Logic and **L**ogical **P**hilosophy
Volume 28 (2019), 537–551
DOI: 10.12775/LLP.2019.004

Wolfgang Lenzen

LEIBNIZ'S LAWS OF CONSISTENCY AND THE PHILOSOPHICAL FOUNDATIONS OF CONNEXIVE LOGIC

Abstract. As an extension of the traditional theory of the syllogism, Leibniz's algebra of concepts is built up from the term-logical operators of conjunction, negation, and the relation of containment.

Leibniz's laws of consistency state that no concept contains its own negation, and that if concept A contains concept B, then A cannot also contain Not-B. Leibniz believed that these principles would be universally valid, but he eventually discovered that they have to be restricted to *self-consistent* concepts.

This result is of utmost importance for the philosophical foundations of connexive logic, i.e. for the question how far either "Aristotle's Thesis", $\neg(\alpha \to \neg\alpha)$, or "Boethius's Thesis", $(\alpha \to \beta) \to \neg(\alpha \to \neg\beta)$, should be accepted as reasonable principles of a logic of conditionals.

Keywords: connexive logic; Leibniz's logic; term logic vs. propositional logic

1. Introduction

Connexive *logic* may briefly be described as a (non-classical) logic in which the *implication operator*, $\to$, is connexive. The latter condition may in turn be explained by the requirement that "no formula provably implies or is implied by its own negation" [see 13]. At some greater length, Pizzi/Williamson put forward the following claims about an "intuitively conceived relation of implication":

(1) No proposition implies its own negation.
(2) No proposition implies each of two contradictory propositions.
(3) No proposition implies every proposition.
(4) No proposition is implied by every proposition. [12, p. 569]

Special Issue: Advances in Connexive Logic. Guest Editors: Hitoshi Omori and Heinrich Wansing

Published online January 28, 2019

Aristotle, Boethius, Chrysippus, and Abelard are often considered as advocates of a connexive conception of implication. In particular, principle

ARIST $\neg(\alpha \to \neg\alpha)$

has been claimed by McCall [11, p. 415] to represent Aristotle's view about implication; therefore it is commonly referred to as "Aristotle's [first] thesis". Closely related to ARIST is the subsequent principle:

BOETH If $(\alpha \to \beta)$, then not (also) $(\alpha \to \neg\beta)$.

It has been dubbed "Boethius' thesis" because, according to McCall, Boethius defended the view that "the two implications 'If q then r' and 'If q then not-r' are incompatible".[1]

In formulas ARIST, BOETH and throughout this paper, '$\neg$' symbolizes propositional *negation* and '$\to$' some kind of *strict* or *logical implication*. Furthermore, we use '$\wedge$' and '$\vee$' as symbols for (propositional) *conjunction* and *disjunction*; '$\Diamond$', '$\Box$' as symbols for the modal operators 'it is *possible* that' and 'it is *necessary* that'; and '$\forall x$' and '$\exists x$' to abbreviate the quantifiers 'for every x' and 'for at least one x'.

It almost goes without saying that the whole issue of connexive logic does *not* concern the operator of *material* implication (symbolized by '$\supset$') because $\neg(\alpha \supset \neg\alpha)$ is truth-functionally equivalent to $(\alpha \wedge \neg\neg\alpha)$, i.e. $\alpha \wedge \alpha$, or simply α. Hence, if the '$\to$' in ARIST were understood in the sense of '$\supset$', "Aristotle's thesis" would amount to the absurd claim that every proposition α is true!

According to Sextus Empiricus, Chrysippus considered a conditional as "sound when the contradictory of its consequent is incompatible with its antecedent". With the help of a binary operator '$\bullet$' denoting that α is compatible with β, McCall formalized this Chrysippian definition of implication as follows:

CHRYS $(\alpha \to \beta) \leftrightarrow \neg(\alpha \bullet \neg\beta)$.[2]

[1] Cf. [11, p. 416]. McCall further pointed out that as one of his "centrepieces of [a] theory of conditionals" Abelard defended the following variant of BOETH: $\neg[(\alpha \to \beta) \wedge (\alpha \to \neg\beta)]$. In [11, p. 417] this principle is stated with '$\sim$' as a symbol of negation, '&' as a symbol of conjunction and 'p' and 'q' as propositional variables instead of 'α' and 'β'.

[2] Cf. [10, p. 435], McCall uses 'A', 'B' instead of 'α', 'β', and he denotes the negation of A by $\bar{A}$.

As McCall showed in [10, p. 435], Chrys entails Boeth provided that one presupposes "the plausible thesis that if α implies β, α is compatible with β", i.e. provided that one accepts the following thesis:

McCall If $(\alpha \to \beta)$, then $(\alpha \bullet \beta)$.

It seems important to scrutinize the historical sources in order to find out whether the ancient logicians really defended a connexive conception of implication as expressed in theses Arist, Boeth, Chrys (and McCall). For reasons of space, however, this rewarding task cannot be carried out here but must be reserved for another occasion. The present paper will instead focus on Leibniz's logic, which sheds some important light on the philosophical foundations of connexive implication.

After giving a brief survey of the traditional theory of the syllogism in Section 2, Leibniz's algebra of concepts will be outlined in Section 3. In Section 4 it will be shown how this *term logic* may be transformed into a system of *propositional logic*. This system constitutes a calculus of strict implication in which the following *restrictions* of Arist and Boeth become provable:

Leib 1 If $\Diamond\alpha$, then $\neg(\alpha \to \neg\alpha)$
Leib 2 If $\Diamond\alpha$, then if $(\alpha \to \beta)$, then not (also) $(\alpha \to \neg\beta)$.

In section 5 it will be argued that the acceptance of the *unrestricted* axioms of connexive implication would be tantamount to assuming that (even) a *contradictory proposition* doesn't entail its own negation:

Conn 1 If $\neg\Diamond\alpha$, then $\neg(\alpha \to \neg\alpha)$.

As a corollary of Conn 1 one obtains in particular:

Conn 2 $\neg((\alpha \wedge \neg\alpha) \to \neg(\alpha \wedge \neg\alpha))$.

Hence any «hardcore» connexivist has to bite the bullet and maintain that the self-contradiction $(\alpha \wedge \neg\alpha)$ does not entail the tautology $\neg(\alpha \wedge \neg\alpha)$!

2. The Theory of the Syllogism

Traditional logic is mainly concerned with the four *categorical forms* of universal (U) and particular (P) propositions which can be either affirmative (A) or negative (N):

UA	Every S is P	UN	No S is P
PA	Some S is P	PN	Some S isn't P.

Here the *subject*, S, and the *predicate*, P, are supposed to be *concepts* such as 'man', 'wise', 'animal', etc.[3] Medieval logicians introduced the vowels a, e, i, and o to abbreviate these forms as:

UA	$S \mathrm{a} P$	UN	$S \mathrm{e} P$
PA	$S \mathrm{i} P$	PN	$S \mathrm{o} P$.[4]

In terms of first order logic, these formulas are nowadays interpreted as:

UA	$\forall x(Sx \supset Px)$	UN	$\forall x(Sx \supset \neg Px)$
PA	$\exists x(Sx \wedge Px)$	PN	$\exists x(Sx \wedge \neg Px)$.

There are mainly three groups of so-called *«simple» laws*. The theory of *opposition* states that the UA and the PA are *contradictory* to each other, just as are the UN and the PA:

Opp 1 $\neg(S \mathrm{a} P) \leftrightarrow (S \mathrm{o} P)$
Opp 2 $\neg(S \mathrm{e} P) \leftrightarrow (S \mathrm{i} P)$.

The theory of *subalternation* states that an (affirmative or negative) *universal* proposition entails its *particular* counterpart:

Sub 1* $S \mathrm{a} P \rightarrow S \mathrm{i} P$
Sub 2* $S \mathrm{e} P \rightarrow S \mathrm{o} P$.[5]

The theory of *conversion* states that the UN and the PA may be converted «simpliciter», while the UA may only be converted «per accidens»[6]:

Conv 1 $S \mathrm{e} P \rightarrow P \mathrm{e} S$
Conv 2 $S \mathrm{i} P \rightarrow P \mathrm{i} S$
Conv 3* $S \mathrm{a} P \rightarrow P \mathrm{i} S$.[7]

---

[3] In addition to *general* terms, sometimes also *singular* terms such as 'Aristotle' are taken into account.

[4] In [2, pp. 244 f], the invention of the so-called mnemonic syllogistic is attributed to Petrus Hispanus.

[5] The '*' behind the «name» of the formula is meant to indicate that the principle is *not* (entirely) *valid*.

[6] The PN, in contrast, does not allow any conversion at all.

[7] Principle Conv 3 is redundant since it follows from Sub 1 and Conv 2; similarly, Conv 1 together with Sub 2 entails that the UN may also be converted «accidentally»: $S \mathrm{e} P \rightarrow P \mathrm{o} S$. Furthermore, Conv 1 might be strengthened into a biconditional: $S \mathrm{e} P \leftrightarrow P \mathrm{e} S$; and from this one easily derives $\neg(S \mathrm{e} P) \leftrightarrow \neg(P \mathrm{e} S)$, i.e. in view of Sub 2: $S \mathrm{i} P \leftrightarrow P \mathrm{i} S$. Hence also Conv 2 may be strengthened into a biconditional.

While the laws of opposition and «simple» conversion are easily verified by the contemporary interpretation of the categorical forms, the principles of subalternation (and of «accidental» conversion) remain somewhat problematic since, e.g., the inference from $\forall x(Sx \supset Px)$ to $\exists x(Sx \wedge Px)$ only holds when the subject is not «empty», i.e. when there exists at least one x such that Sx.

Medieval logicians explicitly acknowledged the use of *negative terms*. Let C be any concept occupying the subject or predicate position of a categorical form. Then its negation Not-C shall be symbolized by means of the operator '$\sim$' as '$\sim C$'. In analogy to the principle for *propositional negation*, $(\neg\neg\alpha \leftrightarrow \alpha)$, the negation operator $\sim$ satisfies the law of double negation

Neg 1 $\quad \sim\sim C = C$.

The laws of *obversion* allow to reduce the negative propositions $S\,\mathrm{e}\,P$ and $S\,\mathrm{o}\,P$ to corresponding affirmative propositions with negated predicates:

Obv 1 $\quad S\,\mathrm{e}\,P \leftrightarrow S\,\mathrm{a}\,{\sim}P$
Obv 2 $\quad S\,\mathrm{o}\,P \leftrightarrow S\,\mathrm{i}\,{\sim}P$.[8]

Furthermore, in analogy to the principle of *propositional contraposition*, $(\alpha \rightarrow \beta) \leftrightarrow (\neg\beta \rightarrow \neg\alpha)$, in the realm of term logic the following law of «conversion by contraposition» is taken to hold:

Contra $\quad S\,\mathrm{a}\,P \leftrightarrow {\sim}P\,\mathrm{a}\,{\sim}S$.

The *syllogistic «moods»* are inferences leading from *two* premises P_1, P_2 to a conclusion Q. The subject, S, and the predicate, P, of the conclusion are called the *minor* and the *major* term, respectively, and each of the premises contains a third, or *middle*, term M, which will be related to S and to P. The most famous moods of the so-called First figure are:

Barbara $\quad M\,\mathrm{a}\,P, S\,\mathrm{a}\,M \Rightarrow S\,\mathrm{a}\,P$[9]
Celarent $\quad M\,\mathrm{e}\,P, S\,\mathrm{a}\,M \Rightarrow S\,\mathrm{e}\,P$
Darii $\quad M\,\mathrm{a}\,P, S\,\mathrm{i}\,M \Rightarrow S\,\mathrm{i}\,P$
Ferio $\quad M\,\mathrm{e}\,P, S\,\mathrm{i}\,M \Rightarrow S\,\mathrm{o}\,P$.

---

[8] Similarly, all affirmative propositions might be transformed into negative propositions by means of the laws $(S\,\mathrm{a}\,P \leftrightarrow S\,\mathrm{e}\,{\sim}P)$ and $(S\,\mathrm{i}\,P \leftrightarrow S\,\mathrm{o}\,{\sim}P)$.

[9] We use the arrow '$\Rightarrow$' instead of '$\rightarrow$' to formalize logical inferences.

Within the theory of the syllogism, *propositional* inferences and transformations are usually carried out only *implicitly*. In particular, the inference of so-called «regress»:

REGR If $P_1, P_2 \Rightarrow Q$, then $P_1, \neg Q \Rightarrow \neg P_2$

is often tacitly presupposed for deriving the moods of the Second and the Third from those of the First Figure.[10] The soundness of REGR follows from the definition of a valid inference which says that $\{P_1, \ldots, P_n\} \Rightarrow Q$ is logically valid if and only if it can't be the case that all premises P_i are true and yet the conclusion Q be false. In the special case with only *one* premise, schema REGR reduces to what later came to be called *modus tollens*:

TOLL If $P \Rightarrow Q$ and if $\neg Q$, then $\neg P$.

3. Leibniz's Algebra of Concepts

As has been shown at some length in [7] and [8], the basic system of Leibniz's logic of concepts uses the following primitive elements:

(a) A (possibly infinite) number of concept letters, A, B, C, ...
(b) The relation of conceptual *containment*, 'A contains B', formally $A \mathrel{\varepsilon} B$.
(c) The operation of conceptual *conjunction* joining two concepts A and B into 'A and B', or formally AB.
(d) The operator of conceptual *negation* which, for any concept A, yields the concept 'Not-A', formally ${\sim}A$.

Two further operators may be defined as follows:

(e) Conceptual *identity* as mutual conceptual containment: $A = B =_{\mathrm{df}} A \mathrel{\varepsilon} B \wedge B \mathrel{\varepsilon} A$
(f) *Possibility* (or self-consistency) of a concept A as obtaining whenever A does not contain a contradictory conjunction like $B \sim B$: $\mathrm{P}(A) =_{\mathrm{df}} \neg(A \mathrel{\varepsilon} B \sim B)$.[11]

The algebra of concepts may be axiomatized, e.g., by means of the fol-

[10] A comprehensive derivation was given by Leibniz in "De formis syllogismorum mathematice definiendis" [cf. 3, pp. 410–416].

[11] A referee of this paper raised doubts whether the definiens $\neg A \mathrel{\varepsilon} B \sim B$ (with 'B' as a free, unbound variable) would be formally *adequate*. He suggested to use instead the explicitly quantified variant "for all B: not: A est $B \sim B$". As a matter of

lowing principles put forward by Leibniz in [8]:

CONT 1	$A \,\varepsilon\, A$
CONT 2	$A \,\varepsilon\, B \wedge B \,\varepsilon\, C \rightarrow A \,\varepsilon\, C$
CONJ 1	$A \,\varepsilon\, BC \leftrightarrow A \,\varepsilon\, B \wedge A \,\varepsilon\, C$
NEG 1	$\sim\sim A = A$
NEG 2	$A \,\varepsilon\, B \leftrightarrow \sim B \,\varepsilon\, \sim A$
NEG 3	$A \,\varepsilon\, B \leftrightarrow \neg \mathrm{P}(A \sim B)$.[12]

All these laws are validated by the following (extensional) semantics which accords with Leibniz's own intuitions:

DEFINITION 1. An *extensional interpretation* of *L1* is a triple $\langle \mathbf{U}, \mathbf{Ext}, \mathbf{Val} \rangle$ such that

1. $\mathbf{U}$ (the universe of discourse) is a non-empty set (intuitively to be regarded as the set of all possible individuals);
2. $\mathbf{Ext}$ is a function which assigns to each concept letter A a subset of $\mathbf{U}$ (to be regarded as the *extension* of concept A) such that
   2.1. $\mathbf{Ext}(\sim A) = \overline{\mathbf{Ext}(A)}$
   2.2. $\mathbf{Ext}(AB) = \mathbf{Ext}(A) \cap \mathbf{Ext}(B)$
3. $\mathbf{Val}$ is a *valuation function* which assigns to each proposition α a truth-value 't' or 'f' such that
   3.1. $\mathbf{Val}(A \,\varepsilon\, B) = t$ if and only if $\mathbf{Ext}(A) \subseteq \mathbf{Ext}(B)$
   3.2. $\mathbf{Val}(\mathrm{P}(A)) = t$ if and only if $\mathbf{Ext}(A) \neq \varnothing$.

Condition (3.1) is the formal counterpart of the principle of the *reciprocity of extension and intension* which Leibniz put forward, e.g., in [6]:

> The common manner of statement concerns individuals, whereas Aristotle's refers rather to ideas or universals. For when I say *Every man is an animal* I mean that all the men are included among all the animals;

---

fact, Leibniz himself expressed this condition alternatively by means of an «indefinite concept» Y as $\neg(A \,\varepsilon\, Y \sim Y)$. Cf. [4, p. 749, fn. 8]: "A non-A *contradictorium* est. *Possibile* est quod non continet contradictorium seu A non-A. Possibile est quod non est: Y non-Y." However, both conditions are provably equivalent, and one may further simplify the definition of $\mathrm{P}(A)$ by requiring that A doesn't contain its own negation: $\neg(A \,\varepsilon \sim A)$.

[12] Cf. the following §§ of [4]: "B is B" (§37); "[...] if A is B and B is C, A will be C" (§19); "That A contains B and A contains C is the same as that A contains BC" (§35); "Not-not-A = A" (§96); "In general, A is B is the same as Not-B is Not-A" (§77); "A Not-B is not a thing is equivalent to the universal affirmative, Every A is B" (§169).

> but at the same time I mean that the idea of animal is included in the idea of man. 'Animal' comprises more individuals than 'man' does, but 'man' comprises more ideas or more attributes: one has more instances, the other more degrees of reality; one has the greater extension, the other the greater intension. [cf. 6, Book IV, ch. XVII, §8; p. 469]

If '$\mathrm{Int}(A)$' and '$\mathrm{Ext}(A)$' abbreviate the «intension» and the extension of concept A, respectively, then the *law of reciprocity* can be formalized as follows:

RECI 1 $\quad \mathrm{Int}(A) \subseteq \mathrm{Int}(B) \leftrightarrow \mathrm{Ext}(A) \supseteq \mathrm{Ext}(B)$.

From this it immediately follows that two concepts have the same «intension» if and only if they have the same extension:

RECI 2 $\quad \mathrm{Int}(A) = \mathrm{Int}(B) \leftrightarrow \mathrm{Ext}(A) = \mathrm{Ext}(B)$.[13]

This somewhat surprising result might seem to unveil an inadequacy of Leibniz's logic. However, «intensionality» in the sense of traditional logic must not be mixed up with intensionality in the modern sense. Furthermore, in Leibniz's view, the extension of concept A is not just the set of *actually existing* individuals, but rather the set of all *possible* individuals that fall under concept A. This observation helps to justify also condition (3.2) according to which the proposition 'A is possible' is evaluated as true if and only if A's extension is not empty. Clearly, if A

[13] A referee of a previous version of this paper was puzzled by the use of the *set-theoretical* operator '$\subseteq$' for formalizing the law of reciprocity. "What properly means that $\mathrm{Int}(A)$ *set-theoretically* includes $\mathrm{Int}(B)$? How can this be possible?" Well, the answer is: If one follows the traditional approach as described, e.g., in the "Logic of Port Royal", then the intension (or «comprehension») of a concept A is the *set* of all «*attributes*» which are contained in A [cf. 1, p. 59]. Thus $\mathrm{Int}(A) \supseteq \mathrm{Int}(B)$ iff for every attribute C: If $C \in \mathrm{Int}(B)$, then $C \in \mathrm{Int}(A)$, i.e. whenever B contains C (in the sense of $B \mathrel{\varepsilon} C$), then A also contains C.

Moreover, the referee thought that the law of reciprocity doesn't cope with negative terms and/or negative propositions. He pointed out that, e.g., 'No man is a dog' is true, according to the extensional point of view, iff the sets Ext(Man) and Ext(Dog) are totally *disjoint*. And then he wondered: "But what is the reciprocal of this from the intensional point of view? The concept of man and that of dog have a lot of properties in common." As a matter of fact, both concepts contain, e.g., the attribute 'living being', so that $\mathrm{Int}(\mathrm{Man}) \cap \mathrm{Int}(\mathrm{Dog}) \neq \varnothing$. But this fact poses no problem for the law of reciprocity. 'No man is a dog' can be paraphrased as 'Every man is a not-dog'; hence $\mathrm{Ext}(\mathrm{Man}) \subseteq \mathrm{Ext}(\mathrm{Not\text{-}Dog})$; therefore, according to RECI 1, $\mathrm{Int}(\mathrm{Man}) \supseteq \mathrm{Int}(\mathrm{Not\text{-}Dog})$, i.e. every $C \in \mathrm{Int}(\mathrm{Not\text{-}Dog})$ must also be an element of Int(Man). This means: whenever $(\mathrm{Not\text{-}Dog}) \mathrel{\varepsilon} C$, then $(\mathrm{Man}) \mathrel{\varepsilon} C$. But this holds simply because $(\mathrm{Man}) \mathrel{\varepsilon} (\mathrm{Not\text{-}Dog})$!

is possible, then there must exist at least one *possible individual* x that falls under that concept!

Let us now consider some laws of the algebra *L1* which are of particular relevance for the issue of connexive logic. In several drafts of his Calculus, Leibniz put forward two laws of *consistency* stating (i) that a concept A will never contain its own negation, and (ii) that if A contains B, A will not also contain $\sim B$:

NEG 4* $\neg(A\,\varepsilon \sim A)$
NEG 5* $A\,\varepsilon\,B \rightarrow \neg(A\,\varepsilon \sim B)$.[14]

NEG 4* was formulated in §43 of [4] as follows: "It is false that B contains Not-B, or B does not contain Not-B". In the subsequent §44, Leibniz added the variant "that it is false that Not-B contains B", and yet another § later he derived the corollary "It is false that B and Not-B coincide", i.e.

NEG 6 $\neg(A = \sim A)$.

Now while NEG 6 actually is a (semantically valid) theorem of *L1*, principles NEG 4* and NEG 5* are not *entirely* valid but have to be *restricted to self-consistent concepts*:

NEG 4 $\mathrm{P}(A) \rightarrow \neg(A\,\varepsilon \sim A)$
NEG 5 $\mathrm{P}(A) \rightarrow (A\,\varepsilon\,B \rightarrow \neg(A\,\varepsilon \sim B)$.

Leibniz himself tried to prove NEG 4* by starting from axiom CONT 1, $A\,\varepsilon\,A$, and inferring that *therefore* $\neg(A\,\varepsilon \sim A)$ because otherwise A would contain both A and $\sim A$, i.e. a contradiction:

> B contains B (by 37); therefore it does not contain Not-B, otherwise it would be impossible (by 32).

While § 37 mentioned in this proof contains just another version of the law of reflexivity of 'ε', § 32 says that a conjunction like A (and) Not-A is always impossible[15]:

NEG 7 $\neg\mathrm{P}(A \sim A)$.

Hence the «laws» NEG 4* and NEG 5*, which at first sight appeared to be *generally* valid, in fact only hold if concept A is self-consistent!

This result sheds some interesting light on the long-standing controversy about *subalternation*! As was mentioned in section 2, traditional

[15] Cf. [4, §32]: "B Not-B is impossible, or if B Not-$B = C$, C will be impossible".

logic always considered it as evident that a universal proposition entails its particular counterpart:

SUB 1* $A \,\mathrm{a}\, B \to A \,\mathrm{i}\, B$
SUB 2* $A \,\mathrm{e}\, B \to A \,\mathrm{o}\, B$.

As a matter of fact, one may even give a syllogistic «*proof*» of these laws, but this «proof» relies on the following two «identities»:

ID 1 $A \,\mathrm{a}\, A$
ID 2* $A \,\mathrm{i}\, A$.[16]

Now while ID 1 indeed is an identity which in terms of 1st order logic amounts to $\forall x(Ax \supset Ax)$, ID 2* amounts to condition $\exists x(Ax \wedge Ax)$ which *fails to be true* in the case where the extension of A is «empty»! Since, in Leibniz's logic, the categorical forms can be represented by the following formulas:

UA	$A \,\varepsilon\, B$	UN	$A \,\varepsilon\, {\sim}B$
PA	$\neg(A \,\varepsilon\, {\sim}B)$	PN	$\neg(A \,\varepsilon\, B)$,

the former principle of *consistency*, NEG 5*, represents a formalized version of principle SUB 1*! So in the same way as NEG 4* and NEG 5* must be restricted to NEG 4 and NEG 5, respectively, so also the laws of subalternation and the second «identity» have to be restricted to self-consistent concepts:

SUB 1 If $\mathrm{P}(A)$, then $(A \,\mathrm{a}\, B \to A \,\mathrm{i}\, B)$
SUB 2 If $\mathrm{P}(A)$, then $(A \,\mathrm{e}\, B \to A \,\mathrm{o}\, B)$
ID 2 If $\mathrm{P}(A)$, then $A \,\mathrm{i}\, A$.

Just remember that in view of the extensional semantics developed above, concept A is self-consistent if and only if A has a non-empty extension (within the domain of all possible individuals, $\mathtt{U}$)!

4. Leibniz's (modal) propositional logic *PL1*

Leibniz discovered a simple, ingenious method to transform the algebra of concepts, *L1*, into an algebra of propositions, *PL1*. Already in [5]

[16] Leibniz put forward these proofs, e.g., in "De formis syllogismorum mathematice definiendis"; [cf. 3, p. 412]: "*Omne A est B. Quoddam A est A. Ergo Quoddam A est B, quod est argumentum in Darii.*"

(written between 1683 and 1685), he pointed out to the parallel between the *containment* relation among *concepts* and the *implication* relation among *propositions*. Just as the categorical proposition 'A is B' is true, "when the predicate [A] is contained in the subject" B, so a *conditional* proposition 'If A is B, then C is D' is true, "when the consequent is contained in the antecedent" [cf. 5, p. 551]. In later works Leibniz compressed this idea into formulations such as "a proposition is true whose predicate is contained in the subject *or more generally* whose consequent is contained in the antecedent" [cf. 3, p. 401]. The most detailed explanation of this idea was given in §§ 75 and 189 of [4]:

> If, as I hope, I can conceive all propositions as terms, and hypotheticals as categoricals and if I can treat all propositions universally, this promises a wonderful ease in my symbolism and analysis of concepts, and will be a discovery of the greatest importance. [...] Our principles, therefore, will be these [...] whatever is said of a term which contains a term can also be said of a proposition from which another proposition follows.

To conceive all propositions in analogy to concepts means in particular that the *conditional* 'If α then β' will be logically treated like the *containment* relation between concepts, 'A contains B'. Furthermore, as Leibniz explained elsewhere, negations and conjunctions of *propositions* are to be conceived just as negations and conjunctions of *concepts*. Thus one obtains the following mapping of the formulas of *L1* into formulas of *PL1*:

$A \,\varepsilon\, B$	$\alpha \rightarrow \beta$
$A = B$	$\alpha \leftrightarrow \beta$
$\sim A$	$\neg\alpha$
AB	$\alpha \wedge \beta$
$\mathrm{P}(A)$	$\Diamond\alpha$.

As Leibniz himself noticed, the fundamental law NEG 3, $A \,\varepsilon\, B \leftrightarrow \neg\mathrm{P}(A{\sim}B)$, not only holds for the containment-relation between concepts but also for the entailment relation between propositions:

> 'A contains B' is a true proposition if 'A Not-B' entails a contradiction. This applies both to categorical and to hypothetical propositions.
>
> [cf. 3, p. 407]

Hence, as a propositional analogue of Neg 3, one obtains formula $(\alpha \to \beta) \leftrightarrow \neg\Diamond(\alpha \wedge \neg\beta)$ which unmistakably shows that Leibniz's conditional operator is not a *material* but rather a *strict* implication.[17]

5. The philosophical foundations of connexive logic

As counterparts of the laws of consistency Neg 4 and Neg 5, Leibniz's propositional logic *PL1* contains the following theorems which are just restrictions of the principles Arist and Boeth to self-consistent propositions:

Leib 1 If $\Diamond\alpha$, then $\neg(\alpha \to \neg\alpha)$
Leib 2 If $\Diamond\alpha$, then $((\alpha \to \beta) \to \neg(\alpha \to \neg\beta))$.

It is important to note that the validity of Leib 1, 2 is not a peculiar feature of *Leibniz's* logic only. These principles rather are theorems of *any* «*normal*» modal logic which has '$\Diamond$' as a possibility-operator, '$\to$' as a relation of strict implication, and in which the counterpart of Neg 7, i.e. $\neg\Diamond(\alpha \wedge \neg\alpha)$, is provable.[18]

If a «hardcore» defender of connexivism insists that *absolutely no* proposition ever entails its own implication, this thesis therefore becomes equivalent to the claim that (even) if α is self-contradictory, α will not entail its own negation:

Conn 1 $\neg\Diamond\alpha \supset \neg(\alpha \to \neg\alpha)$.

As a corollary of Conn 1 the self-contradictory proposition $(\alpha \wedge \neg\alpha)$ doesn't entail its own tautological negation:

Conn 2 $\neg((\alpha \wedge \neg\alpha) \to \neg(\alpha \wedge \neg\alpha))$.

Of course, from a merely formal point of view, it may be worth while investigating the syntactical, proof-theoretical and semantic features of non-classical logics satisfying Conn 1, 2. But the following straightforward refutation makes clear that «hardcore connexivism» is incompatible with some *very elementary* principles of classical logic, to wit:

[17] A closer investigation of calculus *PL1* may be found in [7].

[18] A referee of this paper pointed out that Leib 2 is not a theorem of modal system K (which lacks the «truth-axiom» $\Box\alpha \to \alpha$). Maybe this is correct, but in systems at least as strong as T both «Leibnitian» principles become provable.

CONJ 1	$\alpha \wedge \beta \rightarrow \alpha$
CONJ 2	$\alpha \wedge \beta \rightarrow \beta$
CONTRA	If $(\alpha \rightarrow \beta)$, then $(\neg\beta \rightarrow \neg\alpha)$
TRANS	If $(\alpha \rightarrow \beta)$ and $(\beta \rightarrow \gamma)$, then $(\alpha \rightarrow \gamma)$.

PROOF.
(i) $\alpha \wedge \neg\alpha \rightarrow \alpha$ (CONJ 1)
(ii) $\alpha \wedge \neg\alpha \rightarrow \neg\alpha$ (CONJ 2)
(iii) If $(\alpha \wedge \neg\alpha \rightarrow \alpha)$, then $(\neg\alpha \rightarrow \neg(\alpha \wedge \alpha))$ (CONTRA)
(iv) $\neg\alpha \rightarrow \neg(\alpha \wedge \alpha)$ ((i), (iii), TRANS)
(v) $\alpha \wedge \neg\alpha \rightarrow \neg(\alpha \wedge \alpha)$ ((ii), (iv), TRANS) □

As Christopher Martin kindly informed me, a variant of this refutation of CONN 2 had already been put forward by the 12th century logician Alberic of Paris against "Abelard's second thesis":

ABEL 2 $\neg(\alpha \rightarrow \neg\alpha)$.[19]

Taken for granted that (necessarily) "if something is a human, then it is an animal", Alberic showed that the (impossible) assumption "Socrates is human and Socrates is not an animal" entails its own negation. According to Martin's reconstruction [see 9, pp. 191–192],

> Alberic argued by simplification that
> (1) If Socrates is human and Socrates is not an animal, then Socrates is not an animal, and by contraposition that
> (2) if Socrates is not an animal, then Socrates is not human. But by simplification and contraposition, it follows that
> (3) if Socrates is not human, then it is not the case that Socrates is human and Socrates is not an animal and so by transitivity that
> (4) if Socrates is human and Socrates is not an animal, then it is not the case that Socrates is human and Socrates is not an animal.

Martin further reports that according to a contemporary source "this argument was too much for [the old man] Abelard [so] that he simply accepted the conclusion". But Martin thinks it possible that Abelard might have found a way out of the problem by modifying the "principle of simplification", i.e. basically our conjunction principles CONJ 1, 2. In the *Introductiones Montanae minores* (written by a follower of Alberic) a possible argument against CONJ 2 had been formulated as follows:

> They said that 'if Socrates is human and Socrates is not an animal, then Socrates is not an animal' does not hold because a negation is not

> so powerful (*vehemens*) when joined with an affirmation as it is when it is alone, and something follows from a negation alone which does not follow from it when it is conjoined with an affirmation.
>
> [9, pp. 198–199, Note 76]

This argument represents a typical example of what one nowadays calls a *sophistic solution* of a problem. It is a *possible*, or at least *possibly possible* way out, which, however appears to have been invented entirely *ad hoc* and which is otherwise totally implausible and substantially unfounded.[20]

There is a simple lesson which contemporary logicians can learn from Leibniz's laws of consistency: *The basic idea* of connexive implication *is absolutely fine*:

(1) No «*normal*», i.e. no *self-consistent* proposition implies its own negation.
(2) No «normal» proposition implies each of two contradictory propositions.
(3) No «normal» proposition implies every proposition.

Just *one* exception: «*Abnormal*», *self-contradictory* propositions like $(\alpha \wedge \neg\alpha)$ necessarily *do* entail their own negation!

References

[1] Arnauld, Antoine, and Pierre Nicole, *La Logique ou L'Art de Penser*, 5th edition 1683. Reprint Paris 1965.

[2] Bocheński, Józef M., *Formale Logik*, Verlag Karl Alber GmbH, Freiburg, München, 1956.

[3] Couturat, Louis, *Opuscules et fragments inédits de Leibniz*, Paris, 1903. Reprint Hildesheim 1961.

[4] Leibniz, Gottfried Wilhelm, "Generales inquisitiones de analysi notionum et veritatum", pages 739–788 in *Academy Edition* (i.e. Gottfried Wilhelm Leibniz, *Sämtliche Schriften und Briefe*, hrg. von der Berlin-Brandenburgischen Akademie der Wissenschaften und der Akademie der Wissenschaften in Göttingen), Series VI, vol. 4, Berlin, 1999.

[20] The basic idea of the author of the *Introductiones Montanae minores* consists in restricting Conj 2, $\alpha \wedge \beta \to \beta$, to the case where β is an affirmative proposition, i.e. to deny the validity of the corresponding inference $\alpha \wedge \neg\gamma \to \neg\gamma$. This «way out» rests on the problematic presupposition that one might strictly distinguish between affirmative and negative propositions. But what about, e.g., $\alpha \wedge \neg\neg\delta \to \neg\neg\delta$?

[5] Leibniz, Gottfried Wilhelm, "Notationes generales", pages 550–557 in *Academy Edition*, Series VI, vol. 4, 1999.

[6] Leibniz, Gottfried Wilhelm, *Nouveaux essais sur l'entendement*, in *G. W. Leibniz, Die philosophischen Schriften*, edited by C. J. Gerhard, vol. 5, Berlin, 1882. Reprint Georg Olms Verlag, Hildesheim/New York, 1978.

[7] Lenzen, Wolfgang, "Leibniz's calculus of strict implication", pages 1–35 in J. Srzednicki (ed.), *Initiatives in Logic*, Martinus Nijhoff Publishers, Dordrecht, 1987. DOI: 10.1007/978-94-009-3673-7

[8] Lenzen, Wolfgang, "Leibniz's logic", pages 1–83 in D. Gabbay and J. Woods (eds.), *Handbook of the History of Logic*, Vol. 3, *The Rise of Modern Logic: From Leibniz to Frege*, Elsevier North Holland, Amsterdam, Boston, Heidelberg, London, New York, Oxford, Paris, San Diego, San Francisco, Singapore, Sydney, Tokyo, 2004. DOI: 10.1016/S1874-5857(04)80014-1

[9] Martin, Christopher J., "Logic", pages 158–199 in J. E. Brower and K. Guilfoy (eds.), *The Cambridge Companion to Abelard*, Chapter 5, Cambridge University Press, Cambridge UK, New York, 2006. DOI: 10.1017/CCOL0521772478

[10] McCall, Storrs, "Connexive implication", pages 434–452 in A. R. Anderson and N. D. Belnap (eds.), *Entailment: The Logic of Relevance and Necessity*, Vol. 1, Princeton University Press, Princeton, 1975.

[11] McCall, Storrs, "A history of connexivity", pages 415–449 in D. Gabbay et al. (eds.), *Handbook of the History of Logic*, vol. 11, Elsevier, 2012. DOI: 10.1016/B978-0-444-52937-4.50008-3

[12] Pizzi, C., and T. Williamson, "Strong Boethius' thesis and consequential implication", *Journal of Philosophical Logic* 26, 5 (1997): 569–588. DOI: 10.1023/A:1004230028063

[13] Wansing, Heinrich, "Connexive logic", in E. N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy*, Fall 2014 edition, 2014. http://plato.stanford.edu/archives/fall2014/entries/logic-connexive/

Wolfgang Lenzen
University of Osnabrück
P.O. Box
D-49069 Osnabrück, Germany
lenzen@uos.de