Logic and **L**ogical **P**hilosophy
Volume 28 (2019), 137–155
DOI: 10.12775/LLP.2018.008

Andrzej Indrzejczak

FREGEAN DESCRIPTION THEORY IN PROOF-THEORETICAL SETTING

Abstract. We present a proof-theoretical analysis of the theory of definite descriptions which emerges from Frege's approach and was formally developed by Kalish and Montague. This theory of definite descriptions is based on the assumption that all descriptions are treated as genuine terms. In particular, a special object is chosen as a designatum for all descriptions which fail to designate a unique object. Kalish and Montague provided a semantical treatment of such theory as well as complete axiomatic and natural deduction formalization. In the paper we provide a sequent calculus formalization of this logic and prove cut elimination theorem in the constructive manner.

Keywords: sequent calculus; cut elimination; definite descriptions; Frege

1. Introduction

Since the publication of B. Russell's famous paper "On denoting" [22] many researchers provided deep and detailed studies of the phenomenon of definite descriptions. Yet, despite the long history and variety of proposed solutions we can hardly say that some of them are treated as obvious or commonly acceptable. In particular, a correct treatment of so called improper descriptions (i.e., those which fail to have a unique denotation) is hotly disputed.

It seems that most researchers dealing with the problem of definite descriptions follow the Russellian route and eliminate them in favour of ordinary first-order logic with identity. Such reductionist approach has many disadvantages however and not surprisingly several logicians provided theories in which definite descriptions are treated as genuine terms.

Received February 25, 2018. *Revised* May 26, 2018. *Published online* June 17, 2018

In fact, many solutions were developed on the ground of non-classical logics like free or modal logic. It seems natural that richer resources of such logical systems offer better prospects for development of satisfactory theory of definite descriptions. Bencivenga [1] provides a good survey of theories based on free logics, whereas Fitting and Mendelsohn [7] and Garson [10] offer adequate formalizations of significantly different modal theories of descriptions.

However, even in the framework of classical logic there is a tradition, starting with Frege [8, 9], in which definite descriptions are treated as genuine terms. This account was first elaborated by Carnap [4] under the name of the chosen object theory, and developed in the setting of different versions of set theory by Bernays [3] and Quine [20] where richer languages allow for special treatment. On the ground of pure classical logic, Fregean approach was also formalised by Rosser [21] but in the way which is incomplete in some sense (see Hailperin [11]). Eventually Kalish and Montague [15] provided semantical treatment with adequate axiomatization in the setting of classical first-order logic (FOL). One can also find a tableau characterization of this logic in Bencivenga, Lambert and van Fraasen [2].

In this paper we are not concerned with philosophical problems connected with different approaches but with their proof-theoretical treatment. Although some of the proposed theories are formulated in terms of natural deduction (Kalish and Montague [15], Garson [10]) or tableau systems (Bencivenga, Lambert and van Fraasen [2], Fitting and Mendelsohn [7]) it is quite evident that a structural proof theory, in the sense of Negri and von Plato [19], is still to come. In particular, we think of a formalization provided in terms of sequent calculi with suitably defined rules admitting cut elimination. In this perspective, rules of the mentioned systems are rather hardly suitable for proof-theoretical analysis. In what follows we provide a sequent calculus equivalent to Kalish and Montague system and prove cut elimination theorem for it.

2. Kalish and Montague System

The system of Kalish and Montague was first presented in [15] where semantics for descriptions is provided with completeness proof for axiomatic formalization. Natural deduction system, later extensively used in a slightly modified form in the textbook [16], is also introduced there.

We recall briefly the essential features of their formalization of Frege's theory of descriptions.

The system is formulated in the standard predicate language with identity and with iota-operator forming definite descriptions from formulae of the language. We will use the following categories of expressions denoted by the following symbols:

- denumerably infinite set of variables VAR $= \{x, y, z, \ldots\}$,
- denumerably infinite set of operation symbols of different arities FUN $= \{a, b, c, \ldots, f, g, h, \ldots\}$,
- denumerably infinite set of predicate symbols of different arities PRED $= \{A, B, C, \ldots\}$
- logical constants: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\forall$, $\exists$, $=$, ι.

Since definite descriptions are complex terms involving formulae, a formal characterisation of a term and formula must be done simultaneously, in one definition. One may find such formal recursive definitions in [16] or [10]. Here we omit details for brevity's sake, but underline that the set of terms cover variables, operations formed by means of elements of FUN, and descriptions. The latter are written as $\iota x\varphi$, where φ is a formula in the scope of iota-operator. In the presentation of sequent calculus we adopt the convention, particularly useful in this framework, that variables x, y, ..., may occur only bound by quantifiers or iota-operator. Thus, in contrast to original Kalish and Monatague formulation, in the role of free variables we will be using symbols of nullary operations (constant names) a, b, c. In general, the same symbols are applied in the language and metalanguage but with additional metavariables φ, ψ, χ used for any formulae and Γ, Δ, Π, Σ for their sets and multisets. Moreover, we will use a symbol $\imath$ for denoting a special chosen object being a designatum of all improper descriptions. Metavariables t, t_1, ..., will be applied for any term, including descriptions and $\imath$. $\varphi[x/t]$ is officially used for the operation of correct substitution of a term t for x. However, to simplify matters, we will be also using freely in proof schemata a notation $\varphi(x)$, $\varphi(a)$, $\varphi(t)$. In particular, $\varphi(x)$ will be used to mean that φ being a scope of some operator contains at least one occurrence of (bound) x, whereas $\varphi(a)$ and $\varphi(t)$ will denote the result of substitution. A predicate (including identity) with arguments being any terms, including descriptions, is treated as atomic formula and its complexity is always 1. Complexity of any compound formula is the sum of the number of occurrences of atomic formulae and the number of connectives and quantifiers used in the formation of this formula from atomic components. It is important to note that neither iota-operators nor any logical constants occurring inside scopes of descriptions are counted. Thus the complexity of $A\imath x(Ba \vee \exists yCy)$ is 1 but the complexity of $Ba \vee \exists yCy$ as well as of $B(\imath x(\forall z(Az \rightarrow Rz\imath yCy))) \vee \exists yCy$ is 4.

Kalish and Montague characterised their system both semantically and syntactically. The latter is presented both as axiomatic and Jaśkowski-style natural deduction system. Moreover, in the later textbook presentation there is a small change in the syntactical characterization of improper descriptions. We omit details of their system for FOL with identity and recall only axioms and corresponding natural deduction rules for descriptions:

$$\forall x(\varphi(x) \leftrightarrow x = y) \rightarrow y = \imath x\varphi$$
$$\neg\exists y\forall x(\varphi(x) \leftrightarrow x = y) \rightarrow \imath x\varphi = \imath x(x = x)$$

where in both cases y does not occur in $\imath x\varphi$ and $\imath x(x = x)$ is a fixed improper description. Note that the first axiom by universal generalization (primitive rule in Kalish Montague axiomatization of FOL) is equivalent to $\forall y(\forall x(\varphi(x) \leftrightarrow x = y) \rightarrow y = \imath x\varphi)$. The latter implies $\forall x(\varphi(x) \leftrightarrow x = t) \rightarrow t = \imath x\varphi$ for any term properly substitutable for y which is important for later considerations.

In natural deduction system instead of axioms we have two rules of proper and improper description introduction:

$$\exists y\forall x(\varphi(x) \leftrightarrow x = y) \vdash \varphi(\imath x\varphi) \tag{PD}$$
$$\neg\exists y\forall x(\varphi(x) \leftrightarrow x = y) \vdash \imath x\varphi = \imath x(x = x) \tag{ID}$$

It is easy to demonstrate that (PD) gives the same result as the first axiom thus both systems are equivalent. In [16] a natural deduction system is a little bit changed. Improper descriptions are now characterised by the following rule:

$$\neg\exists y\forall x(\varphi(x) \leftrightarrow x = y) \vdash \imath x\varphi = \imath x(x \neq x) \tag{ID$'$}$$

However, in their system $\imath x(x = x) = \imath x(x \neq x)$ is a thesis, so it is still the same theory. In fact, the way we syntactically represent the chosen object being a denotation of all improper descriptions is inessential. In what follows we will be using just a constant symbol $\imath$ as the name of this fixed denotation. In this respect we follow a solution advocated by Carnap [4].

Kalish and Montague system (KM) is adequate with respect to semantics with models having a fixed object $\imath$ in the domain. A definition of a model $\mathfrak{M}$, denotation function V, valuation of variables v and satisfaction relation $\vDash$ is standard. For descriptions we have the following clause:

> If there is a unique $o \in D$ such that $\mathfrak{M}, v^x_o \vDash \varphi$, then $V(\imath x\varphi) = o$; otherwise $V(\imath x\varphi) = \imath$,

where for any v, v^x_o is an x-variant of v with o assigned to x.

The axiomatic system is proved (weakly) adequate with respect to this semantics.

3. Sequent Calculus

We will use a version of Gentzen's LK calculus but with sequents built not from finite lists but from multisets of formulae and with all rules multiplicative (i.e., context-free in case of many-premiss rules).

$$\text{(AX)}\ \varphi \Rightarrow \varphi \qquad\qquad \text{(Cut)}\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

$$(\text{W}\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad\qquad (\Rightarrow\text{W})\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$$

$$(\text{C}\Rightarrow)\ \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad\qquad (\Rightarrow\text{C})\ \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$$

$$(\neg\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta} \qquad\qquad (\Rightarrow\neg)\ \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$$

$$(\wedge\Rightarrow)\ \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \qquad\qquad (\Rightarrow\wedge)\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Pi \Rightarrow \Sigma, \psi}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \varphi \wedge \psi}$$

$$(\vee\Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Pi \Rightarrow \Sigma}{\varphi \vee \psi, \Gamma, \Pi \Rightarrow \Delta, \Sigma} \qquad\qquad (\Rightarrow\vee)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$(\rightarrow\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Pi \Rightarrow \Sigma}{\varphi \rightarrow \psi, \Gamma, \Pi \Rightarrow \Delta, \Sigma} \qquad\qquad (\Rightarrow\rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi}$$

$$(\leftrightarrow\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \psi \quad \varphi, \psi, \Pi \Rightarrow \Sigma}{\varphi \leftrightarrow \psi, \Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

$$(\Rightarrow\leftrightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta, \psi \quad \psi, \Pi \Rightarrow \Sigma, \varphi}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \varphi \leftrightarrow \psi}$$

$$(\forall\Rightarrow)\ \frac{\varphi[x/t], \Gamma \Rightarrow \Delta}{\forall x\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\forall)\ \frac{\Gamma \Rightarrow \Delta, \varphi[x/a]}{\Gamma \Rightarrow \Delta, \forall x\varphi}$$

where a is not in Γ, Δ, φ

$$(\exists\Rightarrow)\ \frac{\varphi[x/a], \Gamma \Rightarrow \Delta}{\exists x\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\exists)\ \frac{\Gamma \Rightarrow \Delta, \varphi[x/t]}{\Gamma \Rightarrow \Delta, \exists x\varphi}$$

where a is not in Γ, Δ, φ

$$(=\Rightarrow)\ \frac{t = t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$(\Rightarrow=)\ \frac{\Gamma \Rightarrow \Delta, t_1 = t_2 \quad \Pi \Rightarrow \Sigma, \varphi[x/t_1]}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \varphi[x/t_2]}$$

$$(\Rightarrow\imath)\ \frac{\varphi[x/a], \Gamma \Rightarrow \Delta, t = a \quad t = a, \Pi \Rightarrow \Sigma, \varphi[x/a]}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, t = \imath x\varphi}$$

where a is not in Γ, Δ, Π, Σ, φ; and t is not $\imath$

$$(\Rightarrow\imath 1)\ \frac{\varphi[x/a], \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \imath = \imath x\varphi}$$

where a is not in Γ, Δ, φ

$$(\Rightarrow\imath 2)\ \frac{\Gamma \Rightarrow \Delta, \varphi[x/t_1] \qquad \Pi \Rightarrow \Sigma, \varphi[x/t_2] \qquad t_1 = t_2, \Lambda \Rightarrow \Theta}{\Gamma, \Pi, \Lambda \Rightarrow \Delta, \Sigma, \Theta, \imath = \imath x\varphi}$$

The definition of a proof is standard. Similarly for definitions of principal, side and parametric formulae in rule's applications.

4. Equivalence of the Systems

We will show that for every $\Gamma \Rightarrow \Delta$ derivable in our sequent calculus we can provide a proof of $\Gamma \vdash \bigvee \Delta$ (where $\bigvee \Delta$ is a disjunction of elements of Δ) in Kalish and Montague system and conversely. We restrict considerations to rules for descriptions. For this aim let us take two sequents expressing Kalish and Montague axioms (corresponding to theses T401 and T404 in [16]):

$$\Rightarrow \forall y(\forall x(\varphi \leftrightarrow x = y) \rightarrow \imath x\varphi = y)$$

and

$$\Rightarrow \neg\exists y\forall x(\varphi \leftrightarrow x = y) \rightarrow \imath x\varphi = \imath$$

The first one differs from original KM axiom in having $\text{ı}x\varphi = y$ instead of $\varphi(\text{ı}x\varphi)$ but both formulations are equivalent.

THEOREM 1. *If $\Gamma \vdash \varphi$ in KM system, then $\vdash \Gamma \Rightarrow \varphi$.*

PROOF. It is sufficient to show that sequents corresponding to both axioms for ı are provable. For better readability we underline side-formulae of all rule-applications. As for the first see the proof (I) on p. 144. For (ID) see the proof (II) on p. 144. ⊣

THEOREM 2. *If $\vdash \Gamma \Rightarrow \Delta$, then $\Gamma \vdash \bigvee \Delta$ in KM system.*

PROOF. It is enough to demonstrate that three rules for descriptions are derivable in sequent calculus enriched with two additional axiomatic sequents corresponding to Kalish and Montague rules. For easier work note first that the following sequents are derivable from additional axioms:

$$\forall x(\varphi \leftrightarrow t = x) \Rightarrow t = \text{ı}x\varphi$$

and

$$\neg\exists y\forall x(\varphi \leftrightarrow y = x) \Rightarrow \imath = \text{ı}x\varphi$$

Derivability of $(\Rightarrow\text{ı})$ is simple; see Figure (III) on p. 145. Derivability of $(\Rightarrow\imath 1)$ goes like on Figure (IV) on p. 145.

Derivability of $(\Rightarrow\imath 2)$ is the most tedious task. First note that on the basis of the three premisses we can derive:

$$(\Rightarrow\exists)\ \dfrac{(\Rightarrow\wedge)\ \dfrac{(\Rightarrow\wedge)\ \dfrac{\Gamma \Rightarrow \Delta, \underline{\varphi(t_1)} \qquad \Pi \Rightarrow \Sigma, \underline{\varphi(t_2)}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \underline{\varphi(t_1) \wedge \varphi(t_2)}} \qquad \dfrac{\underline{t_1 = t_2}, \Lambda \Rightarrow \Theta}{\Lambda \Rightarrow \Theta, \underline{t_1 \neq t_2}}\ (\Rightarrow\neg)}{\Gamma, \Pi, \Lambda \Rightarrow \Delta, \Sigma, \Theta, \underline{\varphi(t_1) \wedge \varphi(t_2) \wedge t_1 \neq t_2}}}{\Gamma, \Pi, \Lambda \Rightarrow \Delta, \Sigma, \Theta, \exists x, y(\varphi(x) \wedge \varphi(y) \wedge x \neq y)}$$

Now we can prove:

$$(\mathrm{W}\Rightarrow)\ \dfrac{(\forall\Rightarrow)\ \dfrac{(\leftrightarrow\Rightarrow)\ \dfrac{(\Rightarrow\mathrm{W})\ \dfrac{\varphi(b) \Rightarrow \varphi(b)}{\varphi(b) \Rightarrow \underline{\varphi(b), b = c}} \qquad \dfrac{\dfrac{b = c \Rightarrow \underline{b = c} \qquad a = c \Rightarrow \underline{a = c}}{b = c, a = c \Rightarrow a = b}\ (\Rightarrow =)}{\underline{\varphi(b), b = c}, a = c \Rightarrow a = b}\ (\mathrm{W}\Rightarrow)}{\varphi(b), a = c, \underline{\varphi(b) \leftrightarrow b = c} \Rightarrow a = b}}{\varphi(b), a = c, \forall x(\varphi(x) \leftrightarrow x = c) \Rightarrow a = b}}{\varphi(a), a = c, \varphi(b), \forall x(\varphi(x) \leftrightarrow x = c) \Rightarrow a = b}$$

(I)

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{\varphi(b)\Rightarrow\varphi(b)}{\varphi(b)\Rightarrow\underline{\varphi(b),a=b}}
\qquad
\dfrac{a=b\Rightarrow a=b}{\underline{\varphi(b),a=b}\Rightarrow a=b}
}{\underline{\varphi(b)\leftrightarrow a=b},\varphi(b)\Rightarrow a=b}
}{\forall x(\varphi(x)\leftrightarrow a=x),\underline{\varphi(b)}\Rightarrow\underline{a=b}}
\qquad
\dfrac{
\dfrac{
\dfrac{a=b\Rightarrow a=b}{a=b\Rightarrow\underline{\varphi(b),a=b}}
\qquad
\dfrac{\varphi(b)\Rightarrow\varphi(b)}{\underline{\varphi(b),a=b}\Rightarrow\varphi(b)}\;(\mathrm{W})
}{\underline{\varphi(b)\leftrightarrow a=b},a=b\Rightarrow\varphi(b)}\;(\leftrightarrow\Rightarrow)
}{\forall x(\varphi(x)\leftrightarrow a=x),\underline{a=b}\Rightarrow\underline{\varphi(b)}}\;(\forall\Rightarrow)
}{
\dfrac{
\dfrac{\forall x(\varphi(x)\leftrightarrow a=x)\Rightarrow a=\iota x\varphi(x)}{\Rightarrow\forall x(\varphi(x)\leftrightarrow a=x)\rightarrow a=\iota x\varphi(x)}\;(\Rightarrow\rightarrow)
}{\Rightarrow\forall y(\forall x(\varphi(x)\leftrightarrow y=x)\rightarrow y=\iota x\varphi(x))}\;(\Rightarrow\forall)
}\;(\Rightarrow\iota)
$$

where a, b are new.

(II)

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
(\Rightarrow\imath 2)\;\dfrac{\varphi(a)\Rightarrow\underline{\varphi(a)}\qquad\varphi(b)\Rightarrow\underline{\varphi(b)}\qquad\underline{a=b}\Rightarrow a=b}{\varphi(a),\underline{\varphi(b)}\Rightarrow\underline{a=b},\imath=\iota x\varphi(x)}
\qquad
\varphi(a),\underline{a=b}\Rightarrow\underline{\varphi(b)}
}{\underline{\varphi(a)},\varphi(a)\Rightarrow\varphi(b)\leftrightarrow a=b,\imath=\iota x\varphi(x)}\;(\Rightarrow\leftrightarrow)
}{\varphi(a)\Rightarrow\underline{\varphi(b)\leftrightarrow a=b},\imath=\iota x\varphi(x)}\;(\mathrm{C}\Rightarrow)
}{\varphi(a)\Rightarrow\underline{\forall x(\varphi(x)\leftrightarrow a=x)},\imath=\iota x\varphi(x)}\;(\Rightarrow\forall)
}{\underline{\varphi(a)}\Rightarrow\exists y\forall x(\varphi(x)\leftrightarrow y=x),\imath=\iota x\varphi(x)}\;(\Rightarrow\exists)
}{\Rightarrow\exists y\forall x(\varphi(x)\leftrightarrow y=x),\imath=\iota x\varphi(x),\underline{i=\iota x\varphi(x)}}\;(\Rightarrow\imath 1)
}{\Rightarrow\underline{\exists y\forall x(\varphi(x)\leftrightarrow y=x)},\imath=\iota x\varphi(x)}\;(\Rightarrow\mathrm{C})
}{
\dfrac{\neg\exists y\forall x(\varphi(x)\leftrightarrow y=x)\Rightarrow\imath=\iota x\varphi(x)}{\Rightarrow\neg\exists y\forall x(\varphi(x)\leftrightarrow y=x)\rightarrow\imath=\iota x\varphi(x)}\;(\Rightarrow\rightarrow)
}\;(\neg\Rightarrow)
$$

where a and b are new and the rightmost leaf is deduced from two axioms by $(\Rightarrow=)$

(III)

$$\text{(Cut)}\;\dfrac{\text{(}{\Rightarrow}\forall\text{)}\;\dfrac{\text{(}{\Rightarrow}{\leftrightarrow}\text{)}\;\dfrac{\underline{\varphi(a)}, \Gamma \Rightarrow \Delta, \underline{t=a} \qquad \underline{t=a}, \Pi \Rightarrow \Sigma, \underline{\varphi(a)}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \underline{\varphi(a) \leftrightarrow t=a}}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \underline{\forall x(\varphi(x) \leftrightarrow t=x)}} \qquad \underline{\forall x(\varphi(x) \leftrightarrow t=x)} \Rightarrow t = \iota x\varphi(x)}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, t = \iota x\varphi(x)}$$

where a is new.

(IV)

$$\text{(Cut)}\;\dfrac{\text{(}{\Rightarrow}\neg\text{)}\;\dfrac{\text{(}\exists{\Rightarrow}\text{)}\;\dfrac{\text{(}\forall{\Rightarrow}\text{)}\;\dfrac{\text{(}{=}{\Rightarrow}\text{)}\;\dfrac{\text{(}{\leftrightarrow}{\Rightarrow}\text{)}\;\dfrac{\text{(W)}\;\dfrac{a=a \Rightarrow a=a}{a=a \Rightarrow \underline{a=a, \varphi(a)}} \qquad \dfrac{\varphi(a), \Gamma \Rightarrow \Delta}{\underline{\varphi(a), a=a}, \Gamma \Rightarrow \Delta}}{\underline{a=a}, \varphi(a) \leftrightarrow a=a, \Gamma \Rightarrow \Delta}}{\underline{\varphi(a) \leftrightarrow a=a}, \Gamma \Rightarrow \Delta}}{\underline{\forall x(\varphi(x) \leftrightarrow a=x)}, \Gamma \Rightarrow \Delta}}{\underline{\exists y\forall x(\varphi(x) \leftrightarrow y=x)}, \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta, \underline{\neg\exists y\forall x(\varphi(x) \leftrightarrow y=x)}} \qquad \underline{\neg\exists y\forall x(\varphi(x) \leftrightarrow y=x)} \Rightarrow \imath = \iota x\varphi(x)}{\Gamma \Rightarrow \Delta, \imath = \iota x\varphi(x)}$$

where a is new

which continues:

$$
\scriptstyle(\Rightarrow\neg)\ \dfrac{\scriptstyle(\exists\Rightarrow)\ \dfrac{\scriptstyle(\wedge\Rightarrow)\ \dfrac{\scriptstyle(\neg\Rightarrow)\ \dfrac{\scriptstyle(\exists\Rightarrow)\ \dfrac{\scriptstyle(C\Rightarrow)\ \dfrac{\scriptstyle(\forall\Rightarrow)\ \dfrac{\scriptstyle(\leftrightarrow\Rightarrow)\ \dfrac{\scriptstyle(\Rightarrow W)\ \dfrac{\varphi(a) \Rightarrow \varphi(a)}{\varphi(a) \Rightarrow \underline{\varphi(a), a = c}} \qquad \underline{\varphi(a), a = c}, \varphi(b), \forall x(\varphi(x) \leftrightarrow x = c) \Rightarrow a = b}{\varphi(a), \varphi(b), \underline{\varphi(a) \leftrightarrow a = c}, \forall x(\varphi(x) \leftrightarrow x = c) \Rightarrow a = b}}{\varphi(a), \varphi(b), \underline{\forall x(\varphi(x) \leftrightarrow x = c)}, \forall x(\varphi(x) \leftrightarrow x = c) \Rightarrow a = b}}{\varphi(a), \varphi(b), \underline{\forall x(\varphi(x) \leftrightarrow x = c) \Rightarrow a = b}}}{\varphi(a), \varphi(b), \exists y \forall x(\varphi(x) \leftrightarrow x = y) \Rightarrow \underline{a = b}}}{\underline{\varphi(a), \varphi(b), a \neq b}, \exists y \forall x(\varphi(x) \leftrightarrow x = y) \Rightarrow}}{\underline{\varphi(a) \wedge \varphi(b) \wedge a \neq b}, \exists y \forall x(\varphi(x) \leftrightarrow x = y) \Rightarrow}}{\exists x, y(\varphi(x) \wedge \varphi(y) \wedge x \neq y), \underline{\exists y \forall x(\varphi(x) \leftrightarrow x = y)} \Rightarrow}}{\exists x, y(\varphi(x) \wedge \varphi(y) \wedge x \neq y) \Rightarrow \neg \exists y \forall x(\varphi(x) \leftrightarrow x = y)}
$$

where a, b, c are new. It is obvious that by two cuts with $\Gamma, \Pi, \Lambda \Rightarrow \Delta, \Sigma, \Theta, \exists x, y(\varphi(x) \wedge \varphi(y) \wedge x \neq y)$ and $\neg \exists y \forall x(\varphi(x) \leftrightarrow y = x) \Rightarrow \imath = \imath x \varphi(x)$ we derive the conclusion of the rule. ⊣

5. Cut Elimination

As a preliminary step we will prove:

Lemma 1 (Substitution). *If* $\vdash_k \Gamma \Rightarrow \Delta$*, then* $\vdash_k (\Gamma \Rightarrow \Delta)[a/t]$.

Proof. By induction on the height of a proof. It is straightforward but tedious exercise. Note that we provided not sheer admissibility but height-preserving admissibility. ⊣

Moreover, we assume that all proofs satisfy the condition of regularity — every constant which is fresh by side condition on the respective rule, must be fresh in the entire proof not only on the branch where the application of this rule takes place. Clearly, every proof may be systematically transformed into regular proof by Substitution lemma.

Let us define the notions of cut-degree and proof-degree:

1. Cut-degree is the complexity of cut-formula φ and is noted as $d\varphi$.
2. Proof-degree ($d\mathcal{D}$) is the maximal cut-degree in $\mathcal{D}$.

We follow the strategy of proof which was originally introduced for hypersequent calculi by Metcalfe, Olivetti and Gabbay [18] and later extensively used in this framework (see, e.g., Ciabattoni, Metcalfe, Montagna [5], Indrzejczak [12], Kurokawa [17]) but applicable also to standard sequent calculi (see Indrzejczak [13, 14]). The general strategy of proof is somewhat similar to Curry's [6] proof of cut admissibility but simpler in some respects and still based rather on local transformations of proof instead of global ones characteristic for Curry's proof. The proof of cut elimination theorem is based on two lemmata which make a reduction first on the right and next on the left premiss of cut.

LEMMA 2 (Right reduction). *Let* $\mathcal{D}_1 \vdash \Gamma \Rightarrow \Delta, \varphi$ *and* $\mathcal{D}_2 \vdash \varphi^k, \Pi \Rightarrow \Sigma$ *with* $d\mathcal{D}_1, d\mathcal{D}_2 < d\varphi$*, and* φ *principal in* $\Gamma \Rightarrow \Delta, \varphi$*, then we can construct a proof* $\mathcal{D}$ *such that* $\mathcal{D} \vdash \Gamma^k, \Pi \Rightarrow \Delta^k, \Sigma$ *and* $d\mathcal{D} < d\varphi$*.*

PROOF. By induction on the height of $\mathcal{D}_2$. The basis is trivial. Induction step requires consideration of all cases of possible derivation of $\varphi^k, \Pi \Rightarrow \Sigma$ and the role of cut-formula in the transition. In all cases where all occurrences of φ are parametric we simply apply the induction hypotheses to premisses of $\varphi^k, \Pi \Rightarrow \Sigma$ and then apply to them respective rule — it is essentially due to the context independence of almost all rules and regularity of proofs. In the case of troubles with side condition on fresh constants we must first apply Substitution lemma. In the case one of the occurrence of φ in the premise(s) is a side formula of the last rule we must additionally apply weakening to restore the lacking formula before the application of a rule. This situation covers also applications of rules $(=\Rightarrow)$ since active formula is in the antecedents only. Note also that there are no rules introducing identities with descriptions in the antecedents as principal formulae (the same remark applies to $(\Rightarrow =)$) hence in case φ is such a formula everything is obtained by the induction hypothesis.

In the cases where one occurrence of φ in $\varphi^k, \Pi \Rightarrow \Sigma$ is principal we make use of the fact that φ in the left premiss is principal too (note that for C and W it is trivial). We analyse the case of $(\forall \Rightarrow)$:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, \forall x\varphi(x)} \qquad \dfrac{\varphi(t), \forall x\varphi(x)^{k-1}, \Pi \Rightarrow \Sigma}{\forall x\varphi(x)^k, \Pi \Rightarrow \Sigma}}{\Gamma^k, \Pi \Rightarrow \Delta^k, \Sigma}$$

where a is fresh, hence by Substitution Lemma we have:

$\Gamma \Rightarrow \Delta, \varphi(t)$

and by the induction hypothesis we have:

$$\varphi(t), \Gamma^{k-1}, \Pi \Rightarrow \Delta^{k-1}, \Sigma$$

and we can build a proof:

$$\frac{\Gamma \Rightarrow \Delta, \varphi(t) \qquad \varphi(t), \Gamma^{k-1}, \Pi \Rightarrow \Delta^{k-1}, \Sigma}{\Gamma^k, \Pi \Rightarrow \Delta^k, \Sigma}$$

One should notice that even if t is a complex term containing definite descriptions, due to the way we defined complexity of a formula, the new proof has lower cut-degree. ⊣

LEMMA 3 (Left reduction). *Let* $\mathcal{D}_1 \vdash \Gamma \Rightarrow \Delta, \varphi^k$ *and* $\mathcal{D}_2 \vdash \varphi, \Pi \Rightarrow \Sigma$ *with* $d\mathcal{D}_1, d\mathcal{D}_2 < d\varphi$*, then we can construct a proof* $\mathcal{D}$ *such that* $\mathcal{D} \vdash \Gamma, \Pi^k \Rightarrow \Delta, \Sigma^k$ *and* $d\mathcal{D} < d\varphi$*.*

PROOF. The proof of the Left Reduction Lemma is similar but on the height of $\mathcal{D}_1$. The only difference is that in case cut-formula is principal we apply first the induction hypothesis and then the Right Reduction Lemma. For example, if the last sequent is obtained by ($\Rightarrow\iota$) we have:

$$\frac{\varphi(a), \Gamma_1 \Rightarrow \Delta_1, t = \iota x\varphi(x)^i, t = a \qquad t = a, \Gamma_2 \Rightarrow \Delta_2, t = \iota x\varphi(x)^j, \varphi(a)}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, t = \iota x\varphi(x)^k}$$

where a is new, $k = i + j + 1$ and Γ, Δ are multiset unions of Γ_1, Γ_2 and Δ_1, Δ_2 respectively. By the induction hypothesis we obtain:

$$\frac{\varphi(a), \Gamma_1, \Pi^i \Rightarrow \Delta_1, \Sigma^i, t = a \qquad t = a, \Gamma_2, \Pi^j \Rightarrow \Delta_2, \Sigma^j, \varphi(a)}{\Gamma, \Pi^{i+j} \Rightarrow \Delta, \Sigma^{i+j}, t = \iota x\varphi(x)}$$

Since this proof satisfies assumptions of the Right Reduction Lemma, then by the latter (with possibly some weakenings, if there are more than one occurrence of cut-formula in the right premiss) we obtain the result. ⊣

Eventually on the basis of the Left Reduction Lemma we obtain cut elimination by successive decreasing of cut degree. Therefore:

THEOREM 3 (Cut Elimination). *If* $\Gamma \Rightarrow \Delta$ *is provable, then it is provable without application of* (Cut)

6. Concluding Remarks

From the standpoint of several requirements formulated for well-behaved sequent calculi one may raise some objections towards presented system. In our opinion, it provides a satisfactory solution to the problem of devising cut-free sequent system despite of some inelegancies. We have mentioned in the introduction that there is a tableau system due to Bencivenga, Lambert and van Fraasen [2] for essentially the same logic. It is well known that, in principle, tableau systems may be easily transformed into sequent calculi and vice versa (at least in the standard versions of both). Hence one could suspect that these tableau rules may provide better characterization of ɿ-operator. However, in sequent version they are of the form:

$$\frac{\exists y(\forall x(\varphi(x) \leftrightarrow x = y) \land \psi(y)), \Gamma \Rightarrow \Delta \quad \neg\exists y \forall x(\varphi(x) \leftrightarrow x = y) \land \psi(\imath x\, x \neq x), \Gamma \Rightarrow \Delta}{\psi(\imath x \varphi), \Gamma \Rightarrow \Delta}$$

where $\psi(\imath x \varphi)$ is either a predicate or negated predicate with at least one occurrence of $\imath x \varphi$ among its arguments. In particular, it may be a negated identity statement. It is clear that such rules have the form which is hardly suitable for satisfactory proof theoretical treatment. One needs to make some decomposition of the formulae in premisses to obtain more satisfactory result and this is what we have done in our system.

Still it may seem that a better system is possible where descriptions are characterised by the rules which introduce them also to the antecedents of conclusion-sequents, and that introduce them not only as arguments of identity statements but also of more general (atomic) formulae. It is possible to do it in fact and we consider these both questions in what follows.

One may add to the system the following rule:

$$(\imath\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, t = a, \varphi[x/a] \qquad t = a, \varphi[x/a], \Pi \Rightarrow \Sigma \qquad t = i, \Lambda \Rightarrow \Theta}{t = \imath x \varphi, \Gamma, \Pi, \Lambda \Rightarrow \Delta, \Sigma, \Theta}$$

where a is not in Γ, Δ, Π, Σ, Λ, Θ, φ.

The problem is that such a rule is redundant what can be easily shown. To establish derivability of $(\imath\Rightarrow)$ we first derive on the basis of the first two premisses as on Figure (V) on p. 153.

The last sequent together with the axiomatic $t = \imath x \varphi(x) \Rightarrow t = \imath x \varphi(x)$ yields by $(\Rightarrow =)$ $t = \imath x \varphi(x), \Gamma, \Pi \Rightarrow \Delta, \Sigma, t = \imath$. This by cut with the third premiss $t = \imath, \Lambda \Rightarrow \Theta$ yields the conclusion.

In a similar way we can demonstrate redundancy of other rules we might want to have. For example consider the following, apparently more general rule:

$$(\Rightarrow\imath')\ \frac{\varphi[x/a], \Gamma \Rightarrow \Delta, t = a \qquad t = a, \Pi \Rightarrow \Sigma, \varphi[x/a] \qquad \Lambda \Rightarrow \Theta, At}{\Gamma, \Pi, \Lambda \Rightarrow \Delta, \Sigma, \Theta, A\imath x\varphi}$$

where a is new. Such a rule is easily derivable in the following way:

$$(\Rightarrow=)\ \frac{(\Rightarrow\imath)\ \dfrac{\varphi[x/a], \Gamma \Rightarrow \Delta, t = a \qquad t = a, \Pi \Rightarrow \Sigma, \varphi[x/a]}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, t = \imath x\varphi} \qquad \Lambda \Rightarrow \Theta, At}{\Gamma, \Pi, \Lambda \Rightarrow \Delta, \Sigma, \Theta, A\imath x\varphi}$$

One may obtain similar results for other rules introducing proper or improper descriptions to antecedents. Hence, by enriching the system with these rules we do not obtain anything new and in fact it should be not surprising since KM is a complete system. Moreover if we add such rules as primitive to our sequent calculus our cut elimination proof will be lost. At first it seems that everything goes well but several complications inevitably follow. First of all they are connected with the fact that identity statements having descriptions as arguments may be deduced by rules of different sort. In fact, the shape of our rules for identity was dictated by the need of avoiding such troubles. If we would use additional axiomatic sequents of the form $\Rightarrow t = t$ and $t_1 = t_2, \varphi[x/t_1] \Rightarrow \varphi[x/t_2]$ it is not possible to eliminate cuts with at least one premiss being of such form and having definite description as one of the arguments of identity while the other premiss having this identity is deduced by one of the rules for $\imath$. Similarly we could use some other rules for expressing Leibniz Law, like e.g. Negri and von Plato's rule:

$$\frac{t_1 = t_2, \varphi[x/t_1], \varphi[x/t_2], \Gamma \Rightarrow \Delta}{t_1 = t_2, \varphi[x/t_1], \Gamma \Rightarrow \Delta}$$

But such choice also does not work, even in our original system, if $t_1 = t_2$ is cut formula with at least one description as an argument. Consider a situation where it is principal in both premisses but in the left premiss introduced by $(\Rightarrow\imath)$; in such cases there is no possibility of replacing this cut with cut made on premisses. So if we add as primitive rule $(\imath\Rightarrow)$ we have the same problem with cut formula deduced by our $(\Rightarrow=)$ in the left premiss. But this is not a serious problem since we can use some other rules for identity to avoid collision. In fact five more are possible (see Indrzejczak [14]) and among them there is:

$$(\Rightarrow =')\ \frac{\Gamma \Rightarrow \Delta, \varphi[x/t_1] \qquad \Pi \Rightarrow \Sigma, t_1 = t_2 \qquad \varphi[x/t_2], \Lambda \Rightarrow \Theta}{\Gamma, \Pi, \Lambda \Rightarrow \Delta, \Sigma, \Theta}$$

The choice of such rule avoids the problem we described but still a system with such additional primitive rules causes problems. It will be instructive to analyse in detail what happens if we just add $(\imath \Rightarrow)$. If we try to prove the Right Reduction Lemma we need to consider also a case where $\imath x\varphi(x)$ is a principal formula in the right premiss. Specifically we have:

$t = \imath x\varphi(x)^k, \Gamma, \Pi, \Lambda \Rightarrow \Delta, \Sigma, \Theta$

deduced from:

$t = \imath x\varphi(x)^i, \Pi_1 \Rightarrow \Sigma_1, t = a, \varphi(a)$

$t = \imath x\varphi(x)^j, t = a, \varphi(a), \Pi_2 \Rightarrow \Sigma_2$

$t = \imath x\varphi(x)^n, t = \imath, \Pi_3 \Rightarrow \Sigma_3$

where a is new, Π and Σ are multiset unions of parameters from premisses, and $k = i + j + n + 1$. By the induction hypothesis we obtain:

(a) $\Gamma^i, \Pi_1 \Rightarrow \Delta^i, \Sigma_1, t = a, \varphi(a)$
(b) $t = a, \varphi(a), \Gamma^j, \Pi_2 \Rightarrow \Delta^j, \Sigma_2$
(c) $t = \imath, \Gamma^n, \Pi_3 \Rightarrow \Delta^n, \Sigma_3$

In the case t is just $\imath$ we obtain the conclusion simply from the third premiss by $(= \Rightarrow)$ and several weakenings. So the only case to consider is when t is not $\imath$ and the left premiss is deduced by $(\Rightarrow \imath)$:

$$\frac{\varphi(b), \Gamma_1 \Rightarrow \Delta_1, t = b \qquad t = b, \Gamma_2 \Rightarrow \Delta_2, \varphi(b)}{\Gamma \Rightarrow \Delta, t = \imath x\varphi(x)}$$

where b is not in Γ, Δ, φ, and by regularity also a is not. Hence by Substitution Lemma we obtain proofs of:

(d) $\varphi(a), \Gamma_1 \Rightarrow \Delta_1, t = a$
(e) $t = a, \Gamma_2 \Rightarrow \Delta_2, \varphi(a)$

These sequents may be combined in the following way. By cut on (a) and (d) we obtain:

(f) $\Gamma_1, \Gamma^i, \Pi_1 \Rightarrow \Delta_1, \Delta^i, \Sigma_1, t = a, t = a$

and by cut on (b) and (e) we obtain:

(g) $t = a, t = a, \Gamma_2, \Gamma^j, \Pi_2 \Rightarrow \Delta_2, \Delta^j, \Sigma_2$

Eventually, applications of contraction and cut on (f) and (g) yield the result. Apparently, it looks well but, unfortunatelly, it does not work since new cuts are not of lower degree. $\imath x\varphi$ as an argument of identity statement has complexity 0 but even if φ is not a complex formula if it is unpacked it has complexity ≥ 1. One may think that changing a definition of complexity may help but it does not work too. If we count constants and predicates present in scopes of descriptions, then the cases with quantifiers give no guarantee for reduction of complexity since t substituted for x in applications of $(\Rightarrow\exists)$ or $(\forall\Rightarrow)$ may be a complex description. Thus what seems to be one of the virtues of KM system, i.e., the fact that new rules for descriptions do not spoil an old machinery of FOL, in the setting of sequent calculus and cut elimination proof, introduces a serious problem. This is the reason why, in spite of some inelegancies, the provided solution seems to be the best possible if cut elimination is our primary goal.

The last thing which may seem curious is why there are two rules for introduction of improper descriptions in the succedent instead of one, or two — but with the second introducing it in the antecedent. As for the latter the situation is exactly as in the case of rules for proper descriptions. We could add a rule introducing improper descriptions to the antecedent but it is derivable and moreover, its addition as primitive leads to the same complications in cut elimination proof as those reported above.

As for the first question it is in fact possible to use one rule instead of two. Using a strategy mentioned in Negri and von Plato [19] we could introduce a rule of the form:

$$(\Rightarrow\imath)\ \frac{\varphi[x/a], \Gamma \Rightarrow \Delta, \varphi[x/t_1] \qquad \varphi[x/a], \Pi \Rightarrow \Sigma, \varphi[x/t_2] \qquad \varphi[x/a], t_1 = t_2, \Lambda \Rightarrow \Theta}{\Gamma, \Pi, \Lambda \Rightarrow \Delta, \Sigma, \Theta, \imath = \imath x\varphi}$$

where a is not in Γ, Δ and φ.

Both original rules are derivable by means of this rule and weakenings. In the other direction we can show a derivability of the new rule in the way from Figure (VI) on p. 153.

However, having two rules instead of more complex one is a better choice. Two primitive rules naturally correspond to two different sources of improper descriptions: $(\Rightarrow\imath 1)$ to nonexistence of a designatum, $(\Rightarrow\imath 2)$ to existence of more than one designata. Moreover, applications of these two rules significantly simplify proofs.

(V)

$$
(\mathrm{Cut})\,\dfrac{(\Rightarrow\neg)\,\dfrac{(\exists\Rightarrow)\,\dfrac{(\forall\Rightarrow)\,\dfrac{(\leftrightarrow\Rightarrow)\,\dfrac{\Gamma \Rightarrow \Delta, \underline{\varphi(t), a = t} \qquad \underline{\varphi(t), a = t}, \Pi \Rightarrow \Sigma}{\underline{\varphi(t) \leftrightarrow a = t}, \Gamma, \Pi \Rightarrow \Delta, \Sigma}}{\underline{\forall x(\varphi(x) \leftrightarrow a = x)}, \Gamma, \Pi \Rightarrow \Delta, \Sigma}}{\underline{\exists y \forall x(\varphi(x) \leftrightarrow y = x)}, \Gamma, \Pi \Rightarrow \Delta, \Sigma}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \underline{\neg\exists y \forall x(\varphi(x) \leftrightarrow y = x)}} \qquad \underline{\neg\exists y \forall x(\varphi(x) \leftrightarrow y = x)} \Rightarrow \imath = \iota x \varphi(x)}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \imath = \iota x \varphi(x)}
$$

where a is new and the rightmost premiss was already proven

(VI)

$$
(\Rightarrow\mathrm{C})\,\dfrac{(\Rightarrow\imath 1)\,\dfrac{(\mathrm{C}\Rightarrow)\,\dfrac{(\Rightarrow \imath 2)\,\dfrac{\varphi[x/a], \Gamma \Rightarrow \Delta, \varphi[x/t_1] \qquad \varphi[x/a], \Pi \Rightarrow \Sigma, \varphi[x/t_2] \qquad \varphi[x/a], t_1 = t_2, \Lambda \Rightarrow \Theta}{\varphi[x/a], \varphi[x/a], \varphi[x/a], \Gamma, \Pi, \Lambda \Rightarrow \Delta, \Sigma, \Theta, \imath = \iota x \varphi}}{\varphi[x/a], \Gamma, \Pi, \Lambda \Rightarrow \Delta, \Sigma, \Theta, \imath = \iota x \varphi}}{\Gamma, \Pi, \Lambda \Rightarrow \Delta, \Sigma, \Theta, \imath = \iota x \varphi, \imath = \iota x \varphi}}{\Gamma, \Pi, \Lambda \Rightarrow \Delta, \Sigma, \Theta, \imath = \iota x \varphi}
$$

Acknowledgements. The results reported in this paper are supported by the National Science Centre, Poland (grant number: DEC-2017/25/B/HS1/01268).

References

[1] Bencivenga, E., “Free logics”, pages 373–426 in: D. Gabbay, F. Guenthner (eds.), *Handbook of Philosophical Logic*, vol. III, Reidel Publishing Company, Dordrecht 1986. DOI: 10.1007/978-94-009-5203-4_6

[2] Bencivenga, E., K. Lambert and B. C. van Fraasen, *Logic, Bivalence and Denotation*, Ridgeview, Atascadero 1991.

[3] Bernays, P., *Axiomatic Set Theory*, North Holland, Amsterdam 1958.

[4] Carnap, R., *Meaning and Necessity*, Chicago 1947.

[5] Ciabattoni, A., G. Metcalfe and F. Montagna, “Algebraic and proof-theoretic characterizations of truth stressers for MTL and its extensions”, *Fuzzy Sets and Systems* 161, 3 (2010): 369–389. DOI: 10.1016/j.fss.2009.09.001

[6] Curry, H. B., *Foundations of Mathematical Logic*, McGraw-Hill, New York 1963.

[7] Fitting, M., and R. L. Mendelsohn, *First-Order Modal Logic*, Kluwer, Dordrecht 1998. DOI: 10.1007/978-94-011-5292-1

[8] Frege, G., “Über Sinn und Bedeutung”, *Zeitschrift für Philosophie und Philosophische Kritik* 100 (1892): 25–50.

[9] Frege, G., *Grundgesetze der Arithmetic I*, Hermann Pohl, Jena 1893.

[10] Garson, J. W., *Modal Logic for Philosophers*, Cambridge University Press, Cambridge 2006. DOI: 10.1017/CBO9780511617737

[11] Hailperin, T., “Remarks on identity and description in first-order axiom systems”, *Journal of Symbolic Logic* 19 (1954): 14–20. DOI: 10.2307/2267645

[12] Indrzejczak, A., “Eliminability of cut in hypersequent calculi for some modal logics of linear frames”, *Information Processing Letters* 115, 2 (2015): 75–81. DOI: 10.1016/j.ipl.2014.07.002

[13] Indrzejczak, A., “Simple cut elimination proof for hybrid logic”, *Logic and Logical Philosophy* 25, 2 (2016): 129–141. DOI: 10.12775/LLP.2016.004

[14] Indrzejczak, A., ‘Rule-maker theorem and its applications’, submitted.

[15] Kalish, D., and R. Montague, “Remarks on descriptions and natural deduction”, *Archiv. für Mathematische Logik und Grundlagen Forschung* 3 (1957): 50–64, 65–73

[16] Kalish, D., and R. Montague, *Logic. Techniques of Formal Reasoning*, Harcourt, Brace & World, Inc., New York 1964.

[17] Kurokawa, H., "Hypersequent calculi for modal logics extending S4", pages 51–68 in *New Frontiers in Artificial Intelligence (2013)*, Springer, 2014. DOI: 10.1007/978-3-319-10061-6_4

[18] Metcalfe, G., N. Olivetti and D. Gabbay, *Proof Theory for Fuzzy Logics*, Springer, 2008.

[19] Negri, S., and J. von Plato, *Structural Proof Theory*, Cambridge University Press, Cambridge 2001. DOI: 10.1017/CBO9780511527340

[20] Quine, W. V. O., *Mathematical Logic*, W. W. Norton and Company, New York 1940.

[21] Rosser, J. B., *Logic for Mathematicians*, McGraw-Hill Book Company, Inc., New York 1953.

[22] Russell, B., "On denoting", *Mind* 14 (1905), 479–493.

Andrzej Indrzejczak
Department of Logic
University of Łódź
Lindleya 3/5
90-131 Łódź
andrzej.indrzejczak@filozof.uni.lodz.pl