Logic and Logical Philosophy
Volume 28 (2019), 611–627
DOI: 10.12775/LLP.2018.012

Yale Weiss

CONNEXIVE EXTENSIONS OF REGULAR CONDITIONAL LOGIC

Abstract. The object of this paper is to examine half and full connexive extensions of the basic regular conditional logic **CR**. Extensions of this system are of interest because it is among the strongest well-known systems of conditional logic that can be augmented with connexive theses without inconsistency resulting. These connexive extensions are characterized axiomatically and their relations to one another are examined proof-theoretically. Subsequently, algebraic semantics are given and soundness, completeness, and decidability are proved for each system. The semantics is also used to establish independence results. Finally, a deontic interpretation of one of the systems is examined and defended.

Keywords: conditional logic; connexive logic; conditional obligation; deontic logic

1. Introduction

The object of this paper is to examine connexive extensions of regular systems of conditional logic (regular systems will be defined in Section 2). In particular, we develop a half-connexive regular system, which contains weak Boethius' theses,[1] and also a (fully) connexive regular system which contains, in addition, Aristotle's theses.

In Section 2, axiom systems are presented for the basic regular conditional logic **CR** and its half and full connexive extensions, **CR1** and **CR2**. The inconsistency of various extensions of these relatively weak systems is proved. These inconsistency results can profitably be compared with those in [12].

[1] For weak Boethius' theses and their strong counterparts, see [10].

Special Issue: Advances in Connexive Logic. Guest Editors: Hitoshi Omori and Heinrich Wansing
 Published online August 16, 2018

In Section 3, algebraic semantics are given for these systems. These semantics modify and extend semantics developed by Nute [8]. Soundness, completeness, and decidability are proved for each of the systems. In addition, the semantics is used to demonstrate some independence results and prove that the systems discussed are *bereft*: no formula with the conditional as its main connective is a theorem.

We suggest a deontic interpretation of **CR1** in Section 4. In particular, we contend that **CR1** is a plausible logic of conditional obligation. Chellas' work on closely related systems in [2] is discussed as appropriate. Finally, the relation of **CR1** (and an extension of it) to a system of deontic modal logic is discussed. This section, as a whole, demonstrates that connexive extensions of regular conditional logics are not mere technical curiosities.

Concluding remarks are offered in Section 5. There we expand on the relation of this work to previous work on connexive logic and, more particularly, connexive conditional logic. We also discuss avenues for future development based on this work.

2. Systems of Conditional Logic

We build our systems in the language $\mathcal{L}$ of classical propositional logic augmented with a new binary conditional connective $\Box\!\!\rightarrow$.[2] The formation rules are standard (in particular, nesting of $\Box\!\!\rightarrow$ is permitted). Π is the set of all propositional letters (p, q, ...) and Φ is the set of all formulae (ϕ, ψ, ...).

A set $\boldsymbol{L}$ of formulae (in the language $\mathcal{L}$) is a *system of conditional logic* if it contains all classical propositional tautologies and is closed under modus ponens (for material implication):

$$\frac{\phi, \quad \phi \rightarrow \psi}{\psi} \tag{MP}$$

$\vdash_{\boldsymbol{L}} \phi$ (ϕ is a *theorem* of $\boldsymbol{L}$) if and only if $\phi \in \boldsymbol{L}$. $\Gamma \vdash_{\boldsymbol{L}} \phi$ if and only if there is a set $\{\phi_1, \ldots, \phi_n\} \subseteq \Gamma$ ($n \geqslant 0$) such that $\vdash_{\boldsymbol{L}} (\phi_1 \wedge \cdots \wedge \phi_n) \rightarrow \phi$.

Since every system of conditional logic contains classical propositional logic (**PL**), **PL** can be appealed to in axiomatic proofs wherever classical inferences are used. We obtain proper extensions of **PL** (in $\mathcal{L}$)

[2] We take $\rightarrow$, $\wedge$, $\vee$, and $\neg$ as primitive; this entails some redundancy, but does no harm. The biconditional ($\leftrightarrow$) and constants ($\perp$ and $\top$) are defined as usual.

by closing under rules and adding axioms (axiom schemata) such as the following (for each of the rules, the premise is a theorem):

$$\frac{\phi \leftrightarrow \psi}{(\phi \mathrel{\Box\!\!\rightarrow} \chi) \leftrightarrow (\psi \mathrel{\Box\!\!\rightarrow} \chi)} \tag{RCEA}$$

$$\frac{\phi \leftrightarrow \psi}{(\chi \mathrel{\Box\!\!\rightarrow} \phi) \leftrightarrow (\chi \mathrel{\Box\!\!\rightarrow} \psi)} \tag{RCEC}$$

$$\frac{(\phi_1 \wedge \cdots \wedge \phi_n) \rightarrow \phi}{((\psi \mathrel{\Box\!\!\rightarrow} \phi_1) \wedge \cdots \wedge (\psi \mathrel{\Box\!\!\rightarrow} \phi_n)) \rightarrow (\psi \mathrel{\Box\!\!\rightarrow} \phi)} \tag{RCK}$$

$$\phi \mathrel{\Box\!\!\rightarrow} \phi \tag{ID}$$

$$\neg(\phi \mathrel{\Box\!\!\rightarrow} \neg\phi) \tag{AT1}$$

$$\neg(\neg\phi \mathrel{\Box\!\!\rightarrow} \phi) \tag{AT2}$$

$$(\phi \mathrel{\Box\!\!\rightarrow} \psi) \rightarrow \neg(\phi \mathrel{\Box\!\!\rightarrow} \neg\psi) \tag{WBT1}$$

$$(\phi \mathrel{\Box\!\!\rightarrow} \neg\psi) \rightarrow \neg(\phi \mathrel{\Box\!\!\rightarrow} \psi) \tag{WBT2}$$

$$(\phi \mathrel{\Box\!\!\rightarrow} (\psi \wedge \theta)) \rightarrow ((\phi \mathrel{\Box\!\!\rightarrow} \psi) \wedge (\phi \mathrel{\Box\!\!\rightarrow} \theta)) \tag{CM}$$

$$((\phi \mathrel{\Box\!\!\rightarrow} \psi) \wedge (\phi \mathrel{\Box\!\!\rightarrow} \theta)) \rightarrow (\phi \mathrel{\Box\!\!\rightarrow} (\psi \wedge \theta)) \tag{CC}$$

We adopt the standard convention that when $n = 0$, (RCK) licenses the inference from ϕ to $\psi \mathrel{\Box\!\!\rightarrow} \phi$.

A system of conditional logic $\boldsymbol{L}$ closed under (RCEA) and (RCEC) is *classical*. A classical system $\boldsymbol{L}$ is *monotonic* if it contains (CM); is *regular* if it contains (CM) and (CC); and is *normal* if it is closed under (RCK). This terminology is standard [see 1, 8]. A system of conditional logic $\boldsymbol{L}$ is *half-connexive* if it contains (WBT1) and (WBT2). Moreover, $\boldsymbol{L}$ is *connexive* if it contains, in addition, (AT1) and (AT2).

Schemata (WBT1) and (WBT2) are sometimes referred to as weak Boethius' theses [10]. Schemata (AT1) and (AT2) are often referred to as Aristotle's theses [6]. Some logicians characterize connexive systems in terms of stronger versions of Boethius' theses which we will not discuss here. Moreover, it is typically required that formulae such as $(\phi \mathrel{\Box\!\!\rightarrow} \psi) \mathrel{\Box\!\!\rightarrow} (\psi \mathrel{\Box\!\!\rightarrow} \phi)$ not be theorems; this fails to be a theorem in each of the main systems discussed below (see Proposition 3.3).

The following few propositions should be compared with results of Unterhuber [12]. We use these results to characterize the strongest consistent connexive conditional logics.[3]

[3] The terminology and formalism used by Unterhuber [12] are somewhat non-

PROPOSITION 2.1. *Every normal connexive system of conditional logic is inconsistent.*

PROOF.

1	$\neg\bot$	$\vdash$
2	$\bot \boxright \neg\bot$	(RCK) 1
3	$\neg(\bot \boxright \neg\bot)$	(AT1)
4	$\bot$	**PL** 2, 3 □

PROPOSITION 2.2. *Every monotonic connexive system of conditional logic containing* (ID) *is inconsistent.*

PROOF.

1	$\bot \wedge \neg\bot \leftrightarrow \bot$	$\vdash$
2	$\bot \boxright \bot$	(ID)
3	$(\bot \boxright \bot) \leftrightarrow (\bot \boxright (\bot \wedge \neg\bot))$	(RCEC) 1
4	$\bot \boxright (\bot \wedge \neg\bot)$	**PL** 2, 3
5	$(\bot \boxright (\bot \wedge \neg\bot)) \to ((\bot \boxright \bot) \wedge (\bot \boxright \neg\bot))$	(CM)
6	$(\bot \boxright \bot) \wedge (\bot \boxright \neg\bot)$	**PL** 4, 5
7	$\bot \boxright \neg\bot$	**PL** 6
8	$\neg(\bot \boxright \neg\bot)$	(AT1)
9	$\bot$	**PL** 7, 8 □

Though it will not concern us below, a referee points out that a strengthening of Proposition 2.2 is possible [see also 12]: every monotonic half-connexive system of conditional logic containing (ID) is inconsistent.

PROPOSITION 2.3. *A classical system of conditional logic contains* (AT1) *if and only if it contains* (AT2).

PROOF.

1	$\phi \leftrightarrow \neg\neg\phi$	$\vdash$
2	$(\neg\phi \boxright \phi) \leftrightarrow (\neg\phi \boxright \neg\neg\phi)$	(RCEC) 1
3	$\neg(\neg\phi \boxright \neg\neg\phi)$	(AT1)
4	$\neg(\neg\phi \boxright \phi)$	**PL** 2, 3

The converse implication holds by an analogous proof. □

PROPOSITION 2.4. *A classical system of conditional logic contains* (WBT1) *if and only if it contains* (WBT2).

---

standard, so we have thought it valuable to cover some of the same ground here using a more canonical formalism.

PROOF.

1	$\phi \mathrel{\Box\!\!\rightarrow} \neg\psi$	Assumption
2	$(\phi \mathrel{\Box\!\!\rightarrow} \neg\psi) \rightarrow \neg(\phi \mathrel{\Box\!\!\rightarrow} \neg\neg\psi)$	(WBT1)
3	$\neg(\phi \mathrel{\Box\!\!\rightarrow} \neg\neg\psi)$	**PL** 1, 2
4	$\psi \leftrightarrow \neg\neg\psi$	$\vdash$
5	$(\phi \mathrel{\Box\!\!\rightarrow} \psi) \leftrightarrow (\phi \mathrel{\Box\!\!\rightarrow} \neg\neg\psi)$	(RCEC) 4
6	$\neg(\phi \mathrel{\Box\!\!\rightarrow} \psi)$	**PL** 3, 5
7	$(\phi \mathrel{\Box\!\!\rightarrow} \neg\psi) \rightarrow \neg(\phi \mathrel{\Box\!\!\rightarrow} \psi)$	**PL** 1–6

The converse implication holds by an analogous proof. □

PROPOSITION 2.5. *Every regular system of conditional logic containing* (AT1) *also contains* (WBT1).

PROOF.

1	$\bot \leftrightarrow (\phi \wedge \neg\phi)$	$\vdash$
2	$(\phi \mathrel{\Box\!\!\rightarrow} (\phi \wedge \neg\phi)) \rightarrow ((\phi \mathrel{\Box\!\!\rightarrow} \phi) \wedge (\phi \mathrel{\Box\!\!\rightarrow} \neg\phi))$	(CM)
3	$(\phi \mathrel{\Box\!\!\rightarrow} \bot) \leftrightarrow (\phi \mathrel{\Box\!\!\rightarrow} (\phi \wedge \neg\phi))$	(RCEC) 1
4	$(\phi \mathrel{\Box\!\!\rightarrow} \bot) \rightarrow ((\phi \mathrel{\Box\!\!\rightarrow} \phi) \wedge (\phi \mathrel{\Box\!\!\rightarrow} \neg\phi))$	**PL** 2, 3
5	$(\phi \mathrel{\Box\!\!\rightarrow} \bot) \rightarrow (\phi \mathrel{\Box\!\!\rightarrow} \neg\phi)$	**PL** 4
6	$\bot \leftrightarrow (\psi \wedge \neg\psi)$	$\vdash$
7	$(\phi \mathrel{\Box\!\!\rightarrow} \bot) \leftrightarrow (\phi \mathrel{\Box\!\!\rightarrow} (\psi \wedge \neg\psi))$	(RCEC) 6
8	$\neg(\phi \mathrel{\Box\!\!\rightarrow} \neg\phi)$	(AT1)
9	$\neg(\phi \mathrel{\Box\!\!\rightarrow} \bot)$	**PL** 5, 8
10	$\neg(\phi \mathrel{\Box\!\!\rightarrow} (\psi \wedge \neg\psi))$	**PL** 7, 9
11	$((\phi \mathrel{\Box\!\!\rightarrow} \psi) \wedge (\phi \mathrel{\Box\!\!\rightarrow} \neg\psi)) \rightarrow (\phi \mathrel{\Box\!\!\rightarrow} (\psi \wedge \neg\psi))$	(CC)
12	$\neg((\phi \mathrel{\Box\!\!\rightarrow} \psi) \wedge (\phi \mathrel{\Box\!\!\rightarrow} \neg\psi))$	**PL** 10, 11
13	$(\phi \mathrel{\Box\!\!\rightarrow} \psi) \rightarrow \neg(\phi \mathrel{\Box\!\!\rightarrow} \neg\psi))$	**PL** 12 □

It is important to note that the converse of Proposition 2.5 does not hold. This is established using semantic methods in Proposition 3.2.

Proposition 2.1 demonstrates that there are no consistent normal connexive systems of conditional logic. As a result, to avoid triviality, we must turn to subnormal systems, of which (merely) regular systems are among the best known and most logically robust candidates. Since all regular connexive systems are monotonic, they cannot consistently be augmented by (ID), per Proposition 2.2.

Following Chellas [1, p. 138], we call the smallest regular system of conditional logic **CR**. **CR1** is obtained by extending **CR** with (WBT1). By Proposition 2.4, **CR1** is half-connexive. **CR2** is obtained by extending **CR** with (AT1). By Propositions 2.3, 2.4, and 2.5, **CR2** is connexive.

THEOREM 2.1 (Consistency). **CR**, **CR1**, *and* **CR2** *are consistent.*

PROOF. Following Lowe [5, p. 360], we define a translation τ from Φ into the set of formulae of classical propositional logic in the standard language such that $\tau(\phi \mathrel{\Box\!\!\rightarrow} \psi) = \tau(\phi) \wedge \tau(\psi)$ but otherwise τ commutes with the connectives (e.g. $\tau(\neg\phi) = \neg\tau(\phi)$). Then it is easily seen, for each system, that τ maps each axiom to a tautology and the rules preserve this property. Thus, every theorem of each system, under τ, is a tautology. Since $\tau(\bot)$ is not a tautology, $\bot$ is not a theorem of any of the systems. $\square$

3. Algebraic Semantics

In this section, we present algebraic semantics for **CR**, **CR1**, and **CR2**. This work builds on the algebraic semantics given for numerous (non-connexive) conditional logics by Nute [8].[4] We prove soundness, completeness, and decidability for each of the systems and discuss several technical applications of these results.

DEFINITION 3.1. A *basic conditional algebra* is a structure $\mathcal{A} = \langle \mathcal{B}, * \rangle$ in which $\mathcal{B} = \langle \mathbb{B}, 1, 0, -, \cup, \cap \rangle$ is a Boolean algebra and $*$ is a binary operation on $\mathbb{B}$ subject to the condition:

1. $x * (y \cap z) = (x * y) \cap (x * z)$

DEFINITION 3.2. An α-*conditional algebra* is a basic conditional algebra $\mathcal{A} = \langle \mathcal{B}, * \rangle$ in which $*$ is also subject to the condition:

2. $x * y \leq -(x * -y)$

DEFINITION 3.3. A β-*conditional algebra* is a basic conditional algebra $\mathcal{A} = \langle \mathcal{B}, * \rangle$ in which $*$ is also subject to the condition:

2. $x * -x = 0$

A *conditional model* is a structure $\mathfrak{I} = \langle \mathcal{A}, f \rangle$ in which $\mathcal{A}$ is a conditional algebra and $f \colon \Pi \to \mathbb{B}$. Moreover:

1. $f(\neg\phi) = -f(\phi)$
2. $f(\phi \vee \psi) = f(\phi) \cup f(\psi)$

[4] However, it should be noted that the terminology and formalism employed here departs from [8] in a number of ways. We also note that Pizzi uses algebraic semantics to characterize a connexive conditional logic — albeit of a kind rather different than those studied here — in [9].

3. $f(\phi \wedge \psi) = f(\phi) \cap f(\psi)$
4. $f(\phi \to \psi) = -f(\phi) \cup f(\psi)$
5. $f(\phi \mathrel{\Box\!\!\to} \psi) = f(\phi) * f(\psi)$

For a conditional model $\mathfrak{I} = \langle \mathcal{A}, f \rangle$, we say $\models^{\mathfrak{I}} \phi$ if and only if $f(\phi) = 1$. Where $\mathcal{K}$ is a class of conditional models, $\models_{\mathcal{K}} \phi$ if $\models^{\mathfrak{I}} \phi$ for all $\mathfrak{I} \in \mathcal{K}$. If $\models_{\mathcal{K}} \phi$, we say that ϕ is *valid* (in $\mathcal{K}$).

We say that $\mathfrak{I}$ is an α-conditional (resp. β-conditional) model if $\mathcal{A}$ is an α-conditional (resp. β-conditional) algebra. We denote the class of all conditional models by $\mathcal{C}$; of all α-conditional models by $\mathcal{C}_\alpha$; and of all β conditional models by $\mathcal{C}_\beta$.

Immediate from the above definitions and the properties of Boolean algebras we obtain:

PROPOSITION 3.1. *For arbitrary $\mathfrak{I} = \langle \mathcal{A}, f \rangle \in \mathcal{C}$:*

1. $\models^{\mathfrak{I}} \top$
2. $\models^{\mathfrak{I}} \phi \to \psi$ *if and only if* $f(\phi) \leq f(\psi)$
3. $\models^{\mathfrak{I}} \phi \leftrightarrow \psi$ *if and only if* $f(\phi) = f(\psi)$

THEOREM 3.1 (Soundness). **CR**, **CR1**, *and* **CR2** *are sound with respect to $\mathcal{C}$, $\mathcal{C}_\alpha$, and $\mathcal{C}_\beta$ respectively.*

PROOF. The proof proceeds by showing that the axioms are valid and the rules preserve validity in the respective classes of models. We examine the cases of (RCEA), (CM), and (AT1) with respect to $\mathcal{C}_\beta$; the other cases are all straightforward modifications of these.

For (RCEA), let $\mathfrak{I} \in \mathcal{C}_\beta$ be arbitrary such that $\models^{\mathfrak{I}} \phi \leftrightarrow \psi$. By Proposition 3.1, $f(\phi) = f(\psi)$. Then $f(\phi) * f(\chi) = f(\psi) * f(\chi) \Rightarrow f(\phi \mathrel{\Box\!\!\to} \chi) = f(\psi \mathrel{\Box\!\!\to} \chi) \Rightarrow \models^{\mathfrak{I}} (\phi \mathrel{\Box\!\!\to} \chi) \leftrightarrow (\psi \mathrel{\Box\!\!\to} \chi)$. For (CM), let $\mathfrak{I} \in \mathcal{C}_\beta$ be arbitrary such that $\models^{\mathfrak{I}} \phi \mathrel{\Box\!\!\to} (\psi \wedge \theta)$. Then $1 = f(\phi) * (f(\psi) \cap f(\theta)) = (f(\phi) * f(\psi)) \cap (f(\phi) * f(\theta))$ by Definition 3.3. Thus, $\models^{\mathfrak{I}} (\phi \mathrel{\Box\!\!\to} \psi) \wedge (\phi \mathrel{\Box\!\!\to} \theta)$. Finally, for (AT1), let $\mathfrak{I} \in \mathcal{C}_\beta$ be arbitrary and note that $f(\neg(\phi \mathrel{\Box\!\!\to} \neg\phi)) = -(f(\phi) * -f(\phi)) = -0 = 1$ by Definition 3.3. Therefore, $\models^{\mathfrak{I}} \neg(\phi \mathrel{\Box\!\!\to} \neg\phi)$. $\square$

PROPOSITION 3.2. *(AT1) and (AT2) are independent of* **CR1**.

PROOF. Consider the α-conditional algebra $\mathcal{A}$ based on the two element Boolean algebra $\mathcal{B}$ with $*$ given by the following table:

$*$	0	1
0	0	1
1	0	1

It is tedious, though not hard, to verify that $*$ satisfies all of conditions required by Definition 3.2. Now consider a conditional model $\mathfrak{I} = \langle \mathcal{A}, f \rangle$ in which $f(p) = 0$. Then $f(\neg(p \boxright \neg p)) = -(f(p) * -f(p)) = -(0 * -0) = -1 = 0$. Since $\not\models^{\mathfrak{I}} \neg(p \boxright \neg p)$, by Theorem 3.1, $\nvdash_{\mathbf{CR1}} \neg(\phi \boxright \neg\phi)$. $\square$

A system of conditional logic $\boldsymbol{L}$ is *bereft* if no formula of the form $\phi \boxright \psi$ is a theorem.

PROPOSITION 3.3. **CR**, **CR1**, *and* **CR2** *are bereft.*

PROOF. Consider the β-conditional algebra $\mathcal{A}$ based on the two element Boolean algebra $\mathcal{B}$ with $*$ given by the following table:

$*$	0	1
0	0	0
1	0	0

It is easy to see that this algebra satisfies all the conditions required by Definition 3.3. Moreover, it is clear that, for any formula $\phi \boxright \psi$, $\not\models^{\mathfrak{I}} \phi \boxright \psi$ for any $\mathfrak{I}$ based on this algebra. By Theorem 3.1, no formula of the form $\phi \boxright \psi$ is derivable in **CR**, **CR1**, or **CR2**. $\square$

An obvious corollary of Proposition 3.3 is that $\nvdash_{\boldsymbol{L}} (\phi \wedge \psi) \boxright \phi$ (where $\boldsymbol{L}$ is any of the systems mentioned in the proposition). The failure of this law — Simplification — is a common (though not essential) feature of nontrivial connexive logics [see, e.g., 7, p. 112].

To prove completeness, we construct Lindenbaum algebras and canonical models for each system. The techniques and definitions employed here are fairly standard [see 8, Ch. 7, and 2, pp. 243–244].

Where $\boldsymbol{L}$ is a classical system of conditional logic, we write $\phi \sim_{\boldsymbol{L}} \psi$ if and only if $\vdash_{\boldsymbol{L}} \phi \leftrightarrow \psi$. Then $[\phi]^{\boldsymbol{L}} := \{\psi : \phi \sim_{\boldsymbol{L}} \psi\}$ (unless needed to disambiguate, we suppress the superscript).

The *Lindenbaum algebra* for a classical system of conditional logic $\boldsymbol{L}$ is a structure $\mathcal{A}_{\boldsymbol{L}} = \langle\langle \mathbb{B}, 1, 0, -, \cup, \cap\rangle, *\rangle$ defined as follows:

1. $\mathbb{B} = \{[\phi] : \phi \in \Phi\}$
2. $-[\phi] = [\neg\phi]$
3. $[\phi] \cup [\psi] = [\phi \vee \psi]$
4. $[\phi] \cap [\psi] = [\phi \wedge \psi]$
5. $[\phi] * [\psi] = [\phi \boxright \psi]$

Remark. The operations are well-defined. For example, suppose $[\phi] = [\psi]$. Then $\phi \sim_{\boldsymbol{L}} \psi \Rightarrow \vdash_{\boldsymbol{L}} \phi \leftrightarrow \psi \Rightarrow \vdash_{\boldsymbol{L}} \neg\phi \leftrightarrow \neg\psi \Rightarrow [\neg\phi] = [\neg\psi]$

$\Rightarrow -[\phi] = -[\psi]$. Also, note that 1, 0, and $\leq$ can be defined using the operations explicitly defined above.

PROPOSITION 3.4. *Given the Lindenbaum algebra $\mathcal{A}_{\boldsymbol{L}}$ for a classical system of conditional logic $\boldsymbol{L}$:*

1. $\vdash_{\boldsymbol{L}} \phi$ *if and only if* $[\phi] = 1$
2. $\vdash_{\boldsymbol{L}} \phi \rightarrow \psi$ *if and only if* $[\phi] \leq [\psi]$
3. $\vdash_{\boldsymbol{L}} \phi \leftrightarrow \psi$ *if and only if* $[\phi] = [\psi]$

PROOF. It is clear that $\langle \mathbb{B}, 1, 0, -, \cup, \cap \rangle$ is a Boolean algebra. The properties listed above follow straightforwardly as a consequence. $\square$

The *canonical Lindenbaum model* for a classical system of conditional logic $\boldsymbol{L}$ is a structure $\mathfrak{I}_{\boldsymbol{L}} = \langle \mathcal{A}_{\boldsymbol{L}}, f \rangle$ in which $\mathcal{A}_{\boldsymbol{L}}$ is the Lindenbaum algebra for $\boldsymbol{L}$ and $f(p) = [p]$ for any $p \in \Pi$.

LEMMA 3.1 (Truth Lemma). *For all $\phi \in \Phi$, $\vdash_{\boldsymbol{L}} \phi$ if and only if $\models^{\mathfrak{I}_{\boldsymbol{L}}} \phi$.*

PROOF. By induction on the complexity of ϕ, note that for all $\phi \in \Phi$, $[\phi] = f(\phi)$. Then it follows that $[\phi] = 1$ if and only if $f(\phi) = 1$. Therefore, $\vdash_{\boldsymbol{L}} \phi$ if and only if $\models^{\mathfrak{I}_{\boldsymbol{L}}} \phi$. $\square$

LEMMA 3.2. *$\mathfrak{I}_{\mathbf{CR}} \in \mathcal{C}$, $\mathfrak{I}_{\mathbf{CR1}} \in \mathcal{C}_\alpha$, and $\mathfrak{I}_{\mathbf{CR2}} \in \mathcal{C}_\beta$.*

PROOF. We take the case of **CR1**. It has to be shown that the algebra on which $\mathfrak{I}_{\mathbf{CR1}}$ is based — $\mathcal{A}_{\mathbf{CR1}}$ — satisfies the constraints on $*$ given in Definition 3.2. First, note that $\vdash_{\mathbf{CR1}} (\phi \mathrel{\Box\!\!\rightarrow} (\psi \wedge \theta)) \rightarrow ((\phi \mathrel{\Box\!\!\rightarrow} \psi) \wedge (\phi \mathrel{\Box\!\!\rightarrow} \theta))$ and $\vdash_{\mathbf{CR1}} ((\phi \mathrel{\Box\!\!\rightarrow} \psi) \wedge (\phi \mathrel{\Box\!\!\rightarrow} \theta)) \rightarrow (\phi \mathrel{\Box\!\!\rightarrow} (\psi \wedge \theta))$. Therefore, $\vdash_{\mathbf{CR1}} (\phi \mathrel{\Box\!\!\rightarrow} (\psi \wedge \theta)) \leftrightarrow ((\phi \mathrel{\Box\!\!\rightarrow} \psi) \wedge (\phi \mathrel{\Box\!\!\rightarrow} \theta)) \Rightarrow [(\phi \mathrel{\Box\!\!\rightarrow} (\psi \wedge \theta))] = [(\phi \mathrel{\Box\!\!\rightarrow} \psi) \wedge (\phi \mathrel{\Box\!\!\rightarrow} \theta)] \Rightarrow [\phi] * ([\psi] \cap [\theta]) = ([\phi] * [\psi]) \cap ([\phi] * [\theta])$. $\vdash_{\mathbf{CR1}} (\phi \mathrel{\Box\!\!\rightarrow} \psi) \rightarrow \neg(\phi \mathrel{\Box\!\!\rightarrow} \neg\psi)$ implies that $[\phi] * [\psi] \leq -([\phi] * -[\psi])$, by Proposition 3.4. $\square$

THEOREM 3.2 (Completeness). **CR**, **CR1**, *and* **CR2** *are complete with respect to $\mathcal{C}$, $\mathcal{C}_\alpha$, and $\mathcal{C}_\beta$ respectively.*

PROOF. The proof is essentially the same in each of the three cases; take **CR1** for concreteness. $\nvdash_{\mathbf{CR1}} \phi$ implies $\not\models^{\mathfrak{I}_{\mathbf{CR1}}} \phi$ by Lemma 3.1. Since $\mathfrak{I}_{\mathbf{CR1}} \in \mathcal{C}_\alpha$, by Lemma 3.2, it follows that $\not\models_{\mathcal{C}_\alpha} \phi$. $\square$

PROPOSITION 3.5. *Given any conditional algebra $\mathcal{A}$, $x * y \leq -(x * -y)$ if and only if $x * 0 = 0$.*

PROOF. $x*y \leq -(x*-y) \Leftrightarrow (x*y)\cap--(x*-y) = 0 \Leftrightarrow (x*y)\cap(x*-y) = 0 \Leftrightarrow x*(y\cap -y) = 0 \Leftrightarrow x*0 = 0$. □

LEMMA 3.3. *If $\mathcal{A} = \langle\langle \mathbb{B}, 1, 0, -, \cup, \cap\rangle, *\rangle$ is a conditional algebra and $a_1, \ldots, a_n \in \mathbb{B}$, then there is a conditional algebra $\mathcal{A}^\dagger = \langle\langle \mathbb{B}^\dagger, 1, 0, -^\dagger, \cup^\dagger, \cap^\dagger\rangle, *^\dagger\rangle$ of the same type (basic, α, β) with at most 2^{2^n} elements such that*:

1. $a_1, \ldots, a_n \in \mathbb{B}^\dagger$
2. *For any* $x \in \mathbb{B}^\dagger$, $-x = -^\dagger x$
3. *For any* $x, y \in \mathbb{B}^\dagger$, $x \cup y = x \cup^\dagger y$
4. *For any* $x, y \in \mathbb{B}^\dagger$, $x \cap y = x \cap^\dagger y$
5. *For any* $x, y \in \mathbb{B}^\dagger$, *if* $x * y \in \mathbb{B}^\dagger$, *then* $x * y = x *^\dagger y$

PROOF. Given a finite list of elements $a_1, \ldots, a_n \in \mathbb{B}$, we take the Boolean subalgebra of $\mathcal{A}$ generated by that list. In this way, we obtain an algebra $\langle \mathbb{B}^\dagger, 1, 0, -^\dagger, \cup^\dagger, \cap^\dagger\rangle$ with at most 2^{2^n} elements satisfying the first four properties. For $x, y \in \mathbb{B}^\dagger$ and $z \in \mathbb{B}$, we say that y *x-covers* z if and only if $z \leq y$ and $z, x * z \in \mathbb{B}^\dagger$ [as in 8, p. 136]. For $x, y \in \mathbb{B}^\dagger$, define $x *^\dagger y = \bigcup^\dagger \{x * y_i : y \ x\text{-covers } y_i\}$.[5]

The proof that $\mathcal{A}^\dagger$ satisfies the fifth property and is basic (in the sense of Definition 3.1) if $\mathcal{A}$ is basic is exactly the same as in [8, p. 136] and is omitted. It remains to show that $\mathcal{A}^\dagger$ is of type α (β) if $\mathcal{A}$ is of type α (β). Suppose $\mathcal{A}$ is an α-conditional algebra and $x \in \mathbb{B}^\dagger$. Then $x * 0 = 0 \in \mathbb{B}^\dagger$, so by the fifth property $0 = x*0 = x*^\dagger 0$. By Proposition 3.5, $\mathcal{A}^\dagger$ is an α-conditional algebra. Alternatively, suppose $\mathcal{A}$ is a β-conditional algebra and $x \in \mathbb{B}^\dagger$. Then $-x = -^\dagger x \in \mathbb{B}^\dagger$ and $x*-^\dagger x = x*-x = 0 \in \mathbb{B}^\dagger$. By the fifth property, $0 = x * -^\dagger x = x *^\dagger -^\dagger x$. Therefore, $\mathcal{A}^\dagger$ is a β-conditional algebra. □

DEFINITION 3.4. Given a conditional model $\mathfrak{I} = \langle \mathcal{A}, f\rangle$ and a finite set of formulae closed under subformulae Γ, a *filtration* of $\mathfrak{I}$ through Γ, $\mathfrak{I}^\Gamma = \langle \mathcal{A}^\Gamma, f^\Gamma\rangle$, is a structure such that:

1. $\mathcal{A}^\Gamma$ is the conditional algebra $\mathcal{A}^\dagger$ generated by the set $\{f(\phi) : \phi \in \Gamma\}$
2. For any $p \in \Gamma$, $f^\Gamma(p) = f(p)$

LEMMA 3.4. *Given a finite set Γ of formulae closed under subformulae, for all $\phi \in \Gamma$, $\models^{\mathfrak{I}^\Gamma} \phi$ if and only if $\models^{\mathfrak{I}} \phi$.*

PROOF. By induction on the complexity of $\phi \in \Gamma$. Consider the case where ϕ is of the form $\psi \mathrel{\Box\!\!\!\rightarrow} \theta$. By Definition 3.4, $f(\psi)*f(\theta) = f(\psi \mathrel{\Box\!\!\!\rightarrow} \theta)$

[5] It is clear that $\mathbb{B}^\dagger$ is closed under $*^\dagger$ because (by convention) $\bigcup^\dagger \emptyset = 0$.

belongs to $\mathbb{B}^{\Gamma}$. Hence, by the induction hypothesis and Lemma 3.3, $f(\psi \mathrel{\Box\!\!\rightarrow} \theta) = f^{\Gamma}(\psi \mathrel{\Box\!\!\rightarrow} \theta)$. Then clearly $f(\psi \mathrel{\Box\!\!\rightarrow} \theta) = 1$ if and only if $f^{\Gamma}(\psi \mathrel{\Box\!\!\rightarrow} \theta) = 1$. □

Theorem 3.3 (Finite Model Property). *Given any of the three systems $\boldsymbol{L}$, if $\nvdash_{\boldsymbol{L}} \phi$, there is a finite model $\mathfrak{I}^{\Gamma}$ in the appropriate class such that $\not\models^{\mathfrak{I}^{\Gamma}} \phi$.*

Proof. For concreteness, take $\boldsymbol{L}$ to be **CR1**; the proof works the same way in each case. Suppose that $\nvdash_{\mathbf{CR1}} \phi$. By Theorem 3.2, there is a model $\mathfrak{I} \in C_{\alpha}$ such that $\not\models^{\mathfrak{I}} \phi$. Let Γ contain ϕ and be closed under subformulae. Then, by lemmata 3.3 and 3.4, $\mathfrak{I}^{\Gamma} \in C_{\alpha}$, $\mathfrak{I}^{\Gamma}$ is finite, and $\not\models^{\mathfrak{I}^{\Gamma}} \phi$. □

A corollary of Theorem 3.3, given that **CR**, **CR1**, and **CR2** are finitely axiomatizable, is that each system is decidable.

4. Conditional Obligation

In this section, we examine an application of **CR1** to conditional obligation. Obligations are either unconditional ("do not steal") or conditional ("given that you stole, return the stolen goods"). Unconditional obligations can be modeled relatively well using the resources of standard modal logic, but matters become more complicated when it comes to conditional obligation [see 2, p. 201, for a discussion].

Let us read $\phi \mathrel{\Box\!\!\rightarrow} \psi$ as saying that given ϕ, ψ ought to be the case or is obligatory. We can also introduce a connective for conditional permissibility: let us read $\phi \mathrel{\Diamond\!\!\rightarrow} \psi$ as saying that given ϕ, ψ is permissible. We take $\mathrel{\Box\!\!\rightarrow}$ as primitive and adopt the definition:[6]

$$\phi \mathrel{\Diamond\!\!\rightarrow} \psi \;\equiv\; \neg(\phi \mathrel{\Box\!\!\rightarrow} \neg\psi) \qquad (\mathrel{\Diamond\!\!\rightarrow} \text{Def.})$$

Bearing this interpretation in mind, we can now investigate what an appropriate logic is for $\mathrel{\Box\!\!\rightarrow}$.

First, note that (RCK) and (ID) are inappropriate. (RCK) implies that every situation gives rise to obligations of a trivial variety: for any ϕ, $\phi \mathrel{\Box\!\!\rightarrow} \top$ is a theorem. This is implausible. Why is it obligatory, given

[6] Under the counterfactual interpretation of these connectives, this equivalence is of course famously endorsed by Lewis [4, p. 2]. We thank an anonymous referee for pressing us to expand on the issue of conditional permissibility here.

that $2+2=4$, that one either go for a walk or not go for a walk? (ID) is even more objectionable: given that people are starving, surely we do not want to conclude that this *ought* to be the case.

On the other hand, (RCEA), (RCEC), (CM), and (CC) all make decent sense given this interpretation. The basic idea behind why any logic of obligation should be classical is that what determines an obligation is something deeper than a formula, such that in all *equivalent* circumstances, the same obligation arises (and similarly, what is obligatory is some state of affairs, not a formula) [2, p. 273].

If one accepts that 'ought' implies 'can,' then it should be held that nothing impossible is obligatory under any circumstance. Formally, this amounts to an endorsement of the axiom scheme $\neg(\phi \boxright \bot)$. It should already be apparent on the basis of Proposition 3.5 that this is equivalent to (WBT1) in the logic of conditional obligation so far constructed. Nevertheless, the result can easily be proved axiomatically as well:

PROPOSITION 4.1. *A regular system of conditional logic contains the scheme* $\neg(\phi \boxright \bot)$ *if and only if it contains* (WBT1).

PROOF.

1	$\neg(\phi \boxright \bot)$	$\vdash$
2	$\bot \leftrightarrow (\psi \wedge \neg\psi)$	$\vdash$
3	$(\phi \boxright \bot) \leftrightarrow (\phi \boxright (\psi \wedge \neg\psi))$	(RCEC) 2
4	$\neg(\phi \boxright (\psi \wedge \neg\psi))$	**PL** 1, 3
5	$((\phi \boxright \psi) \wedge (\phi \boxright \neg\psi)) \rightarrow (\phi \boxright (\psi \wedge \neg\psi))$	(CC)
6	$\neg((\phi \boxright \psi) \wedge (\phi \boxright \neg\psi))$	**PL** 4, 5
7	$(\phi \boxright \psi) \rightarrow \neg(\phi \boxright \neg\psi)$	**PL** 6

1	$(\phi \boxright \psi) \rightarrow \neg(\phi \boxright \neg\psi)$	(WBT1)
2	$\neg((\phi \boxright \psi) \wedge (\phi \boxright \neg\psi))$	**PL** 1
3	$(\phi \boxright (\psi \wedge \neg\psi)) \rightarrow ((\phi \boxright \psi) \wedge (\phi \boxright \neg\psi))$	(CM)
4	$\neg(\phi \boxright (\psi \wedge \neg\psi))$	**PL** 2, 3
5	$\bot \leftrightarrow (\psi \wedge \neg\psi)$	$\vdash$
6	$(\phi \boxright \bot) \leftrightarrow (\phi \boxright (\psi \wedge \neg\psi))$	(RCEC) 5
7	$\neg(\phi \boxright \bot)$	**PL** 4, 6 $\square$

Chellas [2, p. 273] considers (something equivalent to) the scheme $\neg(\phi \boxright \bot)$, before rejecting it as implausible and opting for a weaker scheme. The substance of his objection is that, given an impossible situation, there might be impossible obligations. In other words, the formula $\neg(\bot \boxright \bot)$ should not be a theorem.

In response to this objection, we think there is some intuitiveness to claiming that, in an impossible situation, everything is permissible and (consequently) nothing is obligatory. The claim that everything is permissible in an impossible situation is captured by the scheme $\bot \mathrel{\Diamond\!\!\rightarrow} \phi$ from which $\neg(\bot \mathrel{\Box\!\!\rightarrow} \bot)$ follows straightforwardly. Finally, note that while $\bot \mathrel{\Diamond\!\!\rightarrow} \phi$ is independent of the logic so far sketched (use the model from Proposition 3.2), it is not hard to see that it can consistently be added (use the translation from Theorem 2.1).[7] Therefore, Chellas' objection to $\neg(\phi \mathrel{\Box\!\!\rightarrow} \bot)$ notwithstanding, both **CR1** and its extension by $\bot \mathrel{\Diamond\!\!\rightarrow} \phi$ seem fairly plausible as logics of conditional obligation.

In considering systems of conditional obligation, a question naturally arises about the possibility of embedding systems of *unconditional* obligation into them. For to say that ϕ is obligatory is, intuitively, just to say that, given something true in any circumstance, ϕ is obligatory. That is, if $\Box\phi$ is understood to mean that ϕ is obligatory (unconditionally), it is entirely reasonable to adopt the definition:[8]

$$\Box\phi \equiv \top \mathrel{\Box\!\!\rightarrow} \phi \tag{$\Box$ Def.}$$

Given our proposed interpretation of **CR1** and its extension by $\bot \mathrel{\Diamond\!\!\rightarrow} \phi$, it is surely a desideratum that they determine plausible deontic modal logics under ($\Box$ Def.). In fact, they determine precisely the same system: the deontic modal logic Lemmon [3] calls **D2**.[9]

To show that **D2** can be embedded into **CR1**, we modify results from [13, pp. 88–90]. First, let us describe the system **D2**. Let the language of **D2** be $\mathcal{L}_\Box$ ($\mathcal{L}$ but with the unary connective $\Box$ instead of $\mathrel{\Box\!\!\rightarrow}$). Then $\Phi_\Box$ is the set of formulae in $\mathcal{L}_\Box$. Defining theoremhood as usual, let **D2** be the smallest system (set of formulae from $\Phi_\Box$) containing **PL** (in $\mathcal{L}_\Box$) closed under (RE) and containing all (instances) of (M), (C), and (D):[10]

$$\frac{\phi \leftrightarrow \psi}{\Box\phi \leftrightarrow \Box\psi} \tag{RE}$$

[7] It should also be noted that (AT1) is independent of this system (**CR1** extended by $\bot \mathrel{\Diamond\!\!\rightarrow} \phi$). We thank Branden Fitelson for obtaining this result with a computer program.

[8] ($\Box$ Def.) is anticipated in the literature on deontic logic at least as far back as von Wright [14, p. 509]. It can also be found in the literature on non-deontic conditional logic [see, e.g., 5, p. 360].

[9] Incidentally, **D2** is also the "inner" modal logic determined by Lewis' system **V** [4, § 6.3].

[10] As before, (RE) applies only if the premise is a theorem.

$$\Box(\phi \wedge \psi) \to (\Box\phi \wedge \Box\psi) \tag{M}$$

$$(\Box\phi \wedge \Box\psi) \to \Box(\phi \wedge \psi) \tag{C}$$

$$\neg\Box\bot \tag{D}$$

The axiomatization of **D2** offered here is different from that given by Lemmon [3, p. 184], but is easily shown to be equivalent.[11] Note that **D2**, which is not closed under the rule of necessitation, is a subnormal modal logic.

Define a function $\sigma\colon \Phi_\Box \to \Phi$ that commutes with the connectives except that, in conformity to ($\Box$ Def.), $\sigma(\Box\phi) = \top \mathrel{\Box\!\!\to} \sigma(\phi)$. Then:

LEMMA 4.1. $\vdash_{\mathbf{D2}} \phi$ *implies* $\vdash_{\mathbf{CR1}} \sigma(\phi)$.

PROOF. By a routine induction on the length of proof; the axiomatization used for **D2** renders the result obvious. □

Define a function $\sigma^{-1}\colon \Phi \to \Phi_\Box$ that commutes with the connectives except that $\sigma^{-1}(\phi \mathrel{\Box\!\!\to} \psi) = \Box(\sigma^{-1}(\phi) \wedge \sigma^{-1}(\psi))$ (cf. τ in Theorem 2.1). Then, again by a routine induction on the length of proof, we obtain:

LEMMA 4.2. $\vdash_{\mathbf{CR1}} \phi$ *implies* $\vdash_{\mathbf{D2}} \sigma^{-1}(\phi)$.

LEMMA 4.3. $\vdash_{\mathbf{D2}} \phi \leftrightarrow \sigma^{-1}(\sigma(\phi))$.

PROOF. By induction on the complexity of ϕ. The only case of interest is that in which ϕ is of the form $\Box\psi$. By the induction hypothesis and definitions of the translation functions, what must be shown is that $\vdash_{\mathbf{D2}} \Box\psi \leftrightarrow \Box(\top \wedge \psi)$. But this is trivial using (RE). □

THEOREM 4.1 (Modal Embedding). $\vdash_{\mathbf{D2}} \phi$ *if and only if* $\vdash_{\mathbf{CR1}} \sigma(\phi)$.

PROOF. The direction from left to right is established by Lemma 4.1. Conversely, suppose that $\vdash_{\mathbf{CR1}} \sigma(\phi)$. By Lemma 4.2, $\vdash_{\mathbf{D2}} \sigma^{-1}(\sigma(\phi))$. Thus, by Lemma 4.3, $\vdash_{\mathbf{D2}} \phi$. □

To see that **D2** also embeds into **CR1** extended by $\bot \mathrel{\Diamond\!\!\to} \phi$, simply observe that $\neg(\bot \mathrel{\Box\!\!\to} \neg\phi)$ is provable in **D2** under σ^{-1}; the rest of the argument goes as before.

Theorem 4.1 buttresses the credibility of the deontic interpretation of **CR1** (and its extension by $\bot \mathrel{\Diamond\!\!\to} \phi$), for it shows that an independently motivated logic of (unconditional) obligation can be naturally

[11] We generally follow the conventions of Chellas [2] in our axiomatization.

interpreted within these systems.[12] Before concluding this section, it is worth briefly noting that **CR2** does not have a plausible deontic interpretation:

> Given that there is starvation, it ought to be the case that there is not starvation

This statement of conditional obligation seems to be true. But in **CR2**, the negation of it is a theorem. Therefore, the logic of conditional obligation, while arguably half-connexive, is not (fully) connexive.

5. Concluding Remarks

Connexive principles like Aristotle's and Boethius' theses have by now been studied in a variety of settings: consistent and inconsistent, classical and nonclassical, etc. Since **CR**, **CR1**, and **CR2** all contain classical logic, this paper belongs to a tradition which examines connexivism in a consistent classical context. More particularly, connexive principles are here situated in classical conditional logic, as pioneered by Stalnaker [11] and Lewis [4] and further systematized by Chellas [1, 2] and Nute [8].

Interest in augmenting classical conditional logic with connexive principles is well-established. As indicated in Section 4, deontic applications of (half) connexive conditional logic were considered by Chellas [1, 2] and earlier authors, though never, apparently, under the connexive label. Another early (non-deontic) contribution in this vein is [5] in which half-connexive systems extending **CR1** are presented. We leave for future work a semantic investigation of Lowe's systems along the lines of Section 3, although this should present no great difficulties. Finally, the most recent work of direct relevance to this paper, as discussed in Section 2, is that of [12].

It is hoped that this work will encourage further investigations of connexive conditional logic, both technical and philosophical. On the technical side, besides algebraically characterizing the systems of Lowe [5], we aim to give worlds semantics for connexive systems in the neighborhood of **CR**. One way this might be done is by modifying Chellas'

[12] It is beyond the scope of this paper to present arguments on behalf of **D2**, but it is worth noting that **D2** (modulo certain technical details) was among the first deontic systems popularly advocated for. See [3, p. 185] for some of the relevant history.

"minimal" semantics from [1]. On the philosophical side, a more thoroughgoing treatment of the deontic interpretation of $\Box\!\!\rightarrow$ (and the history of such interpretations, insofar as they involve connexive theses) is desirable. Finally, non-deontic interpretations of the systems discussed here — especially **CR2**, for which no interpretation has been offered — ought to be explored.

Acknowledgements. We thank two anonymous referees, the editors of *Logic and Logical Philosophy*, Branden Fitelson, and Graham Priest for comments and advice that led to improvements in this paper. An early version of this paper was presented at Professor Priest's doctoral students group and we thank the members for comments and feedback.

References

[1] Chellas, B. F., "Basic conditional logic", *Journal of Philosophical Logic* 4, 2 (1975): 133-153. DOI: 10.1007/BF00693270

[2] Chellas, B. F., *Modal Logic*, Cambridge University Press, 1980. DOI: 10.1017/CBO9780511621192

[3] Lemmon, E. J., "New foundations for Lewis modal systems", *Journal of Symbolic Logic* 22, 2 (1957): 176–186. DOI: 10.2307/2964179

[4] Lewis, D., 1973, *Counterfactuals*, Oxford University Press.

[5] Lowe, E. J., "A simplification of the logic of conditionals", *Notre Dame Journal of Formal Logic* 24, 3 (1983): 357–366. DOI: 10.1305/ndjfl/1093870380

[6] McCall, S., "Connexive implication", *Journal of Symbolic Logic* 31, 3 (1966): 415–433. DOI: 10.2307/2270458

[7] Mortensen, C., "Aristotle's thesis in consistent and inconsistent logics", *Studia Logica* 43, 1–2 (1984): 107–116. DOI: 10.1007/BF00935744

[8] Nute, D., *Topics in Conditional Logic*, D. Reidel Publishing Company, 1980. DOI: 10.1007/978-94-009-8966-5

[9] Pizzi, C., "Boethius' thesis and conditional logic", *Journal of Philosophical Logic* 6, 1 (1977): 283–302. DOI: 10.1007/BF00262063

[10] Pizzi, C., and T. Williamson, "Strong Boethius' thesis and consequential implication", *Journal of Philosophical Logic* 26, 5 (1997): 569–588. DOI: 10.1023/A:1004230028063

[11] Stalnaker, R. C., "A theory of conditionals", pages 41–55 in W. L. Harper, R. Stalnaker, and G. Pearce (eds.), *Ifs*, D. Reidel Publishing Company, 1968. DOI: 10.1007/978-94-009-9117-0_2

[12] Unterhuber, M., "Beyond system **P** – Hilbert-style convergence results for conditional logics with a connexive twist", *IfCoLog Journal of Logics and their Applications* 3, 3 (2016): 377–412.

[13] Williamson, T., "Modal logic within counterfactual logic", pages 81–96 in B. Hale and A. Hoffmann (eds.), *Modality: Metaphysics, Logic, and Epistemology*, Oxford University Press, 2010. DOI: 10.1093/acprof:oso/9780199565818.003.0005

[14] von Wright, G. H., "A note on deontic logic and derived obligation", *Mind* 65, 1 (1956): 507–509. DOI: 10.1093/mind/65.1.507

Yale Weiss
Philosophy Program
The Graduate Center, CUNY
365 5th Ave., Rm. 7113
New York, NY 10016, USA
yweiss@gradcenter.cuny.edu