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WHAT IS DIAGRAMMATIC REASONING IN MATHEMATICS?

Abstract. In recent years, epistemological issues connected with the use of diagrams and visualization in mathematics have been a subject of increasing interest. In particular, it is open to dispute what role diagrams play in justifying mathematical statements. One of the issues that may appear in this context is: what is the character of reasoning that relies in some way on a diagram or visualization and in what way is it distinct from other types of reasoning in mathematics? In this paper it is proposed to distinguish between several ways of using visualization or diagrams in mathematics, each of which could be connected with a different concept of diagrammatic/visual reasoning. Main differences between those types of reasoning are also hinted at. A distinction between visual (diagrammatic) reasoning and visual (diagrammatic) thinking is also considered.

Keywords: diagrammatic reasoning; philosophy of mathematics

1. Introduction

Diagrams, as a types of representation of mathematical objects, are an important tool in mathematics; they appear in textbooks as well as in proofs and various arguments. Despite this fact, it is common to hold that diagrams can only be used as heuristics and should never become an essential part of any justification of a mathematical statement. Two reasons for this view can be given: firstly, in consequence of the development of mathematical logic, it can be said that every theorem of which we have a proof can be proved within some purely symbolic formal system; in other words, diagrams are, at least in theory, never indispensable in a proof. Secondly, diagrams and reasonings that rely on them have considerable weaknesses. Let us name only three of them: diagrams are singular in a sense that they represent concrete mathematical objects

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(e.g., a concrete right-angled triangle); they are over-specific in that on every diagram more information is presented than is needed to represent an object or carry out a reasoning; finally, many diagrams are inexact. In consequence, reasoning using diagrams may put the mathematician at risk of making mistakes, such as performing an unjustified generalization.

In recent years we are witnessing a revival of interest in the role of visualization in mathematical practice. Philosophers of mathematics specializing in the subject notice that diagrammatic representation (especially computer graphics which allows visualization of very complicated mathematical objects) is used in increasingly many disciplines such as graph theory, category theory, differential equations or even number theory. It is also pointed out that diagrams can play (and often do) a significant role in mathematical reasoning and that the danger of making mistakes using diagrams can be avoided by their careful use. Nonetheless, there are many problematic issues connected with the use of diagrams in mathematics. Here are some of the most important ones: what role do diagrams play in justification of mathematical statements or in other epistemic activities such as learning or discovering? In what way can the dangers connected with the use of diagrams be avoided? Can some diagrams be seen as constituting the proof of a mathematical statement? The aim of this paper is a critical evaluation of the concept of "diagrammatic reasoning". The main question that will be raised in this context is the following: what is distinct about reasoning that may be called "diagrammatic" or "visual", as opposed to familiar types of mathematical reasoning that make use only of symbolic representation? To make an attempt at answering it a closer look at how visualization is used in mathematics is needed or as Marcus Giaquinto puts it, at "the nature of the causal route from visual experience (of sight or imagination) to mathematical belief" [Giaquinto, 2007, p. 2]. This, however, is not an easy task, since visualization is used in very different ways in mathematics. In this paper, several possible ways of understanding the concept of visual or diagrammatic reasoning are considered. This is done by pointing at three general ways in which visualization is used in mathematical practice, each of which may be connected with a different concept of diagrammatic reasoning. It is also suggested that in some contexts it is better to apply the concept of visual *thinking* rather than visual reasoning.[1]

[1] It is not the aim this paper to consider the validity of different types of diagrammatic reasoning, but rather different ways of understanding this concept itself.

In the next section some necessary comments on the terminology used in the paper are made, the most important concepts being those of a diagram, visualization and reasoning.

2. Some comments on the terminology used

The two most important concepts used in this paper are those of a diagram and visualization. A diagram is taken here very broadly to be any collection of dots, segments or other physical marks or shapes created in order to represent mathematical objects, in a way that assigns mathematical meaning to spatial relations between those shapes. Diagrams are usually contrasted with symbols, which may be simply understood as letters, numbers and other signs whose meaning is specified within a certain formal system, as well as with sentences that are built from them. It is, however, not an easy task to give a precise definition of the concept of a diagram. Mixed kinds of representations are present in many branches of mathematics and some authors admit many intermediary types of representations, which may be said to have properties traditionally attributed to symbols and to diagrams.[2] We will not further deal with those complex semiotic issues, taking mixed representations, consisting of both symbols and mentioned physical marks, to be diagrams as well. Diagrams are thus such mathematical representations as a graph, labeled geometrical diagram, computer graphics and arrow-diagram in algebra or category theory.

The second crucial term, that of a "visualization" is a broader one. We will take a visualization to be a diagram as well as an inner visualization, performed in mental space and dynamic visualization such as computer animation.

It is further not an easy task to define what reasoning (or argument) is in general. An argument may generally be understood as a series of propositions of which some are claimed to follow from others; moreover,

[2] Marcus Giaquinto names three properties of mathematical representation that are usually attributed to symbols and three properties that are usually attributed to diagrams. Assuming that every representation may have a given property or not, we get 64 types of representations, with one "purely diagrammatic", one "purely symbolic", and the remaining ones being intermediary between them [see Giaquinto, 2007, pp. 240–249]. Another possibility is to dismiss the distinction symbol/diagram altogether and take all representations in mathematics to be diagrams, as it was done by Ch. S. Peirce.

an argument is expressed outwardly, usually with an aim of convincing someone of the truth of some statement. Definitions of the term "reasoning" usually tend to be broader. Jesse Norman takes it to be a "personal-level psychological process, consisting of inferences" [Normann, 2003, p. 18]. In a paper on diagrammatic reasoning, Zenon Kulpa proposes to understand the term "reasoning", as any kind of "knowledge processing" [Kulpa, 2009, pp. 75–76]. We will follow this broad understanding of the term, taking "reasoning" to be any knowledge processing occurring on a personal level that consists of inferences and results in stating of a theses. This broad account will allow us to generally understand "diagrammatic" or "visual" reasoning as one in case of which some of the inferences refer somehow to a given visualization, making it a significant factor in the occurrence of a mathematical belief. It has to be stressed, that the assumption that every reasoning results in formulating a thesis or in forming a certain belief-state is important for further considerations. It will allow differentiating between those mental processes (possibly written down using symbols and/or diagrams) that may be called "reasoning" and those that should perhaps rather be taken as instances of visual "thinking", involving only a certain way of looking at mathematical objects without necessarily compelling one to formulate any specific claims about them.[3]

Analyzing the notion of diagrammatic reasoning it is interesting to note its relation to the concept of "intuition". The distinction symbol/diagram, or sentence/diagram may namely be (and often is) linked with the distinction between conceptual and intuitive knowledge. The crucial characteristic usually attributed to intuitive knowledge or cognition is that it is an immediate cognition of its object, one that effects in belief-states without any articulation of their grounds, in particular without the need to carry out a reasoning. Visualization may also be seen as giving intuitive knowledge of mathematical objects. What could be meant by this is, that a visualization inspires a sudden occurrence of a mathematical belief just by looking at a diagram, and without a

[3] One of the main problematic issues connected with the use of diagrams in mathematics is connected with what is often called "visual proofs" or "picture proofs". Some researchers, like, e.g., J. R. Brown, claim that "some visual representations can constitute proofs in and of themselves, rendering any further traditional proof unnecessary" [Hanna and Sidoli, 2007, p. 74]. This problem will not be discussed in my paper, which focuses on the nature of reasoning that somehow relies on diagrams rather than on stipulation as to which representations can count as a proof.

need to perform any additional analysis, thinking or deduction. This may happen when we look at a geometrical diagram, and understand "immediately" the truth of a given statement. This, however, puts the relevance of the very notion of diagrammatic reasoning in question which, by the the given definition, consists of inference steps. In what follows, I will not analyze the role of intuition in the use of diagrams, and assume that the characteristic nature of diagrammatic reasoning is to be found elsewhere. However, this is not to say, that diagrammatic reasoning cannot have certain characteristics usually attributed to intuitive knowledge. One of them is immediacy understood not as an absence of reasoning, but of certain reasoning steps or lack of necessity to use certain concepts in the process of reasoning. Diagrammatic (or visual) reasoning may also be conceived as non-conceptual in the sense of being non-sentential as opposed to traditional reasoning, that consists of sequences of sentences.

3. Static diagrams treated as adjuncts to a proof

The first use of diagrams that will be singled out involves analysis of a single diagram whose physical characteristics carry some information about the object represented by them. In this case the diagram is not necessarily meant to be an integral part of a proof, but rather as a heuristic that may convince, contribute to discovery and understanding or otherwise result in appearance of some belief states. In particular, it may inspire certain types of reasoning.

Let us first take a look at two simple and familiar examples. The first diagram (Figure 1) can be said to justify (*via* the interpretation of numbers as areas) the following equality:

$$\sum_{n=1}^{\infty} \frac{1}{2^{2n}} = \frac{1}{3}$$

On the second diagram (Figure 1a) numbers are represented as colored circles.[4] Observing it, one can notice that each of the colored triangles represents a sum $1+2+3+4+5$; on the other hand (since the

[4] Clearly, they could as well be represented as dots, small squares, etc. In other words, the specific shape and geometrical properties of those objects is not significant, but rather their relation in space. It should be noted, that on this diagram additional information is also carried by the colors.

Figure 1.

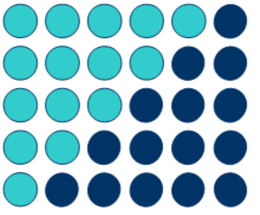

Figure 1a.

colored circles form a rectangle) there are all together $5 * 6 = 30$ circles. The result of this observation: $1 + 2 + 3 + 4 + 5 = (5 * 6)/2$ can then be generalized into $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$ without a great difficulty.

In each of the examples, the mathematical facts can be in a sense "read off" the diagram. Many more examples of such diagrams could be given. It should also be stressed that in each case, some degree of mathematical knowledge and skill is needed to interpret the diagram properly and draw the correct conclusions.

In some cases, construction and analysis of the diagram enable one to infer statements in an even more direct and "effortless" manner. This occurs when after entering the premises on the diagram, certain statements can be "read off" the diagram using just the conventions used to construct it (such as: "circles represents sets", etc.). A simple example given Atsushi Shimojima can be cited [Shimojima, 2015, p. 23]: Let us assume that the two facts are to be presented on an Euler diagram: $A \subset B$ and $C \cap B = \emptyset$ (see Figure 2). Introducing those facts onto the diagram makes it possible to simply observe that $A \cap C = \emptyset$ without further argument. In such cases, as Shimojima puts it, "expressing a set of information in diagrams can result in the expression of other, consequential information", calling it diagrams' potential for "Free Ride" in inference [Shimojima, 2015, p. 13]. It should be stressed, that whereas diagrams 1 and 1a would typically be constructed by somebody who knew a given theorem in advance and intended to present it in a suggestive way, in the second case the construction of the diagram may allow discovering (or "reading of the diagram") truths that were not known before the construction was carried out.

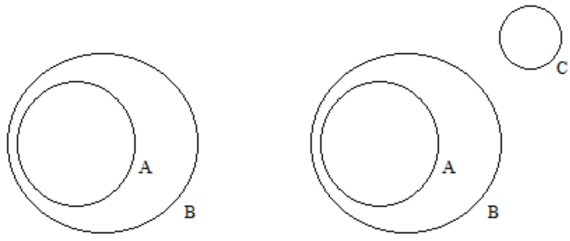

Figure 2.

In general, an analysis of a static diagram may be modeled by splitting it up into four stages. The first stage is the construction of a diagram. It has to be noted, that in order to represent a mathematical object on a diagram, rules of interpretation of this object or its parts or aspects as physical marks on a diagram have to be stated. In other words, a translation of the symbolic language into a visual language must somehow be carried out.[5] The second stage is the observation of the diagram and noticing the relation between its parts. The third stage consists in interpretation of the observed facts and relations by translating them back into the initial mathematical language. The final stage consists in stating of a thesis.

How can one place the notion of "diagrammatic reasoning" in this context? One possibility is to use this term to denote the whole process described, with all of its stages (after all, each of them somehow relates to the diagram). Such reasoning is not "purely diagrammatic" in the sense that it could not be carried out without mathematical concepts and some previous mathematical knowledge; some parts of the process, like a generalization of the analyzed thesis, demand the use of skills that go beyond the diagram. However, one may further ask: which parts or aspects of the thinking process can be said to involve distinctly diagrammatic characteristics, allowing, in particular, to discover new facts within the mentioned second stage of the reasoning process? It seems that one way of answering the above question is to point at the characteristics of

[5] For example, graph of a continuous function is represented by a non-interrupted line, in pebble-arithmetic diagrams numbers are represented by dots (or other simple objects like circles) and in graph theory – the vertices by dots and the edges by segments. It is common to translate the "non-visual" concepts into geometrical ones, such as area, segment, etc., but it is in general possible to represent them by any physical marks.

diagrams that distinguish them from symbols and sentences. Many such characteristics can be named (they are widely discussed in literature, e.g. in Shimojima [2015] or Giaquinto [2007]), here let us only mention those that seem to be the most important. The first consists in the structural similarity of the diagram to the represented object. Typically, the physical parts of the diagram and spatial relations between them correspond in some way to the properties and relations of the mathematical object being represented (for example, there is a structural similarity between Euler diagrams and sets, or between graphs defined in a set-theoretic fashion and those drawn with use of dots and segments). Another crucial feature of diagrams is that they allow the presentation of a great amount of information simultaneously. Diagrams can represent in a concise form what would often require large amount of sentences which is especially true in case of computer graphics [see, e.g., Barwise and Etchemendy, 1996, p. 18]. Such simultaneous display of information can make it easier to notice certain relations between the parts of a diagram (and corresponding mathematical objects), as was the case with Figure 2, depicting set-theoretic objects.

One can mention further consequences of the two characteristics of diagrams mentioned above that contribute to the unique character of diagrammatic reasoning. Firstly, the analysis of diagrams seems to have a non-linear character, one that is not determined in advance. This is pointed out by Peter Borwein and Loki Jörgenson, who suggest (referring in this case to computer graphics) that in case of visualization "the path through the information is usually indeterminate, leaving the viewer to establish what is important(and what is not) and in what order the dependencies should be assessed" [Borwein and Jörgenson, 2001, p. 899]. A second "visual technique" allowed by the two properties of diagrams is sometimes called *aspect shifting*; it enables seeing the elements of the diagram in two different ways, or more generally seeing "single expression as an instance of two distinct forms" [Giaquinto, 2007, p. 262]. Such an attentional shift may enable one to discover mathematical truths by observation, comparison and drawing from them the appropriate conclusions. Such aspect shift is employed in the analysis of Figure 1a — here the reasoning involves seeing the circles as a rectangle with sides 5 and 6 and, at the same time, as consisting of two triangles.

4. Modifications of diagrams and dynamic visualization

The use of static diagrams can be distinguished from manipulation of diagrams, as well as from the use of dynamic visualization (possibly visualized in our inner mental space). Both will be analyzed in this section.

Manipulation of diagrams, as well as the production of multiple diagrams and their comparison is a common method in mathematics. Such modifications of diagrams often contribute to learning about mathematical objects and might broaden our knowledge about them. This may happen when one draws various graphs with an aim of better understanding properties of certain functions, or relations between some functions. This may also happen when we generate various computer graphics, "playing around" with different types of visualization of mathematical objects (as it is often done when visualization is used to study differential equations or e.g., patterns in number theory). Can such activity be called reasoning? The answer could be "yes", if we agree that manipulation of diagrams can result in learning about properties of some mathematical objects, and contribute to formulating mathematical claims. A philosopher that stressed this aspect of mathematical reasoning very strongly was Ch. S. Peirce, for whom all reasoning in mathematics "depends directly or indirectly on diagrams" [Shin, 2002, p. 19]. Observation and experimentation with diagrams is a crucial aspect of mathematical practice and enables us to acquire knowledge about them (remembering that Peirce took all kinds of mathematical representation to be diagrams). As Peirce further elaborates,

> by diagrammatic reasoning, I mean reasoning which constructs a diagram according to a precept expressed in general terms, performs experiments upon this diagram, notes their results, assures itself that similar experiments performed upon any diagram constructed according to the same precept would have the same results, and expresses this in general terms.
>
> Peirce, Ch. S., *Collected Papers*, quoted in: [Shin, 2002, p. 19]

We will not comment in detail on the above passage and on Peirce's complex philosophy of mathematics in general. It may be added, that some contemporary accounts of diagrammatic reasoning echoe that of Peirce's; one of them has been formulated by Arthur Bakker, who claims that diagrammatic reasoning employs three steps:

> (1) construct a diagram that represents necessary and significant relationships, (2) experiment with the diagram based on actions that are permitted by the representational system that enabled its construction in the first place; and (3) observe and reflect upon the results in order to articulate important relationships.
>
> [Bakker, 2007, pp. 17–18], quoted in [Rivera, 2011, p. 229]

It is clear that this account of what may be called "diagrammatic reasoning" is similar to the one that was given in the previous section. In this case, however, not only analysis of a single diagram is involved, but also its manipulations and experiments performed on it. It seems that such reasoning may be called "diagrammatic" in a stronger sense than the first one. Firstly, the connection between visual data and their mathematical counterparts seems here less straightforward. Secondly, results of a reasoning are not directly "read off" the diagram; it is rather the comparison of diagrams or possibility of their modification that might convince the reasoner of the truth of a given statement. It is important to note that for Peirce and Willibald Dörfler experimentation with diagrams is a crucial characteristic of diagrammatic reasoning. For Dörfler diagrams that "do nothing else but represent do not convey diagrammatic reasoning" [Rivera, 2011, p. 229].

In course of carrying out a reasoning one might also be required to perform a dynamic visualization, i.e., imagine certain rearrangement or displacement of some objects. For the case of simplicity, let us name two historical examples: the first one is the controversial proof of theorem I.4. from Euclid's *Elements*, in which two triangles are first constructed and then superposed to each another in order to prove that they are congruent. Another example is the "proof" of the 5th Postulate by the Arab mathematician Ibn al-Haytham (865–901), in which an analysis of a movement of a point is part of the deduction.[6] Both proofs were criticized by many mathematicians, mainly because introducing motion to mathematics was widely thought to be in conflict with often presupposed eternal and non-sensual character of mathematical truths. In contemporary mathematics reasoning that refers to the visualization of motion may be found, e.g., in topology and in all those disciplines that

[6] The "proof" reads as follows: "given point P not on line L_1, the shortest distance between them is the line PS, which goes through point P and perpendicular to line L_1. If we start moving PS along the line L_1, a new line L_2 will be constructed. L_2 is parallel to L_1 since the distance between P and L_1 remains unchanged" [De Cruz, 2007, p. 199].

make use of computer graphics. One way of pointing at a distinctive nature of this use of visualization would be to notice that it is not the shape of the diagram that plays a role in the reasoning, but the movement of the given shape itself; one could say that in some cases it is the very possibility of such movement that proves a given thesis or at least convinces the mathematician of its truth. Let us note that in consequence of the suggested way of understanding the terms "diagram" and "visualization", those types of reasoning should perhaps be called "visual" rather than "diagrammatic".

5. Static diagrams as integral parts of a proof

The third way of using a diagram that is suggested to be discussed separately is one where the diagram is not only a heuristic or adjunct, but becomes an integral part of a proof (or reasoning), which refers to the diagram in some if its deductive steps. This may happen when the proof partly relies on the diagram, or in particular within proof systems in which the use of diagrams is fully formalized. In both cases the reasoning may clearly be called diagrammatic inasmuch it relies on diagrams in some of its steps.

As is well known, proofs that make use of diagrams as their integral parts, making it impossible to follow the reasoning without them, can be found in Euclid's Elements. In order to avoid the many logical flaws of Euclidean arguments, as well as of other kinds of geometric reasoning, various formalizations of geometrical arguments have been suggested. Let us briefly mention one such systematization presented by Kulpa who proposes to "transfer to diagrams a few (. . .) techniques from ordinary mathematical reasoning, after appropriate adaptation, like the divergence method and the Theorem of Constants" [Kulpa, 2009, p. 94]. In order to allow diagrams with their specific features to be used in proofs in a rigorous an reliable way, Kulpa formulates rules that show how to recognize and represent variables on diagrams as well as "similar rules for finding sets over which the variables vary and quantifiers range" [Kulpa, 2009, p. 94]. In this case, the analysis of diagrams is made more strict and systematic using the tools of predicate logic. One advantage of this is that reasonings are more reliable, for example helping to avoid the mistake of unjustified generalization.

Diagrammatic reasoning that seems to be epistemically significantly different from the ones mentioned above is to be found within formalized diagrammatic proof systems. In this case diagrams are parts of the given system's syntax, their shape and possible manipulations of them being strictly defined. If the system is heterogenous, that is it allows both diagrams and "traditional" symbols in its syntax, it may be the case that "rules of inference allow one to make inferences form sentences to sentences, from sentences to diagrams, from diagrams to diagrams, and from diagrams to sentences", as is for example the case in Eric Hammer's formalization of the use of Euler's diagrams [Hammer, 1994, p. 74]. Giving details of any such system is beyond the scope of this paper. It may only be noted, that within such formalizations a diagram is not treated as a representation of a mathematical object, whose specific physical parts are to represent mathematical objects or relations. Rather, diagrams are treated as separate symbols themselves. Thus reasoning that refers to diagrams does not in this case consist in noticing properties of the represented object by making use of the diagram's structural similarity to that object, or by using the fact that it displays a lot of information. Rather, the reasoning is a purely mechanical and linear process that makes use only of the fixed rules of inference.

6. Visual thinking

As was stressed in the beginning, diagrammatic (or visual) reasoning may be contrasted with visual thinking. The latter is ubiquitous in mathematics. One can generally say that we think visually of a mathematical object, if we have a tendency to visualize a diagram associated with that object. Visual content is linked with many mathematical concepts, not only geometrical ones but also such concepts as "graph", "lattice", etc. Giaquinto shows (referring to some empirical investigations) that we also have a tendency to visualize certain arithmetic and algebraic concepts and operations, e.g., concept of number line or matrix multiplication [Giaquinto, 2007, pp. 127–129, 216]. We may think of continuous functions as curves that are uninterupted lines, of negative numbers as being "on the left" of the number line, or we may imagine movement of numbers in space when multiplying matrices or performing other types of calculation. However, thinking visually of an object does not have to result in stating any thesis or carrying out any reasoning.

In addition to that, in many cases the contact with a diagram results with visual experience of an object rather than visual thinking. This seems to be the case with visualization of fractals like the famous Mandelbrot's set, which may be experienced as interesting or beautiful, without necessarily inspiring any strictly mathematical considerations.

7. Conclusion

In this paper I argued that it is possible to point at three general ways in which visualization is used in mathematical reasoning, taken as a thinking process that leads to stating of a thesis. In the first case a static diagram is analysed. Conviction of the truth of a given statement may be reached either by discovering it by an inventive analysis of a diagram or by almost effortless "reading the result off the diagram" also called a "free ride". The main reason why this is possible seems to be the characteristic properties of diagrams, some of them being the possibility to display on them large amounts of information simultaneously and their structural similarity to the mathematical object represented. The second type of the usage of diagrams may be also splitted into two subcases: experimentation with diagrams and dynamic visualization such as computer animation. The main characteristic of this type of use of a diagram is, that it is the comparison of different diagrams or their movement that may convince us of the truth of a given statement, rather than the characteristics of one specific diagram. In all the cases mentioned above the overall structure of the thinking process may be modeled as consisting of four stages: translation from the symbolic language into diagrammatic one (construction stage), observation of a diagram and possibly its modifications, translation of the observed relations back into the symbolic language and stating of a thesis (with a remark, however, that it may be difficult to give a strict symbolic interpretation of the movement of diagrams or shapes). In case of formalized diagrammatic proof systems the reasoning does not at all have to make use of the two highlighted properties of diagrams; the result may be reached at by transformation of diagrams carried out using the syntactic rules that are stated in advance. In this case the overall structure of the reasoning is also different than in the first two cases. There is no translation between the usual symbolic language and the diagrammatic one — rather, the language used consists of the diagrams as its integral part. This makes this type of reasoning more similar to the "usual" arguments that consist of a series of sentences, than the ones discussed in the previous sections.

The above classification cannot be said to fully reflect the variety of ways in which diagrams are used in mathematics. There is a great number of different types of diagrams beginning with geometric figures, through pebble- and arrow-diagrams to computer animation. The way in which they are used varies from one mathematical discipline to another, different visualization may also be used for the same mathematica objects. It is thus difficult to give a general characterization of how we reason using visualization. In consequence, the classification proposed in this paper can only be a rough one. There is, for example, a great variety of ways in which the stage of observing the diagram may look like. Explanation of the difference between the usage of static and dynamic diagrams is also a topic for further research. My aim in this paper was to suggest a means of understanding the main differences between ways in which the use of visualization may lead to mathematical convictions. The difference seems to be epistemically significant and may be taken as a starting point for further research on the nature diagrammatic reasoning.

Finally, two other issues should be noted in relation with the concept of diagrammatic reasoning. First of all, the vague boundary between the notions of a diagram and a symbol make it difficult to sharply distinguish between "thinking (reasoning) with diagrams" and "thinking (reasoning) with symbols". This is strongly stressed by Giaquinto, for whom "the diversity of visual thinking in mathematics outruns any twofold classification" [Giaquinto, 2007, p. 264], such as the distinctions "thinking with diagrams vs thinking with symbols" or "algebraic thinking vs geometric thinking". The use of symbols often involves imagining their movement in space, on the other hand no diagrammatic reasoning can, as it seems, be fully carried out without symbols and strictly defined mathematical concepts. The second point, that was generally dismissed in this paper, is the validity of reasoning with diagrams. It is known that the use of diagrams can lead to mistakes and various techniques should be used in order to prevent them (like the technique using notions of predicate logic to analyze the diagram). The most important issue that appears in this context is the problem of the generality of reasoning using diagrams. Thus, the various types of diagrammatic reasoning analysed in this paper should also be analysed and assessed from the perspective of their validity and the possible mistakes their usage may lead to.

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