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REMARKS ON STABLE FORMULAS IN INTUITIONISTIC LOGIC

Abstract. ONNILLI-formulas were introduced in [2] and were shown to be the set of formulas that are preserved under monotonic images of descriptive or Kripke frames. As a result, ONNILLI is a syntactically defined set of formulas that axiomatize all stable logics. In this paper, among other things, by proving the uniform interpolation property for ONNILLI we show that ONNILLI is exactly the set of formulas that are preserved in monotonic bijections of descriptive or (finite) Kripke models. This resolves an open problem in [2].

Keywords: intuitionistic logic; intermediate logic; subframe logics; monotonic maps; stable logics; uniform interpolation property

Mathematics Subject Classification: 03B55, 03B20

Preliminaries

Subframe formulas for modal logic were first introduced by K. Fine [4], and for intuitionistic logic were defined by M. Zakharyaschev [6]. For an overview of the results, see [3]. Subframe logics are intermediate logics axiomatizable by subframe formulas. M. Zakharyaschev in [6, 7] (see also [3]) showed that subframe logics are exactly those logics axiomatized by $[\wedge, \to]$-formulas. He also showed that subframe logics are exactly the ones whose frames are closed under taking subframes.

In order to provide a syntactical equivalent for subframe formulas, in [2], N. Bezhanishvili and D. de Jongh used the NNIL-formulas of [5]. It

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was shown that NNIL-formulas are (up to frame equivalence) the formulas that are preserved under taking (descriptive or Kripke) subframes. As a result it was obtained that NNIL (up to frame equivalence) coincides with the class of subframe formulas. NNIL was defined in [5], and stands for **N**o **N**esting of **I**mplications to the **L**eft. In [5] it was shown that NNIL-formulas satisfy left and right approximation and uniform interpolation property, and that NNIL is exactly the set of formulas that are preserved under taking submodels of Kripke models.

Stable formulas for intuitionistic logic were defined by G. Bezhanishvili and N. Bezhanishvili in [1]. They showed that stable logics, i.e., intermediate logics axiomatizable by stable formulas, are exactly the ones whose class of rooted frames is preserved under monotonic images.

To syntactically define the set of stable formulas, N. Bezhanishvili and D. de Jongh introduced ONNILLI-formulas in [2], and showed that they are (up to frame equivalence) the formulas that are preserved in monotonic images of rooted (descriptive or Kripke) frames. As a result, ONNILLI is (up to frame equivalence) the class of stable formulas. As for the name, ONNILLI stands for **O**nly **NNIL** to the **L**eft of **I**mplications.

Stable formulas (ONNILLI, up to frame equivalence) play the same role for stable logics that subframe formulas (NNIL, up to frame equivalence) play for subframe logics. Also the role that subframes play for subframe formulas is played by monotonic images for stable formulas.

Whether ONNILLI is exactly the set of formulas that are preserved in monotonic images of rooted (descriptive or Kripke) models was left as an open problem in [2].

We use a similar method to that of [5] and prove analogous results for ONNILLI. To characterize the stable formulas syntactically, we first need to show that ONNILLI satisfies the uniform interpolation property, and hence left and right approximation. Using the acquired results, we prove that if a formula is preserved under surjective monotonic maps of rooted models, then it is provably equivalent to its left approximant. As a consequence, we prove a stronger version of the proposed problem in [2]; that is, ONNILLI is exactly the set of formulas preserved under bijective or surjective monotonic maps of rooted descriptive or (finite) Kripke models.

In the sequel, we briefly review some definitions and fix some notations. For the definition and facts about intuitionistic propositional logic IPC we refer to [3]. Our notations mostly coincide with those of [5]. Let $\mathcal{L}$ be the language of IPC consisting of PV, a fixed set of propositional variables p_0, p_1, ..., propositional connectives $\wedge$, $\vee$, $\rightarrow$, and a propositional constant $\bot$. We assume that p, q, r, ... range over propositional variables, ϕ, ψ, χ, ... range over arbitrary formulas, and $\vec{p}$, $\vec{q}$, $\vec{r}$, ... range over finite sets of propositional variables. For $\vec{p}$ and $\vec{q}$, we abbreviate $\vec{p} \cup \vec{q}$ by $\vec{p}, \vec{q}$. $\neg\phi$ is defined as $\phi \rightarrow \bot$, $\phi \leftrightarrow \psi$ is defined as $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$, and $\top$ is defined as $\neg\bot$. PV together with $\top$ and $\bot$ is the set of *atoms*. For a set of propositional variables $\mathcal{P}$, we write $\mathcal{L}(\mathcal{P})$ for $\mathcal{L}$ restricted to $\mathcal{P}$. Similar notations will be used for other classes of formulas. We use $\vdash \phi$, whenever ϕ is a theorem of IPC.

A set $R \subseteq X \times X$ is called a (binary) relation on X. We use xRy when $\langle x, y\rangle \in R$. Let $\mathrm{Dom}(R) := \{x \in X \mid \exists y\ \langle x, y\rangle \in R\}$ and $\mathrm{Im}(R) := \{y \in X \mid \exists x\ \langle x, y\rangle \in R\}$. Define $\hat{R} := \{\langle y, x\rangle \in X \times X \mid xRy\}$. Also, let $R(X) := \{y \in X \mid \exists x \in X\ xRy\}$ and $R(x) := R(\{x\})$. We also define $R\restriction_X := \{\langle x, y\rangle \in R \mid x \in X\}$. A reflexive, transitive relation is called a *quasi-order*. An antisymmetric quasi-order is called a *partial order*. If $\leq$ is a partial order over W, $U \subseteq W$ is said to be *upset* (with respect to $\leq$) if $u \in U$ and $u \leq v$ imply $v \in U$.

A *Kripke frame* is a structure $\mathfrak{F} = (W, \leq)$, where $W \neq \emptyset$ and $\leq$ is a partial order. In this paper, all our frames and models are automatically rooted unless explicitly mentioned, hence there exists $r \in W$ such that $\forall w \in W\ r \leq w$. We show the root of $\mathfrak{F}$ by $r_{\mathfrak{F}}$. A *Kripke model* is a structure $\mathfrak{M} = (\mathfrak{F}, \models)$, where $\mathfrak{F}$ is a Kripke frame and $\models$ is the atomic forcing relation on PV. Depending on the context, we may use $m \models_{\mathfrak{M}} p$ or simply $m \models p$ to denote it. When the forcing relation is restricted to a set $\mathcal{P} \subseteq PV$, $\mathfrak{M}$ is called a $\mathcal{P}$-model. Forcing should satisfy the following condition:

$$m \leq m' \text{ and } m \models p \Rightarrow m' \models p. \qquad \text{(Persistence)}$$

We say $\mathfrak{M} \models p$ when $\forall m \in W\ m \models p$, or equivalently $r \models p$. For a $\mathcal{P}$-model $\mathfrak{M}$ and $\phi \in \mathcal{L}(\mathcal{P})$, $\mathfrak{M} \models \phi$ is defined in the standard way (see, e.g., [3]). It is well-known that intuitionistic logic is sound and complete with respect to its finite Kripke models.

For a Kripke model $\mathfrak{M} = (W, \leq, \models)$, we denote W by $\mathfrak{M}$ when clear from the context. Since the underlying frame $\mathfrak{F} = (W, \leq)$ is rooted, so will be the model $\mathfrak{M}$, with $r_{\mathfrak{M}} := r_{\mathfrak{F}}$. Note that our Kripke models (frames) are what is usually called *rooted Kripke models (frames)*. Given a $\mathcal{P}$-model $\mathfrak{M}$ and $\mathcal{Q} \subseteq \mathcal{P}$, we write $\mathfrak{M}(\mathcal{Q})$ for the result of restricting the atomic forcing of $\mathfrak{M}$ to $\mathcal{Q}$.

For a set of formulas Φ, we define $\mathrm{Th}_{\Phi}(\mathfrak{M}) := \{\phi \in \Phi \mid \mathfrak{M} \models \phi\}$ and $\mathrm{Th}(\mathfrak{M}) := \mathrm{Th}_{\mathcal{L}}(\mathfrak{M})$. We use $\mathrm{Th}_{\Phi}(m)$ and $\mathrm{Th}_{\Phi}(\mathfrak{M}[m])$ interchangeably, where for $m \in \mathfrak{M}$, $\mathfrak{M}[m]$ is the generated submodel of $\mathfrak{M}$ with the root m. Whenever we talk about (partial) maps between Kripke models, we assume that they are *valuation preserving*: Given $\mathcal{P}$-models $\mathfrak{M}$ and $\mathfrak{N}$, a (partial) map $f : \mathfrak{M} \to \mathfrak{N}$ is valuation preserving when $\forall p \in \mathcal{P}\ \forall m \in \mathrm{Dom}(f)(m \models p \Leftrightarrow f(m) \models p)$.

For undefined notions such as general and descriptive models we refer the reader to [3] or [2].

NNIL-formulas in *normal form* are defined by

$$\phi := \bot \mid p \mid \phi \wedge \phi \mid \phi \vee \phi \mid p \to \phi.$$

The class ONNILLI is defined as the closure of $\{\phi \to \psi \mid \phi \in \mathrm{NNIL}, \psi \in \mathrm{BASIC}\}$ under conjunction and disjunction. Where BASIC is the closure of the set of the atoms under conjunction and disjunction.

We notice that, neither NNIL nor ONNILLI contains the other and that atoms and their negations belong to both NNIL and ONNLLI. Also, one can easily see that ONNILLI($\vec{p}$) is finite (modulo provable equivalence).

1. Subsimulations and Robustness

The connection between NNIL and zig-subsimulations was studied in [5]. In this section, we observe that zag-subsimulations will play the same role for ONNILLI. The obtained results will help us establish the uniform interpolation property for ONNILLI in Section 2. We will come back to subsimulations later when we prove IPC-equivalence of ONNILLI and the set of robust formulas under total zags.

We start by defining different kinds of subsimulations. The definitions are the same as [5], but the notations have been changed for convenience in our context.

Definition 1.1. Let $\mathfrak{M}$ and $\mathfrak{N}$ be $\mathcal{P}$-models. A relation R on $\mathfrak{M} \times \mathfrak{N}$ is called a *zag-subsimulation* (or shortly, *zag*) of $\mathfrak{M}$ in $\mathfrak{N}$ if it satisfies the following conditions:

- $mRn \Rightarrow \forall p \in \mathcal{P}\ (m \models p \Leftrightarrow n \models p)$,
- (back) $mRn \leq n' \Rightarrow \exists m'\ m \leq m'Rn'$.

When $\hat{R}$ is zag, we say R is *zig*, i.e., it satisfies the "forth" condition instead of "back":

- (forth) $m' \geq mRn \Rightarrow \exists n' \; m'Rn' \geq n$.

R is said to be *total* if its domain is $\mathfrak{M}$. If a total, surjective R is both zig and zag, it is called a *bisimulation*. If a zag (zig) R is root-preserving, it is called a *+-zag* (*+-zig*). A +-zag (+-zig) is automatically surjective (total). Let us fix some notations as follows

- $R : \mathfrak{M} \preceq \mathfrak{N} :\Leftrightarrow R$ is a total zag of $\mathfrak{M}$ in $\mathfrak{N}$,
- $R : \mathfrak{M} \preceq^{+} \mathfrak{N} :\Leftrightarrow R$ is a total +-zag of $\mathfrak{M}$ in $\mathfrak{N}$,
- $\mathfrak{M} \equiv \mathfrak{N} :\Leftrightarrow \mathfrak{M} \preceq \mathfrak{N}$ and $\mathfrak{N} \preceq \mathfrak{M}$,
- $R : \mathfrak{M} \simeq \mathfrak{N} :\Leftrightarrow R$ is a bisimulation.

We use $\mathfrak{M} \preceq \mathfrak{N}$ to show that there exists R such that $R\colon \mathfrak{M} \preceq \mathfrak{N}$, and analogous notation is used for other relations, too. The analogous relations for zigs are defined by a "zig" subscript, e.g., $R : \mathfrak{M} \preceq_{zig} \mathfrak{N}$ means that R is a total zig of $\mathfrak{M}$ in $\mathfrak{N}$, and so forth.

It is well-known that bisimulations preserve the truth of all formulas.

DEFINITION 1.2. Let $\mathfrak{M}$ and $\mathfrak{N}$ be $\mathcal{P}$-models. We say $\mathfrak{M} \ll^{(p)} \mathfrak{N}$ when $\mathfrak{M}$ is (partial) monotonic image of $\mathfrak{N}$, i.e., there exists a (partial) $f : \mathfrak{N} \to \mathfrak{M}$ such that

- f is valuation preserving,
- f is surjective,
- f is monotonic, i.e., $\forall n, n' \in \mathrm{Dom}(f)(n \leq_{\mathfrak{N}} n' \Rightarrow f(n) \leq_{\mathfrak{M}} f(n'))$,
- $\mathrm{Dom}(f)$ is upset.

Note that domains of functions are automatically upset. Similarly, we use $\mathfrak{M} \precsim^{(p)} \mathfrak{N}$ to show that $\mathfrak{M}$ is a (partial) bijective monotonic image of $\mathfrak{N}$, i.e., the (partial) function f has the above conditions as well as being a bijection.

We also note that whenever we talk about partial functions throughout the text, it is assumed that their domain is upset.

Remark 1.1. Relations $\ll^{(p)}$ and $\precsim^{(p)}$ are quasi-orders. Trivially $\ll \subsetneq \ll^{p}$, $\precsim \subsetneq \precsim^{p}$, and $\precsim^{(p)} \subsetneq ll^{(p)}$. Its easy to see that the inverse of a (partial) surjective monotonic map $f\colon \mathfrak{N} \to \mathfrak{M}$ is a total zag of $\mathfrak{M}$ in $\mathfrak{N}$. Therefore, $\precsim^{(p)} \subsetneq \ll^{(p)} \subsetneq \preceq$. That the second equality does not hold can be seen in Figure 1.

Remark 1.1 asserts that $\preceq$, $\ll^{(p)}$ and $\precsim^{(p)}$ are not equivalent; however, we shall see that as far as robustness is concerned, they are equiva-

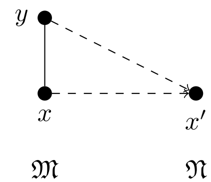

Figure 1. $\mathfrak{M} \preceq \mathfrak{N}$ but $\mathfrak{M} \not\ll^{(p)} \mathfrak{N}$ (dashed arrows show a total zag)

lent. Robustness is defined for a set of formulas with respect to a relation between Kripke models as follows.

DEFINITION 1.3. Let $\trianglelefteq$ be a relation between Kripke models. A set of formulas $\varPhi$ is said to be $\trianglelefteq$-*robust* (*finitely* $\trianglelefteq$-*robust*) if for all $\phi \in \varPhi$, and (finite) Kripke models $\mathfrak{M}$ and $\mathfrak{N}$, $\mathfrak{M} \trianglelefteq \mathfrak{N}$ and $\mathfrak{N} \models \phi$ imply $\mathfrak{M} \models \phi$.

THEOREM 1.1 ([5]). *For arbitrary Kripke models $\mathfrak{M}$ and $\mathfrak{N}$:*

1. *If R is a zig of $\mathfrak{M}$ in $\mathfrak{N}$, then $mRn \Rightarrow \mathrm{Th}_{\mathrm{NNIL}}(n) \subseteq \mathrm{Th}_{\mathrm{NNIL}}(m)$.*
2. $\mathfrak{M} \preceq_{zig} \mathfrak{N} \Rightarrow \mathrm{Th}_{\mathrm{NNIL}}(\mathfrak{N}) \subseteq \mathrm{Th}_{\mathrm{NNIL}}(\mathfrak{M})$.

The next theorem is the analog of Theorem 1.1(2) for ONNILLI.

THEOREM 1.2. *For arbitrary Kripke models $\mathfrak{M}$ and $\mathfrak{N}$:*

$$\mathfrak{M} \preceq \mathfrak{N} \Rightarrow \mathrm{Th}_{\mathrm{ONNILLI}}(\mathfrak{N}) \subseteq \mathrm{Th}_{\mathrm{ONNILLI}}(\mathfrak{M}).$$

PROOF. Suppose $R\colon \mathfrak{M} \preceq \mathfrak{N}$ and $\mathfrak{M} \not\models \phi$ where $\phi \in$ ONNILLI. Since conjunctions and disjunctions preserve robustness, it suffices to assume that there exist $\psi \in$ NNIL and $\theta \in$ BASIC such that $\mathfrak{M} \not\models \psi \to \theta$. Therefore, for some $m \in \mathfrak{M}$, $m \models \psi$ but $m \not\models \theta$. Consider $n \in R(m)$, due to totality, such n exists. Clearly $n \not\models \theta$. Since $\hat{R}$ is zig, from Theorem 1.1 it follows that $n \models \psi$. Therefore $n \not\models \psi \to \theta$, implying $\mathfrak{N} \not\models \psi \to \theta$, or equivalently $\mathfrak{N} \not\models \phi$. ⊣

COROLLARY 1.3. ONNILLI *is $\preceq$-robust, $\ll^{(p)}$-robust and $\precsim^{(p)}$-robust.*

Remark 1.2. The analog of Theorem 1.1(1) does not hold for ONNILLI (as shown in Figure 2).

The Lifting Theorem for total zigs was proved in [5]. In this section we present an analogous version for total zags. In order to prove the Lifting Theorem, we prove lemmas 1.4 and 1.5 first. The proof of the former is highly influenced by that of Theorem 6.6 (Lifting Theorem) in

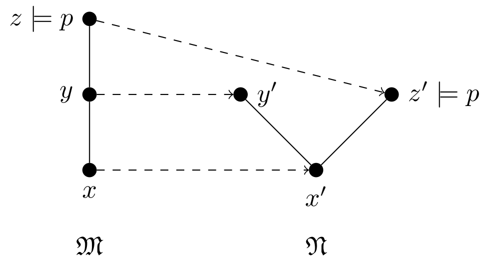

Figure 2. R is shown by dashed arrows, where $R : \mathfrak{M} \preceq \mathfrak{N}$. Here yRy' and $y' \models \neg p$, but $y \not\models \neg p$. Since $\neg p \in$ ONNILLI, $\mathrm{Th}_{\mathrm{ONNILLI}}(y') \not\subseteq \mathrm{Th}_{\mathrm{ONNILLI}}(y)$.

[5]. The difference between them lies in the nature of the defined models and relations.

In the following lemmas and the Lifting Theorem, some of $\vec{p}$, $\vec{q}$, and $\vec{r}$ might be infinite.

Lemma 1.4. *Let $\vec{q}$, $\vec{p}$ and $\vec{r}$ be disjoint sets of variables. Let $\mathfrak{M}$ be a $\vec{q}, \vec{p}$-model and $\mathfrak{N}$ be a $\vec{p}, \vec{r}$-model. Suppose $\mathfrak{M}(\vec{p}) \preceq^+ \mathfrak{N}(\vec{p})$. Then there are $\vec{q}, \vec{p}, \vec{r}$-models $\mathfrak{M}'$ and $\mathfrak{N}'$ such that $\mathfrak{M} \simeq \mathfrak{M}'(\vec{q}, \vec{p})$, $\mathfrak{M}' \precsim \mathfrak{N}'$ and $\mathfrak{N} \simeq \mathfrak{N}'(\vec{p}, \vec{r})$.*

Proof. Assume $R : \mathfrak{M}(\vec{p}) \preceq^+ \mathfrak{N}(\vec{p})$. Define $\vec{q}, \vec{p}, \vec{r}$-models $\mathfrak{M}'$ and $\mathfrak{N}'$ as follows:

- $\mathfrak{M}' := \{\langle m, n \rangle \mid mRn\}$;
- $\langle m, n \rangle \leq_{\mathfrak{M}'} \langle m', n' \rangle :\Leftrightarrow m <_{\mathfrak{M}} m'$ or ($m = m'$ and $n \leq_{\mathfrak{N}} n'$);
- $r_{\mathfrak{M}'} := \langle r_{\mathfrak{M}}, r_{\mathfrak{N}} \rangle$; $\langle m, n \rangle \models_{\mathfrak{M}'} s :\Leftrightarrow m \models_{\mathfrak{M}} s$ or $n \models_{\mathfrak{N}} s$;
- $\mathfrak{N}' := \{\langle m, n \rangle \mid mRn\}$;
- $\langle m, n \rangle \leq_{\mathfrak{N}'} \langle m', n' \rangle :\Leftrightarrow m \leq_{\mathfrak{M}} m'$ and $n \leq_{\mathfrak{N}} n'$;
- $r_{\mathfrak{N}'} := \langle r_{\mathfrak{M}}, r_{\mathfrak{N}} \rangle$; $\langle m, n \rangle \models_{\mathfrak{N}'} s :\Leftrightarrow m \models_{\mathfrak{M}} s$ or $n \models_{\mathfrak{N}} s$.

It is easy to see that for $s \in \vec{q}, \vec{p}$ we have $\langle m, n \rangle \models_{\mathfrak{M}'} s \Leftrightarrow \langle m, n \rangle \models_{\mathfrak{N}'} s \Leftrightarrow m \models s$. And for $s \in \vec{p}, \vec{r}$ we have $\langle m, n \rangle \models_{\mathfrak{M}'} s \Leftrightarrow \langle m, n \rangle \models_{\mathfrak{N}'} s \Leftrightarrow n \models s$. Particularly, $\mathfrak{M}' \models s \Leftrightarrow \mathfrak{N}' \models s$ for $s \in \vec{q}, \vec{p}, \vec{r}$.

Define B by $mB\langle m', n \rangle$ if $m = m'$, and C by $nC\langle m, n' \rangle$ if $n = n'$. For both B and C totality and the back condition trivially hold. We proceed by checking the forth condition. For B, assume that $mB\langle m, n \rangle$ and $m \leq_{\mathfrak{M}} m'$. By totality of R, there exists n' such that $m'Rn'$. Therefore $m'B\langle m', n' \rangle$ where $\langle m, n \rangle \leq_{\mathfrak{M}'} \langle m', n' \rangle$. For C, assume that $nB\langle m, n \rangle$ and $n \leq_{\mathfrak{N}} n'$. Since R is zag, there exists m' such that $m \leq_{\mathfrak{M}} m'Rn'$. Therefore $n'C\langle m', n' \rangle$ where $\langle m, n \rangle \leq_{\mathfrak{N}'} \langle m', n' \rangle$.

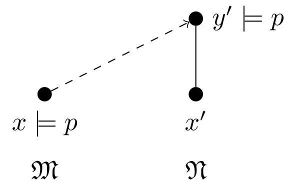

Figure 3. R as shown with dashed arrows is a total zag, hence $\mathfrak{M}(\vec{p}) \preceq \mathfrak{N}(\vec{p})$. But for all $\mathfrak{M}'$ and $\mathfrak{N}'$ such that $\mathfrak{M}'(\vec{p}) \simeq \mathfrak{M}$ and $\mathfrak{N}'(\vec{p}) \simeq \mathfrak{N}$, we have $\mathfrak{M}' \not\preceq^+ \mathfrak{N}'$ (therefore, $\mathfrak{M}' \not\ll \mathfrak{N}'$ and $\mathfrak{M}' \not\precsim \mathfrak{N}'$).

So $B\colon \mathfrak{M} \simeq \mathfrak{M}'(\vec{q}, \vec{p})$ and $C : \mathfrak{N} \simeq \mathfrak{N}'(\vec{p}, \vec{r})$. The map $f : \mathfrak{N}' \to \mathfrak{M}'$ defined as $f(\langle m, n\rangle) = \langle m, n\rangle$ is bijective, monotonic, and truth preserving for atoms. Therefore, $\mathfrak{M}' \precsim \mathfrak{N}'$. $\dashv$

Remark 1.3. Lemma 1.4 does not hold when we take out the "+", even if we replace $\precsim$ with the weaker relations $\ll$ or $\preceq^+$ (as shown in Lemma 3). However, as we shall see, it holds for $\precsim^p$ (therefore, $\ll^p$ and $\preceq$, too).

Lemma 1.5. *Let $\vec{q}$, $\vec{p}$ and $\vec{r}$ be disjoint sets of variables. Let $\mathfrak{M}$ be a $\vec{q}, \vec{p}$-model and $\mathfrak{N}$ be a $\vec{p}, \vec{r}$-model. Suppose $R\colon \mathfrak{M}(\vec{p}) \preceq \mathfrak{N}(\vec{p})$ where R is surjective. Then there are $\vec{q}, \vec{p}, \vec{r}$-models $\mathfrak{M}'$ and $\mathfrak{N}'$ such that: $\mathfrak{M} \simeq \mathfrak{M}'(\vec{q}, \vec{p})$, $\mathfrak{M}' \precsim^p \mathfrak{N}'$ and $\mathfrak{N} \simeq \mathfrak{N}'(\vec{p}, \vec{r})$.*

Proof. By totality and surjectivity, there exist $n \in \mathfrak{N}$ and $m \in \mathfrak{M}$ such that $r_{\mathfrak{M}} R n$ and $m R r_{\mathfrak{N}}$. Therefore, forcing over $\vec{p}$ at $r_{\mathfrak{M}}$ is identical with forcing over $\vec{p}$ at $r_{\mathfrak{N}}$. It is easy to see that $R \cup \{\langle r_{\mathfrak{M}}, r_{\mathfrak{N}}\rangle\}$ is a $+$-zag and the result follows from Lemma 1.4. $\dashv$

We are now ready to prove the Lifting Theorem. It will be used in this section to give us the equivalence of robustness under the defined quasi-orders. Later, in Lemma 2 it will result in the uniform interpolation property. Finally, in Lemma 3 it will be used to obtain the desired IPC-equivalence of ONNILLI and $\preceq$-robust formulas.

Theorem 1.6 (Lifting). *Let $\vec{q}$, $\vec{p}$ and $\vec{r}$ be disjoint sets of variables. Let $\mathfrak{M}$ be a $\vec{q}, \vec{p}$-model and $\mathfrak{N}$ be a $\vec{p}, \vec{r}$-model. Suppose $\mathfrak{M}(\vec{p}) \preceq \mathfrak{N}(\vec{p})$. Then there are $\vec{q}, \vec{p}, \vec{r}$-models $\mathfrak{M}'$ and $\mathfrak{N}'$ such that: $\mathfrak{M} \simeq \mathfrak{M}'(\vec{q}, \vec{p})$, $\mathfrak{M}' \precsim^p \mathfrak{N}'$ and $\mathfrak{N} \simeq \mathfrak{N}'(\vec{p}, \vec{r})$.*

Proof. Assume that $R : \mathfrak{M} \preceq \mathfrak{N}$. By Lemma 1.5, it suffices to consider non-surjective zags only. Also, without loss of generalization assume that $R(r_{\mathfrak{M}})$ has a minimum element n_0.

Define $\mathfrak{M}'$ and $\mathfrak{N}'$ as in the proof of Lemma 1.4, except that now $r_{\mathfrak{M}'} = \langle r_{\mathfrak{M}}, n_0\rangle$ and $\mathfrak{N}'$ is possibly unrooted. Since R is not surjective, $r_{\mathfrak{N}} \notin \mathrm{Im}(R)$. Let

- $\mathfrak{N}'' := \mathfrak{N}' \cup \{\langle n\rangle \mid n \notin \mathrm{Im}(R)\}$,
- $\leq_{\mathfrak{N}''} := \leq_{\mathfrak{N}'} \cup \{\langle\langle n\rangle, \langle n'\rangle\rangle \mid n \leq_{\mathfrak{N}} n'\} \cup \{\langle\langle n\rangle, \langle m, n'\rangle\rangle \mid n \leq_{\mathfrak{N}} n'\}$,
- $r_{\mathfrak{N}''} := \langle r_{\mathfrak{N}}\rangle$,
- $\models_{\mathfrak{N}''} := \models_{\mathfrak{N}'} \cup \{\langle\langle n\rangle, s\rangle \mid n \models s\}$.

Define B and C as in the proof of Lemma 1.4, also let $C' := C \cup \{\langle n, \langle n\rangle\rangle \mid n \notin \mathrm{Im}(R)\}$. Clearly, $B : \mathfrak{M} \simeq \mathfrak{M}'(\vec{q}, \vec{p})$ and $C' : \mathfrak{N} \simeq \mathfrak{N}''(\vec{p}, \vec{r})$.

It is easy to see that by defining the partial map $f\colon \mathfrak{N}'' \to \mathfrak{M}'$ as $f(\langle m, n\rangle) = \langle m, n\rangle$, we have $\mathfrak{M} \simeq \mathfrak{M}'(\vec{q}, \vec{p})$, $\mathfrak{M}' \precsim^p \mathfrak{N}''$, and $\mathfrak{N} \simeq \mathfrak{N}''(\vec{p}, \vec{r})$. $\dashv$

Note that in lemmas 1.4 and 1.5, and the Lifting Theorem, when $\mathfrak{M}$ and $\mathfrak{N}$ are finite, so will be the constructed models $\mathfrak{M}'$ and $\mathfrak{N}'$. We are now ready to prove the promised equivalence between robustness under $\preceq$, $\ll^{(p)}$ and $\precsim^{(p)}$.

Theorem 1.7. *The following are equivalent*:

(i) ϕ *is* (*finitely*) $\preceq$*-robust,*
(ii) ϕ *is* (*finitely*) $\ll$*-robust,*
(iii) ϕ *is* (*finitely*) $\ll^p$*-robust,*
(iv) ϕ *is* (*finitely*) $\precsim$*-robust,*
(v) ϕ *is* (*finitely*) $\precsim^p$*-robust.*

Proof. We only prove "(v)⇒(i)". Suppose ϕ is (finitely) $\precsim^p$-robust and $\mathfrak{M}, \mathfrak{N}$ are (finite) Kripke models such that $\mathfrak{M} \preceq \mathfrak{N} \models \phi$. By the Lifting Theorem, there are (finite) models $\mathfrak{M}'$ and $\mathfrak{N}'$ such that $\mathfrak{M} \simeq \mathfrak{M}' \precsim^p \mathfrak{N}' \simeq \mathfrak{N}$. We have $\mathfrak{N} \models \phi \Rightarrow \mathfrak{N}' \models \phi \Rightarrow \mathfrak{M}' \models \phi \Rightarrow \mathfrak{M} \models \phi$. $\dashv$

That the finite and general cases in Theorem 1.7 are equivalent will be established later, in Section 3, when we relate both to ONNILLI.

2. The Uniform Interpolation Property

In this section, we prove that ONNILLI satisfies the uniform interpolation property. Our method is similar to that of [5] for NNIL. The result will be used later in Section 3 for the promised IPC-equivalence between stable formulas and ONNILLI.

We start by defining certain formulas for a model $\mathfrak{M}$ and a finite set of atoms $\vec{p}$. Formulas $\nu_{\mathfrak{M}}(\vec{p})$, $\mu_{\mathfrak{M}}(\vec{p})$, $\rho_{\mathfrak{M}}(\vec{p})$ and $\pi_{\mathfrak{M}}(\vec{p})$ are defined as in [5].

DEFINITION 2.1. Let $\mathfrak{M}$ be a Kripke model and $\vec{p}$ a finite set of atoms. Define

- $\nu_{\mathfrak{M}}(\vec{p}) := \bigvee\{\phi \in \mathrm{NNIL}(\vec{p}) \mid \mathfrak{M} \not\models \phi\}$,
- $\mu_{\mathfrak{M}}(\vec{p}) := \bigwedge\{\phi \in \mathrm{NNIL}(\vec{p}) \mid \mathfrak{M} \models \phi\}$,
- $\gamma_{\mathfrak{M}}(\vec{p}) := \bigvee\{\phi \in \mathrm{ONNILLI}(\vec{p}) \mid \mathfrak{M} \not\models \phi\}$,
- $\delta_{\mathfrak{M}}(\vec{p}) := \bigwedge\{\phi \in \mathrm{ONNILLI}(\vec{p}) \mid \mathfrak{M} \models \phi\}$,
- $\rho_{\mathfrak{M}}(\vec{p}) := \bigvee\{p \in \vec{p} \mid \mathfrak{M} \not\models \phi\}$,
- $\pi_{\mathfrak{M}}(\vec{p}) := \bigwedge\{p \in \vec{p} \mid \mathfrak{M} \models \phi\}$.

THEOREM 2.1 ([5]). *Let $\mathfrak{M}$ and $\mathfrak{N}$ be $\vec{p}$-models. Then*

$$\mathfrak{M} \preceq_{zig} \mathfrak{N} \Leftrightarrow \mathrm{Th}_{\mathrm{NNIL}(\vec{p})}(\mathfrak{N}) \subseteq \mathrm{Th}_{\mathrm{NNIL}(\vec{p})}(\mathfrak{M}).$$

What we need to prove the analog of Theorem 2.1 for ONNILLI (Theorem 2.3) is the next lemma, implicit in the proof of Theorem 7.1.2 in [5].

LEMMA 2.2 ([5]). *For $\vec{p}$-models $\mathfrak{M}$ and $\mathfrak{N}$ such that $\mathrm{Th}_{\mathrm{NNIL}(\vec{p})}(\mathfrak{N}) \subseteq \mathrm{Th}_{\mathrm{NNIL}(\vec{p})}(\mathfrak{M})$, define R as mRn if $\mathrm{Th}_{\mathrm{NNIL}(\vec{p})}(n) \subseteq \mathrm{Th}_{\mathrm{NNIL}(\vec{p})}(m)$ and $\mathrm{Th}_{\vec{p}}(m) \subseteq \mathrm{Th}_{\vec{p}}(n)$. Then R is a total zig.*

THEOREM 2.3. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be $\vec{p}$-models. Then*

$$\mathfrak{M} \preceq \mathfrak{N} \Leftrightarrow \mathrm{Th}_{\mathrm{ONNILLI}(\vec{p})}(\mathfrak{N}) \subseteq \mathrm{Th}_{\mathrm{ONNILLI}(\vec{p})}(\mathfrak{M}).$$

PROOF. "$\Rightarrow$" is immediate from Theorem 1.2. For the converse, suppose that $\mathrm{Th}_{\mathrm{ONNILLI}(\vec{p})}(\mathfrak{N}) \subseteq \mathrm{Th}_{\mathrm{ONNILLI}(\vec{p})}(\mathfrak{M})$. Define R as mRn if $\mathrm{Th}_{\mathrm{NNIL}(\vec{p})}(m) \subseteq \mathrm{Th}_{\mathrm{NNIL}(\vec{p})}(n)$ and $\mathrm{Th}_{\vec{p}}(n) \subseteq \mathrm{Th}_{\vec{p}}(m)$. That R preserves truth of atoms follows from the fact that all atoms are in NNIL.

To show that R is total, consider $m \in \mathfrak{M}$. $m \not\models \mu_m(\vec{p}) \to \rho_m(\vec{p})$, which belongs to ONNILLI. So $\mathfrak{N} \not\models \mu_m(\vec{p}) \to \rho_m(\vec{p})$. Hence there exists $n \in \mathfrak{N}$ such that $n \models \mu_m(\vec{p})$ and $n \not\models \rho_m(\vec{p})$. Therefore, $\mathrm{Th}_{\mathrm{NNIL}(\vec{p})}(m) \subseteq \mathrm{Th}_{\mathrm{NNIL}(\vec{p})}(n)$ and $\mathrm{Th}_{\vec{p}}(n) \subseteq \mathrm{Th}_{\vec{p}}(m)$, or equivalently mRn.

For the back condition, suppose that $mRn \leq n'$. By definition, we have $\mathrm{Th}_{\mathrm{NNIL}(\vec{p})}(m) \subseteq \mathrm{Th}_{\mathrm{NNIL}(\vec{p})}(n)$. From Lemma 2.2, $\hat{R}\restriction_{\mathfrak{N}[n]}$ is a total zig. Therefore there exists m' such that $n'\hat{R}m' \geq m$, or equivalently, $m \leq m'Rn'$. $\dashv$

Corollary 2.4. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be $\vec{p}$-models. Then*

$$\mathfrak{M} \preceq \mathfrak{N} \Leftrightarrow \mathfrak{N} \not\models \gamma_{\mathfrak{M}}(\vec{p}) \Leftrightarrow \mathfrak{M} \models \delta_{\mathfrak{N}}(\vec{p}).$$

Theorem 2.5. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be $\vec{p}$-models. Then*

$$\mathfrak{M} \preceq \mathfrak{N} \Leftrightarrow \vdash \gamma_{\mathfrak{M}}(\vec{p}) \to \gamma_{\mathfrak{N}}(\vec{p}) \Leftrightarrow \vdash \delta_{\mathfrak{M}}(\vec{p}) \to \delta_{\mathfrak{N}}(\vec{p}).$$

Proof. Suppose that $\mathfrak{M} \preceq \mathfrak{N}$ and $\mathfrak{K} \not\models \gamma_{\mathfrak{N}}(\vec{p})$. Corollary 2.4 results in $\mathfrak{N} \preceq \mathfrak{K}$. Therefore, $\mathfrak{M} \preceq \mathfrak{K}$, so $\mathfrak{K} \not\models \gamma_{\mathfrak{M}}(\vec{p})$. Then, $\vdash \gamma_{\mathfrak{M}}(\vec{p}) \to \gamma_{\mathfrak{N}}(\vec{p})$. Conversely, assume $\vdash \gamma_{\mathfrak{M}}(\vec{p}) \to \gamma_{\mathfrak{N}}(\vec{p})$. By $\mathfrak{N} \not\models \gamma_{\mathfrak{N}}(\vec{p})$, we have $\mathfrak{N} \not\models \gamma_{\mathfrak{M}}(\vec{p})$. From Corollary 2.4 it follows that $\mathfrak{M} \preceq \mathfrak{N}$.

Now let $\mathfrak{M} \preceq \mathfrak{N}$ and $\mathfrak{K} \models \delta_{\mathfrak{M}}(\vec{p})$. Corollary 2.4 results in $\mathfrak{K} \preceq \mathfrak{M}$. Therefore, $\mathfrak{K} \preceq \mathfrak{N}$, so $\mathfrak{K} \models \delta_{\mathfrak{N}}(\vec{p})$, which implies that $\vdash \delta_{\mathfrak{M}}(\vec{p}) \to \delta_{\mathfrak{N}}(\vec{p})$. Conversely, assume $\vdash \delta_{\mathfrak{M}}(\vec{p}) \to \delta_{\mathfrak{N}}(\vec{p})$. By $\mathfrak{M} \models \delta_{\mathfrak{M}}(\vec{p})$, we have $\mathfrak{M} \models \delta_{\mathfrak{N}}(\vec{p})$. Hence, by Corollary 2.4, $\mathfrak{M} \preceq \mathfrak{N}$. ⊣

Corollary 2.6. *The number of $\equiv$-equivalence classes of $\vec{p}$-models is finite.*

We are now ready to prove the uniform interpolation property for ONNILLI. First, we start by definition of uniform interpolation. We also define left and right approximation.

Definition 2.2. Let $\varPhi$ be a set of formulas.

- For a given ϕ and $\mathcal{P} \subseteq \mathrm{PV}(\phi)$, we say $\psi \in \mathcal{L}(\mathcal{P})$ is the *uniform $\varPhi$ left-interpolant* of ϕ, when $\vdash \psi \to \phi$ and for $\chi \in \varPhi$, $\mathrm{PV}(\chi) \cap \mathrm{PV}(\phi) \subseteq \mathcal{P}$ and $\vdash \chi \to \phi$ imply $\vdash \chi \to \psi$.
- For a given ϕ and $\mathcal{P} \subseteq \mathrm{PV}(\phi)$, we say $\psi \in \mathcal{L}(\mathcal{P})$ is the *uniform $\varPhi$ right-interpolant* of ϕ, when $\vdash \phi \to \psi$ and for $\chi \in \varPhi$, $\mathrm{PV}(\chi) \cap \mathrm{PV}(\phi) \subseteq \mathcal{P}$ and $\vdash \phi \to \chi$ imply $\vdash \psi \to \chi$.
- $\varPhi$ is said to have the *uniform interpolation property*, if for all $\phi \in \mathcal{L}$ and $\mathcal{P} \subseteq \mathrm{PV}(\phi)$, both its uniform $\varPhi$ left-interpolant and right-interpolant exist.
- $\varPhi$ is said to satisfy left approximation if for all $\phi \in \mathcal{L}$, then there exists ϕ^* such that for all $\psi \in \varPhi$ we have $\vdash \psi \to \phi^*$ iff $\vdash \psi \to \phi$.
- $\varPhi$ is said to satisfy right approximation if for all $\phi \in \mathcal{L}$, then there exists ϕ° such that for all $\psi \in \varPhi$ we have $\vdash \phi^\circ \to \psi$ iff $\vdash \phi \to \psi$.

We also define

- $\phi^*(\vec{p}) := \bigvee\{\psi \in \mathrm{ONNILLI}(\vec{p}) \mid \vdash \psi \to \phi\}$.
- $\phi^\circ(\vec{p}) := \bigwedge\{\psi \in \mathrm{ONNILLI}(\vec{p}) \mid \vdash \phi \to \psi\}$.

THEOREM 2.7 (Uniform Interpolation). ONNILLI *satisfies the uniform interpolation property.*

PROOF. Suppose that $\psi \in \mathcal{L}(\vec{q}, \vec{p})$ is given. We show that $\psi^*(\vec{p})$ is the uniform ONNILLI left-interpolant of ψ. Clearly, $\vdash \psi^*(\vec{p}) \to \psi$. Let $\phi \in \mathcal{L}(\vec{p}, \vec{r})$ be in ONNILLI such that $\vdash \phi \to \psi$. If $\nvdash \phi \to \psi^*(\vec{p})$, there exists a $\vec{p}, \vec{r}$-model $\mathfrak{N}$ such that $\mathfrak{N} \models \phi$ but $\mathfrak{N} \not\models \psi^*(\vec{p})$. For a contradiction, suppose $\delta_{\mathfrak{N}}(\vec{p}) \vdash \psi$. Since $\delta_{\mathfrak{N}}(\vec{p}) \in \text{ONNILLI}(\vec{p})$, $\delta_{\mathfrak{N}}(\vec{p}) \vdash \psi^*(\vec{p})$ and hence $\mathfrak{N} \models \psi^*(\vec{p})$ which is a contradiction. So $\delta_{\mathfrak{N}}(\vec{p}) \nvdash \psi$. So there exists a $\vec{q}, \vec{p}$-model $\mathfrak{M}$ such that $\mathfrak{M} \models \delta_{\mathfrak{N}}(\vec{p})$ but $\mathfrak{M} \not\models \psi$. From Corollary 2.4 we have $\mathfrak{M}(\vec{p}) \preceq \mathfrak{N}(\vec{p})$. From the Lifting Theorem there exist $\vec{p}, \vec{q}, \vec{r}$-models $\mathfrak{M}'$ and $\mathfrak{N}'$ such that $\mathfrak{M} \simeq \mathfrak{M}'(\vec{q}, \vec{p})$, $\mathfrak{M}' \preceq \mathfrak{N}'$ and $\mathfrak{N} \simeq \mathfrak{N}'(\vec{p}, \vec{r})$. By bisimulation, $\mathfrak{M}' \not\models \psi$ and $\mathfrak{N}' \models \phi$. From Corollary 1.3 we conclude $\mathfrak{M}' \models \phi$. Hence $\mathfrak{M}' \not\models \phi \to \psi$ which is a contradiction. Therefore, $\vdash \phi \to \psi^*(\vec{p})$.

Similarly, it can be shown that $\phi^\circ(\vec{p})$ is the uniform ONNILLI right-interpolant of ϕ. ⊣

COROLLARY 2.8. ONNILLI *satisfies left and right approximation. Furthermore, for $\phi \in \mathcal{L}(\vec{p})$ we have*

- $\vdash \phi^* \leftrightarrow \phi^*(\vec{p})$.
- $\vdash \phi^\circ \leftrightarrow \phi^\circ(\vec{p})$.

In the following theorem we give an alternative characterizations of $\phi^*(\vec{p})$ and $\phi^\circ(\vec{p})$.

THEOREM 2.9. *Let $\phi \in \mathcal{L}(\vec{p}, \vec{q})$. Then we have*

1. $\vdash \phi^*(\vec{p}) \leftrightarrow \bigwedge\{\gamma_{\mathfrak{M}}(\vec{p}) \mid \mathfrak{M}$ *is finite*, $\mathfrak{M} \not\models \phi\}$,
2. $\vdash \phi^\circ(\vec{p}) \leftrightarrow \bigvee\{\delta_{\mathfrak{M}}(\vec{p}) \mid \mathfrak{M}$ *is finite*, $\mathfrak{M} \models \phi\}$.

PROOF. *Ad* 1 "$\to$": Suppose $\mathfrak{M} \not\models \phi$ for finite $\mathfrak{M}$. Then $\mathfrak{M} \not\models \phi^*(\vec{p})$. By definition of $\gamma_{\mathfrak{M}}(\vec{p})$, we have $\vdash \phi^*(\vec{p}) \to \gamma_{\mathfrak{M}}(\vec{p})$.

"$\leftarrow$": For convenience, we denote $\bigwedge\{\gamma_{\mathfrak{M}}(\vec{p}) \mid \mathfrak{M}$ is finite, $\mathfrak{M} \not\models \phi\}$ by $\Phi^*(\vec{p})$. Assume that $\nvdash \Phi^*(\vec{p}) \to \phi$. Then there exists a finite model $\mathfrak{N}$ such that $\mathfrak{N} \models \Phi^*(\vec{p})$ but $\mathfrak{N} \not\models \phi$, and hence $\mathfrak{N} \models \gamma_{\mathfrak{N}}(\vec{p})$ which is a contradiction. Therefore, $\vdash \Phi^*(\vec{p}) \to \phi$. By definition of $\phi^*(\vec{p})$, we have $\vdash \Phi^*(\vec{p}) \to \phi^*(\vec{p})$.

Ad 2. Similar to 1. ⊣

3. Main Result

In this section we use the established results in previous sections to prove our main result, that ONNILLI is exactly the set of (finitely) robust formulas under $\preceq$, $\ll^{(p)}$, or $\precsim^{(p)}$.

We start by defining modal operators in fashion of [5]. Our notations coincide with those of [5] for convenience.

Definition 3.1. Let $\mathfrak{M}$ and $\mathfrak{N}$ be Kripke models. Define

- $\mathfrak{N} \models \bigcirc\phi :\Leftrightarrow \mathfrak{M} \models \phi$, for all finite $\mathfrak{M} \preceq \mathfrak{N}$,
- $\mathfrak{N} \models \Diamond\phi :\Leftrightarrow \mathfrak{M} \models \phi$, for some finite $\mathfrak{M} \succeq \mathfrak{N}$.

Theorem 3.1. *Let $\mathfrak{N}$ be a finite Kripke model and $\phi \in \mathcal{L}(\vec{p})$. Then we have*

1. $\mathfrak{N} \models \bigcirc\phi \Leftrightarrow \mathfrak{N} \models \phi^*$,
2. $\mathfrak{N} \models \Diamond\phi \Leftrightarrow \mathfrak{N} \models \phi^\circ$.

Proof. *Ad* 1. "$\Leftarrow$" is immediate from Corollary 1.3. Conversely, suppose $\mathfrak{N} \not\models \phi^*$. Then $\mathfrak{N} \not\models \phi^*(\vec{p})$. From Theorem 2.9 there exists finite $\mathfrak{M}$ such that $\mathfrak{M} \not\models \phi$ and $\mathfrak{N} \not\models \gamma_{\mathfrak{M}}(\vec{p})$. We may assume that $\mathfrak{M}$ is a $\vec{p}$-model. From Corollary 2.4 we have $\mathfrak{M} \preceq \mathfrak{N}(\vec{p})$. For some (possibly infinite) $\vec{r}$, disjoint from $\vec{p}$, $\mathfrak{N}$ is a $\vec{p},\vec{r}$-model. By the Lifting Theorem, there exists a finite $\vec{p},\vec{r}$-model $\mathfrak{M}'$ such that $\mathfrak{M} \simeq \mathfrak{M}'(\vec{p})$ and $\mathfrak{M}' \preceq \mathfrak{N}$. By bisimulation $\mathfrak{M}' \not\models \phi$. Therefore, $\mathfrak{N} \not\models \bigcirc\phi$.

Ad 2. Similar to 1. $\dashv$

We are now ready to prove our main results.

Theorem 3.2. *Let $\phi \in \mathcal{L}(\vec{p})$. Then*

$$\phi \textit{ is (finitely) } \preceq\textit{-robust} \Leftrightarrow \phi \textit{ is in } \text{ONNILLI}.$$

Proof. "$\Leftarrow$" is immediate from Corollary 1.3. Conversely, let ϕ be a (finitely) $\preceq$-robust formula in $\mathcal{L}(\vec{p})$. Let $\mathfrak{M}$ be a finite Kripke model such that $\mathfrak{M} \models \phi$. By (finite) robustness, $\mathfrak{M} \models \bigcirc\phi$. It follows from Theorem 3.1 that $\mathfrak{M} \models \phi^*$. Then, $\vdash \phi \leftrightarrow \phi^*$ and so ϕ is in ONNILLI. $\dashv$

Corollary 3.3. *The finite and general cases in Theorem 1.7 are equivalent.*

Theorem 3.4. *The following are equivalent:*

(i) *ϕ is preserved under bijective (resp. surjective) monotonic maps of Kripke models,*

(ii) ϕ *is preserved under bijective* (*resp. surjective*) *monotonic maps of descriptive models,*

(iii) ϕ *is preserved under bijective* (*resp. surjective*) *monotonic maps of finite Kripke models.*

PROOF. "(i) $\Rightarrow$ (ii)" is trivial. "(ii) $\Rightarrow$ (iii)" follows from the fact that finite Kripke models are automatically descriptive. Finally, "(iii) $\Rightarrow$ (i)" is immediate from Corollary 3.3. $\dashv$

We can sum it up as the next theorem, which resolves the open problem of [2].

THEOREM 3.5. ONNILLI *is exactly the set of formulas preserved under bijective or surjective monotonic maps of descriptive or* (*finite*) *Kripke models.*

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