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ALL PROPERTIES ARE DIVINE OR GOD EXISTS
The Sacred Thesis and its Ontological Argument

To the union of Åslaug Hegstad and Lars Eivind Lervåg

Abstract. A metaphysical system engendered by a third order quantified modal logic S5 plus impredicative comprehension principles is used to isolate a third order predicate $\mathfrak{D}$, and by being able to impredicatively take a second order predicate G to hold of an individual just if the individual necessarily has all second order properties which are $\mathfrak{D}$ we in Section 2 derive the thesis (40) that all properties are $\mathfrak{D}$ or some individual is G. In Section 3 theorems 1 to 3 suggest a sufficient kinship to Gödelian ontological arguments so as to think of thesis (40) in terms of *divine property* and *Godly being*; *divine* replaces *positive* with Gödel and others. Thesis (40), the *sacred thesis*, supports the ontological argument that *God exists because some property is not divine*. In Section 4 a fixed point analysis is used as diagnosis so that atheists may settle for the minimal fixed point. Theorem 3 shows it consistent to postulate theistic fixed points, and a monotheistic result follows if one assumes theism and that it is divine to be identical with a deity. Theorem 4 (the Monotheorem) states that if Gg and it is divine to be identical with g, then necessarily all objects which are G are identical with g. The impredicative origin of $\mathfrak{D}$ suggests weakened Gaunilo-like objections that offer related theses for other second order properties and their associated diverse presumptive individual bearers. Nevertheless, in the last section we finesse these Gaunilo-like objections by adopting what we call an *apathiatheistic* opinion which suggest that the best concepts ‘God’ allow thorough indifference as to whether God exists or not.

Keywords: modal logic; ontological argument; Gödel; higher order logic; philosophy of religion; theology

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1. Introduction

We employ the word “divine” instead of “positive” as in foregoing literature to expose certain conceptual connections which diverges from the Leibnizian philosophy that influenced Gödel. The term “the sacred thesis” is used as a metonym for the main title. Section 2 gives an account of the syntax and the evaluation semantics of the metaphysical system we presuppose, and it contains philosophical matters as well as a general discussion of higher order modal logic. Section 3 shows how impredicative comprehension with a third order modal logic suffices to define the second order property *divine* and the first order property *God* so as to establish the sacred thesis. Section 4 isolates some theorems which relate the work of Section 2 to modern work on modal ontological arguments originating with Gödel’s work divulged in 1970 and accounted for most thoroughly in [15]. The conceptual and historical connections evidenced suggest that the use of terms as in the sacred thesis is justified. Section 5 points out how the sacred thesis induces an affiliated ontological argument, and it offers some considerations pertinent to the suggested reasoning and related ontological arguments. The concluding Section 6 proposes an *apathiatheistic* attitude to whether or not there is a God, and the advocated indifference gives room for admitting the soundness of the advanced ontological argument without committing to epistemological superstition.

2. Preliminaries

In this section we seek to make a fairly precise exposition of the system we use, and we relate some references to the literature in order to facilitate comparisons for the reader, and we put forward some philosophical comments.

2.1. Syntax

We let the set of symbols $\mathcal{A} = \{\forall, \downarrow, x, X, \mathfrak{X}, a, A, \mathfrak{A}, ', \Box\}$ be the alphabet of the formal language $\mathcal{L}$, and define the constants, variables and terms of order one, two and three:

(I) x (X, $\mathfrak{X}$) is a first (second, third) order variable.
(II) If y (Y, $\mathfrak{Y}$) is a first (second, third) order variable then y', (Y', $\mathfrak{Y}'$) is a first (second, third) order variable.
(III) a (A, $\mathfrak{A}$) is a first (second, third) order constant.

(IV) If b (B, $\mathfrak{B}$) is a first (second, third) order constant then b', (B', $\mathfrak{B}'$) is a first (second, third) order constant.
(V) There are no more constants or variables.
(VI) All and only first (second, third) order constants and variables are first (second, third) order terms.

We define the set of formulas in a Polish manner:

(i) If T is a second order and t a first order term then Tt is a formula.
(ii) If $\mathfrak{T}$ is a third order and T a second order term then $\mathfrak{T}T$ is a formula.
(iii) If α is a formula then $\Box\alpha$ is a formula.
(iv) If α and β are formulas then $\downarrow\alpha\beta$ is a formula.
(v) if y is a first order variable and α is a formula then $\forall y\alpha$ is a formula.
(vi) If Y is a second order variable and α is a formula then $\forall Y\alpha$ is a formula.
(vii) If $\mathfrak{Y}$ is a third order variable and α is a formula then $\forall\mathfrak{Y}\alpha$ is a formula.
(viii) Nothing else is a formula.

On the basis of the definitions above wherein parentheses are not needed, we use the following metalinguistic conventions to facilitate reading:

(A) Instead of t (T, $\mathfrak{T}$) followed by a finite number of $'$ as per (i) ((ii), (iii)) above we use succeeding letters of the appropriate alphabet for first (second, third) order terms in the exposition.
(B) $(\neg\alpha)$ is short for $\downarrow\alpha\alpha$,
(C) $(\alpha\vee\beta)$ is short for $\downarrow\downarrow\alpha\beta\downarrow\alpha\beta$,
(D) $(\alpha\wedge\beta)$ is short for $\downarrow\downarrow\alpha\alpha\downarrow\beta\beta$,
(E) $(\alpha\rightarrow\beta)$ is short for $(\neg\alpha\vee\beta)$,
(F) $(\alpha\leftrightarrow\beta)$ is short for $((\alpha\rightarrow\beta)\wedge(\beta\rightarrow\alpha))$,
(G) $(\exists y)\alpha$ is short for $\neg(\forall y)\neg\alpha$,
(H) $(\exists B)\alpha$ is short for $\neg(\forall B)\neg\alpha$,
(I) $(\exists\mathfrak{B})\alpha$ is short for $\neg(\forall\mathfrak{B})\neg\alpha$ and
(J) $\Diamond\alpha$ is short for $\neg\Box\neg\alpha$.
(K) Parentheses which are not needed to disambiguate are supressed for legibility as the occasion allows.

We define the set of free first (second, third) order variables in a formula:

(a) $\mathrm{FV1}(Ta)=\emptyset$,

(b) $\mathrm{FV1}(Ty) = \{y\}$,
(c) $\mathrm{FV1}(\Box\alpha) = \mathrm{FV1}(\alpha)$,
(d) $\mathrm{FV1}(\downarrow\alpha\beta) = \mathrm{FV1}(\alpha) \cup \mathrm{FV1}(\beta)$,
(e) $\mathrm{FV1}(\forall y\alpha) = \mathrm{FV1}(\alpha) \setminus \{y\}$,
(f) $\mathrm{FV2}(\mathfrak{T}A) = \emptyset$,
(g) $\mathrm{FV2}(\mathfrak{T}Y) = \{Y\}$,
(h) $\mathrm{FV2}(\Box\alpha) = \mathrm{FV2}(\alpha)$,
(i) $\mathrm{FV2}(\downarrow\alpha\beta) = \mathrm{FV2}(\alpha) \cup \mathrm{FV2}(\beta)$,
(j) $\mathrm{FV2}(\forall B\alpha) = \mathrm{FV2}(\alpha) \setminus \{B\}$,
(k) $\mathrm{FV3}(\mathfrak{A}T) = \emptyset$,
(l) $\mathrm{FV3}(\mathfrak{Y}T) = \{\mathfrak{Y}\}$,
(m) $\mathrm{FV3}(\Box\alpha) = \mathrm{FV3}(\alpha)$,
(n) $\mathrm{FV3}(\downarrow\alpha\beta) = \mathrm{FV3}(\alpha) \cup \mathrm{FV1}(\beta)$,
(o) $\mathrm{FV3}(\forall\mathfrak{Y}\alpha) = \mathrm{FV3}(\alpha) \setminus \{\mathfrak{Y}\}$.

We let $\alpha\lceil t/y\rceil$ ($\alpha\lceil T/Y\rceil$, $\alpha\lceil\mathfrak{T}/\mathfrak{Y}\rceil$) denote the formula obtained by substituting all free occurrences of the first (second, third) order variable y (Y, $\mathfrak{Y}$) in α with the first (second, third) order term t (T, $\mathfrak{T}$). A first (second, third) order variable y (Y, $\mathfrak{Y}$) is substitutable for first (second, third) order variable z (Z, $\mathfrak{Z}$) in α just if y (Y, $\mathfrak{Y}$) is free in $\alpha\lceil y/z\rceil$ ($\alpha\lceil Y/Z\rceil$, $\alpha\lceil\mathfrak{Y}/\mathfrak{Z}\rceil$) precisely where z (Z, $\mathfrak{Z}$) is free in α, and a first (second, third) order constant is a closed term so substitutable for any first (second, third) order variable in any formula.

The presence of Barcan formulas and their converses of all orders facilitate the definition of prenex normal form of formulas made use of in defining a formula as Δ^1_0 iff it is equivalent to a formula that has no second order or third order quantifiers, and as Σ^1_1 (Π^1_1) iff it is equivalent to a formula with just existential (universal) second order quantifiers to begin with in its prenex normal form. We write that a formula schema is *properly parametrized* if it just contains second order or third order constants that have already been introduced earlier in a derivation where it occurs.

2.2. Axiom schemas for the logic $\mathbf{Q^3S5}$

The superscript 3 on Q indicates that third order quantification is included, and we take the inclusion of the Barcan formulas at any order as default.

A1 $\alpha \to (\beta \to \alpha)$
A2 $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$
A3 $(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$
K $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$
4 $\Box\alpha \to \Box\Box\alpha$
B $\alpha \to \Box\Diamond\alpha$
T $\Box\alpha \to \alpha$
$\mathbf{Q}_0^0$ $\forall x\Box\alpha \leftrightarrow \Box\forall x\alpha$
$\mathbf{Q}_0^1$ $\forall X\Box\alpha \leftrightarrow \Box\forall X\alpha$
$\mathbf{Q}_0^2$ $\forall \mathfrak{X}\Box\alpha \leftrightarrow \Box\forall \mathfrak{X}\alpha$
$\mathbf{Q}_1^0$ $\forall x\alpha \to \alpha\lceil t/x\rceil$, provided t is substitutable for x in α.
$\mathbf{Q}_1^1$ $\forall X\alpha \to \alpha\lceil T/X\rceil$, provided T is substitutable for A in α.
$\mathbf{Q}_1^2$ $\forall \mathfrak{X}\alpha \to \alpha\lceil \mathfrak{T}/\mathfrak{X}\rceil$, provided $\mathfrak{T}$ is substitutable for $\mathfrak{X}$ in α.
$\mathbf{Q}_2^0$ $\forall x(\alpha \to \beta) \to (\forall x\alpha \to \forall x\beta)$
$\mathbf{Q}_2^1$ $\forall X(\alpha \to \beta) \to (\forall X\alpha \to \forall X\beta)$
$\mathbf{Q}_2^2$ $\forall \mathfrak{X}(\alpha \to \beta) \to (\forall \mathfrak{X}\alpha \to \forall \mathfrak{X}\beta)$
$\mathbf{Q}_3^0$ $\alpha \to \forall x\alpha$, provided x is not in FV1(α).
$\mathbf{Q}_3^1$ $\alpha \to \forall X\alpha$, provided X is not in FV2(α).
$\mathbf{Q}_3^2$ $\alpha \to \forall \mathfrak{X}\alpha$, provided $\mathfrak{X}$ is not in FV3(α).

2.3. Inference rules and a derivation rule for Q^3S5

We write $\vdash \alpha$ for the statement that α is a thesis of Q^3S5, and all axioms of subsection 2.2 (stated without parameters) are theses. We first state the five inference rules for Q^3S5:

Modus ponens: $\vdash \alpha \,\&\, \vdash (\alpha \to \beta) \Rightarrow \vdash \beta$
Necessitation: $\vdash \alpha \Rightarrow \vdash \Box\alpha$
1-Generalization: $\vdash \alpha \Rightarrow \vdash \forall x\alpha$
2-Generalization: $\vdash \alpha \Rightarrow \vdash \forall X\alpha$
3-Generalization: $\vdash \alpha \Rightarrow \vdash \forall \mathfrak{X}\alpha$

A derivation $\triangle$ of Q^3S5 is a finite sequence of formulas each of which is an instance of an axiom of Q^3S5 or resulting from earlier elements of $\triangle$ by means of one of the five inference rules above or by a use of the derivation rule *Existential Instantiation* which requires that just if one in constructing a derivation $\triangle$ reaches a first (second, third) order formula

$\exists x\alpha$ ($\exists X\alpha$, $\exists\mathfrak{X}\alpha$) and constant b (B, $\mathfrak{B}$) has not been used earlier in $\triangle$ may one existentially instantiate with said constant and in a succeeding step of the derivation $\triangle$ put $\alpha\lceil b/x\rceil$ ($\alpha\lceil B/X\rceil$, $\alpha\lceil\mathfrak{B}/\mathfrak{X}\rceil$).

All the inference rules Modus Ponens, Necessitation, 1-Generalization, 2-Generalization and 3-Generalization are as well derivation rules in that they may be appealed to unrestrictedly in any derivation. As pointed out we avail ourselves of the derivation rule *Existential Instantiation* which we consider proper as a specific instantiation of a constant that it may allow at a stage in a given derivation $\triangle$ may be prohibited at stages in derivation $\triangle'$.

We illustrate the use of constants in deriving Σ^1_1-comprehension from Π^1_1-comprehension for a given formula ψ where we have that $\exists Y\Box\forall x(Yx \leftrightarrow \forall Z\psi)$. Existentially instantiate with a constant A so that $\Box\forall x(Ax \leftrightarrow \forall Z\psi)$. Use the constant A as a parameter so we have $\exists W\Box\forall x(Wx \leftrightarrow \forall Z\neg Ax)$. The last quantifier is superfluous as $\exists W\Box\forall x(Wx \leftrightarrow \neg Ax)$, and we next existentially instantiate with B to obtain $\Box\forall x(Bx \leftrightarrow \neg Ax)$; as $\Box\forall x(Ax \leftrightarrow \forall Z\psi)$, we derive $\Box\forall x(Bx \leftrightarrow \neg\forall Z\psi)$ and so by letting ϕ be $\neg\psi$ and using existential generalization justified by $\mathbf{Q}^1_1$ we get $\exists Y\Box\forall x(Yx \leftrightarrow \exists Z\phi)$.

2.4. Semantics

2.4.1. Philosophical remarks

In order to account for the meaning of the axiom schemas just given and the derivation and inference rules, we will elaborate upon the *evaluation semantics* for modal logic introduced by the author in [3] as this simplifies some matters. A translation to an account in a more standard possible worlds semantics may be undertaken, but the evaluation semantics we give has some independent interest which we point out. Notice well that the evaluation semantics we propose differs from Carnap's state description semantics and it does not as the latter validate the formula $\Diamond\alpha$ for any atomic formula α; for more on this arguably unwelcome feature of Carnap's state description semantics, confer [9].

It is true that a variety of entities can take the place of *possible worlds* in accounting for modal logics, and indeed a lot of theories have been offered even to clarify the meaning of the term "possible world". However, it is vitally important that the semantical apparatus presupposed also does justice to the way we speak as well as to the ontological economy of our commitments. As I see it, the evaluation semantics is quite advantageous on these scores as it simplifies matters ontologically and operates with just one domain of discourse. The evaluation semantics does thus not in the semantics commit to anything like *possible worlds*, which have shown themselves prone to lead philosophical and related discourse astray. Instead the *valuations* of the evaluation of any model are intuitively just the *properties*, *states* or *situations* the world may be in; model theoretically the valuations are just properties of formulas or ordered pairs from variables and objects with an appropriate set theoretical relation to the domain of discourse. As I see it there is nothing ontologically untoward by a metalogical commitment to properties which are not instantiated by the world, and we do not intend the evaluation semantics to account for what it means to be a property.

Accordingly the evaluationist semantics induces a replacement of the slogan that necessary truths are to be identified with "truth in all possible worlds" with the motto that "a necessary truth is a statement that is true in all possible states of the world".

2.4.2. Evaluation models

A novelty with the evaluation semantics is the unusual conditions *rhombicity* and *quadrangularity* for order 1 through 3 which justify the Barcan formula and its converse at that order. Pay also attention to the fact that we do not invoke the power set of the domain of discourse in our statement of the semantics. We suppress the semantical consideration of constants in derivations in this exposition. In the next subsection we illustrate how the semantics validates a selection of axioms and inference rules of the third order logic Q^3S5, and we further below introduce a moderate level of second order and third order comprehension.

Let an evaluation model $\mathcal{M}$ be a triple $\langle \mathbf{D}, \mathbf{E}, \mathbf{R} \rangle$ where $\mathbf{D}$ is the domain of discourse, $\mathbf{E}$ is an *evaluation* and $\mathbf{R} \subset \mathbf{E}^2$ is the *accessibility relation* on $\mathbf{E}$. An evaluation $\mathbf{E}$ is a set of *valuations*, and the latter we denote by V, V' and its kins. Valuations are monadic *properties* or *attributes* or *states* which only hold of some formulas of $\mathcal{L}$ and some ordered pairs $\langle y, d \rangle$ ($\langle Y, \delta \rangle$, $\langle \mathfrak{Y}, \Delta \rangle$)where y (Y, $\mathfrak{Y}$) is a first, (second, third) order variable of $\mathcal{L}$ and d (δ, Δ) is intuitively the value the valuation assigns the said variable relative to the domain of discourse $\mathbf{D}$: If $V \in \mathbf{E}$ and y is a first order variable of $\mathcal{L}$ then $\mathrm{V}(\langle y, d \rangle)$ only if $d \in \mathbf{D}$. If $V \in \mathbf{E}$ and Y is a second order variable then $\mathrm{V}(\langle Y, \delta \rangle)$ only if $\delta \subset \mathbf{D}$. If $V \in \mathbf{E}$ and $\mathfrak{Y}$ is a third order variable then $\mathrm{V}(\langle \mathfrak{Y}, \Delta \rangle)$ only if $\forall x(x \in \Delta \Rightarrow x \subset \mathbf{D})$.

Valuations are to be functional in that for $V \in \mathbf{E}$ and y a first order variable V($\langle y, d\rangle$) and V($\langle y, d`\rangle$) only if $d$ is $d`$, so we write $V(y) = d$ for $V(\langle y, d\rangle)$; we for the same reason write $V(Y) = \delta$ for $V(\langle y, \delta\rangle)$ and $V(\mathfrak{Y}) = \Delta$ for $V(\langle \mathfrak{Y}, \Delta\rangle)$.

For any first (second, third) order variable y (Y, $\mathfrak{Y}$) we write $V \textcircled{y} V'$ ($V \textcircled{Y} V'$, $V \textcircled{$\mathfrak{Y}$} V'$) to express that V and V' at most differ in that $V(y) \neq V'(y)$.

To save space we use Σ and Π for existential and universal generalisation at the metalevel, and for the same reason we use $\sim$ for negation, $\Rightarrow$ for implication. $\Leftrightarrow$ for biimplication and & for conjunction at the metalevel.

With these notions we require for any model $\mathcal{M} = \langle \mathbf{D}, \mathbf{E}, \mathbf{R}\rangle$ and $V \in \mathbf{E}$ and first order variable y, second order variable Y, third order variable $\mathfrak{Y}$ and formulas $\alpha, \beta \in \mathcal{L}$:

1-*denotation*:	$\Sigma d(d \in \mathbf{D} \ \&\ \ V(y) = d)$
2-*denotation*:	$\Sigma \delta(\delta \subset \mathbf{D} \ \&\ \ V(Y) = \delta)$
3-*denotation*:	$\Sigma \Delta(\Pi y(y \in \Delta \Rightarrow (y \subset \mathbf{D})) \ \&\ \ V(\mathfrak{Y}) = \Delta)$
Higher atoms:	$V(\mathfrak{Y}Y) \Leftrightarrow (V(Y) \in V(\mathfrak{Y}))$
Lower atoms :	$V(Yy) \Leftrightarrow\ (V(y) \in V(Y))$
NOR:	$V(\downarrow \alpha\beta) \Leftrightarrow\ (\sim V(\alpha) \,\&\, \sim V(\beta))$
1-Generality:	$V(\forall y\alpha) \Leftrightarrow \Pi V'(V \textcircled{y} V' \Rightarrow V'(\alpha))$
2-Generality:	$V(\forall Y\alpha) \Leftrightarrow \Pi V'(V \textcircled{Y} V' \Rightarrow V'(\alpha))$
3-Generality:	$V(\forall \mathfrak{Y}\alpha) \Leftrightarrow \Pi V'(V \textcircled{$\mathfrak{Y}$} V' \Rightarrow V'(\alpha))$
Apodicticity:	$V(\Box\alpha) \Leftrightarrow \Pi V'(V\mathbf{R}V' \Rightarrow V'(\alpha))$
1-Rhombicity:	$(V\mathbf{R}V' \& V' \textcircled{y} V'') \Rightarrow \Sigma V'''(V \textcircled{y} V''' \& V'''\mathbf{R}V'')$
1-Quadrangularity:	$(V \textcircled{y} V' \& V'\mathbf{R}V'') \Rightarrow \Sigma V'''(V\mathbf{R}V''' \& V''' \textcircled{y} V'')$
2-Rhombicity:	$(V\mathbf{R}V' \& V' \textcircled{Y} V'') \Rightarrow \Sigma V'''(V \textcircled{Y} V''' \& V'''\mathbf{R}V'')$
2-Quadrangularity:	$(V \textcircled{Y} V' \& V'\mathbf{R}V'') \Rightarrow \Sigma V'''(V\mathbf{R}V''' \& V''' \textcircled{Y} V'')$
3-Rhombicity:	$(V\mathbf{R}V' \& V' \textcircled{$\mathfrak{Y}$} V'') \Rightarrow \Sigma V'''(V \textcircled{$\mathfrak{Y}$} V''' \& V'''\mathbf{R}V'')$
3-Quadrangularity:	$(V \textcircled{$\mathfrak{Y}$} V' \& V'\mathbf{R}V'') \Rightarrow \Sigma V'''(V\mathbf{R}V''' \& V''' \textcircled{$\mathfrak{Y}$} V'')$
1-Indifference:	$(V(\alpha)\ \&\ x \notin \mathrm{FV}(\alpha)) \Rightarrow \Pi V'(V \textcircled{x} V' \Rightarrow V'(\alpha))$
2-Indifference:	$(V(\alpha)\ \&\ X \notin \mathrm{FV}(\alpha)) \Rightarrow \Pi V'(V \textcircled{X} V' \Rightarrow V'(\alpha))$
3-Indifference:	$(V(\alpha)\ \&\ \mathfrak{X} \notin \mathrm{FV}(\alpha)) \Rightarrow \Pi V'(V \textcircled{$\mathfrak{X}$} V' \Rightarrow V'(\alpha))$

If 1-Rhombicity and 1-Quadrangularity had not been assumed and we had deleted references to principles of higher order, we would have generated a Kripke style semantics as in [19] for first order modal logic with the peculiar restrictions on the use of open sentences. As I see it, 1-Rhombicity and 1-Quadrangularity are plausible principles, but the associated first order Barcan Formula and its converse are not needed in the derivation of the Sacred Thesis below.

I mention en passant that I welcome the necessitist consequences of adopting the first order Barcan formula and its converse. The necessitist point of view that all things exist necessarily has been most extensively defended by Timothy Williamson in [30] and foreshadowed by [31], and it was also implicit in [20]. The author independently came up with a necessitist point of view in the lecture [7]. The argument for necessiticm in the evaluationist framework is different, and appeals to there being such a plenitude of valuations in the evaluation so that 1-Rhombicity and 1-Quadrangularity obtain.

2.4.3. Semantical validations

We first establish the basic truth functional conditions for valuations given the definitions (B) to (F) in section 2.1 and the conditions for NOR in section 2.4.2 and our metalinguistic conventions. We presuppose the truth functional logic of the meta language and take previous steps into account as we proceed:

NOT $V(\neg\alpha) \Leftrightarrow V(\downarrow \alpha\alpha) \Leftrightarrow\ ({\sim} V(\alpha)\ \&\ {\sim} V(\alpha)) \Leftrightarrow \text{not } V(\alpha)$
OR $V(\alpha \vee \beta) \Leftrightarrow V(\downarrow\downarrow \alpha\beta \downarrow \alpha\beta) \Leftrightarrow\ ({\sim} V(\downarrow \alpha\beta)\ \&\ {\sim} V(\downarrow \alpha\beta))$
$\Leftrightarrow \text{not } V(\downarrow \alpha\beta) \Leftrightarrow \text{not } ({\sim} V(\alpha)\ \&\ {\sim} V(\beta)) \Leftrightarrow (V(\alpha) \text{ or } V(\beta))$
AND $V(\alpha \wedge \beta) \Leftrightarrow V(\downarrow\downarrow \alpha\alpha \downarrow \beta\beta) \Leftrightarrow\ ({\sim} V(\downarrow \alpha\alpha)\ \&\ {\sim} V(\downarrow \beta\beta)))$
$\Leftrightarrow ({\sim} V(\neg\alpha) \text{ and} {\sim} V(\neg\beta)) \Leftrightarrow (V(\alpha) \text{ and } V(\beta))$
IF $V(\alpha \to \beta) \Leftrightarrow V(\neg\alpha \vee \beta) \Leftrightarrow (\text{not-}V(\alpha) \text{ or } V(\beta))$
IFF $V(\alpha \leftrightarrow \beta) \Leftrightarrow (V(\alpha \to \beta) \text{ and } V(\beta \to \alpha))$

In validating the axioms of Q^3S5 we in each case assume that there is an evaluation model to the contrary and derive a contradiction, and we assume that the relation $\mathbf{R}$ in the model $\mathcal{M} = \langle \mathbf{D}, \mathbf{E}, \mathbf{R} \rangle$ is an equivalence relation.

We now validate the axiom schemas of Q^3S5 as provided in 2.2:

A1-3 These follow from **NOT**, **OR**, **AND**, **IF** and **IFF**.

K Suppose a valuation V has $V(\Box\alpha)$ and $V(\Box(\alpha \to \beta))$ and not $V(\Box\beta)$. From the latter, $V(\neg\Box\beta)$ so that $V(\Diamond\neg\beta)$ and so $\Sigma V'(VRV'\ \&\ {\sim} V'(\beta))$, and we let V^* be a representative so that $VRV^*\ \&\ {\sim} V^*(\beta))$. From the former $\Pi V'(VRV' \Rightarrow V'(\alpha))$ and $\Pi V(VRV' \Rightarrow V'(\alpha \to \beta))$ so that by truth functionality of valuations $\Pi V'(VRV' \Rightarrow V'(\beta))$, and so by instantiation with the representative V^* we get $(VRV^* \Rightarrow V^*(\beta))$ which contradicts VRV^* & $mathop{\sim}V^*(\beta)$.

4 Assume $V(\Box(\alpha))$ and not $V(\Box\Box(\alpha))$. From these we derive that $\Pi V'(V\mathbf{R}V' \Rightarrow V'(\alpha))$ and $\Sigma V', V''(V\mathbf{R}V' \,\&\, V'\mathbf{R}V'' \,\&\, {\sim} V''(\alpha))$. As $\mathbf{R}$ is transitive it follows that $\Sigma V'(V\mathbf{R}V' \,\&\, {\sim} V'(\alpha))$, so we have a contradiction.

B Assume $V(\alpha)$ and not $V(\Box\Diamond\alpha)$. It follows that $V(\alpha)$ and $V(\Diamond\Box\neg\alpha)$, so that $V(\alpha)$ and $\Sigma V'(V\mathbf{R}V' \& \Pi V''(V'\mathbf{R}V'' \Rightarrow\ {\sim} V''(\alpha))))$. By a choice of instantiation in the latter we have that $V\mathbf{R}V' \,\&\, V'\mathbf{R}V \Rightarrow\ \sim\ V(\alpha)))$, and that $V(\alpha)$. As $\mathbf{R}$ is supposed to be symmetric we derive that $V'\mathbf{R}V$ and the contradiction that $V(\alpha)$ and ${\sim} V(\alpha)$.

T Assume to the contrary that $V(\Box\alpha)$ and not $V(\alpha))$. Frome these we have that $\Pi V'(V\mathbf{R}V' \Rightarrow V'(\alpha))$ and not $V(\alpha)$, and so by instantiation $V\mathbf{R}V \Rightarrow V(\alpha)$ and not $V(\alpha)$. As $\mathbf{R}$ is reflexive we have a contradiction.

$\mathbf{Q_0^0}$ a) Assume $V(\forall x\Box\alpha)$ and not $V(\Box\forall x\alpha)$. From the former we deduce $\Pi V'(V \otimes V' \Rightarrow \Pi V''(V'\mathbf{R}V'' \Rightarrow V''(\alpha)))$ and from the latter we derive that $\Sigma V'(V\mathbf{R}V' \,\&\, \Sigma V''(V' \otimes V'' \,\&\, {\sim} V''(\alpha)))$. Let V^* and V^{**} be representatives so that $V\mathbf{R}V^* \,\&\, V^* \otimes V^{**} \,\&\, {\sim} V^{**}(\alpha))$. By 1-Rhombicity we find a V^{***} such that $(V \otimes V^{***} \,\&\, V^{***}\mathbf{R}V^{**} \,\&\, {\sim} V^{**}(\alpha))$. If we now instantiate with with V^{***} and then with V^{**} in $\Pi V'(V \otimes V' \Rightarrow \Pi V''(V'\mathbf{R}V'' \Rightarrow V''(\alpha)))$ aabsurdity follows. b) Assume $V(\Box\forall x\alpha)$ and not $V(\forall x\Box\alpha)$. A contradiction is derived as in a) by invoking 1-Quadrangularity.

$\mathbf{Q_0^1}$ As $\mathbf{Q_0^0}$, but invoking 2-Rhombicity and 2-Quadrangularity.

$\mathbf{Q_0^2}$ As $\mathbf{Q_0^0}$, but invoking 3-Rhombicity and 3-Quadrangularity.

$\mathbf{Q_1^0}$ Suppose $V((\forall x)\alpha)$ and not $V(\alpha\lceil t/x\rceil)$, where t is substitutable for x in α. We then have $\Pi V'(V \otimes V' \Rightarrow V'(\alpha))$ and not $V(\alpha\lceil t/x\rceil)$. Let $V \otimes V'$ and $V'(x) = V(t)$, which entails that $V'(\alpha)$ and that $V'(\alpha) \Leftrightarrow V(\alpha\lceil t/x\rceil)$. A contradiction follows as we also assumed not $V(\alpha\lceil t/x\rceil)$.

$\mathbf{Q_1^1}$ As $\mathbf{Q_1^0}$.

$\mathbf{Q_1^2}$ As $\mathbf{Q_1^0}$.

$\mathbf{Q_2^0}$ Suppose $V(\forall x(\alpha \to \beta))$, $V(\forall x\alpha)$ and not $V(\forall x\beta))$. It follows that $\Pi V'(V \otimes V' \Rightarrow V'(\alpha \to \beta))$, $\Pi V'(V \otimes V' \Rightarrow V'(\alpha))$ and $\Sigma V'(V \otimes V' \,\&\, {\sim} V(\beta))$. Let $V \otimes V^*$ & not $V^*(\beta)$, and instantiate in the first to get $V \otimes V^* \Rightarrow V^*(\alpha \to \beta)$ and $V \otimes V^* \Rightarrow V^*(\alpha)$. By detachment and repetition we have $V^*(\alpha)$ and $V^*(\alpha \to \beta)$ and not $V^*(\beta)$, so a contradiction follows by the truth functionality of V^*.

$\mathbf{Q_2^1}$ As $\mathbf{Q_2^0}$.

$\mathbf{Q_2^2}$ As $\mathbf{Q_2^0}$.

$\mathbf{Q_3^0}$ By 1-Indifference.

$\mathbf{Q_3^1}$ By 2-Indifference.

$\mathbf{Q_3^2}$ By 3-Indifference.

2.5. The comprehensive metaphysical system $\mathbf{Q^3S5\Pi^{1,2}_{1,1}}$

The metaphysical system $\mathrm{Q^3S5\Pi^{1,2}_{1,1}}$ comes about by adding the following impredicative parametrized second and third order comprehension schemas **2C** and **3C** to the logic $\mathrm{Q^3S5}$, as we to make space employ conventional vector notation to abbreviate that we may have several succeeding variables:

2C $\forall\vec{x}\forall\vec{X}\exists Y\forall y(Yy \leftrightarrow \forall\vec{Z}\beta(\vec{x},y,\vec{X},\vec{Z}))$ for β any Δ^1_0 formula.

3C $\forall\vec{x}\forall\vec{X}\forall\vec{\mathfrak{X}}\exists\mathfrak{Y}\forall Y(\mathfrak{Y}Y \leftrightarrow \forall\vec{\mathfrak{Z}}\beta(\vec{x},Y,\vec{X},\vec{\mathfrak{X}},\vec{\mathfrak{Z}}))$ for β any Σ^1_1 formula.

2.6. Higher order modal logics

Higher order modal logic was appealed to in the work of Montague to account for phenomena in the semantics of natural language in a series of very influential papers [23, 24, 25] which were also reprinted in the collection [29]. The monograph [14] of Montague's student Gallin develops Montague's *Intensional Logic* further. In the useful survey article [26] Muskens also relate the work of Bressan on higher order modal logics in the monograph [8] which was motivated by axiomatizing classical mechanics.

In a standard semantics for a second order logic the second order quantifiers are taken to range over the full power set $\wp(D)$ of the domain D presupposed for the firt order quantifyers. In a Henkin semantics it is instead presupposed that the second order quantifiers range over a *general structure* which extends *pre-structures* built from the domain D such that all second order comprehension principles are validated.

Our investigation shall be carried through with a third order monadic modal predicate logic extending a second order monadic modal predicate logic which in its turn extends a first order modal predicate logic.

We do not need first order UI or EG in the derivation of the Sacred Theses. So we might have presupposed a free logic as discussed towards the end of 2.4.2, though extended with higher order principles as indicated.

We mention that Hájek in [16, 17] uses moderate cautious comprehension whereas Sobel appeals to full comprehension in the criticism of Gödel, and also Anderson uses full second order comprehension in his rectification of Gödel's proof. Also Hájek's "vorsichtige Komprehension" is impredicative.

3. Deriving the Sacred Thesis

The terms *K-axiom*, *T-axiom*, *B-axiom*, 4-*axiom* and 5-*axiom* have their expected meaning as from the literature (cfr. also section 2.2); we suppress the appendage "schema" for brevity, and overlook that the axiom schemas so named are not independent of each other. We assume that gothic majuscules stand for monadic third order predicate letters, and $\alpha\lceil\mathfrak{M}/\mathfrak{N}\rceil$ denotes the formula obtained from replacing all occurrences of third order predicate letter $\mathfrak{N}$ in formula α with third order predicate letter $\mathfrak{M}$. We consider formula α *modally operative in* $\mathfrak{C}$ iff $\Box\forall H(\mathfrak{A}H \rightarrow \mathfrak{B}H)$ entails that $\Box(\alpha\lceil\mathfrak{A}/\mathfrak{C}\rceil \rightarrow \alpha\lceil\mathfrak{B}/\mathfrak{C}\rceil)$. If we disregard the contribution modal operators make to complexity and comprehensive strength it turns out that just an instance of Π^2_1-comprehension to obtain a third order property $\mathfrak{D}$ and an instance of Π^1_1-comprehension with $\mathfrak{D}$ as parameter to isolate second order property G are the impredicative principles needed for the crucial definitions needed to obtain the sacred thesis.

Use the modal Π^2_1-comprehension principle of $\mathrm{Q}^3\mathrm{S5}\Pi^{1,2}_{1,1}$ to obtain:

(0) $\exists\mathfrak{X}\forall X(\mathfrak{X}X \leftrightarrow \forall\mathfrak{Y}((\Box\forall x(\forall Y(\mathfrak{Y}Y \rightarrow \Box Yx) \rightarrow Xx) \rightarrow \Box\mathfrak{Y}X) \rightarrow \mathfrak{Y}X))$

Existentially instantiate with the constant $\mathfrak{D}$:

(1) $\forall X(\mathfrak{D}X \leftrightarrow \forall\mathfrak{Y}((\Box\forall x(\forall Y(\mathfrak{Y}Y \rightarrow \Box Yx) \rightarrow Xx) \rightarrow \Box\mathfrak{Y}X) \rightarrow \mathfrak{Y}X))$

It follows that:

(2) $\forall X\forall\mathfrak{Y}((\Box\forall x(\forall Y(\mathfrak{Y}Y \rightarrow \Box Yx) \rightarrow Xx) \rightarrow \Box\mathfrak{Y}X) \rightarrow (\mathfrak{D}X \rightarrow \mathfrak{Y}X))$

In third order $S5$ the following is a thesis:

(3) $\forall X\forall\mathfrak{Y}(\Box(\Box\forall x(\forall Y(\mathfrak{Y}Y \rightarrow \Box Yx) \rightarrow Xx) \rightarrow \Box\mathfrak{Y}X) \leftrightarrow$
$(\Box\forall x(\forall Y(\mathfrak{Y}Y \rightarrow \Box Yx) \rightarrow Xx) \rightarrow \Box\mathfrak{Y}X))$

So by a hypothetical syllogism the following is a thesis of third order $S5$:

(4) $\forall X\forall\mathfrak{Y}(\Box(\Box\forall x(\forall Y(\mathfrak{Y}Y \rightarrow \Box Yx) \rightarrow Xx) \rightarrow \Box\mathfrak{Y}X) \rightarrow (\mathfrak{D}X \rightarrow \mathfrak{Y}X))$

By necessitation and the second and third order converse Barcan formulas:

$$(5)\quad \forall X \forall \mathfrak{P} \Box(\Box(\Box\forall x(\forall Y(\mathfrak{P}Y \to \Box Yx) \to Xx) \to \Box\mathfrak{P}X) \to (\mathfrak{D}X \to \mathfrak{P}X))$$

By the K-axiom and an application of the 4-axiom it follows that:

$$(6)\quad \forall X \forall \mathfrak{P}(\Box(\Box\forall x(\forall Y(\mathfrak{P}Y \to \Box Yx) \to Xx) \to \Box\mathfrak{P}X) \to \Box(\mathfrak{D}X \to \mathfrak{P}X))$$

On account of (3) and (6):

$$(7)\quad \forall X \forall \mathfrak{P}((\Box\forall x(\forall Y(\mathfrak{P}Y \to \Box Yx) \to Xx) \to \Box\mathfrak{P}X) \to \Box(\mathfrak{D}X \to \mathfrak{P}X))$$

By the modal operativity of $\Box\forall x(\forall Y(\mathfrak{B}Y \to \Box Yx) \to Xx)$ in $\mathfrak{B}$:

$$(8)\quad \forall X \forall \mathfrak{P}(\Box(\mathfrak{D}X \to \mathfrak{P}X) \to \\ (\Box\forall x(\forall Y(\mathfrak{D}Y \to \Box Yx) \to Xx) \to \Box\forall x(\forall Y(\mathfrak{P}Y \to \Box Yx) \to Xx)))$$

From (7) and (8) by a hypothetical syllogism:

$$(9)\quad \forall X \forall \mathfrak{P}((\Box\forall x(\forall Y(\mathfrak{P}Y \to \Box Yx) \to Xx) \to \Box\mathfrak{P}X) \to \\ (\Box\forall x(\forall Y(\mathfrak{D}Y \to \Box Yx) \to Xx) \to \Box\forall x(\forall Y(\mathfrak{P}Y \to \Box Yx) \to Xx)))$$

By a truth functional argument on (9):

$$(10)\quad \forall X \forall \mathfrak{P}((\Box\forall x(\forall Y(\mathfrak{P}Y \to \Box Yx) \to Xx) \to \Box\mathfrak{P}X) \to \\ (\Box\forall x(\forall Y(\mathfrak{D}Y \to \Box Yx) \to Xx) \to \Box\mathfrak{P}X))$$

By universal instantiation and change of order:

$$(11)\quad \Box\forall x(\forall Y(\mathfrak{D}Y \to \Box Yx) \to Xx) \to \\ \forall \mathfrak{P}((\Box\forall x(\forall Y(\mathfrak{P}Y \to \Box Yx) \to Xx) \to \Box\mathfrak{P}X) \to \Box\mathfrak{P}X))$$

By the T-axiom:

$$(12)\quad \Box\forall x(\forall Y(\mathfrak{D}Y \to \Box Yx) \to Xx) \to \\ \forall \mathfrak{P}((\Box\forall x(\forall Y(\mathfrak{P}Y \to \Box Yx) \to Xx) \to \Box\mathfrak{P}X) \to \mathfrak{P}X)$$

By (1) and (12) and universal generalization and necessitation:

$$(13)\quad \Box\forall X(\Box\forall x(\forall Y(\mathfrak{D}Y \to \Box Yx) \to Xx) \to \mathfrak{D}X)$$

Use third order comprehension with $\mathfrak{D}$ as parameter and existential instantiation to define $\mathfrak{F}$ so that:

$$(14)\quad \Box\forall X(\mathfrak{F}X \leftrightarrow \Box\forall x(\forall Y(\mathfrak{D}Y \to \Box Yx) \to Xx))$$

From (13) and (14) we have:

$$(15)\quad \Box\forall X(\mathfrak{F}X \to \mathfrak{D}X)$$

By the modal operativity of $\Box\forall x(\forall Y(\mathfrak{Y}Y \to \Box Yx) \to Xx)$ in $\mathfrak{Y}$ we have:

(16) $\Box\forall x(\forall I(\mathfrak{F}I \to \Box Ix) \to Xx) \to \Box\forall x(\forall I(\mathfrak{D}I \to \Box Ix) \to Xx)$

By (14) and (16) we have:

(17) $\Box\forall x(\forall Y(\mathfrak{F}Y \to \Box Yx) \to Xx) \to \mathfrak{F}X$

By the 4-axiom of third order S5 we have:

(18) $\Box\forall x(\forall Y(\mathfrak{F}Y \to \Box Yx) \to Xx) \to \Box\mathfrak{F}X$

By (18) and an instance of (1) it follows:

(19) $\Box\forall X(\mathfrak{D}X \to \mathfrak{F}X)$

Combining (15) and (19) we have:

(20) $\Box\forall X(\mathfrak{D}X \leftrightarrow \mathfrak{F}X)$

Using (14), this is to say:

(21) $\Box\forall X(\mathfrak{D}X \leftrightarrow \Box\forall x(\forall Y(\mathfrak{D}Y \to \Box Yx) \to Xx))$

Define, by using second order comprehension with parameter $\mathfrak{D}$:

(22) $\exists X\Box\forall x(Xx \leftrightarrow \forall Y(\mathfrak{D}Y \to \Box Yx))$

Existentially instantiate with second order constant G:

(23) $\Box\forall x(Gx \leftrightarrow \forall Y(\mathfrak{D}Y \to \Box Yx))$

From substitution of (23) in (21):

(24) $\Box\forall X(\mathfrak{D}X \leftrightarrow \Box\forall x(Gx \to Xx))$

From (24) and the fact that $\Box\forall x(Gx \to Gx)$ is a thesis:

(25) $\mathfrak{D}\mathrm{G}$

By the 4-axiom:

(26) $\Box\mathfrak{D}G$

From (24) and the fact that $\Diamond\exists x(Gx \wedge \neg Xx) \to \Diamond\exists x(Gx)$:

(27) $\neg\mathfrak{D}X \to \Diamond\exists x(Gx)$

From (23) by instantiation, simplification and permutation:

(28) $\Box(\mathfrak{D}G \to \forall x(Gx \to \Box Gx))$

From (26) and (28) and the K-axiom:

(29) $\Box\forall x(Gx \to \Box Gx)$

From (29) and quantifier rules:

(30) $\Box(\exists xGx \to \exists x\Box Gx)$

The following is a theorem of all quantified modal logics:

(31) $\Box(\exists x\Box Gx \to \Box\exists xGx)$

From (30) and (31):

(32) $\Box(\exists xGx \to \Box\exists xGx)$

From (32) and the 4-axiom:

(33) $\Box(\exists xGx \to \Box\Box\exists xGx)$

From (33) using an equivalent of the K-axiom:

(34) $\Diamond\exists xGx \to \Diamond\Box\Box\exists xGx$

From the B-axiom:

(35) $\Diamond\Box\Box\exists xGx \to \Box\exists xGx$

From (34) and (35):

(36) $\Diamond\exists xGx \to \Box\exists xGx$

From (27) and (36):

(37) $\neg\mathfrak{D}X \to \Box\exists xGx$

As X is arbitrary, by generalization:

(38) $\forall X(\neg\mathfrak{D}X \to \Box\exists xGx)$

From the quantifier rules:

(39) $\exists X\neg\mathfrak{D}X \to \Box\exists xGx$

By connective rules and interdefinability we have:

(40) $\forall X\mathfrak{D}X \lor \Box\exists xGx$

4. Bonds and theorems

This is not the occasion to expound the extensive literature initiated by the circulation of Gödel's handwritten note, and such an endeavor would at any rate fall outside the scope of the author's competence. In [28] Sobel showed that the assumptions of Gödel in [15] lead to modal collapse in the sense that $\alpha \leftrightarrow \Box\alpha$ becomes a thesis with Gödel for all formulas α; but confer Hájek's qualification referred to in the next section. In [1] Anderson presented a rectified ontological argument that avoids the problem pointed out by Sobel.

We list a translation of Gödel's Definition 1 (**DG1**), Anderson's first definition (**DA1**) and the first four of Anderson's five axiomatic assumptions (**AA1-4**) into our idiolect with *divine* instead of *positive* and a use of overline for denoting the complement of a property; we stress that just AA3 is a proper axiom:

DG1 $Gx \triangleq \forall I(\mathfrak{D}I \to Ix)$

DA1 $Gx \triangleq \forall I(\mathfrak{D}I \leftrightarrow \Box Ix)$

AA1 $\mathfrak{D}A \to \neg\mathfrak{D}\bar{A}$

AA2 $\mathfrak{D}A \wedge \Box\forall x(Ax \to Bx) \to \mathfrak{D}B$

AA3 $\mathfrak{D}G$

AA4 $\mathfrak{D}A \to \Box\mathfrak{D}A$

In our idiolect Gödel presupposed an axiom **AG1** stating that $\mathfrak{D}A \leftrightarrow \neg\mathfrak{D}\bar{A}$ instead of Anderson's **AA1**, and the translation of Gödel's **AG2** to **AG4** would be as **AA2** to **AA4**. Anderson and Gödel further presupposed axiomatically that some defined property akin to *necessary existence* is divine, and below we refer to these respective assumptions as **AA5** and **AG5**.

We use a modification of Gödel's DG1 and Anderson's DA1:

D $\quad Gx \triangleq \forall I(\mathfrak{D}I \to \Box Ix)$

THEOREM 0. (i) *The biconditional of definition* **D**,(ii) **AA2**, (iii) **AA3**, *and* (iv) **AA4** *are theses of* $\mathrm{S5}\Pi^{1,2}_{1,1}$.

PROOF. (i) By step (23) of Section 3. (ii) From step (24) of Section 3 and the transitivity of strict implication. (iii) By step (25) of Section 3. (iv) By step (21) and other steps of Section 3 and the 4-axiom of $\mathrm{Q}^3\mathrm{S5}\Pi^{1,2}_{1,1}$ $\Box$

We leave it as an exercise to establish:

THEOREM 1. $\mathrm{Q}^3\mathrm{S5}\Pi^{1,2}_{1,1} + \exists X\neg\mathfrak{D}X \vdash$ **AA1** *and*
$\mathrm{Q}^3\mathrm{S5}\Pi^{1,2}_{1,1} +$ **AA1** $\vdash \exists X\neg\mathfrak{D}X$

We next establish that $Q^3S5\Pi^{1;2}_{1,1} + \exists X \neg \mathfrak{D} X$ does not have all sentences as consequences:

THEOREM 2. $Q^3S5\Pi^{1;2}_{1,1} + \exists X \neg \mathfrak{D} X$ *has a model.*

PROOF. Let the set $\mathbf{E}$ of valuations be a singleton set $\{V\}$, the domain $D = D_0 \cup D_1 \cup D_2$ where $D_0 = \{g\}$ is the domain for the first order quantifiers, $D_1 = \{\{g\}, \emptyset\}$ is the domain for the second order quantifiers and the domain for the third order quantifiers is $D_2 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{g\}\}, \{\{g\}\}\}$. The accessibility relation $\mathbf{R}$ on $\mathbf{E}$ is $\{\langle V, V\rangle\}$. We leave it as an exercise to verify that a model with these ingredients serves. $\square$

Theorems 0 to 2 justify us in thinking of the derivation of the salient conclusion of $Q^3S5\Pi^{1;2}_{1,1} + \exists X \neg \mathfrak{D} X$ in such terms as have been used in the literature on modal ontological argument in the tradition from [15].

We define $a = b$ in a Leibnizian-Russellian manner as given by the formula $\forall F(Fa \rightarrow Fb)$. It is straightforward to show that identity so taken is symmetric given the impredicative second order comprehension principle. Identity of second order predicates can be isolated analogously by an appeal to our third order impredicative comprehension.

In the following we make use of Church's lambda operator so that $\lambda x Fx$ is an alternative name for the property F.

THEOREM 3 (The Monotheorem).
If Gg *and* $\mathfrak{D}\lambda x(x = g)$, *then* $\Box\forall x(Gx \rightarrow x = g)$.

PROOF. Given (24) in Section 3 we have $\mathfrak{D}\lambda x(x = g) \leftrightarrow \Box\forall x(Gx \rightarrow \lambda x(x = g)x)$. As we assume $\mathfrak{D}\lambda x(x = g)$ we have $\Box\forall x(Gx \rightarrow \lambda x(x = g)x)$, so that by beta abstraction we have $\Box\forall x(Gx \rightarrow x = g)$. $\square$

5. Pinpoints and diagnoses

We will in this paper not scrutinize possible objections towards the use of the logical and metaphysical apparatus invoked, and now just mention *en passant* that the use of systems such as our third order modal logic has been pointed to e.g. in [14, 30].

It nevertheless bears mentioning that we have invoked a rather impredicative system that some philosopers with a predicativist bent are prone to find dubious. Precisely the quasi circularity with impredicative principles such as we have invoked has been an aversion to predicativists, and those in the mathemathical minority who want to press predicativist points of views may raise similar doubts concerning other ontological arguments. Feferman in [11] gives a good introduction to the predicativist point of view and its tradition back to Cohen, Poincaré and Russell at the beginning of the previous century. It is popular to disagree with predicativist strictures for being too prohibitive as concerns self-reference.

In my opinion, a respectable framework for thinking about modal matters should also deal properly with modal paradoxes, as isolated by Montague in [22] and others, in some comprehensive way that also deals with the set theoretical paradoxes and thus accounts more appropriately for the foundation of mathematics. Some readers may at this point find it relevant to consult the author's take on the paradoxes in his alternative set theory £ as so far set out in [4, 5] and in some preceding work.

The definiens of definition **D** in Section 4 is employed crucially in step (1) of Section 3, and the definiens of Anderson's definition **DA1** could not have been so presupposed for the formula $\Box\forall x(\forall I(\mathfrak{G}I \leftrightarrow \Box Ix) \rightarrow Hx)$ is not modally operative in $\mathfrak{G}$ as the latter has negative occurrences in the former.

In [16, 17] Hájek improves upon an, as it turns out, just slightly insufficient model theoretic argument by Magari in [21] to the effect that Gödel's **AG1**, **AG2** and **AG3** already suffice for the derivation of the theistic main theorem. Magari's idea is confirmed as applied to Anderson's argument for Hájek shows that Anderson's **AA3** and **AA4** are superfluous in the presence of full second order modal comprehension with the underlying logic. Hájek also shows that such a comprehensive Andersonian argument is interpretable in Gödel's original set of axioms with a *cautious* comprehension principle, and that such a cautious version of Gödel's argument does not lead to the modal collapse which Sobel derived presupposing full comprehension in [28]. The reader is sent to Hájek's articles for more details.

In [2] the author showed that a simplification using just a part of Gödel's first axiom and some version of his last axiom suffices for the theistic conclusion, and it was noted that a combination with Hájek's independent result can be obtained to simplify matters further. This article delivers upon the promise of such an improved result.

We mention that [2] was referred to by André Fuhrmann in [13], and he correctly related the point that we in [2] take the property of being Godlike as primitive unlike Gödel's more Leibnizian approach of taking the second order property *positive* (with us *divine*) as primitive. This issue as to what is more primitive of *divine* and *God* disappears in our third order framework.

One may use the negation of the first disjunct of the sacred thesis as premise in an ontological argument for the necessary existence of a God, and we say that that argument is the ontological argument *supported by the sacred thesis*. The said ontological argument is naturally analyzed by a fixed point strategy, and we summarily state some upshots. A least fixed point is atheist, and atheists are committed to the point of view that all properties are divine in the sense of our definition. There may be theistic fixed points which commit to properties which are not divine, and such fixed points have simple models as in Theorem 2. The content of Theorem 3, the Monotheorem, is that if there is a God and *being identical with a God* is a divine property then also monotheism is true. On account of the given fixed point diagnosis one may resist the objection by [27] towards Gödelian ontological arguments that arbitrary second order properties may be substituted for *divine*.

It bears noticing that the sacred thesis and its ontological argument do not depend upon there being some maximum for properties like *being positive* or *being perfect*, as with Anselm or Leibniz and others. Nevertheless, somewhat Gaunilo-like objections remain and corresponding to the sacred thesis there is e.g. a similar *diabolical thesis* and related theses for other second order properties and their associated presumptive bearers. Statements as the sacred thesis and kindred theses do not by themselves carry ontological commitments, however. Notice that by our results the fixed point analysis propounded here applies as well to related modal ontological arguments.

6. Apathiatheistic appendage

Our attitude to arguments such as the ontological argument supported by the sacred thesis carries information concerning our attitude to ratiocination or ontology which may be of some wider interest. As I see it, the word "God" is well nigh maximally imprecise so that it seems rather unhelpful to be a militant atheist without clarifying what interpretation one has of the word; by the same token, it is not so useful to be a zealous theist. We are certainly all militant atheists with respect to some interpretations of the word "God" and no serious person is a theist with respect to more than a small range of interpretations of the word. Notice that one initially better takes "God" to express a predicate so as to not favor monotheism and so as to avoid certain befuddlements concerning supposedly non-referring names.

I have, as opposed to the *theophilic* and *theophobic* attitudes considered by Magari in [21], come to adopt what I call an *apathiatheistic* opinion according to which the best concepts 'God' cum understandings of reality are such that the question as to whether there is a God or not is academic in a sense similar to the question as to whether there are holes or just holed things. If God is such an object as apathiatheism requires then there is, as I see it, not sufficient untowardness ratiocinatively or ontologically with accepting the soundness of the ontological argument supported by the sacred thesis for this to be ruled out. From an apathiatheistic point of view atheist thinkers may as well engage in religious discourse as theological anti realists who take "God" as a theoretical term, and I do not see that apathiatheism is in conflict with reasonable theologies. Arguably, the most important religious question is not whether there is a God but whether something ultimately rectifies the unsayable sufferings in the world; I have scrutinized this and related religious and philosophical questions in [6].

However these matters are, from my apatiatheistic point of view the world may be as if there is a God even though there is no God and it may be as if there is no God even though there is a God.

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