal logics.[1] This development follows the lines of [7] and [17], where results about the class of definable connectives were given with respect to intuitionistic logic and various constructive modal logics with strong negation. We will first restrict the class of expressible metalogical verification and falsification conditions, and then show that within the so defined class, all metalogically expressible verification and falsification conditions can be expressed by object language formulas of the respective logics.

The second approach considers expressibility in terms of mutual

[1] There are several other papers that investigate some notion of functional completeness. However, despite its title, the paper [3] is not about the definability of logical operations in systems of modal logic, but rather about embedding a deductive metatheory into the object language. In [15, 16] a space of possible connectives is given neither by a class of truth-functions, nor by a class of verification and falsification conditions, but instead by schemata for left and right introduction rules in sequent calculi.

faithful embeddability of one logic into another logic. Here, a distinction is drawn between definitional equivalence, which is defined with respect to a pair of structural translations between two languages, and weak definitional equivalence, which is defined with respect to a weaker notion of translations, namely with respect to weak structural translations. Sufficient conditions will be stated under which weak definitional equivalence implies definitional equivalence, and it will be observed that the presence of the constants b and n for the glutty and the gappy truth value, respectively, of **FDE** may lead to definitional equivalence. In particular, the logics $\mathbf{BK}^{\Box+\Box-}$, $\mathbf{BK}^{\mathsf{FS}}_{bl}$, and **MBL** turn out to be definitionally equivalent. In order to stretch the results about the definability of logical connectives to these logics, we will extend the metalogical vocabulary for defining verification and falsification conditions to contain metalanguage counterparts of b and n as well as operations standing for the lattice meet and lattice join with respect to the information order in logical bilattices, cf. [1].

The paper is organized as follows. In Section 1 we will recall the semantical definitions of the systems $\mathbf{K_{FDE}}$, **KN4**, $\mathbf{BK}^{\Box-}$, $\mathbf{BK}^{\Box}$, **BK**, and $\mathbf{BK}^{\Box+\Box-}$, $\mathbf{BK}^{\mathsf{FS}}_{bl}$, **MBL**. In Section 2 we will explore the classes of definable logical connectives for $\mathbf{K_{FDE}}$, **KN4**, $\mathbf{BK}^{\Box-}$, $\mathbf{BK}^{\Box}$ and **BK**. In Section 3 we will proof definitional equivalence for some modal extensions of **FDE** and in Section 4 we will investigate the classes of definable logical connectives for $\mathbf{BK}^{\Box+\Box-}$, $\mathbf{BK}^{\mathsf{FS}}_{bl}$ and **MBL**. Finally, Section 5 contains some concluding remarks.

1. FDE-based modal logics

Almost all logics considered in this section were studied in [10], and although their presentation here will be self-contained, we refer the interested reader to that paper for further information, including presentations of some axiomatizations and tableau calculi.

In order to semantically define the respective **FDE**-based modal logic, the metalogical language is a two-sorted first-order language containing:

- all formulas in the language of L as the first sort of individual variables, where $\mathsf{L} \in \{\mathbf{K_{FDE}}, \mathbf{KN4}, \mathbf{BK}^{\Box-}, \mathbf{BK}^{\Box}, \mathbf{BK}, \mathbf{BK}^{\Box+\Box-}, \mathbf{BK}^{\mathsf{FS}}_{bl}, \mathbf{MBL}\}$,
- a non-empty denumerable set V of information state variables as the second sort of variables,
- the classical connectives $⩕$, $⩖$, $\neg$, $\twoheadrightarrow$,

- the classical quantifiers $\mathbin{\rotatebox[origin=c]{180}{A}}$ and $\exists$,
- the binary predicate symbols $\Vdash^+$, $\Vdash^-$, and R.

The metalanguage is then defined as follows:

- state variables: $w \in V$,
- object language formula variables: A,
- atomic formulas of the metalanguage: α,
- formulas of the metalanguage: φ,
- $\alpha ::= w \Vdash^+ A \mid w \Vdash^- A \mid wRw$,
- $\varphi ::= \alpha \mid \neg\varphi \mid \varphi \curlywedge \varphi \mid \varphi \curlyvee \varphi \mid \varphi \twoheadrightarrow \varphi \mid \forall\,\varphi \mid \exists\,\varphi$.

Bi-implication, $\twoheadleftarrow\!\twoheadrightarrow$, is defined as usual.

1.1. Semantics for $\mathbf{K_{FDE}}$, KN4, $\mathbf{BK}^{\Box-}$, $\mathbf{BK}^{\Box}$, and BK

The languages $\mathcal{L}_{\mathbf{K_{FDE}}} = \{\vee, \wedge, \sim, \Box\}$, $\mathcal{L}_{\mathbf{KN4}} = \{\vee, \wedge, \Rightarrow, \sim, \Box\}$, $\mathcal{L}_{\mathbf{BK}^{\Box-}} = \{\vee, \wedge, \rightarrow, \sim, \Box\}$, $\mathcal{L}_{\mathbf{BK}^{\Box}} = \{\vee, \wedge, \rightarrow, \sim, \Box, \bot\}$ and $\mathcal{L}_{BK} = \{\vee, \wedge, \rightarrow, \sim, \Box, \Diamond, \bot\}$ are based on a non-empty countable set of atomic propositions Prop. We denote by $\mathsf{Form}(\mathsf{L})$, where L stands for the respective logic, the set of formulas defined as usual, formulas by A, B, C, etc., and sets of formulas by Γ, Δ, Σ, etc.

An L-model is a tuple $\mathcal{M} = \langle W, R, v^+, v^- \rangle$, where W is a non-empty set of information states (possible worlds), $R \subseteq W \times W$ is an accessibility relation on W, and v^+ and v^- are functions $v^+, v^- \colon \mathsf{Prop} \to 2^W$. We now define verification and falsification relations $\Vdash^+$ and $\Vdash^-$ between worlds and formulas in a model $\mathcal{M}$ as follows:[2]

$$
\begin{aligned}
w \Vdash^+ p &\text{ iff } w \in v^+(p);\\
w \Vdash^- p &\text{ iff } w \in v^-(p);\\
w \Vdash^+ A \wedge B &\text{ iff } (w \Vdash^+ A \curlywedge w \Vdash^+ B);\\
w \Vdash^- A \wedge B &\text{ iff } (w \Vdash^- A \curlyvee w \Vdash^- B);\\
w \Vdash^+ A \vee B &\text{ iff } (w \Vdash^+ A \curlyvee w \Vdash^+ B);\\
w \Vdash^- A \vee B &\text{ iff } (w \Vdash^- A \curlywedge w \Vdash^- B);\\
w \Vdash^+ {\sim}A &\text{ iff } w \Vdash^- A;\\
w \Vdash^- {\sim}A &\text{ iff } w \Vdash^+ A;\\
w \Vdash^+ A \rightarrow B &\text{ iff } (w \Vdash^+ A \twoheadrightarrow w \Vdash^+ B);\\
w \Vdash^- A \rightarrow B &\text{ iff } (w \Vdash^+ A \curlywedge w \Vdash^- B);
\end{aligned}
$$

[2] Since the metalanguage is classical, all classical equivalences hold in the metalanguage.

$w \Vdash^+ A \Rightarrow B$ iff $((w \Vdash^+ A \twoheadrightarrow w \Vdash^+ B) \mathbin{\&\!\!\&} (w \Vdash^- B \twoheadrightarrow w \Vdash^- A))$;
$w \Vdash^- A \Rightarrow B$ iff $(w \Vdash^+ A \mathbin{\&\!\!\&} w \Vdash^- B)$;
$\neg(w \Vdash^+ \bot) \qquad w \Vdash^- \bot$;
$w \Vdash^+ \Box A$ iff $\forall u(wRu \twoheadrightarrow u \Vdash^+ A)$;
$w \Vdash^- \Box A$ iff $\exists u(wRu \mathbin{\&\!\!\&} u \Vdash^- A)$;
$w \Vdash^+ \Diamond A$ iff $\exists u(wRu \mathbin{\&\!\!\&} u \Vdash^+ A)$;
$w \Vdash^- \Diamond A$ iff $\forall u(wRu \twoheadrightarrow u \Vdash^- A)$.

We say that a formula A is true at world w in an L-model $\mathcal{M}$ iff $w \Vdash^+ A$. We say that a formula A is true in an L-model, $\mathcal{M} \Vdash^+ A$, iff A is true at every world w from $\mathcal{M}$'s set of worlds. A formula A is L-valid, $\vDash_{\mathsf{L}} A$, iff A is true in every L-model. Finally, a set of formulas Γ entails a formula A in L, $\Gamma \vDash_{\mathsf{L}} A$, iff for all L-models $\mathcal{M}$ and worlds w, if $w \Vdash^+ B$, for all $B \in \Gamma$, then $w \Vdash^+ A$.

1.2. Semantics for $\mathbf{BK}^{\Box_+\Box_-}$, $\mathbf{BK}^{\mathsf{FS}}_{bl}$, and MBL

The study of modal bilattice logic, **MBL**, was motivated in [12] by obtaining a modal extension of the four-valued logic **FDE** characterized by possible worlds models with a four-valued accessibility relation between possible worlds. In [10] it was shown that **MBL** can be faithfully embedded into a logic characterized by the class of models of the fusion $\mathbf{BK}^{\Box} \times \mathbf{BK}^{\Box}$ of two copies of $\mathbf{BK}^{\Box}$, enriched with the binary connectives $\otimes$ and $\oplus$ and the constants $\top$, b, and n from bilattice logic. Moreover, it was shown that $\mathbf{BK}^{\Box} \times \mathbf{BK}^{\Box}$ and the Fischer Servi-style modal logic $\mathbf{BK}^{\mathsf{FS}}$ are weakly definitionally equivalent. We shall refer to $\mathbf{BK}^{\Box} \times \mathbf{BK}^{\Box}$ and $\mathbf{BK}^{\mathsf{FS}}$ over the set of bilattice connectives as $\mathbf{BK}^{\Box_+\Box_-}$ and $\mathbf{BK}^{\mathsf{FS}}_{bl}$, respectively.[3] It will be shown that $\mathbf{BK}^{\Box_+\Box_-}$, $\mathbf{BK}^{\mathsf{FS}}_{bl}$, and **MBL** are definitionally equivalent.

The languages $\mathcal{L}_{\mathbf{BK}^{\Box_+\Box_-}} = \{\vee, \wedge, \otimes, \oplus, \rightarrow, \sim, \Box_+, \Box_-, \bot, \top, \mathsf{b}, \mathsf{n}\}$, $\mathcal{L}_{\mathbf{BK}^{\mathsf{FS}}_{bl}} = \{\vee, \wedge, \otimes, \oplus, \rightarrow, \sim, \Box_{\mathsf{FS}}, \Diamond_{\mathsf{FS}}, \bot, \top, \mathsf{b}, \mathsf{n}\}$ and $\mathcal{L}_{\mathbf{MBL}} = \{\vee, \wedge, \otimes, \oplus, \rightarrow, \sim, \boxplus, \bot, \top, \mathsf{b}, \mathsf{n}\}$ are based, as above, on a non-empty countable set of atomic propositions Prop. Again, we denote by $\mathsf{Form}(\mathsf{L})$ the set of formulas defined as usual, formulas by A, B, C, etc., and sets of formulas by Γ, Δ, Σ, etc.

$\mathbf{BK}^{\Box_+\Box_-}$-, $\mathbf{BK}^{\mathsf{FS}}_{bl}$- and **MBL**-models are tuples $\mathcal{M} = \langle W, R_+, R_-, v^+, v^-\rangle$, where $R_+, R_- \subseteq W \times W$ are accessibility relations on W, and the

[3] Note that $\mathbf{BK}^{\mathsf{FS}}_{bl}$ ($\mathbf{BK}^{\Box_+\Box_-}$) is a conservative extension of $\mathbf{BK}^{\mathsf{FS}}$ ($\mathbf{BK}^{\Box} \times \mathbf{BK}^{\Box}$).

rest is analogously defined as above. For the connectives and constants not considered so far, we have the following verification and falsification conditions:

$$
\begin{array}{lll}
w \Vdash^{+} A \otimes B & \text{iff} & (w \Vdash^{+} A \mathrel{\&\!\!\&} w \Vdash^{+} B);\\
w \Vdash^{-} A \otimes B & \text{iff} & (w \Vdash^{-} A \mathrel{\&\!\!\&} w \Vdash^{-} B);\\
w \Vdash^{+} A \oplus B & \text{iff} & (w \Vdash^{+} A \Cup w \Vdash^{+} B);\\
w \Vdash^{-} A \oplus B & \text{iff} & (w \Vdash^{-} A \Cup w \Vdash^{-} B);\\
w \Vdash^{+} \top & & \neg(w \Vdash^{-} \top);\\
w \Vdash^{+} \mathsf{b} & & w \Vdash^{-} \mathsf{b};\\
\neg(w \Vdash^{+} \mathsf{n}) & & \neg(w \Vdash^{-} \mathsf{n});\\
w \Vdash^{+} \Box_{+} A & \text{iff} & \forall u(wR_{+}u \twoheadrightarrow u \Vdash^{+} A);\\
w \Vdash^{-} \Box_{+} A & \text{iff} & \exists u(wR_{+}u \mathrel{\&\!\!\&} u \Vdash^{-} A);\\
w \Vdash^{+} \Box_{-} A & \text{iff} & \forall u(wR_{-}u \twoheadrightarrow u \Vdash^{+} A);\\
w \Vdash^{-} \Box_{-} A & \text{iff} & \exists u(wR_{-}u \mathrel{\&\!\!\&} u \Vdash^{-} A);\\
w \Vdash^{+} \Box_{\mathsf{FS}} A & \text{iff} & \forall u(wR_{+}u \twoheadrightarrow u \Vdash^{+} A);\\
w \Vdash^{-} \Box_{\mathsf{FS}} A & \text{iff} & \exists u(wR_{+}u \mathrel{\&\!\!\&} u \Vdash^{-} A);\\
w \Vdash^{+} \Diamond_{\mathsf{FS}} A & \text{iff} & \exists u(wR_{+}u \mathrel{\&\!\!\&} u \Vdash^{+} A);\\
w \Vdash^{-} \Diamond_{\mathsf{FS}} A & \text{iff} & \forall u(wR_{-}u \twoheadrightarrow u \Vdash^{-} A);\\
w \Vdash^{+} \boxplus A & \text{iff} & \forall u(wR_{+}u \twoheadrightarrow u \Vdash^{+} A) \mathrel{\&\!\!\&} \forall u(wR_{-}u \twoheadrightarrow \neg(u \Vdash^{-} A));\\
w \Vdash^{-} \boxplus A & \text{iff} & \exists u(wR_{+}u \mathrel{\&\!\!\&} u \Vdash^{-} A).
\end{array}
$$

Truth at a world, truth in a model, validity and entailment are defined in analogy to the definitions of these notions in Subsection 1.1.

2. Logical Connectives for $\mathbf{K_{FDE}}$, KN4, $\mathbf{BK}^{\Box-}$, $\mathbf{BK}^{\Box}$, and BK

In this section we will follow [17], where in turn the results from [7] were extended. Note that at first we will use almost the same definitions as in [17]. We will make this clear by referring to the original definitions and proofs. As already remarked in Introduction, the results presented here do not establish functional completeness in the usual model theoretical sense, even though we will use this term, but are results about the class of connectives definable in the various modal extensions of **FDE**. As in [7] and [17] we begin by restricting the expressiveness of our metalanguage.

If a metalogical formula φ contains the free variables w, A_1, ..., A_n, we will write this as $\varphi(w, A_1, \ldots, A_n)$. Formulas wRu are called *relational atoms*. In formulas $\forall u(wRu \twoheadrightarrow \psi(u))$ and $\exists u(wRu \mathrel{\&\!\!\&} \psi(u))$ the relational atom wRu is said *to occur as a lower bound* on the quantifier

$\forall u$ ($\exists u$), and the quantifier $\forall u$ ($\exists u$) is said *to be bounded below* by wRu. Note that in the subformulas $\psi(u)$, u is the only free variable.

Definition 2.1 ([17, Definition 1, p. 471]). A formula φ is a *regular metalogical formula* iff

1. φ contains at most one free state variable,
2. all relational atoms occur as a lower bound on a quantifier,
3. every quantifier is bounded below by a relational atom.

Observation 2.1 (cf. Lemma 1 in [17, p. 473]). *Every regular metalogical formula φ is of such shape that every quantified subformula of φ has the shape $\forall u(wRu \twoheadrightarrow \theta(u))$ or $\exists u(wRu \mathbin{\&} \theta(u))$, where θ has no free state variables other than u.*

Definition 2.2 ([17, Definition 2, p. 471]). Let φ be a regular metalogical formula. The formula $\overline{\varphi}$ is inductively defined as follows:

$$\begin{array}{rcl}
\overline{\alpha} & = & \alpha,\ \alpha \text{ is an atom} \\
\overline{\neg wRu} & = & \neg wRu \\
\overline{\neg w \Vdash^{+} A} & = & w \Vdash^{-} A \\
\overline{\neg w \Vdash^{-} A} & = & w \Vdash^{+} A \\
\overline{\neg\neg\theta} & = & \overline{\theta} \\
\overline{\psi \circ \theta} & = & \overline{\psi} \circ \overline{\theta}, \circ \in \{\mathbin{\&}, \vee, \twoheadrightarrow\} \\
\overline{\neg(\psi \mathbin{\&} \theta)} & = & \overline{\neg\psi} \vee \overline{\neg\theta} \\
\overline{\neg(\psi \vee \theta)} & = & \overline{\neg\psi} \mathbin{\&} \overline{\neg\theta} \\
\overline{\neg(\psi \twoheadrightarrow \theta)} & = & \overline{\psi} \mathbin{\&} \overline{\neg\theta} \\
\overline{\forall u(wRu \twoheadrightarrow \theta(u))} & = & \forall u\overline{(wRu \twoheadrightarrow \theta(u))} \\
\overline{\exists w\psi} & = & \exists u\overline{(wRu \mathbin{\&} \theta(u))} \\
\overline{\neg\forall u(wRu \twoheadrightarrow \theta(u))} & = & \exists u\overline{\neg(wRu \twoheadrightarrow \theta(u))} \\
\overline{\neg\exists u(wRu \mathbin{\&} \theta(u))} & = & \forall u\overline{\neg(wRu \mathbin{\&} \theta(u))}
\end{array}$$

If φ is a regular metalogical formula, then $\overline{\varphi}$ is said to be an *L-regular metalogical formula* for $L \in \{\mathbf{KN4}, \mathbf{BK}^{\square-}, \mathbf{BK}^{\square}, \mathbf{BK}\}$.

Remark 2.1. The notion of $\mathbf{K_{FDE}}$-regular formulas requires another condition for $\twoheadrightarrow$ if ψ is not a relational atom:

$$\overline{\psi \twoheadrightarrow \theta} \;=\; \overline{\neg\psi} \vee \overline{\theta}$$

This additional condition restricts the set of $\mathbf{K_{FDE}}$-regular connectives since the classical metalogical implication cannot be expressed in the object language of $\mathbf{K_{FDE}}$. The condition also needs to be used in [17] to make the proof work there.

Definition 2.3 ([17, Definition 3, p. 472]). A definition of an n-place ($n \geqslant 1$) connective $\bigstar$ has the form: for every model $\mathcal{M} = \langle W, R, v^+, v^- \rangle$ and every $w \in W$,

$$w \Vdash^+ \bigstar(A_1, \ldots, A_n) \text{ iff } \overline{\psi}(w, (A_1, \ldots, A_n)) \text{ and } w \Vdash^- \bigstar(A_1, \ldots, A_n) \\ \text{iff } \overline{\neg\psi}(w, (A_1, \ldots, A_n)),$$

where $\overline{\psi}$ is an L-regular metalogical formula, by which $\bigstar$ is said to be defined.

Definition 2.4 ([17, Definition 5, p. 472]). The *degree of quantification* $d(\varphi)$ of a metalogical formula φ is inductively defined as follows:

- $d(\varphi) = 0$, if φ is quantifier-free;
- $d(\varphi) = \max(n_i + 1)$ for $1 \leqslant i \leqslant j$, if φ has j quantifiers, the i-th quantifier ranges over the subformula δ_i of φ, and n_i is the degree of quantification of δ_i.

Observation 2.2 (cf. Lemma 2 in [17, p. 474]). *Every L-regular metalogical formula φ is of such shape that every quantified subformula of φ has the form $\forall u(wRu \twoheadrightarrow \theta(u))$ or $\exists u(wRu \mathbin{\&} \theta(u))$, where θ has no free state variable other than u.*

Proof. The translation $\bar{\cdot}$ does not generate new free variables. ⊣

Observation 2.3 (cf. Corollary 1 in [17, p. 474]). *Let φ be an L-regular metalogical formula. For every subformula of φ of the shape $\forall u(wRu \twoheadrightarrow \theta(u))$ or $\exists u(wRu \mathbin{\&} \theta(u))$, θ is L-regular.*

Observation 2.4 (cf. Observation 2 in [17, p. 474]). *For each L-formula A, $\overline{\varphi}$ iff $w \Vdash^+ A$ just in case $\overline{\neg\varphi}$ iff $w \Vdash^- A$.*

Proof. The proof is by induction on A and follows the proof of Observation 2 in [17, p. 474] (neglecting the constructive implication). We only need to prove the observation for the operators $\bot, \to, \Rightarrow, \Box$, and $\Diamond$.

Suppose $A = A \to B$:[4]

$$\begin{aligned} \overline{\varphi} &\text{ iff } w \Vdash^+ A \to B \\ &\text{ iff } \overline{w \Vdash^+ A \twoheadrightarrow w \Vdash^+ B} \\ \therefore\ \overline{\neg\varphi} &\text{ iff } \overline{\neg(w \Vdash^+ A \twoheadrightarrow w \Vdash^+ B)} \\ &\text{ iff } \overline{w \Vdash^+ A} \mathbin{\&} \overline{\neg w \Vdash^+ B} \end{aligned}$$

[4] We will make use of "∴" to denote the classical consequence relation in the metalanguage

$$
\begin{aligned}
&\text{iff } w \Vdash^{+} A \Wedge w \Vdash^{-} B\\
&\text{iff } w \Vdash^{-} A \to B
\end{aligned}
$$

$$
\begin{aligned}
\overline{\neg\varphi}\ &\text{iff } w \Vdash^{-} A \to B\\
&\text{iff } \overline{w \Vdash^{+} A \Wedge w \Vdash^{-} B}\\
&\text{iff } \overline{\neg(\neg w \Vdash^{+} A \Vee \neg w \Vdash^{-} B)}\\
\therefore\ \overline{\varphi}\ &\text{iff } \overline{\neg w \Vdash^{+} A \Vee \neg w \Vdash^{-} B}\\
&\text{iff } \overline{w \Vdash^{+} A \twoheadrightarrow \neg w \Vdash^{-} B}\\
&\text{iff } w \Vdash^{+} A \twoheadrightarrow w \Vdash^{+} B\\
&\text{iff } w \Vdash^{+} A \to B
\end{aligned}
$$

Now, suppose $A = A \Rightarrow B$:

$$
\begin{aligned}
\overline{\varphi}\ &\text{iff } w \Vdash^{+} A \Rightarrow B\\
&\text{iff } \overline{(w \Vdash^{+} A \twoheadrightarrow w \Vdash^{+} B) \Wedge (w \Vdash^{-} B \twoheadrightarrow w \Vdash^{-} A)}\\
\therefore\ \overline{\neg\varphi}\ &\text{iff } \overline{\neg((w \Vdash^{+} A \twoheadrightarrow w \Vdash^{+} B) \Wedge (w \Vdash^{-} B \twoheadrightarrow w \Vdash^{-} A))}\\
&\text{iff } \overline{\neg(w \Vdash^{+} A \twoheadrightarrow w \Vdash^{+} B)} \Vee \overline{\neg(w \Vdash^{-} B \twoheadrightarrow w \Vdash^{-} A)}\\
&\text{iff } (\overline{w \Vdash^{+} A} \Wedge \overline{\neg w \Vdash^{+} B}) \Vee (\overline{w \Vdash^{-} B} \Wedge \overline{\neg w \Vdash^{-} A})\\
&\text{iff } (w \Vdash^{+} A \Wedge w \Vdash^{-} B) \Vee (w \Vdash^{-} B \Wedge w \Vdash^{+} A)\\
&\text{iff } w \Vdash^{+} A \Wedge w \Vdash^{-} B\\
&\text{iff } w \Vdash^{-} A \Rightarrow B
\end{aligned}
$$

$$
\begin{aligned}
\overline{\neg\varphi}\ &\text{iff } w \Vdash^{-} A \Rightarrow B\\
&\text{iff } \overline{w \Vdash^{+} A \Wedge w \Vdash^{-} B}\\
&\text{iff } \overline{(w \Vdash^{+} A \Wedge w \Vdash^{-} B) \Vee (w \Vdash^{+} A \Wedge w \Vdash^{-} B)}\\
&\text{iff } \overline{\neg(\neg(w \Vdash^{+} A \Wedge w \Vdash^{-} B) \Wedge \neg(w \Vdash^{+} A \Wedge w \Vdash^{-} B))}\\
\therefore\ \overline{\varphi}\ &\text{iff } \overline{\neg(w \Vdash^{+} A \Wedge w \Vdash^{-} B) \Wedge \neg(w \Vdash^{+} A \Wedge w \Vdash^{-} B)}\\
&\text{iff } \overline{(\neg w \Vdash^{+} A \Vee \neg w \Vdash^{-} B) \Wedge (\neg w \Vdash^{+} A \Vee \neg w \Vdash^{-} B)}\\
&\text{iff } \overline{(\neg w \Vdash^{+} A \Vee w \Vdash^{+} B)} \Wedge \overline{(w \Vdash^{-} A \Vee \neg w \Vdash^{-} B)}\\
&\text{iff } \overline{(w \Vdash^{+} A \twoheadrightarrow w \Vdash^{+} B)} \Wedge \overline{(w \Vdash^{-} B \twoheadrightarrow w \Vdash^{-} A)}\\
&\text{iff } w \Vdash^{+} A \Rightarrow B
\end{aligned}
$$

Now, suppose $A = \Box A$:

$$
\begin{aligned}
\overline{\varphi}\ &\text{iff } w \Vdash^{+} \Box A\\
&\text{iff } \overline{\forall u(wRu \twoheadrightarrow u \Vdash^{+} A)};
\end{aligned}
$$

$$
\begin{aligned}
\therefore \overline{\neg\varphi} \text{ iff } & \overline{\neg(\forall\!\!\!\forall\, u(wRu \twoheadrightarrow u \Vdash^{+} A)} \\
\text{iff } & \overline{\exists\!\!\!\exists\, u \neg(wRu \twoheadrightarrow u \Vdash^{+} A)} \\
\text{iff } & \overline{\exists\!\!\!\exists\, u(wRu \mathbin{\wedge\!\!\!\wedge} \neg u \Vdash^{+} A)} \\
\text{iff } & w \Vdash^{-} \Box A
\end{aligned}
$$

and vice versa. The proof for $\Diamond A$ is similar.

The proof for $A = \bot$ is a degenerate case, since $\neg(w \Vdash^{+} \bot)$ and $w \Vdash^{-} \bot$, but neither $\neg(w \Vdash^{-} \bot)$ nor $w \Vdash^{+} \bot$, hence the equivalence is trivially true. $\dashv$

Definition 2.5. A logical connective that can be defined by means of an L-regular metalogical formula is said to be an L-*regular connective.*

Observation 2.5. *The sets of connectives* $\{\vee, \wedge, \sim, \Box\}$, $\{\vee, \wedge, \sim, \Rightarrow, \Box\}$, $\{\vee, \wedge, \sim, \rightarrow, \Box\}$, $\{\vee, \wedge, \sim, \rightarrow, \bot, \Box\}$ *and* $\{\vee, \wedge, \sim, \rightarrow, \bot, \Box, \Diamond\}$ *are sets of* $\mathbf{K_{FDE}}$-, **KN4**-, $\mathbf{BK}^{\Box-}$-, $\mathbf{BK}^{\Box}$- *and* **BK**-*regular connectives, respectively.*

Theorem 2.6. *In the class of* L-*regular connectives the respective sets of connectives are functionally complete. In other words, if an* n-*ary* ($n \geqslant 1$) *connective* $\bigstar$ *is defined by means of an* L-*regular metalogical formula* $\overline{\varphi}$ *, then there is an* L-*formula* A *such that the following holds:* $\overline{\varphi} \twoheadleftarrow\!\!\!\twoheadrightarrow w \Vdash^{+} A$ *(and* $\overline{\neg\varphi} \twoheadleftarrow\!\!\!\twoheadrightarrow w \Vdash^{-} A$*).*

Proof. By induction on the degree of quantification of $\overline{\varphi}$. Let $\Vdash^{\pm}$ stand uniformly either for $+$ or for $-$.

Suppose $d(\overline{\varphi}) = 0$, then every atomic subformula of $\overline{\varphi}$ is a non-relational atom and every metalogical operator occurring in $\overline{\varphi}$ is either $\mathbin{\wedge\!\!\!\wedge}$, $\mathbin{\vee\!\!\!\vee}$, or $\twoheadrightarrow$. Let A be the result of replacing every occurrence of $\mathbin{\wedge\!\!\!\wedge}$ by $\wedge$, every occurrence of $\mathbin{\vee\!\!\!\vee}$ by $\vee$, every occurrence of $w \Vdash^{+} B$ by B, and every occurrence of $w \Vdash^{-} B$ by $\sim B$. In case of $\mathbf{BK}^{\Box-}$, $\mathbf{BK}^{\Box}$ and **BK** we furthermore replace every occurrence of $w \Vdash^{\pm} A \twoheadrightarrow w \Vdash^{\pm} B$ by $A^{*} \rightarrow B^{*}$ and in case of **KN4** every occurrence of $w \Vdash^{\pm} A \twoheadrightarrow w \Vdash^{\pm} B$ by $(A^{*} \Rightarrow (A^{*} \Rightarrow B^{*})) \vee B^{*}$, where A^{*} and B^{*} are the respective replacements of $w \Vdash^{\pm} A$ and $w \Vdash^{\pm} B$. In case of $\mathbf{BK}^{\Box}$ and **BK** we furthermore replace every occurrence of $w \Vdash^{+} \bot$ by $\sim\bot$ and $w \Vdash^{-} \bot$ by $\bot$. Then $\overline{\varphi} \twoheadleftarrow\!\!\!\twoheadrightarrow w \Vdash^{+} A$ (as well as $\overline{\neg\varphi} \twoheadleftarrow\!\!\!\twoheadrightarrow w \Vdash^{-} A$), cf. Observation 2.4.

Now, let $d(\overline{\varphi}) > 0$. Then there is a subformula of $\overline{\varphi}$ of the shape $\forall u(wRu \twoheadrightarrow \theta)$ or $\exists u(wRu \mathbin{\&} \theta)$, where θ is quantifier free. By the induction base case, $\theta \twoheadleftarrow\!\!\twoheadrightarrow A$ for some $\mathcal{L}$-formula A. Now we have:

$$\begin{array}{rll} \forall u(wRu \twoheadrightarrow \theta) & \text{iff} & w \Vdash^{+} \Box A \\ \exists u(wRu \mathbin{\&} \theta) & \text{iff} & w \Vdash^{+} {\sim}\Box{\sim}A \end{array}$$

Moreover, additionally for **BK** we have:

$$\exists u(wRu \mathbin{\&} \theta) \quad \text{iff} \quad w \Vdash^{+} \Diamond A$$

If the subformulas $\forall u(wRu \twoheadrightarrow \theta)$ and $\exists u(wRu \mathbin{\&} \theta)$ in $\overline{\varphi}$ are replaced by their respective equivalents, the result is an $\mathcal{L}$-regular metalogical formula which has one quantifier less than $\overline{\varphi}$, and hence the induction hypothesis can be used. $\dashv$

Remark 2.2. Since $\bigstar(A_1, \ldots, A_n)$ and the $\mathcal{L}$- formula A that is guaranteed to exist by Theorem 2.6 share their verification *and* their falsification conditions, the two formulas are provably strongly equivalent in every logic under consideration in which strong implication, $\Rightarrow$, is either primitive or definable. In [9, 10] it was shown that provable strong equivalence is a congruence relation in **BK** and in $\mathbf{BK}^{\Box}$, so that $\bigstar(A_1, \ldots, A_n)$ and A are mutually replaceable in deductive contexts, which justifies considering A as *definiens* of $\bigstar(A_1, \ldots, A_n)$. The same replacement property holds for $\mathbf{K_{FDE}}$, **KN4**, and $\mathbf{BK}^{\Box-}$.

Remark 2.3. The result presented above is in need of some clarification. At first glance, Theorem 2.6 seems to state the rather trivial result that everything that is expressible is expressible. However, the result not only shows what *is* expressible, but also what is *not*. The proof of functional completeness heavily relies 1) on the notion of $\mathcal{L}$-regular connectives and 2) on the requirement that verification and falsification can be expressed by one metalogical formula as in Observation 2.4. The verification and falsification conditions coming with the connectives for the languages considered so far are thus *symmetrical* in the sense that they are not really independent from each other. As for 1), in the class of $\mathcal{L}$-regular connectives it is, for example, not possible to distinguish between $\neg w \Vdash^{+} A$ and $w \Vdash^{-} A$ ($\neg w \Vdash^{-} A$ and $w \Vdash^{+} A$). This means that operators like $\circlearrowleft$ which would make **FDE** functionally complete in the usual sense, cf. [11], with the following verification and falsification conditions: $w \Vdash^{+} \circlearrowleft A$ iff $(w \Vdash^{+} A \mathbin{\&} \neg w \Vdash^{-} A) \vee (\neg w \Vdash^{+} A \mathbin{\&} \neg w \Vdash^{-} A)$ and $w \Vdash^{-} \circlearrowleft A$ iff $(w \Vdash^{+} A \mathbin{\&} \neg w \Vdash^{-} A) \vee (w \Vdash^{+} A \mathbin{\&} w \Vdash^{-} A)$ cannot be expressed by means of the given languages, as one perhaps desires. As for 2), the requirement that verification and falsification need to be expressed by one metalogical formula limits the class of languages to which the presented method can be applied to. Similar things can be said about the results in [17]. Therefore, in languages that contain connectives for which the verification and falsification conditions can be seen as *asymmetrical*, we need to alter our method to obtain results about the classes of definable connectives.

Remark 2.4. Moreover, Remark 2.3 leads to the conclusion that the presented method cannot be applied to $\mathbf{BK}^{\Box+\Box-}$, $\mathbf{BK}^{\mathsf{FS}}_{bl}$, and **MBL**, as the following observation shows.

Observation 2.7. *For some* $\mathbf{BK}^{\Box+\Box-}$-, $\mathbf{BK}^{\mathsf{FS}}_{bl}$-, *and* ***MBL***-*formula* A, $\overline{\varphi}$ *iff* $w \Vdash^{+} A$ *not just in case* $\overline{\neg\varphi}$ *iff* $w \Vdash^{-} A$.

Proof. Suppose $A = A \otimes B$:

$$\begin{array}{rcl}
\overline{\varphi} & \text{iff} & w \Vdash^{+} A \otimes B \\
 & \text{iff} & \overline{w \Vdash^{+} A \Wedge w \Vdash^{+} B} \\
\therefore \overline{\neg\varphi} & \text{iff} & \overline{\neg(w \Vdash^{+} A \Wedge w \Vdash^{+} B)} \\
 & \text{iff} & \overline{\neg w \Vdash^{+} A} \Vee \overline{\neg w \Vdash^{+} B} \\
 & \text{iff} & w \Vdash^{-} A \Vee w \Vdash^{-} B \\
 & \text{not iff} & w \Vdash^{-} A \otimes B
\end{array}$$

$$\begin{array}{rcl}
\overline{\neg\varphi} & \text{iff} & w \Vdash^{-} A \otimes B \\
 & \text{iff} & \overline{w \Vdash^{-} A \Wedge w \Vdash^{-} B} \\
 & \text{iff} & \overline{\neg(\neg w \Vdash^{-} A \Vee \neg w \Vdash^{-} B)} \\
\therefore \overline{\varphi} & \text{iff} & \overline{\neg w \Vdash^{-} A \Vee \neg w \Vdash^{-} B} \\
 & \text{iff} & \overline{\neg w \Vdash^{-} A} \Vee \overline{\neg w \Vdash^{-} B} \\
 & \text{iff} & w \Vdash^{+} A \Vee w \Vdash^{+} B \\
 & \text{not iff} & w \Vdash^{+} A \otimes B
\end{array}$$

The case for $\oplus$ is similar. $\dashv$

Remark 2.5. Even though the verification and falsification conditions for $\otimes$ and $\oplus$, on their own, are regular metalogical formulas in the sense of Definition 2.1 and even $\mathbf{BK}^{\Box+\Box-}$-, $\mathbf{BK}^{\mathsf{FS}}_{bl}$-, and **MBL**-regular, there is no $\mathbf{BK}^{\Box+\Box-}$-, no $\mathbf{BK}^{\mathsf{FS}}_{bl}$-, and no **MBL**-regular formula φ, such that $\overline{\varphi} \nLeftrightarrow w \Vdash^{+} A \circ B$ (and $\overline{\neg\varphi} \nLeftrightarrow w \Vdash^{-} A \circ B$), with $\circ \in \{\otimes, \oplus\}$.

In the following we will therefore adjust the presented method for obtaining definable connectives. In particular, we will first prove results about definitional equivalence between $\mathbf{BK}^{\Box+\Box-}$, $\mathbf{BK}^{\mathsf{FS}}_{bl}$, and **MBL** and then make some changes in our metalanguage in order to prove results about functional completeness.

3. Weak definitional equivalence in bilattice languages

In this section we will use the following abbreviations:

$$A \Leftrightarrow B := (A \Rightarrow B) \wedge (B \Rightarrow A),$$
$$A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A),$$
$$\neg A := A \rightarrow \bot.$$

The connectives $\Leftrightarrow$ and $\leftrightarrow$ are called strong equivalence and weak equivalence, respectively. The weak equivalence respects the verification of formulas, for any world w of any of the models under consideration we have:

$$w \Vdash^{+} A \leftrightarrow B \ \text{ iff } \ (w \Vdash^{+} A \ \text{ iff } \ w \Vdash^{+} B).$$

The strong equivalence connective is strong insofar as it respects also the falsification of formulas:

$$w \Vdash^{+} A \Leftrightarrow B \ \text{ iff } \ (w \Vdash^{+} A \ \text{ iff } \ w \Vdash^{+} B) \mathbin{\&\!\&} (w \Vdash^{-} A \ \text{ iff } \ w \Vdash^{-} B).$$

We call $\neg$ the *classical negation*. The verification and falsification clauses for the classical negation look as follows:

$$w \Vdash^{+} \neg A \ \text{ iff } \ \sim\!(w \Vdash^{+} A); \quad w \Vdash^{-} \neg A \ \text{ iff } \ w \Vdash^{+} A.$$

3.1. Weak structural translations and weak definitional equivalence

First we recall the notion of definitional equivalence of logics from [5], which was also considered as basic in [13] and [10].

Definition 3.1. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are propositional languages, then an arbitrary mapping $\theta\colon \mathsf{Form}(\mathcal{L}_1) \rightarrow \mathsf{Form}(\mathcal{L}_2)$ is called a *translation from* language $\mathcal{L}_1$ *to* $\mathcal{L}_2$. A translation θ is *structural* if it is induced by some mapping α, which sends every n-ary connective c of $\mathcal{L}_1$ to a formula $\alpha(c)(p_1, \dots, p_n)$ from $\mathsf{Form}(\mathcal{L}_2)$, in the following way:

$$\theta(p) = p, \ p \in \mathsf{Prop}; \quad \theta(c(A_1, \dots, A_n)) = \alpha(c)(p_1/\theta(A_1), \dots, p_n/\theta(A_n)),$$

where $A_1, \dots, A_n \in \mathsf{Form}(\mathcal{L}_1)$.

Notice that the definition of structural translation assumes that formulas of both languages $\mathcal{L}_1$ and $\mathcal{L}_2$ are constructed starting from the same set of propositional variables Prop. This is not an essential restriction for our goals. Notice also that the condition on θ to preserve propositional variables is present also in [8].

In the following we assume that the notion of a logic has been defined such that a logic L determines a Tarskian consequence relation $\vdash_{\mathsf{L}}$ between sets of formulas and single formulas. For all logics defined in Section 2, we mean by $\vdash_{\mathsf{L}}$ the local consequence relation $\models_{\mathsf{L}}$ defined semantically at the end of Subsection 2.2. Moreover, whereas classical negation satisfies contraposition (if $A \models_{\mathsf{L}} B$, then $\neg B \models_{\mathsf{L}} \neg A$), it is characteristic of strong negation, that it does not satisfy contraposition in general, which may lead to failure of the replacement rule for provably weakly equivalent formulas.

Definition 3.2. (cf. [5, 13]) Let L_i be a logic in the propositional language $\mathcal{L}_i$, $i = 1, 2$.[5] Let $\theta\colon \mathsf{Form}(\mathcal{L}_1) \to \mathsf{Form}(\mathcal{L}_2)$ and $\rho\colon \mathsf{Form}(\mathcal{L}_2) \to \mathsf{Form}(\mathcal{L}_1)$ be two structural translations. We say that L_1 and L_2 are *definitionally equivalent* (*d-equivalent*) *via* translations θ and ρ if the following conditions hold:

1. For $\Gamma \cup \{A\} \subseteq \mathsf{Form}(\mathcal{L}_1)$, the relation $\Gamma \vdash_{\mathsf{L}_1} A$ implies $\theta(\Gamma) \vdash_{\mathsf{L}_2} \theta(A)$, where $\theta(\Gamma) := \{\theta(A) \mid A \in \Gamma\}$.
2. For $\Gamma \cup \{A\} \subseteq \mathsf{Form}(\mathcal{L}_2)$, the relation $\Gamma \vdash_{\mathsf{L}_2} A$ implies $\rho(\Gamma) \vdash_{\mathsf{L}_1} \rho(A)$.
3. For every $A \in \mathsf{Form}(\mathcal{L}_1)$ and $B \in \mathsf{Form}(\mathcal{L}_2)$,
$$A \Leftrightarrow \rho\theta(A) \in \mathsf{L}_1 \quad \text{and} \quad B \Leftrightarrow \theta\rho(B) \in \mathsf{L}_2.$$

As compared to [5, 13], we simplified the definition of structural translations, omitting peculiarities connected with the translation of propositional constants via formulas with variables and with the presence of additional variables in formulas $\alpha(c)$. In Item 3 of the original definition from [5], conditions $A \Leftrightarrow \rho\theta(A) \in \mathsf{L}_1$ and $B \Leftrightarrow \theta\rho(B) \in \mathsf{L}_2$ are replaced by conditions $(A, \rho\theta(A)) \in \tilde{\Omega}(\mathsf{L}_1)$ and respectively $(B, \theta\rho(B)) \in \tilde{\Omega}(\mathsf{L}_2)$, where $\tilde{\Omega}(\mathsf{L}_i)$ denotes the so called *Tarski's congruence* over logic L_i, i.e., the largest congruence on the algebra of formulas with the universe $\mathsf{Form}(\mathcal{L}_i)$ that respects all theories over L_i (see [2, 4] for details). However, in many logics L with strong negation, in particular in all logics considered in this article, its Tarski's congruence is determined by the condition $A \Leftrightarrow B \in \mathsf{L}$ (see [10] for details).

[5] We assume that either the strong equivalence connective is present in both languages $\mathcal{L}_1$ and $\mathcal{L}_2$ or $\{\Rightarrow, \wedge\} \subseteq \mathcal{L}_i$, $i = 1, 2$ and $\Leftrightarrow$ is treated as an abbreviation.

In [10] it was suggested to weaken the notion of definitional equivalence for logics with strong negation $\sim$ in the language. Namely, it was admitted that the translations involved in the previous definition are not structural for $\sim$. Moreover, in Item 3 of Definition 3.2 the strong equivalence was replaced by the weak one. We now slightly generalize the notion of a weak structural translation as compared to [10], making it more symmetrical.

Definition 3.3. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be propositional languages containing $\sim$. A translation $\theta\colon \mathsf{Form}(\mathcal{L}_1) \to \mathsf{Form}(\mathcal{L}_2)$ is called *weakly structural* if there are two mappings α and β, which send every n-ary connective c of $\mathcal{L}_1 \setminus \{\sim\}$ to formulas $\alpha(c)(p_1, q_1 \dots, p_n, q_n)$ and $\beta(c)(p_1, q_1 \dots, p_n, q_n)$. The translation θ is determined by α and β as follows:

$\theta(p) = p$, $\theta(\sim p) = \sim p$, $p \in \mathsf{Prop}$; $\theta(\sim\sim A) = \theta(A)$;
$\theta(c(A_1, ..., A_n)) = \alpha(c)(p_1/\theta(A_1),\ q_1/\theta(\sim A_1), ..., p_n/\theta(A_n), q_n/\theta(\sim A_n))$;
$\theta(\sim c(A_1, ..., A_n)){=}\beta(c)(p_1/\theta(A_1), q_1/\theta(\sim A_1), ..., p_n/\theta(A_n), q_n/\theta(\sim A_n))$,
where $A_1, \dots, A_n \in \mathsf{Form}(\mathcal{L}_1)$.

Definition 3.4. Let L_i be a logic in the propositional language $\mathcal{L}_i$, $i = 1, 2$. Let $\theta\colon \mathsf{Form}(\mathcal{L}_1) \to \mathsf{Form}(\mathcal{L}_2)$ and $\rho\colon \mathsf{Form}(\mathcal{L}_2) \to \mathsf{Form}(\mathcal{L}_1)$ be two weak structural translations. We say that L_1 and L_2 are *weakly definitionally equivalent* (*weakly d-equivalent*) *via* translations θ and ρ if the following conditions hold:

1. For $\Gamma \cup \{A\} \subseteq \mathsf{Form}(\mathcal{L}_1)$, the relation $\Gamma \vdash_{\mathsf{L}_1} A$ implies $\theta(\Gamma) \vdash_{\mathsf{L}_2} \theta(A)$.
2. For $\Gamma \cup \{A\} \subseteq \mathsf{Form}(\mathcal{L}_2)$, the relation $\Gamma \vdash_{\mathsf{L}_2} A$ implies $\rho(\Gamma) \vdash_{\mathsf{L}_1} \rho(A)$.
3. For every $A \in \mathsf{Form}(\mathcal{L}_1)$ and $B \in \mathsf{Form}(\mathcal{L}_2)$,

$$A \leftrightarrow \rho\theta(A) \in \mathsf{L}_1 \quad \text{and} \quad B \leftrightarrow \theta\rho(B) \in \mathsf{L}_2.$$

To prove the main theorem of this section we will need weak structural translations satisfying some additional conditions.

Definition 3.5. Let $\mathcal{L}$ be a propositional language containing $\sim$, L a logic in language $\mathcal{L}$, and $A(p_1, \dots, p_n) \in \mathsf{Form}(\mathcal{L})$. We say that $C(p_1, \dots, p_n)$ *agrees with* the weak equivalence of L if for formulas $A_i, B_i \in \mathsf{Form}(\mathcal{L})$, $1 \leqslant i \leqslant n$, the relations

$$A_i \leftrightarrow B_i \in \mathsf{L},\ 1 \leqslant i \leqslant n$$

imply

$$C(A_1, \dots, A_n,) \leftrightarrow C(B_1, \dots, B_n, D_n) \in \mathsf{L}.$$

In other words, a formula $C(p_1, \ldots, p_n)$ agrees with the weak equivalence of L if L is closed under the rule

$$\frac{p_1 \leftrightarrow q_1, \ldots, p_1 \leftrightarrow q_1}{C(p_1, \ldots, p_n) \leftrightarrow C(q_1, \ldots, q_n)}.$$

This condition is not trivial for logics with strong negation. Usually, such logics admit the replacement of weak equivalents in $\sim$-free formulas, but such replacement rule fails in the general case.

Definition 3.6. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be propositional languages containing $\sim$. Let L_2 be a logic in language $\mathcal{L}_2$. We say that a weak structural translation $\theta\colon \mathsf{Form}(\mathcal{L}_1) \to \mathsf{Form}(\mathcal{L}_2)$ *agrees with* the weak equivalence of L_2 if the mappings α and β determining θ are such that for every n-ary connective c of $\mathcal{L}_1 \setminus \{\sim\}$, the formulas

$$\alpha(c)(p_1, q_1, \ldots, p_n, q_n) \text{ and } \beta(c)(p_1, q_1, \ldots, p_n, q_n)$$

agree with the weak equivalence of L_2.

3.2. From weak d-equivalence to d-equivalence

The aim of this section is to formulate conditions on logics under which weak d-equivalence implies d-equivalence.

Definition 3.7. A propositional logic L formulated in the language $\mathcal{L}$ is called a *bl-logic* if it satisfies the following properties:

1. The weak equivalence connective detaches in L, i.e., for any formulas A and B, B', we have
$$A, A \leftrightarrow B \vdash_{\mathsf{L}} B.$$
2. There is a formula $\odot(p, q) \in \mathsf{Form}(\mathcal{L})$ such that for any two formulas $A, B \in \mathsf{Form}(\mathcal{L})$, we have
$$\vdash_{\mathsf{L}} \odot(A, B) \leftrightarrow A \quad \text{and} \quad \vdash_{\mathsf{L}} \sim \odot(A, B) \leftrightarrow B.$$

Item 1 is natural for every truth (verification) preserving biconditional. Item 2 is typical for logics based on a bilattice language, by which we mean a language extending $\{\vee, \wedge, \otimes, \oplus, \to, \sim, \bot, \top, \mathsf{b}, \mathsf{n}\}$, and which explains the choice of the term "bl-logic". This condition was inspired by the results of [1] on mutual definability of different connectives in a bilattice language. To some extent the existence of a combinator formula $\odot(p, q)$, which allows to combine an arbitrary truth condition with an arbitrary falsity condition, corresponds to the fact that a bilattice (with operators) can be represented as a *full* twist-structure over a lattice (with operators)(see the representation theorems in [1, 6, 12]). Recall that a full twist-structure is an algebra defined on the direct power of the universe of another algebra, but the new operations are not defined componentwise, they are somehow "twisted". Roughly speaking, the first component of a twist-structure is a measure of truth, and the second component is a measure of falsity. An arbitrary twist-structure is a subalgebra of the full one. Algebraic models of many logics with strong negation can be represented as twist-structures. These twist-structures are not necessarily full. For example, models of explosive Nelson's logic **N3** are isomorphic to twist-structures over Heyting algebras consisting of pairs (a, b) such that $a \wedge b = 0$ (see [14]). This is not so in the case of bilattice-based logics, whose algebraic models are necessarily full twist-structures. The presence of bilattice connectives in the language guarantees, in particular, that a full twist-structure has no proper subalgebras.

We say that two formulas A and B are L-equivalent if $\vdash_{\mathsf{L}} A \leftrightarrow B$.

Theorem 3.1. *Let L_i be a bl-logic formulated in language $\mathcal{L}_i$, $i = 1, 2$. Assume that logics L_1 and L_2 are weakly d-equivalent via translations θ and ρ such that θ agrees with the weak equivalence of L_2, and ρ agrees with the weak equivalence of L_1. Further, we assume that θ and ρ commute with the weak equivalence connective $\leftrightarrow$. Then there are structural translations θ' and ρ' such that they commute with strong negation and logics L_1 and L_2 are d-equivalent via translations θ' and ρ'.*

Proof. Assume that the functions α and β determine a weak structural translation θ, whereas the functions γ and δ determine ρ. In this way, for every $c \in \mathcal{L}_1 \setminus \{\sim\}$ we have

$$\theta(c(A_1, \ldots, A_n)) = \alpha(c)(\theta(A_1), \theta(\sim A_1), \ldots, \theta(A_n), \theta(\sim A_n)),$$
$$\theta(\sim c(A_1, \ldots, A_n)) = \beta(c)(\theta(A_1), \theta(\sim A_1), \ldots, \theta(A_n), \theta(\sim A_n)),$$

where $A_1, \ldots, A_n \in \mathsf{Form}(\mathcal{L}_1)$. Moreover, for every $c \in \mathcal{L}_2 \setminus \{\sim\}$ we have

$$\rho(c(B_1, \ldots, B_n)) = \gamma(c)(\theta(B_1), \theta(\sim B_1), \ldots, \theta(B_n), \theta(\sim B_n)),$$
$$\rho(\sim c(B_1, \ldots, B_n)) = \delta(c)(\theta(B_1), \theta(\sim B_1), \ldots, \theta(B_n), \theta(\sim B_n)),$$

where $B_1, \ldots, B_n \in \mathsf{Form}(\mathcal{L}_2)$.

Let $\odot_i(p, q)$ be an $\mathcal{L}_i$-formula such that

$$\vdash_{\mathsf{L}_i} \odot_i(A, B) \leftrightarrow A \quad \text{and} \quad \vdash_{\mathsf{L}_i} \sim \odot_i (A, B) \leftrightarrow B$$

for all $\mathcal{L}_i$-formulas A and B. We define two mappings $\zeta : \mathcal{L}_1 \to \mathsf{Form}(\mathcal{L}_2)$ and $\eta : \mathcal{L}_2 \to \mathsf{Form}(\mathcal{L}_1)$ as follows:

$$\zeta(c^n) = \odot_2(\alpha(c)(p_1, {\sim}p_1, \ldots, p_n, {\sim}p_n), \beta(c)(p_1, {\sim}p_1, \ldots, p_n, {\sim}p_n)),$$

for $c \in \mathcal{L}_1 \setminus \{\sim\}$; and

$$\eta(c^n) = \odot_1(\gamma(c)(p_1, {\sim}p_1, \ldots, p_n, {\sim}p_n), \delta(c)(p_1, {\sim}p_1, \ldots, p_n, {\sim}p_n)),$$

for $c \in \mathcal{L}_2 \setminus \{\sim\}$; and

$$\zeta(\sim) = {\sim}p \quad \text{and} \quad \eta(\sim) = {\sim}p.$$

Let θ' and ρ' be structural translations induced by mappings ζ and η respectively. It remains to prove that θ' and ρ' are translations as required, i.e., that θ' and ρ' commute with negation and that L_1 and L_2 are d-equivalent via θ' and ρ'. The former is obvious, to prove the latter we establish first the following fact.

Lemma 3.2. *For every $\mathcal{L}_1$-formula A and $\mathcal{L}_2$-formula B, we have*

$$\vdash_{\mathsf{L}_2} \theta(A) \leftrightarrow \theta'(A) \ \ \textit{and} \ \ \vdash_{\mathsf{L}_1} \rho(B) \leftrightarrow \rho'(B).$$

Proof. We prove only the first of the two equivalences by induction on the structure of formulas, the proof of the second is similar. The base of induction is obvious in view of

$$\theta(p) = \theta'(p) = p \ \ \text{and} \ \ \theta({\sim}p) = \theta'({\sim}p) = {\sim}p.$$

Let c be an n-ary connective of $\mathcal{L}_1$. Assume that for $\mathcal{L}_1$-formulas A_1, …, A_n we just proved

$$\vdash_{\mathsf{L}_2} \theta(A_i) \leftrightarrow \theta'(A_i) \quad \text{and} \quad \vdash_{\mathsf{L}_2} \theta({\sim}A_i) \leftrightarrow \theta'({\sim}A_i),\ i = 1, \ldots, n.$$

We have then

$$\theta'(c(A_1, \ldots, A_n)) = \\ \odot_2 \left(\alpha(c)(\theta'(A_1), {\sim}\theta'(A_1), \ldots), \beta(c)(\theta'(A_1), {\sim}\theta'(A_1), \ldots)\right),$$

which is L_2-equivalent by the properties of $\odot_2(p, q)$ to

$$\alpha(c)(\theta'(A_1), {\sim}\theta'(A_1), \ldots, \theta'(A_n), {\sim}\theta'(A_n)) = \\ \alpha(c)(\theta'(A_1), \theta'({\sim}A_1), \ldots, \theta'(A_n), \theta'({\sim}A_n)).$$

The last equality is due to $\zeta(\sim) = {\sim}p$. Applying the induction hypothesis and the fact that $\alpha(c)$ agrees with the weak equivalence of $\mathsf{L_2}$, we conclude that the last formula is $\mathsf{L_2}$-equivalent to

$$\alpha(c)(\theta(A_1), \theta({\sim}A_1), \ldots, \theta(A_n), \theta({\sim}A_n)) = \theta(c(A_1, \ldots, A_n)).$$

For a negated formula ${\sim}c(A_1, \ldots, A_n)$ we have

$$\theta'({\sim}c(A_1, \ldots, A_n)) \\ \sim \odot_2 (\alpha(c)(\theta'(A_1), {\sim}\theta'(A_1), \ldots), \beta(c)(\theta'(A_1), {\sim}\theta'(A_1), \ldots)).$$

Applying again the properties of $\odot_2(p, q)$, the fact that $\beta(c)$ agrees with the weak equivalence of $\mathsf{L_2}$, and the induction hypothesis, we conclude that this formula is $\mathsf{L_2}$-equivalent to

$$\beta(c)(\theta(A_1), \theta({\sim}A_1), \ldots, \theta(A_n), \theta({\sim}A_n)) = \theta({\sim}c(A_1, \ldots, A_n)). \qquad \dashv$$

We come back to the proof of the theorem. By the previous lemma, the weak equivalences

$$\theta(B) \leftrightarrow \theta'(B),\ \theta(A_1) \leftrightarrow \theta'(A_1), \ldots,\ \theta(A_n) \leftrightarrow \theta'(A_n)$$

are provable in $\mathsf{L_2}$ for any $\mathcal{L}_1$-formulas B, A_1, ..., A_n. This fact, Item 1 of Definition 3.7, and the transitivity of $\vdash_{\mathsf{L_2}}$ entail that the relation

$$\theta(A_1), \ldots, \theta(A_n) \vdash_{\mathsf{L_2}} \theta(B) \text{ implies } \theta'(A_1), \ldots, \theta'(A_n) \vdash_{\mathsf{L_2}} \theta'(B).$$

In this way the fact that θ' embeds $\mathsf{L_1}$ into $\mathsf{L_2}$ follows from the fact that θ embeds $\mathsf{L_1}$ into $\mathsf{L_2}$. Similarly, we prove that ρ' embeds $\mathsf{L_2}$ into $\mathsf{L_1}$.

It remains to prove that the translations θ' and ρ' are mutually inverse up to weak equivalence. Let $A \in \mathsf{Form}(\mathcal{L}_1)$. By assumption we have $\vdash_{\mathsf{L_1}} A \leftrightarrow \rho\theta(A)$, and by Lemma 3.2, $\vdash_{\mathsf{L_2}} \theta(A) \leftrightarrow \theta'(A)$. Since ρ embeds $\mathsf{L_2}$ into $\mathsf{L_1}$ and commutes with $\leftrightarrow$, we obtain $\vdash_{\mathsf{L_1}} \rho\theta(A) \leftrightarrow \rho\theta'(A)$. Again by Lemma 3.2, $\vdash_{\mathsf{L_1}} \rho\theta'(A) \leftrightarrow \rho'\theta'(A)$. Finally, applying the transitivity of weak equivalence, we conclude $\vdash_{\mathsf{L_1}} A \leftrightarrow \rho'\theta'(A)$. In a similar way one can establish that $\vdash_{\mathsf{L_2}} A \leftrightarrow \theta\rho(A)$ for $A \in \mathsf{Form}(\mathcal{L}_2)$. $\dashv$

3.3. Definitional equivalence of $\mathbf{BK}^{\Box_+\Box_-}$, $\mathbf{BK}^{\mathsf{FS}}_{bl}$, and MBL

In [9], it was proved that $\mathbf{BK}^{\mathsf{FS}}$ and the fusion $\mathbf{BK}^{\Box}\times\mathbf{BK}^{\Box}$ are weakly d-equivalent. If we denote the modal operators of $\mathbf{BK}^{\Box}\times\mathbf{BK}^{\Box}$ as $\Box_+$ and $\Box_-$, then $\mathbf{BK}^{\Box_+\Box_-}$ becomes a conservative extension of $\mathbf{BK}^{\Box}\times\mathbf{BK}^{\Box}$, and we can adopt the proof from [9] to establish the weak d-equivalence of $\mathbf{BK}^{\mathsf{FS}}_{bl}$ and $\mathbf{BK}^{\Box_+\Box_-}$. The weakly structural translation $\theta\colon \mathsf{Form}(\mathcal{L}_{\mathbf{BK}^{\mathsf{FS}}_{bl}}) \longrightarrow \mathsf{Form}(\mathcal{L}_{\mathbf{BK}^{\Box_+\Box_-}})$ is defined so that it preserves propositional variables and constants, commutes with all non-modal connectives, whereas for modalities we have

$$\theta(\Box_{\mathsf{FS}}A) = \Box_+\theta(A) \ \text{ and } \ \theta(\Diamond_{\mathsf{FS}}A) = {\sim}\Box_+{\sim}\theta(A).$$

For strongly negated formulas, we define θ as follows:

$$\begin{gathered}
\theta({\sim}p) = {\sim}p,\ \theta({\sim}\bot) = {\sim}\bot,\ \theta({\sim}\top) = {\sim}\top,\ \theta({\sim}\mathsf{b}) = \mathsf{b},\ \theta({\sim}\mathsf{n}) = \mathsf{n},\\
\theta({\sim}(A\vee B)) = \theta({\sim}A)\wedge\theta({\sim}B), \quad \theta({\sim}(A\wedge B)) = \theta({\sim}A)\vee\theta({\sim}B),\\
\theta({\sim}(A\oplus B)) = \theta({\sim}A)\oplus\theta({\sim}B), \quad \theta({\sim}(A\otimes B)) = \theta({\sim}A)\otimes\theta({\sim}B),\\
\theta({\sim}(A\to B)) = \theta(A)\wedge\theta({\sim}B), \quad \theta({\sim}{\sim}A) = \theta(A),\\
\theta({\sim}\Box_{\mathsf{FS}}A) = {\sim}\Box_+{\sim}\theta({\sim}A), \quad \theta({\sim}\Diamond_{\mathsf{FS}}A) = \Box_-\theta({\sim}A).
\end{gathered}$$

The inverse translation $\rho\colon \mathsf{Form}(\mathcal{L}_{\mathbf{BK}^{\Box_+\Box_-}}) \longrightarrow \mathsf{Form}(\mathcal{L}_{\mathbf{BK}^{\mathsf{FS}}_{bl}})$ also preserves propositional variables and constants and commutes with the non-modal connectives. For modal operators, we put

$$\rho(\Box_+A) = \Box_{\mathsf{FS}}\rho(A) \ \text{ and } \ \rho(\Box_-A) = {\sim}\Diamond_{\mathsf{FS}}{\sim}\rho(A).$$

For strongly negated formulas, ρ is defined as follows:

$$\begin{gathered}
\rho({\sim}p) = {\sim}p, \quad \rho({\sim}\bot) = {\sim}\bot, \quad \rho({\sim}\top) = {\sim}\top, \quad \rho({\sim}\mathsf{b}) = \mathsf{b}, \quad \rho({\sim}\mathsf{n}) = \mathsf{n},\\
\rho({\sim}(A\vee B)) = \rho({\sim}A)\wedge\rho({\sim}B), \quad \rho({\sim}(A\wedge B)) = \rho({\sim}A)\vee\rho({\sim}B),\\
\rho({\sim}(A\oplus B)) = \rho({\sim}A)\oplus\rho({\sim}B), \quad \rho({\sim}(A\otimes B)) = \rho({\sim}A)\otimes\rho({\sim}B),\\
\rho({\sim}(A\to B)) = \rho(A)\wedge\rho({\sim}B), \quad \rho({\sim}{\sim}A) = \rho(A),\\
\rho({\sim}\Box_+A) = {\sim}\Box_{\mathsf{FS}}{\sim}\rho({\sim}A), \quad \rho({\sim}\Box_-A) = \neg{\sim}\Diamond_{\mathsf{FS}}{\sim}\neg\rho({\sim}A).
\end{gathered}$$

Theorem 3.3. *The logics $\mathbf{BK}^{\mathsf{FS}}_{bl}$ and $\mathbf{BK}^{\Box_+\Box_-}$ are weakly d-equivalent via translations θ and ρ.*

Proof. Taking into account that both logics were defined over the same class of models, the weak d-equivalence of logics $\mathbf{BK}^{\mathsf{FS}}_{bl}$ and $\mathbf{BK}^{\Box_+\Box_-}$ easily follows from the next lemma.

Lemma 3.4. *For every model* $\mathcal{M} = \langle W, R, R', v^+, v^- \rangle$, $w \in W$, *and formulas* $A \in \mathsf{Form}(\mathcal{L}_{\mathbf{BK}^{\Box+\Box-}})$, $B \in \mathsf{Form}(\mathcal{L}_{\mathbf{BK}^{\mathsf{FS}}_{bl}})$, *the following equivalences hold*:

$$\mathcal{M}, w \models^+ A \textit{ iff } \mathcal{M}, w \models^+ \theta(A),$$
$$\mathcal{M}, w \models^+ \rho(B) \textit{ iff } \mathcal{M}, w \models^+ B.$$

Proof. Both equivalences were proved by induction on the structure of formulas in Lemma 4.6 of [10] for a weaker language not containing the connectives $\oplus$, $\otimes$ and the constants b, n, $\top$. Let us consider the cases for $\sim(A \oplus B)$ and $\sim(A \otimes B)$.

$w \Vdash^+ \theta(\sim(A \oplus B))$
iff $w \Vdash^+ \theta(\sim A) \oplus \theta(\sim B)$ (by definition of θ)
iff $(w \Vdash^+ \theta(\sim A) \Cup w \Vdash^+ \theta(\sim B))$
iff $(w \Vdash^+ \sim A \Cup w \Vdash^+ \sim B)$ (by induction hypothesis)
iff $(w \Vdash^- A \Cup w \Vdash^- B)$
iff $w \Vdash^- A \oplus B$
iff $w \Vdash^+ \sim(A \oplus B)$.

$w \Vdash^+ \theta(\sim(A \otimes B))$
iff $w \Vdash^+ \theta(\sim A) \otimes \theta(\sim B)$ (by definition of θ)
iff $(w \Vdash^+ \theta(\sim A) \Cap w \Vdash^+ \theta(\sim B))$
iff $(w \Vdash^+ \sim A \Cap w \Vdash^+ \sim B)$ (by induction hypothesis)
iff $(w \Vdash^- A \Cap w \Vdash^- B)$
iff $w \Vdash^- A \otimes B$
iff $w \Vdash^+ \sim(A \otimes B)$. ⊣

⊣

Corollary 3.5. *The logics* $\mathbf{BK}^{\mathsf{FS}}_{bl}$ *and* $\mathbf{BK}^{\Box+\Box-}$ *are d-equivalent.*

Proof. To apply Theorem 3.1, we have to check that $\mathbf{BK}^{\mathsf{FS}}_{bl}$ and $\mathbf{BK}^{\Box+\Box-}$ are bl-logics, that translations θ and ρ agree with weak the equivalences of $\mathbf{BK}^{\Box+\Box-}$ and $\mathbf{BK}^{\mathsf{FS}}_{bl}$ respectively, and that both θ and ρ commute with $\leftrightarrow$. The last fact is obvious. The fact that θ and ρ agree with the weak equivalences of $\mathbf{BK}^{\Box+\Box-}$ and $\mathbf{BK}^{\mathsf{FS}}_{bl}$ easily follows from the definition of translations and the definition of verification and falsification of formulas. Item (1) of Definition 3.7 obviously holds for both logics. As for $\odot(p, q)$, one can choose in both cases the formula

$$\odot(p, q) = (p \wedge \mathsf{b}) \vee (\sim q \wedge \mathsf{n}).$$ ⊣

It turns out that **MBL** is also a bl-logic, and we prove now that **MBL** and $\mathbf{BK}^{\Box_+\Box_-}$ are d-equivalent. The weakly structural translation $\zeta\colon \mathsf{Form}(\mathcal{L}_{MBL}) \longrightarrow \mathsf{Form}(\mathcal{L}_{\mathbf{BK}^{\Box_+\Box_-}})$ is a modification of the translation of **MBL**-formulas into $\mathsf{BK}^{\Box_+\Box_-}$-formulas from [10]. This translation preserves propositional variables and the four constants, and commutes with the connectives $\wedge$, $\vee$, $\otimes$, $\oplus$. The modality $\boxplus$ and its strong negation are treated by ζ as follows:

$$\zeta(\boxplus A) = \Box_+\zeta(A) \wedge \Box_-\neg\zeta(\sim A), \qquad \zeta(\sim\boxplus A) = \sim\Box_+\sim\zeta(\sim A).$$

For other strongly negated formulas, ζ is defined analogously to the translation θ (see above).

We define the inverse weakly structural translation $\eta\colon \mathsf{Form}(\mathcal{L}_{\mathbf{BK}^{\Box_+\Box_-}}) \longrightarrow \mathsf{Form}(\mathcal{L}_{MBL})$ so that it preserves again propositional variables and the four constants, and commutes with the connectives $\wedge$, $\vee$, $\otimes$, $\oplus$, $\to$. The modal connectives of $\mathbf{BK}^{\Box_+\Box_-}$ and their strong negations are treated by η as follows:

$$\eta(\Box_+A) = \boxplus(\eta(A) \vee \mathsf{n}), \qquad \eta(\sim\Box_+A) = \sim\boxplus\sim\eta(\sim A),$$
$$\eta(\Box_-A) = \boxplus(\sim\neg\eta(A) \vee \mathsf{b}), \qquad \eta(\sim\Box_-A) = \neg\boxplus(\sim\eta(\sim A) \vee \mathsf{b}).$$

The translations of other strongly negated formulas are defined as above.

Theorem 3.6. *The logics* **MBL** *and* $\mathbf{BK}^{\Box_+\Box_-}$ *are weakly d-equivalent via translations* ζ *and* η.

Proof. Again we have two logics defined over the same class of models, so all what we need is to prove an analogue of Lemma 3.4.

Lemma 3.7. *For every model* $\mathcal{M} = \langle W, R, R', v^+, v^-\rangle$, $w \in W$, *and formulas* $A \in \mathsf{Form}(\mathcal{L}_{\mathbf{MBL}})$, $B \in \mathsf{Form}(\mathcal{L}_{\mathbf{BK}^{\Box_+\Box_-}})$, *the following equivalences hold*:

$$\mathcal{M}, w \models^+ A \text{ iff } \mathcal{M}, w \models^+ \zeta(A),$$
$$\mathcal{M}, w \models^+ \eta(B) \text{ iff } \mathcal{M}, w \models^+ B.$$

Proof. The first of this equivalences was established in the proof of Theorem 5.7 of [10] stating that ζ faithfully embeds **MBL** into $\mathbf{BK}^{\Box_+\Box_-}$.

By induction on the structure of formulas we prove the second equivalence. We only check the cases for modalities and for strong negations of modalities.

$w \Vdash^+ \eta(\Box_+B)$
iff $w \Vdash^+ \boxplus(\eta(B) \vee \mathsf{n})$ (by definition of η)

iff $(\forall u(wR_+u \twoheadrightarrow u \Vdash^+ \eta(B) \vee \mathsf{n}) \mathbin{\&} \forall u(wR_-u \twoheadrightarrow \neg(u \Vdash^- \eta(B) \vee \mathsf{n})))$
iff $(\forall u(wR_+u \twoheadrightarrow (u \Vdash^+ \eta(B) \vee u \Vdash^+ \mathsf{n})) \mathbin{\&} \forall u(wR_-u \twoheadrightarrow \neg(u \Vdash^- \eta(B) \mathbin{\&} u \Vdash^- \mathsf{n})))$
iff $\forall u(wR_+u \twoheadrightarrow u \Vdash^+ \eta(B))$ (in view of $\neg(u \Vdash^+ \mathsf{n})$ and $\neg(u \Vdash^- \mathsf{n})$)
iff $\forall u(wR_+u \twoheadrightarrow u \Vdash^+ B)$ (by induction hypothesis)
iff $w \Vdash^+ \Box_+B$.

$w \Vdash^+ \eta(\sim\Box_+B)$
iff $w \Vdash^+ \sim \boxplus \sim\eta(\sim B)$ (by definition of η)
iff $w \Vdash^- \boxplus\sim\eta(\sim B)$
iff $\exists u(wR_+u \mathbin{\&} u \Vdash^- \sim\eta(\sim B))$
iff $\exists u(wR_+u \mathbin{\&} u \Vdash^+ \eta(\sim B))$
iff $\exists u(wR_+u \mathbin{\&} u \Vdash^+ \sim B)$ (by induction hypothesis)
iff $\exists u(wR_+u \mathbin{\&} u \Vdash^- B)$
iff $w \Vdash^- \Box_+B$.
iff $w \Vdash^+ \sim\Box_+B$.

$w \Vdash^+ \eta(\Box_-B)$
iff $w \Vdash^+ \boxplus(\sim\neg\eta(B) \vee \mathsf{b})$ (by definition of η)
iff $\forall u(wR_+u \twoheadrightarrow u \Vdash^+ \sim\neg\eta(B) \vee \mathsf{b}) \mathbin{\&} \forall u(wR_-u \twoheadrightarrow \neg(u \Vdash^- \sim\neg\eta(B) \vee \mathsf{b}))$
iff $\forall u(wR_+u \twoheadrightarrow (u \Vdash^+ \sim\neg\eta(B) \vee u \Vdash^+ \mathsf{b})) \mathbin{\&} \forall u(wR_-u \twoheadrightarrow \neg(u \Vdash^- \sim\neg\eta(B) \mathbin{\&} u \Vdash^- \mathsf{b}))$
iff $\forall u(wR_-u \twoheadrightarrow \neg(u \Vdash^- \sim\neg\eta(B)))$ (in view of $u \Vdash^+ \mathsf{b}$ and $u \Vdash^- \mathsf{b}$)
iff $\forall u(wR_-u \twoheadrightarrow \neg(u \Vdash^+ \neg\eta(B)))$
iff $\forall u(wR_-u \twoheadrightarrow \neg\neg(u \Vdash^+ \eta(B)))$
iff $\forall u(wR_-u \twoheadrightarrow u \Vdash^+ \eta(B))$
iff $\forall u(wR_-u \twoheadrightarrow u \Vdash^+ B)$ (by induction hypothesis)
iff $w \Vdash^+ \Box_-B$.

$w \Vdash^+ \eta(\sim\Box_-B)$
iff $w \Vdash^+ \neg \boxplus (\sim\eta(\sim B) \vee \mathsf{b})$ (by definition of η)
iff $\neg(w \Vdash^+ \boxplus(\sim\eta(\sim B) \vee \mathsf{b}))$
iff $\neg(\forall u(wR_+u \twoheadrightarrow u \Vdash^+ \sim\eta(\sim B) \vee \mathsf{b}) \mathbin{\&} \forall u(wR_-u \twoheadrightarrow \neg(u \Vdash^- \sim\eta(\sim B) \vee \mathsf{b})))$
iff $\neg(\forall u(wR_+u \twoheadrightarrow (u \Vdash^+ \sim\eta(\sim B) \vee u \Vdash^+ \mathsf{b})) \mathbin{\&} \forall u(wR_-u \twoheadrightarrow \neg(u \Vdash^- \sim\eta(\sim B) \mathbin{\&} u \Vdash^- \mathsf{b})))$
iff $\neg(\forall u(wR_-u \twoheadrightarrow \neg(u \Vdash^- \sim\eta(\sim B)))$ (in view of $u \Vdash^+ \mathsf{b}$ and $u \Vdash^- \mathsf{b}$)
iff $\exists u(wR_-u \mathbin{\&} u \Vdash^- \sim\eta(\sim B)$
iff $\exists u(wR_-u \mathbin{\&} u \Vdash^+ \eta(\sim B)$
iff $\exists u(wR_-u \mathbin{\&} u \Vdash^+ \sim B)$ (by induction hypothesis)

iff $\exists u(wR_{-}u \curlywedge u \Vdash^{-} B)$
iff $w \Vdash^{-} \Box_{-}B$
iff $w \Vdash^{+} {\sim}\Box_{-}B$. ⊣

⊣

Corollary 3.8. *The logics* **MBL** *and* $\mathbf{BK}^{\Box_{+}\Box_{-}}$ *are d-equivalent.*

Proof. We know that $\mathbf{BK}^{\Box_{+}\Box_{-}}$ is a bl-logic. It is easy to see that

$$\odot(p, q) = (p \wedge \mathsf{b}) \vee ({\sim}q \wedge \mathsf{n})$$

can serve as a combinator formula for **MBL**, too. In this way, **MBL** is also a bl-logic. Obviously, both translations ζ and η commute with $\leftrightarrow$. It remains to check that ζ and η agree with the weak equivalences of $\mathbf{BK}^{\Box_{+}\Box_{-}}$ and **MBL**. The case of η is more complicated. Let α and β be mappings that determine the weakly structural translation η. It is easy to see from the definition of $\Vdash^{+}$ that **MBL** is closed under the replacement of weak equivalents in formulas which are $\sim$-free and $\boxplus$-free. Therefore, we have to check the replacement of weak equivalents for the following formulas:

$$\alpha(\Box_{+}) = \boxplus(p \vee \mathsf{n}), \qquad \alpha(\Box_{-}) = \boxplus({\sim}\neg p \vee \mathsf{b}),$$
$$\beta(\Box_{+}) = {\sim}\boxplus{\sim}q, \qquad \beta(\Box_{-}) = \neg\boxplus({\sim}q \vee \mathsf{b}).$$

By proving Lemma 3.7, we established in particular the following:

$$w \Vdash^{+} \boxplus(p \vee \mathsf{n}) \text{ iff } \forall u(wR_{+}u \twoheadrightarrow u \Vdash^{+} p),$$
$$w \Vdash^{+} \boxplus({\sim}\neg p \vee \mathsf{b}) \text{ iff } \forall u(wR_{-}u \twoheadrightarrow u \Vdash^{+} p),$$
$$w \Vdash^{+} {\sim}\boxplus{\sim}q \text{ iff } \exists u(wR_{+}u \curlywedge u \Vdash^{+} q),$$
$$w \Vdash^{+} \neg\boxplus({\sim}q \vee \mathsf{b}) \text{ iff } \exists u(wR_{-}u \curlywedge u \Vdash^{+} q).$$

It follows easily from these equivalences that **MBL** is closed under the replacement rules:

$$\frac{p \leftrightarrow q}{\boxplus(p \vee \mathsf{n}) \leftrightarrow \boxplus(q \vee \mathsf{n})} \qquad \frac{p \leftrightarrow q}{\boxplus({\sim}\neg p \vee \mathsf{b}) \leftrightarrow \boxplus({\sim}\neg q \vee \mathsf{b})}$$

$$\frac{p \leftrightarrow q}{{\sim}\boxplus{\sim}p \leftrightarrow {\sim}\boxplus{\sim}q} \qquad \frac{p \leftrightarrow q}{\neg\boxplus({\sim}p \vee \mathsf{b}) \leftrightarrow \neg\boxplus({\sim}q \vee \mathsf{b})}$$

We have thus proved that η agrees with the weak equivalence of **MBL**. So we can apply Theorem 3.1 to infer the desired conclusion. ⊣

From the transitivity of d-equivalence we obtain.

Corollary 3.9. *The logics* **MBL** *and* $\mathbf{BK}^{\mathsf{FS}}_{bl}$ *are d-equivalent.*

Thus, despite essentially different definitions of modalities, all three logics **MBL**, $\mathbf{BK}^{\mathsf{FS}}_{bl}$, and $\mathbf{BK}^{\Box+\Box-}$ are d-equivalent.

4. Logical Connectives for $\mathbf{BK}^{\Box+\Box-}$, $\mathbf{BK}^{\mathsf{FS}}_{bl}$, and MBL

Since the logics $\mathbf{BK}^{\Box+\Box-}$, $\mathbf{BK}^{\mathsf{FS}}_{bl}$, and **MBL** are definitionally equivalent, in this section we will only discuss the logical connectives for $\mathbf{BK}^{\Box+\Box-}$. As for $\mathbf{BK}^{\mathsf{FS}}_{bl}$ and **MBL**, the result of this section will carry over.

In Remarks 2.3 and 2.4 we showed that the presented method cannot be applied to logics which comprise connectives like $\oplus$ or $\otimes$ because they do not satisfy Observation 2.4. Furthermore, models for $\mathbf{BK}^{\Box+\Box-}$, $\mathbf{BK}^{\mathsf{FS}}_{bl}$, and **MBL** contain two different accessibility relations, R^+ and R^-. To apply the presented method for obtaining functional completeness to $\mathbf{BK}^{\Box+\Box-}$, we first enrich the metalanguage by two binary predicate symbols, R^+ and R^-, and hence omit the predicate symbol R, and by four new operations, $\circledast$, $\oplus\!\!\!\!\oplus$, $\mathbb{b}$, and $\mathbb{n}$. The definitions of regular metalogical formulas and L-regular metalogical formulas have to be changed accordingly. Furthermore, we need to make some adjustments of the verification and falsification conditions for $\otimes$, $\oplus$, b, and n:

$$
\begin{array}{lll}
w \Vdash^+ A \otimes B & \text{iff} & w \Vdash^+ A \circledast w \Vdash^+ B; \\
w \Vdash^- A \otimes B & \text{iff} & w \Vdash^- A \circledast w \Vdash^- B; \\
w \Vdash^+ A \oplus B & \text{iff} & w \Vdash^+ A \oplus\!\!\!\!\oplus\, w \Vdash^+ B; \\
w \Vdash^- A \oplus B & \text{iff} & w \Vdash^- A \oplus\!\!\!\!\oplus\, w \Vdash^- B. \\
w \Vdash^\pm \mathsf{b} & \text{iff} & \mathbb{b}; \\
\neg(w \Vdash^\pm \mathsf{n}) & \text{iff} & \mathbb{n},
\end{array}
$$

where $w \Vdash^\pm A \circledast w \Vdash^\pm B$, $w \Vdash^\pm A \oplus\!\!\!\!\oplus\, w \Vdash^\pm B$ are understood as $w \Vdash^\pm A \mathbin{\bigwedge\!\!\!\!\bigwedge} w \Vdash^\pm B$ and $w \Vdash^\pm A \mathbin{\bigvee\!\!\!\!\bigvee} w \Vdash^\pm B$, respectively. Note that even though we treat $\circledast$ and $\oplus\!\!\!\!\oplus$ as some sort of metalanguage conjunction or disjunction, they are independent operators, as the following definition shows.

Note also that the treatment of $\mathbb{b}$ and $\mathbb{n}$ is similar, i.e., they are understood as $w \Vdash^+ \mathsf{b} \mathbin{\bigwedge\!\!\!\!\bigwedge} w \Vdash^- \mathsf{b}$ and $\neg(w \Vdash^+ \mathsf{n}) \mathbin{\bigwedge\!\!\!\!\bigwedge} \neg(w \Vdash^- \mathsf{n})$, respectively.

Definition 4.1. Let φ be a regular metalogical formula. The formula $\overline{\varphi}$ is now inductively defined as follows:[6]

$$
\begin{array}{rcl}
\overline{\neg wR^{\pm}u} & = & \neg wR^{\pm}u \\
\overline{\neg w \Vdash^{+} A} & = & w \Vdash^{-} A, A \notin \{\mathsf{b}, \mathsf{n}\} \\
\overline{\neg w \Vdash^{-} A} & = & w \Vdash^{+} A, A \notin \{\mathsf{b}, \mathsf{n}\} \\
\overline{\neg \boldsymbol{\mathsf{b}}} & = & \boldsymbol{\mathsf{b}} \\
\overline{\neg \boldsymbol{\mathsf{n}}} & = & \boldsymbol{\mathsf{n}} \\
\overline{\psi \bigotimes \theta} & = & \overline{\psi} \bigotimes \overline{\theta} \\
\overline{\neg(\psi \bigotimes \theta)} & = & \overline{\neg\psi} \bigotimes \overline{\neg\theta} \\
\overline{\psi \bigoplus \theta} & = & \overline{\psi} \bigoplus \overline{\theta} \\
\overline{\neg(\psi \bigoplus \theta)} & = & \overline{\neg\psi} \bigoplus \overline{\neg\theta}
\end{array}
$$

If φ is a regular metalogical formula, then $\overline{\varphi}$ is said to be an *L-regular metalogical formula.*

Observation 4.1. *For each* $\mathbf{BK}\Box_{+}\Box_{-}$*-formula A, $\overline{\varphi}$ iff $w \Vdash^{+} A$ just in case $\overline{\neg\varphi}$ iff $w \Vdash^{-} A$.*

Proof. The proof is by induction on A and follows the proof of Observation 2.4. We only need to prove the observation for the operators $\otimes$ and $\oplus$.

Suppose $A = A \otimes B$:

$$
\begin{array}{rll}
\overline{\varphi} & \text{iff} & w \Vdash^{+} A \otimes B \\
 & \text{iff} & \overline{w \Vdash^{+} A \bigotimes w \Vdash^{+} B} \\
\therefore \overline{\neg\varphi} & \text{iff} & \overline{\neg(w \Vdash^{+} A \bigotimes w \Vdash^{+} B)} \\
 & \text{iff} & \overline{\neg w \Vdash^{+} A} \bigotimes \overline{\neg w \Vdash^{+} B} \\
 & \text{iff} & w \Vdash^{-} A \bigotimes w \Vdash^{-} B \\
 & \text{iff} & w \Vdash^{-} A \otimes B
\end{array}
\qquad
\begin{array}{rll}
\overline{\neg\varphi} & \text{iff} & w \Vdash^{-} A \otimes B \\
 & \text{iff} & \overline{w \Vdash^{-} A \bigotimes w \Vdash^{-} B} \\
 & \text{iff} & \overline{\neg w \Vdash^{+} A \bigotimes \neg w \Vdash^{+} B} \\
 & \text{iff} & \overline{\neg(w \Vdash^{+} A \bigotimes w \Vdash^{+} B)} \\
\therefore \overline{\varphi} & \text{iff} & \overline{w \Vdash^{+} A \bigotimes w \Vdash^{+} B} \\
 & \text{iff} & w \Vdash^{+} A \otimes B
\end{array}
$$

The proof for $A = A \oplus B$ is similar.

The proof for $A = \top$ is a degenerate case, since $\neg(w \Vdash^{-} \top)$ and $w \Vdash^{+} \top$, but neither $\neg(w \Vdash^{+} \top)$ nor $w \Vdash^{-} \top$, hence the equivalence is trivially true. As for $A = \mathsf{b}$ and $A = \mathsf{n}$, the equivalences are trivially true, as well. $\dashv$

Observation 4.2. *The set of connectives* $\{\vee, \wedge, \otimes, \oplus, \rightarrow, \sim, \Box_{+}, \Box_{-}, \bot, \top, \mathsf{b}, \mathsf{n}\}$ *is a set of* $\mathbf{BK}^{\Box_{+}\Box_{-}}$*-regular connectives.*

[6] The definitions for the other metalogical connectives are just like in Definition 2.2.

Theorem 4.3. *In the class of* $\mathbf{BK}^{\Box_+\Box_-}$*-regular connectives the set of connectives* $\{\vee, \wedge, \otimes, \oplus, \rightarrow, \sim, \Box_+, \Box_-, \bot, \top, \mathsf{b}, \mathsf{n}\}$ *is functionally complete. In other words, if an* n*-ary* $(n \geqslant 1)$ *connective* $\bigstar$ *is defined by means of a* $\mathbf{BK}^{\Box_+\Box_-}$*-regular metalogical formula* $\overline{\varphi}$*, then there is a* $\mathbf{BK}^{\Box_+\Box_-}$*-formula* A *such that the following holds:* $\overline{\varphi} \twoheadleftarrow\!\!\twoheadrightarrow w \Vdash^+ A$ *(and* $\overline{\neg\varphi} \twoheadleftarrow\!\!\twoheadrightarrow w \Vdash^- A$*).*

Proof. Since the following proof works like the proof for Theorem 2.6, we only show the theorem by induction on the degree of quantification of $\overline{\varphi}$ for the additional operators.

Suppose $d(\overline{\varphi}) = 0$, then let A be the result of replacing every occurrence of $\circledast$ by $\otimes$ and every occurrence of $\boxplus$ by $\oplus$. Furthermore, we replace every occurrence of $w \Vdash^+ \top$ by $\top$, $w \Vdash^- \top$ by $\sim\top$, and every occurrence of $\mathbb{b}$ by b. Finally, we replace every occurrence of n by $\mathbb{n}$.

Now, let $d(\overline{\varphi}) > 0$. Then there is a subformula of $\overline{\varphi}$ of the shape $\forall\!\!\!\forall\, u(wR^{\pm}u \twoheadrightarrow \theta)$ or $\exists\!\!\!\exists\, u(wR^{\pm}u \mathbin{\&\!\!\&} \theta)$, where θ is quantifier free. By the induction base case, $\theta \twoheadleftarrow\!\!\twoheadrightarrow A$ for some $\mathcal{L}$-formula. Now we have:

$$\begin{array}{rll} \forall\!\!\!\forall\, u(wR^{\pm}u \twoheadrightarrow \theta) & \text{iff} & w \Vdash^+ \Box_{\pm} A \\ \exists\!\!\!\exists\, u(wR^{\pm}u \mathbin{\&\!\!\&} \theta) & \text{iff} & w \Vdash^+ \sim\Box_{\pm}\sim A \end{array}$$

If the subformulas $\forall\!\!\!\forall\, u(wR^{\pm}u \twoheadrightarrow \theta)$ and $\exists\!\!\!\exists\, u(wR^{\pm}u \mathbin{\&\!\!\&} \theta)$ in $\overline{\varphi}$ are replaced by their respective equivalents, the result is a $\mathbf{BK}^{\Box_+\Box_-}$-regular metalogical formula which has one quantifier less than $\overline{\varphi}$, and hence the induction hypothesis can be used. ⊣

5. Discussion of the results

The main result of Section 3, Theorem 3.1, provides a method of proving definitional equivalence for logics in certain languages with strong negation. In particular, we could apply this method to prove the definitional equivalence of certain modal logics extending the non-modal fragment $\{\vee, \wedge, \otimes, \oplus, \rightarrow, \sim, \bot, \top, \mathsf{b}, \mathsf{n}\}$ of $\mathcal{L}_{\mathbf{BK}^{\Box_+\Box_-}}$.

Note that $\{\vee, \wedge, \otimes, \oplus, \rightarrow, \sim, \bot, \top, \mathsf{b}, \mathsf{n}\}$ is functionally complete in the usual sense, cf. [11], whereas $\mathcal{L}_{\mathbf{BK}^{\Box_+\Box_-}}$ is functionally complete only in the class of $\mathbf{BK}^{\Box_+\Box_-}$-regular connectives. Moreover, the two notions of functional completeness do not coincide. The $\mathbf{BK}^{\Box_+\Box_-}$-regular connectives cannot define every truth-function definable from $\{\vee, \wedge, \otimes, \oplus, \rightarrow, \sim, \bot, \top, \mathsf{b}, \mathsf{n}\}$. Take for example again the verification and falsification clauses for $\circlearrowright$: $w \Vdash^+ \circlearrowright A$ iff $(w \Vdash^+ A \mathbin{\Cap} \neg w \Vdash^- A) \mathbin{\Cup} (\neg w \Vdash^+ A \mathbin{\Cap} \neg w \Vdash^- A)$ and $w \Vdash^- \circlearrowright A$ iff $(w \Vdash^+ A \mathbin{\Cap} \neg w \Vdash^- A) \mathbin{\Cup} (w \Vdash^+ A \mathbin{\Cap} w \Vdash^- A)$. It is easy to see that $\circlearrowright$ is not a $\mathbf{BK}^{\Box^+\Box^-}$-regular connective, since the negation in front of a relational atom will disappear and, hence, this connective cannot be expressed as a regular connective. This gives rise to the question of how to close the gap between the functional completeness of the non-modal fragment of $\mathbf{BK}^{\Box^+\Box^-}$ and the expressive incompleteness of the logic in terms of definability by regular metalogical connectives.

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