Logic and **L**ogical **P**hilosophy
Volume 27 (2018), 3–52
DOI: 10.12775/LLP.2017.012

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TEMPORAL ALETHIC DYADIC DEONTIC LOGIC AND THE CONTRARY-TO-DUTY OBLIGATION PARADOX

Abstract. A contrary-to-duty obligation (sometimes called a reparational duty) is a conditional obligation where the condition is forbidden, e.g. "if you have hurt your friend, you should apologise", "if he is guilty, he should confess", and "if she will not keep her promise to you, she ought to call you". It has proven very difficult to find plausible formalisations of such obligations in most deontic systems. In this paper, we will introduce and explore a set of temporal alethic dyadic deontic systems, i.e., systems that include temporal, alethic and dyadic deontic operators. We will then show how it is possible to use our formal apparatus to symbolise contrary-to-duty obligations and to solve the so-called contrary-to-duty (obligation) paradox, a problem well known in deontic logic. We will argue that this response to the puzzle has many attractive features. Semantic tableaux are used to characterise our systems proof theoretically and a kind of possible world semantics, inspired by the so-called T × W semantics, to characterise them semantically. Our models contain several different accessibility relations and a preference relation between possible worlds, which are used in the definitions of the truth conditions for the various operators. Soundness results are obtained for every tableau system and completeness results for a subclass of them.

Keywords: T × W logics; temporal logic; modal logic; dyadic deontic logic; semantic tableaux; conditional norms; contrary-to-duty obligations, the contrary-to-duty (obligation) paradox

1. Introduction

This paper will introduce and explore a set of temporal alethic dyadic deontic systems, i.e., systems that include temporal, alethic and dyadic

Received September 22, 2016. *Revised* March 17, 2017. *Published online* May 1, 2017

deontic operators. Semantic tableaux (see Section 4) are used to characterise our systems proof theoretically and a kind of possible world semantics, inspired by the so-called T × W semantics (see Section 3), to characterise them semantically. Our models contain several different accessibility relations and a preference relation between possible worlds, which are used in the definitions of the truth conditions for the various operators. Soundness results are obtained for every tableau system and completeness results for a subclass of them.

The systems developed in this essay are modifications and extensions of the dyadic deontic systems and temporal alethic-deontic systems introduced by Rönnedal [43] and Rönnedal [44]. (See also Rönnedal [46].) The task of combining these systems is not trivial for several reasons. Firstly, if we want to obtain logics that are philosophically plausible, we have to add an alethic accessibility relation to our frames and models, an accessibility relation that is relativised to moments in time. In other words, the alethic accessibility relation is a ternary relation. A possible world ω' might be accessible from a possible world ω at time τ, even though ω' is not accessible from ω at time τ'. This accessibility relation plays an important part in the formulation of the alternative truth conditions for the various normative sentences introduced in Section 3.1.3. [43] does not include any alethic accessibility relation. Secondly, in our so-called supplemented frames and models, we relativise our preference relation between possible worlds to possible worlds; i.e., our preference relation between possible worlds is treated as a ternary relation in this paper, not as a binary relation as in [43]. The reason for this is connected to the fact that we do not assume that every possible world is alethically accessible from every possible world (at every moment of time) (see Section 3.2.3). Thirdly, we must modify some semantic conditions (e.g. *Cα3*, *C-Dα3* in this paper) and tableau rules (e.g. *Tα3*, *T-Dα3* in this paper) introduced by [43] and [44]; we have to delete some semantic conditions (e.g. *Ca6*) and also add some new conditions (e.g. *C-Dδ6*) and rules (e.g. *T-DMO*) to our systems (sections 3 and 4). In addition, we consider some tableau rules that do not have any counterparts in [43], [44] or [46] (*T-Dd4*, *T-Dd5*, *T-Dd7*, *T-Dad4*, *T-Dad5*) and we prove several interesting theorems. We can, for example, show that all of the following conditional obligations are equivalent in some systems (see Section 5): $\mathbf{O}[A]B$, $\mathbf{O}(A \to B)$, and $A \to \mathbf{O}B$, given that A is “non-future”. Nothing similar can be proved in the dyadic deontic systems in [43]. Furthermore, in this paper we explore how the so-called ordinary (unsupplemented) models are related to the so-called supplemented models (see Theorem 1) in detail; some relationships of this kind were only hinted at in [43]. Therefore, the results in this paper are logically interesting.

Sven Danielsson [20], Bengt Hansson [28], Bas van Fraassen [54, 55], David Lewis [34, 35], Frans von Kutschera [56], Lennart Åqvist [61, 62, 64], Nicholas Rescher [41] and Georg Henrik von Wright [58] are some of the pioneers in dyadic deontic logic. Many of the systems introduced by these logicians are included in our temporal alethic dyadic deontic systems (see [43], Theorems 17 and 18, for more information about this[1]).

Several philosophers and logicians have developed logical systems that deal with various combinations of the conditions governing temporal, alethic and deontic elements (e.g. Chellas [14], Bailhache [3, 4, 5, 6], van Eck [51], Thomason [48, 49], Åqvist and Hoepelman [68], Åqvist [67], Bartha [7], Horty [30], Belnap, Perloff and Xu [8], Brown [9, 10, 11]). However, as far as we know, no one has developed any tableau systems of the kind introduced in this paper. All of the systems in this essay are entirely new.

Many temporal alethic-deontic logicians use an axiomatic proof theory; semantically, they describe some kind of tree-like structure. An early contribution to this tradition is Prior [40]. In this paper, we use semantic tableaux instead.

Other works that deal with combinations of tense and modality include Ciuni and Zanardo [17], DiMaio and Zanardo [21], von Kutschera [57], Zanardo [60], Åqvist [65] and Wölfl [59]. For some informal philosophical reflections, see e.g. [1] and [22]. For more information on various mono-modal systems and on how to combine different systems, see e.g. [4, 5, 6], [12], [15], [26, 27], [32], [42], [63].

Roughly, temporal alethic-deontic logicians have used three kinds of semantics: $\mathrm{T} \times \mathrm{W}$ semantics (e.g. [3, 4, 5, 6, 14, 50, 68, 67]), moment-based (branching time) semantics (e.g. [7, 30, 8]) and branch-based semantics (e.g. [9, 10]). We use a kind of $\mathrm{T} \times \mathrm{W}$ semantics in this essay. According to this approach, truth is relativised to world-moment pairs. A sentence may be true in one possible world at a time and false in another possible world at the same time, or true in one possible world

[1] Similar theorems can be proved about our temporal alethic dyadic deontic systems. This is straightforward, but the proofs are tedious and we will not labour the details.

at a time and false in the same world at another time. This leads to a quite rich and flexible semantics.

There are many philosophically good reasons to be interested in the systems in this paper. It seems, for instance, that we need systems of this kind to be able to adequately formalise so-called contrary-to-duty obligations and overridable obligations, and to solve Arthur Prior's [39] paradoxes of derived obligation and Roderick M. Chisholm's so-called contrary-to-duty (obligation or imperative) paradox [16]. Of these puzzles, the second is probably the more important. Many different solutions to the contrary-to-duty paradox have been suggested in the literature (for an introduction, see e.g. [13]). But it seems to us that one of the most plausible is the one that uses some kind of temporal alethic dyadic deontic logic. In Section 6, we will briefly describe the contrary-to-duty (obligation) paradox and we will show how we can use our formal systems to solve this puzzle. We will also consider some of the reasons why this is a plausible response to the paradox in comparison to many other solutions. Another nice feature of the systems discussed in this paper is that we can introduce some interesting definitions of the comparative evaluative expressions "better than", "at least as good as", and "equally good as" in our systems.

The essay is divided into seven sections. In Section 2, we describe the syntax of our systems and in Section 3, their semantics. Section 4 deals with the proof-theoretic characterisation of our logics, and Section 5 includes some examples of theorems. In Section 6, we briefly describe the contrary-to-duty (obligation) paradox and we show how our formal systems can be used to solve this puzzle. Finally, Section 7 contains soundness proofs for every system and completeness proofs for a subclass of them. Our conjecture is that at least all of the systems based on ordinary models are complete, but we have not been able to prove this.

2. Syntax

Alphabet. 1. A denumerably infinite set Prop of proposition letters p, q, r, s, p_1, q_1, r_1, s_1, p_2, q_2, r_2, s_2, ...

2. A denumerably infinite set NT of names of times t_0, t_1, t_2, t_3 ...

3. The primitive truth functional connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (material implication) and $\leftrightarrow$ (material equivalence).

4. The alethic operators **U**, **M**, $\Box$, and $\Diamond$.
5. The temporal operators R, A, S, G, H, F, and P.
6. The (dyadic) deontic operators **O**, **P**, and **F**.
7. $\top$ (verum), $\bot$ (falsum) and the brackets), (,], and [.

Language. The language L is the set of well-formed formulas (wffs) generated by the usual clauses for proposition letters and propositionally compound sentences, and the following clauses:

1. If A is a wff, then **U**A ("it is universally (or absolutely) necessary that A"), **M**A ("it is universally (or absolutely) possible that A"), $\Box A$ ("it is historically necessary (or settled) that A"), $\Diamond A$ ("it is historically possible that A").

2. If A is a wff, then AA ("it is always the case that A"), SA ("it is some time the case that A"), GA ("it is always going to be the case that A"), HA ("it has always been the case that A"), FA ("it will some time in the future be the case that A"), PA ("it was some time in the past the case that A"), are wffs.

3. If A is a wff and t is in NT, then RtA ("it is realised at time t that A") is a wff.

4. If A and B are wffs, so are **O**$[B]A$ ("it ought to be the case that (it is obligatory that) A given B"), **P**$[B]A$ ("it is permitted that A given B"), and **F**$[B]A$ ("it is forbidden that A given B").

4. Nothing else is a wff.

Capital letters "A", "B", "C" . . . are used to represent arbitrary (not necessarily atomic) formulas of the object language. The upper case Greek letter Γ represents an arbitrary set of formulas. Brackets around sentences are usually dropped if the result is not ambiguous.

Definitions.

1. $\Diamond\!\!\!\!-A$ ("it is historically impossible that A") $=_{\text{df}} \neg\Diamond A$
2. ∇A ("it is historically contingent that A") $=_{\text{df}} \Diamond A \wedge \Diamond\neg A$
3. $\triangle\ A$ ("it is historically non-contingent that A") $=_{\text{df}} \neg\nabla A$ (or $\Box A \vee \Box\neg A$)
4. $[\mathsf{G}]A =_{\text{df}} A \wedge \mathsf{G}A$
5. $\langle\mathsf{F}\rangle A =_{\text{df}} \neg[G]\neg A$ (or $A \vee \mathsf{F}A$)
6. $[\mathsf{H}]A =_{\text{df}} A \wedge \mathsf{H}A$
7. $\langle\mathsf{P}\rangle A =_{\text{df}} \neg[H]\neg A$ (or $A \vee \mathsf{P}A$)
8. **O**A ("it ought to be the case that A") $=_{\text{df}}$ **O**$[\top]A$

9. $\mathsf{P}A$ ("it is permitted that A") $=_{\text{df}}$ $\mathsf{P}[\top]A$
10. $\mathsf{F}A$ ("it is forbidden that A") $=_{\text{df}}$ $\mathsf{F}[\top]A$
11. $\mathsf{O}'[B]A =_{\text{df}} \mathsf{P}[B]\top \wedge \mathsf{O}[B]A$
12. $\mathsf{P}'[B]A =_{\text{df}} \neg\mathsf{O}'[B]\neg A$ (or $\mathsf{O}[B]\bot \vee \mathsf{P}[B]A$)
13. $\mathsf{F}'[B]A =_{\text{df}} \neg\mathsf{P}'[B]A$ (or $\mathsf{O}'[B]\neg A$ or $(\mathsf{P}[B]\top \wedge \mathsf{F}[B]A)$)
14. $A \geq B$ ("A is at least as good as B") $=_{\text{df}}$ $\mathsf{O}[A \vee B]\bot \vee \mathsf{P}[A \vee B]A$ (or $\mathsf{P}'[A \vee B]A$)
15. $A > B$ ("A is better than B") $=_{\text{df}}$ $\mathsf{P}[A \vee B]\top \wedge \mathsf{O}[A \vee B]\neg B$ (or $\mathsf{O}'[A \vee B]\neg B$)
16. $A = B$ ("A is as good as B") $=_{\text{df}}$ $\mathsf{O}[A \vee B]\bot \vee (\mathsf{P}[A \vee B]A \wedge \mathsf{P}[A \vee B]B)$ (or $\mathsf{P}'[A \vee B]A \wedge \mathsf{P}'[A \vee B]B$)[2]

3. Semantics

3.1. Basic concepts

3.1.1. Temporal alethic dyadic deontic frame

We will consider two kinds of frames in this essay: ordinary and supplemented (temporal alethic dyadic deontic) frames. An ordinary (temporal alethic dyadic deontic) frame F is a relational structure $\langle W, T, <, R, \{S_A : A \in L\}\rangle$, where W is a non-empty set of possible worlds, T is a non-empty set of times, $<$ is a binary relation on T ($< \subseteq T \times T$), R is a ternary alethic accessibility relation ($R \subseteq W \times W \times T$), and $\{S_A : A \in L\}$ is a set of ternary dyadic deontic accessibility relations, one for each sentence, A, in L ($S_A \subseteq W \times W \times T$).

A supplemented (temporal alethic dyadic deontic) frame F_s is a relational structure $\langle W, T, <, R, \{S_A : A \in L\} \geq\rangle$, where W, T, $<$, R, and $\{S_A : A \in L\}$ are exactly as in an ordinary frame, and $\geq$ is a ternary preference relation defined over the elements in W ($\geq \subseteq W \times W \times W$).

[2] Definitions 8–16 have been suggested by several logicians, see e.g. [64]. $\mathsf{O}'[B]A$ is an alternative explication of the expression "It is obligatory that A given B"; $\mathsf{P}'[B]A$ is an alternative explication of the expression "It is permitted that A given B", etc. It should be noted that $\mathsf{P}[A]\top$ is equivalent to $\Diamond A$ in many systems, e.g. in **STADDL** (See Section 4.3). Hence, $\mathsf{P}[A \vee B]\top$ is equivalent to $\Diamond(A \vee B)$ in those systems. Furthermore, in many systems, e.g. in **STADDL**, $\mathsf{O}[A \vee B]\neg B$ entails $\mathsf{O}[A \vee B]A$. Intuitively, this means that "$A > B$" (A is better than B) is true in the possible world ω at the time τ just in case A or B is true at τ in some possible world that is alethically accessible from ω at τ, and A is true and B false in all the best "A or B"-worlds that are alethically accessible from ω at τ.

If it is clear that we are talking about a supplemented frame, we will sometimes drop the subscript.

R "corresponds" to the alethic operators $\Box$ and $\Diamond$; $<$ to the temporal operators G, F, H, and P; S_A and $\geq$ to the dyadic deontic operators $\mathbf{O}$, $\mathbf{P}$, and $\mathbf{F}$. Informally, $\tau < \tau'$ says that the time τ is before the time τ' (or that τ' is later than τ), $R\omega\omega'\tau$ says that the possible world ω' is alethically accessible from the possible world ω at time τ, $S_A\omega\omega'\tau$ says that the possible world ω' is A-accessible from the possible world ω at time τ, and $\omega \geq_{\omega'} \omega''$ says that the possible world ω is at least as good as the possible world ω'' (in, according to, or relative to the possible world ω'). To "decide" whether $\omega \geq_{\omega'} \omega''$, we place ourselves in world ω', and ask which world is better, ω or ω''. We can think of $\omega \geq_{\omega'} \omega''$ as a dyadic preference relation relativised to ω'. In temporal alethic dyadic deontic logic it is also possible to relativise our preference relation to points in time (or just points in time or both). However, we will not do this in the present paper. Here we are interested in the comparative goodness over whole possible worlds, not in the comparative goodness over possible worlds at particular moments in time. If the different possible worlds are ordered in the same way in every possible world, we can treat $\geq$ as a binary preference relation between possible worlds and not as a ternary relation. Intuitively, $\omega \geq \omega''$ then says that the possible world ω is at least as good as the possible world ω'' (see condition *C-Dδ6* in Section 3.2.6).

3.1.2. Temporal alethic dyadic deontic model

We will use two kinds of models in this essay: ordinary and supplemented (temporal alethic dyadic deontic) models. An ordinary model M is a triple $\langle F, V, v \rangle$, where: (i) F is a temporal alethic dyadic deontic frame; (ii) V is a valuation or interpretation function, which to every proposition letter p in Prop assigns a subset of $W \times T$, i.e., a set of ordered pairs $\langle \omega, \tau \rangle$, where $\omega \in W$ and $\tau \in T$; and (iii) v is a function which to each temporal name in NT assigns a time in T.

A supplemented model M_s is a triple $\langle F_s, V, v \rangle$ where: F_s is a supplemented frame, and V and v are exactly as in an ordinary model.

We will sometimes drop the subscript if it is clear from the context that we are talking about supplemented models (or if we are talking about any model whatsoever, ordinary or supplemented).

Let $M, \omega, \tau \Vdash A$ abbreviate "A is true in the possible world ω at the time τ in the model M" (or "A is true at the pair $\langle \omega, \tau \rangle$ in M"), and let ω, ω', ω'', etc., be possible worlds in W. In a supplemented model, the dyadic deontic accessibility relations can be defined in terms of the preference relation over our possible worlds in the following way:

C-Dγ0 For every A: $S_A\omega\omega'\tau$ if and only if (iff) $R\omega\omega'\tau$ and $M, \omega', \tau \Vdash A$ and $\forall\omega''(R\omega\omega''\tau \wedge M, \omega'', \tau \Vdash A \to \omega' \geq_\omega \omega'')$.

Intuitively, this condition says that the possible world ω' is A-accessible from the possible world ω at the time τ iff ω' is one of the best alethically accessible worlds according to ω in which A is true at τ. The "free" variables are taken to be implicitly bound by universal quantifiers.[3]

Note that $\mathcal{M}$ stands for a class of models and $\mathcal{F}$ for a class of frames.

3.1.3. Truth in a model, validity, satisfiability, logical consequence, etc.

Let M be any (ordinary or supplemented) temporal alethic dyadic deontic model, based on a frame $\langle W, T, <, R, \{S_A : A \in L\}\rangle$ ($\langle W, T, <, R, \{S_A : A \in L\}, \geq\rangle$). Let $\omega \in W$, $\tau \in T$ and let A be a well-formed sentence in L. The truth conditions for proposition letters and complex sentences are given in the following list. Those for truth functional connectives are the usual ones (illustrated by conjunction):

$$\begin{array}{rll}
M, \omega, \tau \Vdash p & \text{iff} & \langle \omega, \tau \rangle \in V(p) \text{ for any } p \text{ in Prop} \\
M, \omega, \tau \Vdash A \wedge B & \text{iff} & M, \omega, \tau \Vdash A \text{ and } M, \omega, \tau \Vdash B \\
M, \omega, \tau \Vdash \Box A & \text{iff} & \forall \omega' \in W \text{ s.t. } R\omega\omega'\tau\text{: } M, \omega', \tau \Vdash A \\
M, \omega, \tau \Vdash \Diamond A & \text{iff} & \exists \omega' \in W \text{ s.t. } R\omega\omega'\tau\text{: } M, \omega', \tau \Vdash A \\
M, \omega, \tau \Vdash \mathsf{A}A & \text{iff} & \forall \tau' \in T\text{: } M, \omega, \tau' \Vdash A \\
M, \omega, \tau \Vdash \mathsf{S}A & \text{iff} & \exists \tau' \in T\text{: } M, \omega, \tau' \Vdash A \\
M, \omega, \tau \Vdash \mathsf{G}A & \text{iff} & \forall \tau' \in T \text{ s.t. } \tau < \tau'\text{: } M, \omega, \tau' \Vdash A \\
M, \omega, \tau \Vdash \mathsf{F}A & \text{iff} & \exists \tau' \in T \text{ s.t. } \tau < \tau'\text{: } M, \omega, \tau' \Vdash A \\
M, \omega, \tau \Vdash \mathsf{H}A & \text{iff} & \forall \tau' \in T \text{ s.t. } \tau' < \tau\text{: } : M, \omega, \tau' \Vdash A \\
M, \omega, \tau \Vdash \mathsf{P}A & \text{iff} & \exists \tau' \in T \text{ s.t. } \tau' < \tau\text{: } M, \omega, \tau' \Vdash A
\end{array}$$

[3] Some deontic logicians might want to reject the condition (*C-Dγ0*), and also the condition (*C-Dα1*) that follows from (*C-Dγ0*) (see Theorem 1). For if we accept (*C-Dγ0*) we can prove that $\mathbf{O}[A]A$ is valid, and $\mathbf{O}[A]A$ might seem counterintuitive. See e.g. [18] for some critique of this sentence. Personally, however, we are inclined to believe that this is a reasonable theorem if it is interpreted in the "right" way. Intuitively, the formula only says that A is true in all the best possible A-worlds, which seems quite plausible. Note that $\mathbf{O}[A]A$ is not equivalent to $A \to \mathbf{O}A$, and from $\mathbf{O}[A]A$ and A we cannot, in general, derive the unconditional obligation $\mathbf{O}A$.

$$
\begin{aligned}
M,\omega,\tau \Vdash \mathsf{R}tA &\text{ iff } M,\omega,v(t) \Vdash A, \text{ for all } t \in \mathrm{NT}\\
M,\omega,\tau \Vdash \mathbf{U}A &\text{ iff } \forall\omega' \in W \text{ and } \forall\tau' \in T\colon M,\omega',\tau' \Vdash A\\
M,\omega,\tau \Vdash \mathbf{M}A &\text{ iff } \exists\omega' \in W \text{ and } \exists\tau' \in T\colon M,\omega',\tau' \Vdash A\\
M,\omega,\tau \Vdash \mathsf{O}[A]B &\text{ iff } \forall\omega' \in W \text{ s.t. } S_A\omega\omega'\tau\colon M,\omega',\tau \Vdash B\\
M,\omega,\tau \Vdash \mathsf{P}[A]B &\text{ iff } \exists\omega' \in W \text{ s.t. } S_A\omega\omega'\tau\colon M,\omega',\tau \Vdash B\\
M,\omega,\tau \Vdash \mathsf{F}[A]B &\text{ iff } \forall\omega' \in W \text{ s.t. } S_A\omega\omega'\tau\colon M,\omega',\tau \Vdash \neg B
\end{aligned}
$$

$\mathsf{O}[A]B$ is true in a possible world ω at time τ iff B is true in all possible worlds that are A-accessible from ω at τ. $\mathsf{P}[A]B$ is true in a possible world ω at time τ iff B is true in at least one possible world that is A-accessible from ω at τ, etc. If we define the dyadic deontic accessibility relations in terms of the preference relation as in (C-$D\gamma 0$), we can derive the following truth conditions:

- $M,\omega,\tau \Vdash \mathsf{O}[A]B$ iff $\forall\omega'$ s.t. $R\omega\omega'\tau$, and $M,\omega',\tau \Vdash A$, and $\forall\omega''(R\omega\omega''\tau \wedge M,\omega'',\tau \Vdash A \to \omega' \geq_\omega \omega'')$: $M,\omega',\tau \Vdash B$.
- $M,\omega,\tau \Vdash \mathsf{P}[A]B$ iff $\exists\omega'$ s.t. $R\omega\omega'\tau$, and $M,\omega',\tau \Vdash A$, and $\forall\omega''(R\omega\omega''\tau \wedge M,\omega'',\tau \Vdash A \to \omega' \geq_\omega \omega'')$: $M,\omega',\tau \Vdash B$.
- $M,\omega,\tau \Vdash \mathsf{F}[A]B$ iff $\forall\omega'$ s.t. $R\omega\omega'\tau$, and $M,\omega',\tau \Vdash A$, and $\forall\omega''(R\omega\omega''\tau \wedge M,\omega'',\tau \Vdash A \to \omega' \geq_\omega \omega'')$: $M,\omega',\tau \Vdash \neg B$.

In ("non-temporal") dyadic deontic logic the truth conditions for the various dyadic deontic operators are often defined in such a way that $\mathsf{O}[A]B$ is true in a possible world iff B is true in all the best A-worlds, where an A-world is a world where A is true. $\mathsf{P}[A]B$ is true in a possible world iff B is true in at least one of the best A-worlds. And $\mathsf{F}[A]B$ is true in a possible world iff B is false in all the best A-worlds. In temporal alethic dyadic deontic logic, of the kind we are considering in this paper, we are not interested in all the best A-worlds. Instead we focus on all the best A-worlds that are alethically accessible from a particular world at a particular time. Roughly, $\mathsf{O}[A]B$ is true in the world ω at the time τ iff B is true at τ in all the best A-worlds that are alethically accessible from ω at τ. $\mathsf{P}[A]B$ is true in the world ω at the time τ iff B is true at τ in at least one of the best A-worlds that are alethically accessible from ω at τ. And $\mathsf{F}[A]B$ is true in the world ω at the time τ iff B is false at τ in all the best A-worlds that are alethically accessible from ω at τ.

Other basic semantic concepts like validity, satisfiability, logical consequence, etc. are defined as in [44].

3.2. Conditions on frames and models

In this section, we will consider several different frame- and model-conditions that can be used to characterise and classify our temporal alethic dyadic deontic frames and models. Many of these conditions are modifications of conditions introduced by [43], [44], or [46].

The symbols $\wedge$, $\rightarrow$, $\leftrightarrow$, $\forall$ and $\exists$ in tables 1–5 are used as metalogical symbols in the standard way. Let F be a temporal alethic dyadic deontic frame, M a temporal alethic dyadic deontic model based on F, $\{S_A : A \in L\}$ the set of dyadic deontic accessibility relations and $\geq$ the preference relation in F.

If for all S_A in $\{S_A : A \in L\}$, $\forall\tau\forall\omega\forall\omega'(S_A\omega\omega'\tau \rightarrow \ M, \omega', \tau \Vdash A)$, we say that S_A satisfies or fulfills condition *C-Dα1* and also that M satisfies or fulfills condition *C-Dα1* and similarly in all other cases. *C-Dα1* is called "*C-Dα1*" because the tableau rule *T-Dα1* "corresponds" to *C-Dα1* and the sentence *TDα1* is valid in the class of all models that satisfy condition *C-Dα1* and similarly in all other cases. Let C be any of the conditions we explore. Then a C-model is a model that satisfies condition C and similarly for the frame-conditions. If it is clear that we are talking about a condition, the initial C will often be dropped.

3.2.1. Conditions on the alethic accessibility relation R and the temporal accessibility relation $<$

We use the same conditions on the alethic accessibility relation R, and the temporal accessibility relation $<$ as in [44] and [46]. Intuitively, it is often reasonable to treat the alethic accessibility relation R as an equivalence relation. This relation "corresponds" to historical necessity, possibility, etc.

3.2.2. Conditions on the dyadic deontic accessibility relations S_A

The conditions on the dyadic deontic accessibility relations S_A are presented in Table 1. In this table "*Dd*" stands for "Dyadic deontic". The condition *C-Dd4* is similar to the familiar condition *4* in monomodal logic, which says that the ordinary modal accessibility relation is transitive. The same goes for the other conditions. If it is clear from the context that we are talking about dyadic deontic conditions, we can drop "*C-D*" and talk about *d4*, *d5* etc. instead. We will sometimes abbreviate other semantic conditions in a similar way. *d4′*, *d5′*, *dT′* and *dB′* are "temporal" versions of similar conditions introduced by [43].

Condition	Formalisation of condition
C-Dd4	$\forall\tau\forall\omega\forall\omega'\forall\omega''((S_A\omega\omega'\tau \wedge S_A\omega'\omega''\tau) \to S_A\omega\omega''\tau)$
C-Dd4'	$\forall\tau\forall\omega\forall\omega'\forall\omega''((S_A\omega\omega'\tau \wedge S_B\omega'\omega''\tau) \to S_B\omega\omega''\tau)$
C-Dd5	$\forall\tau\forall\omega\forall\omega'\forall\omega''((S_A\omega\omega'\tau \wedge S_A\omega\omega''\tau) \to S_A\omega'\omega''\tau)$
C-Dd5'	$\forall\tau\forall\omega\forall\omega'\forall\omega''((S_A\omega\omega'\tau \wedge S_B\omega\omega''\tau) \to S_B\omega'\omega''\tau)$
C-DdT'	$\forall\tau\forall\omega\forall\omega'(S_A\omega\omega'\tau \to S_A\omega'\omega'\tau)$
C-DdB'	$\forall\tau\forall\omega\forall\omega'\forall\omega''((S_A\omega\omega'\tau \wedge S_A\omega'\omega''\tau) \to S_A\omega''\omega'\tau)$
C-Dd7	$\forall\tau\forall\omega\forall\omega'\forall\omega''((S_A\omega\omega'\tau \wedge S_A\omega\omega''\tau) \to \omega' = \omega'')$

Table 1. Conditions for relations S_A

Condition	Formalisation of condition
C-DMO	$\forall\tau\forall\omega\forall\omega'(S_A\omega\omega'\tau \to R\omega\omega'\tau)$
C-Dad4	$\forall\tau\forall\omega\forall\omega'\forall\omega''((R\omega\omega'\tau \wedge S_A\omega'\omega''\tau) \to S_A\omega\omega''\tau)$
C-Dad5	$\forall\tau\forall\omega\forall\omega'\forall\omega''((R\omega\omega'\tau \wedge S_A\omega\omega''\tau) \to S_A\omega'\omega''\tau)$

Table 2. Conditions concerning the relation between R and S_A

Note that *C-Dd4'* entails *C-Dd4*, but not vice versa. *C-Dd5'* entails *C-Dd5*, but not vice versa. $\mathbf{O}[B]A \to \mathbf{O}[B]\mathbf{O}[B]A$ is valid in every model that satisfies *C-Dd4*; and $\mathbf{O}[B]A \to \mathbf{O}[C]\mathbf{O}[B]A$ is valid in every model that satisfies *C-Dd4'*. $\mathbf{P}[B]A \to \mathbf{O}[B]\mathbf{P}[B]A$ is valid in every model that satisfies *C-Dd5*; and $\mathbf{P}[B]A \to \mathbf{O}[C]\mathbf{P}[B]A$ is valid in every model that satisfies *C-Dd5'*. *C-Dd5'* entails *C-DdT'* and *C-DdB'*; and *C-Dd5* entails *C-DdT'* and *C-DdB'*.

Condition *C-Dd7* is theoretically interesting but intuitively problematic. $\mathbf{O}[B]A \vee \mathbf{F}[B]A$, $\mathbf{O}[B]A \vee \mathbf{O}[B]\neg A$, and $\mathbf{P}[B]A \to \mathbf{O}[B]A$ are valid in every model that satisfies this condition. In every model that satisfies *C-Dα3* (see Table 3), $\Diamond B \to (\mathbf{O}[B]A \to \mathbf{P}[B]A)$ is valid. So, in models that satisfy *C-Dd7* and *C-Dα3*, the distinction between permissions and obligations collapses, nothing is optional and everything is either obligatory or forbidden, given that the condition B is possible. This amounts to a kind of "black and white thinking" or "moral rigorism".

3.2.3. Conditions concerning the relation between R and S_A

The conditions concerning the relation between R and S_A are presented in Table 2.

Note that $\mathbf{O}[B]A \to \Box\mathbf{O}[B]A$ and $\mathbf{F}[B]A \to \Box\mathbf{F}[B]A$ are valid in every model that satisfies *C-Dad4* ("Dyadic alethic deontic 4"). And $\mathbf{P}[B]A \to \Box\mathbf{P}[B]A$ is valid in every model that satisfies *C-Dad5* ("Dyadic alethic deontic 5”). So, in all models that satisfy *C-Dad4* and *C-Dad5*, all norms are historically necessary. Even though they are historically necessary, they are not necessarily absolutely necessary; neither $\mathbf{O}[B]A \rightarrow \mathbf{UO}[B]A$, $\mathbf{P}[B]A \rightarrow \mathbf{UP}[B]A$, nor $\mathbf{F}[B]A \rightarrow \mathbf{UF}[B]A$ is valid. According to *C-DMO*, every dyadic deontic accessibility relation is included in the alethic accessibility relation. This is a reasonable condition if we assume that norms primarily concern things that are historically possible. $\Box A \rightarrow \mathbf{O}[B]A$[4] and $(\mathbf{O}A \wedge \Box(A \rightarrow B)) \rightarrow \mathbf{O}B$ are valid in every model that satisfies *C-DMO* (“Dyadic Must-Ought”). So, in models of this kind, a version of the means-end principle is valid. According to the means-end principle every necessary consequence of something that ought to be itself ought to be.

C-Dad4 and *C-DMO* entail *C-Dd4′*. *C-Dad5* and *C-DMO* entail *C-Dd5′*. *C-Dd4′* and *C-Dd5′* follow if we assume that $S_A\omega\omega'\tau$ means that ω' is one of the best accessible A-worlds from ω at τ, that the ordering of the possible worlds is the same in every possible world, and that *C-a4* (the alethic accessibility relation R is transitive) and *C-a5* (the alethic accessibility relation R is Euclidean) hold. [43] introduces a condition called “*C-a6*” ($= \forall\omega\forall\omega'\forall\omega''(S_A\omega\omega' \rightarrow S_A\omega''\omega')$). *C-a6′* ($= \forall\tau\forall\omega\forall\omega'\forall\omega''(S_A\omega\omega'\tau \rightarrow S_A\omega''\omega'\tau)$) is a temporal version of *C-a6*. *C-a6′* entails *C-Dd4′*, *C-Dd5′*, *C-Dad4* and *C-Dad5*. *C-a6* might be a reasonable condition in (non-temporal) dyadic deontic logic, given certain interpretations. However, *C-a6′* is no longer intuitively plausible. For not all possible worlds are necessarily alethically accessible from all possible worlds at a particular point in time. ω''' might, for instance, be one of the best alethically accessible worlds from ω' at τ in which A is true, even though ω''' is not alethically accessible from ω'' at τ, and

[4] Some might think that this formula is unreasonable and, hence, that we should reject *C-DMO*. They might think so because $\Box A \rightarrow \mathbf{O}[B]A$ seems to have counterintuitive instances. Consider, for example, the following instance: “If it is necessary that $2 + 2 = 4$, then it is obligatory that $2 + 2 = 4$ given that the earth is flat”. It might be interesting to note that we can avoid conclusions of this kind by using a trick inspired by Alan Ross Anderson [2]. We can define a new deontic operator in terms of $\mathbf{O}$ in the following way: $\underline{\mathbf{O}}[B]A =_{\text{df}} \mathbf{O}[B]A \wedge \neg\Box A$. Then we can use this new operator to symbolise the expression “It is obligatory that A given B”. If we use this operator instead, we avoid counterintuitive implications of the kind mentioned above. However, personally we are inclined to believe that the formula $\Box A \rightarrow \mathbf{O}[B]A$ can be defended, even though we will not try to do so in the present paper. Many things that sound very strange may nevertheless be true. So, we are not sure we need this new operator.

	For all A and for all B
C-$D\alpha 0$	$\forall\tau\forall\omega\forall\omega'((\|A\|^{M,\omega,\tau} = \|B\|^{M,\omega,\tau}) \to (S_A\omega\omega'\tau \leftrightarrow S_B\omega\omega'\tau))$
C-$D\alpha 1$	$\forall\tau\forall\omega\forall\omega'(S_A\omega\omega'\tau \to M,\omega',\tau \Vdash A)$
C-$D\alpha 2$	$\forall\tau\forall\omega\forall\omega'((S_A\omega\omega'\tau \ \wedge M,\omega',\tau \Vdash B) \to S_{A\wedge B}\omega\omega'\tau)$
C-$D\alpha 3$	$\forall\tau\forall\omega(\exists\omega'(R\omega\omega'\tau \ \wedge M,\omega',\tau \Vdash A) \to \exists\omega'' S_A\omega\omega''\tau)$
C-$D\alpha 4$	$\forall\tau\forall\omega\forall\omega'\forall\omega''((S_A\omega\omega'\tau \ \wedge M,\omega',\tau \Vdash B)$ $\to (S_{A\wedge B}\omega\omega''\tau \to (S_A\omega\omega''\tau \ \wedge M,\omega'',\tau \Vdash B)))$

Table 3. Conditions concerning the relation between R, S_A and V

hence not one of the best alethically accessible A-worlds in ω'' at τ. So, we will not say anything more about C-$a6'$ in the present paper.

3.2.4. Conditions concerning the relation between R, S_A and V

The conditions concerning the relation between R, S_A and V are presented in Table 3. These conditions are "temporal" versions of conditions introduced by [43]. Note that $\|A\|^{M,\omega,\tau}$ is the set of all alethically accessible A-worlds from ω at τ (in M). Hence, $\|A\|^{M,\omega,\tau} = \|B\|^{M,\omega,\tau}$ is true iff $M,\omega,\tau \Vdash \Box(A \leftrightarrow B)$. Furthermore, note that in C-$D\alpha 3$ we require that the world ω' in which A is true must be alethically accessible from the world ω to be able to "derive" the consequent. It is not enough that A is true in some world ω'' at τ; for ω'' might not be alethically accessible from ω at τ. In this sense, C-$D\alpha 3$ in Table 3 is different from the corresponding condition in [43]. $\Diamond B \to (\mathbf{O}[B]A \to \mathbf{P}[B]A)$ is valid, but $\mathbf{M}B \to (\mathbf{O}[B]A \to \mathbf{P}[B]A)$ is not valid, in every model that satisfies C-$D\alpha 3$.

3.2.5. Conditions concerning the relation between R, $<$ and V

The conditions concerning the relation between R, $<$ and V (see Table 4) are exactly the same as in [44] and [46]. C-SP, C-FT and C-BT correspond to the tableau rules T-SP, T-FT and T-BT (see Table 11). See Table 14 for some examples of theorems that can be proved in some systems that include T-FT and T-BT. [44] includes some examples of provable sentences in systems containing T-SP.

3.2.6. Conditions involving the preference relation *at least as good as* $\geq$

The conditions involving the preference relation *at least as good as* $\geq$ are presented in Table 5. Some of these conditions are similar to some

Condition	Formalisation of condition
C-SP	$\forall\tau\forall\tau'\forall\omega\forall\omega'((\tau < \tau' \wedge R\omega\omega'\tau') \to R\omega\omega'\tau)$
C-FT	If $R\omega'\omega''\tau$ and A is an atomic sentence true in ω' at τ, then A is true in ω'' at τ.
C-BT	If $R\omega'\omega''\tau$ and A is an atomic sentence true in ω'' at τ, then A is true in ω' at τ.

Table 4. Conditions concerning the relation between R, $<$ and V

	For all A and for all B
C-Dγ0	$\forall\tau\forall\omega\forall\omega'(S_A\omega\omega'\tau \leftrightarrow (R\omega\omega'\tau \wedge M,\omega',\tau \Vdash A \wedge \forall\omega''(R\omega\omega''\tau \wedge M,\omega'',\tau \Vdash A \to \omega' \geq_\omega \omega'')))$
C-Dδ2	$\forall\omega\forall\omega'\forall\omega''((\omega' \geq_\omega \omega'' \vee \omega'' \geq_\omega \omega') \vee (\omega' \geq_\omega \omega'' \wedge \omega'' \geq_\omega \omega'))$
C-Dδ3	If $\|A\|^{M,\omega,\tau} \neq \emptyset$ then $\{\omega' \in \|A\|^{M,\omega,\tau} : (\forall\omega'' \in \|A\|^{M,\omega,\tau})\ \omega' \geq_\omega \omega''\} \neq \emptyset$
C-Dδ4	$\forall\omega\forall\omega'\forall\omega''\forall\omega'''((\omega' \geq_\omega \omega'' \wedge \omega'' \geq_\omega \omega''') \to \omega' \geq_\omega \omega''')$
C-Dδ5	$\forall\omega\forall\omega'\omega \geq_{\omega'} \omega$
C-Dδ6	$\forall\omega\forall\omega'\forall\omega''\forall\omega'''(\omega \geq_{\omega''} \omega' \to \omega \geq_{\omega'''} \omega')$
C-Dδ7	$\forall\omega\forall\omega'\forall\omega''((\omega' \geq_\omega \omega'' \wedge \omega'' \geq_\omega \omega') \to \omega' = \omega'')$

Table 5. Conditions concerning $\geq$

conditions introduced by [43]. However, in this paper we use a ternary instead of a binary preference relation. *C-Dδ6* and *C-Dδ7* have no counterparts in [43].

We have already mentioned the condition *C-Dγ0* (see Section 3.1.2). *C-Dγ0* entails *C-DMO*, *C-Dα0*, *C-Dα1* and *C-Dα2* (see Theorem 1 below). In any *C-Dγ0*-model, the following schemas are valid: $\Box A \to \mathbf{O}[B]A$, $\Box(A \leftrightarrow B) \to (\mathbf{O}[A]C \leftrightarrow \mathbf{O}[B]C)$, $\mathbf{O}[A]A$, $\Box(A \to B) \to \mathbf{O}[A]B$, and $\mathbf{O}[A \wedge B]C \to \mathbf{O}[A](B \to C)$.

Intuitively, *C-Dδ2* means that $\geq_\omega$ is "complete" (strongly connected, total), i.e., world ω' is at least as good as world ω'' (relative to world ω) or ω'' is at least as good as ω' (relative to ω) (or ω' and ω'' are equally good (relative to ω)). $\forall\omega\forall\omega'\forall\omega''((\omega' \geq_\omega \omega'' \vee \omega'' \geq_\omega \omega') \vee (\omega' \geq_\omega \omega'' \wedge \omega'' \geq_\omega \omega'))$ is equivalent to $\forall\omega\forall\omega'\forall\omega''(\omega' \geq_\omega \omega'' \vee \omega'' \geq_\omega \omega')$.

Condition *C-Dδ3* contains some free variables. These are supposed to be implicitly bound by universal quantifiers. So, if the condition holds, it holds for all ω, ω', τ, etc. It follows that if the condition is true, then $\forall\omega'\forall\tau((\exists\omega R\omega'\omega\tau \wedge M,\omega,\tau \Vdash A) \to \exists\omega''(R\omega'\omega''\tau \wedge M,\omega'',\tau \Vdash A \wedge \forall\omega'''((R\omega'\omega'''\tau \wedge M,\omega''',\tau \Vdash A) \to \omega'' \geq_{\omega'} \omega''')))$. Roughly, this condition says that if A is true in at least one alethically accessible world, then A is true in at least one of the best alethically accessible worlds (in which A is true). The assumption prohibits that there are no best alethically accessible A-worlds, only an infinite sequence of better and better alethically accessible A-worlds (given that there is at least one alethically accessible A-world), so to speak. *C-Dγ0* and *C-Dδ3* entail *C-Dα3* (see Theorem 1). $\Diamond B \to (\mathbf{O}[B]A \to \mathbf{P}[B]A)$ is valid in any *γ0δ3*-model.

According to *C-Dδ4*, $\geq_\omega$ is "transitive", i.e., if world ω' is at least as good as world ω'' (relative to world ω) and ω'' is at least as good as world ω''' (relative to ω), then ω' is at least as good as ω''' (relative to ω).

Intuitively, *C-Dδ5* says that $\geq_{\omega'}$ is "reflexive", i.e., every world ω is at least as good as itself (relative to any world ω').

According to the condition *C-Dδ6*, the possible worlds are ordered in the same way in every possible world. More precisely, the condition says that for all possible worlds ω, ω', ω'' and ω''': if ω is at least as good as ω' (according to ω''), then ω is at least as good as ω' (according to ω'''). In one sense we can say that this condition entails that the ordering of possible worlds is not relative to possible worlds. For if ω is at least as good as ω' in some world, then ω is at least as good as ω' in any world according to *C-Dδ6*. *C-Dγ0*, *C-Dδ6*, *C-a4* and *C-a5* entail *C-ad4* and *C-ad5* (see Theorem 1). *C-a4* and *C-a5* are introduced by [44]. *C-a4* says that R is transitive and *C-a5* says that R is Euclidean. $\mathbf{O}[B]A \to \Box\mathbf{O}[B]A$, $\mathbf{F}[B]A \to \Box\mathbf{F}[B]A$ and $\mathbf{P}[B]A \to \Box\mathbf{P}[B]A$ are valid in any model that satisfies *C-Dγ0*, *C-Dδ6*, *C-a4* and *C-a5*.

Intuitively *C-Dδ7* says that $\geq_\omega$ is "antisymmetric". If world ω' is at least as good as world ω'' (relative to the world ω) and ω'' is at least as good as ω' (relative to ω), then ω' is identical to ω'', i.e., there are no two distinct worlds that are equally good relative to a possible world ω. *C-Dγ0* and *C-Dδ7* entail *C-Dd7*. $\mathbf{O}[B]A \vee \mathbf{F}[B]A$, $\mathbf{O}[B]A \vee \mathbf{O}[B]\neg A$, and $\mathbf{P}[B]A \to \mathbf{O}[B]A$ are valid in every model that satisfies *C-Dγ0* and *C-Dδ7*.

3.3. Classification of model classes and the logic of a class of models

The conditions on our models listed in tables 1–5 can be used to obtain a categorisation of the set of all models into various kinds. We shall say that $\mathcal{M}(C_1, \ldots, C_n)$ is the class of all models that satisfies the conditions

$C_1, \ldots, C_n$. E.g. $\mathcal{M}(C\text{-}Dd4, C\text{-}DMO, C\text{-}D\alpha 1)$ is the class of all models that satisfies *C-Dd4*, *C-DMO*, and *C-Dα1*. If it is clear from the context that we are speaking about models, we can abbreviate these expressions by omitting redundant letters. A *γ0δ6a45*-model is, for instance, a model that satisfies *C-Dγ0*, *C-Dδ6*, *C-a4*, and *C-a5*.

We shall say that a **T**-model is a model that satisfies all conditions in tables 1–5, and whose alethic accessibility relation is an equivalence relation (at every moment in time), and whose temporal accessibility relation is transitive and comparable and does not branch towards the future or the past. The tableau system **T** is sound with respect to the class of all **T**-models (see Section 4.3 and Soundness Theorem II).

The set of all sentences (in L) that are valid in a class of models $\mathcal{M}$ is called the logical system of (the system of or the logic of) $\mathcal{M}$, in symbols $S(\mathcal{M}) = \{A \in L : \mathcal{M} \Vdash A\}$. E.g. $S(\mathcal{M}(C\text{-}Dd4, C\text{-}DMO, C\text{-}D\alpha 1))$ is the set of all sentences that are valid in the class of all models that satisfies *C-Dd4*, *C-DMO*, and *C-Dα1*.

The following theorem tells us something about the relations between the conditions introduced in Section 3.2.

Theorem 1. 1. *Every supplemented γ0-model satisfies the conditions DMO, α0, α1, and α2.*
2. *Every supplemented γ0δ3-model satisfies α3.*
3. *Every supplemented γ0δ4-model satisfies α4.*
4. *Every supplemented γ0δ6a45-model satisfies ad4 and ad5.*
5. *Every supplemented γ0δ7-model satisfies Dd7.*

Proof. In the following proof, "CL" means that the step is valid in "classical logic".

Ad 1. For *DMO*: Trivial.

For *α0*: Suppose that $||A||^{M,\omega,\tau} = ||B||^{M,\omega,\tau}$ and $S_A\omega\omega'\tau$. Then $R\omega\omega'\tau \wedge M, \omega', \tau \Vdash A \wedge \forall\omega''(R\omega\omega''\tau \wedge M, \omega'', \tau \Vdash A \to \omega' \geq_\omega \omega'')$, by *C-Dγ0*. Hence, by CL and the definition of $||A||^{M,\omega,\tau}$, we have $R\omega\omega'\tau \wedge M, \omega', \tau \Vdash B \wedge \forall\omega''(R\omega\omega''\tau \wedge M, \omega'', \tau \Vdash B \to \omega' \geq_\omega \omega'')$. So $S_B\omega\omega'\tau$, by *C-Dγ0*. Thus, by CL, $S_A\omega\omega'\tau \to S_B\omega\omega'\tau$; $S_B\omega\omega'\tau \to S_A\omega\omega'\tau$; $S_A\omega\omega'\tau \leftrightarrow S_B\omega\omega'\tau$. So if $||A||^{M,\omega,\tau} = ||B||^{M,\omega,\tau}$ then $S_A\omega\omega'\tau \leftrightarrow S_B\omega\omega'\tau$.

For *α1*: Assume that $S_A\omega\omega'\tau$. Then $R\omega\omega'\tau \wedge M, \omega', \tau \Vdash A \wedge \forall\omega''(R\omega\omega''\tau \wedge M, \omega'', \tau \Vdash A \to \omega' \geq_\omega \omega'')$, by *C-Dγ0*. Hence $S_A\omega\omega'\tau \to M, \omega', \tau \Vdash A$, by CL. So $\forall\tau\forall\omega\forall\omega'(S_A\omega\omega'\tau \to M, \omega', \tau \Vdash A)$, by CL.

For *α2*: Assume that $S_A\omega'\omega\tau$, $M, \omega, \tau \Vdash B$, and not-$S_{A\wedge B}\omega'\omega\tau$. Then $R\omega'\omega\tau$ and $M, \omega, \tau \Vdash A$, by *C-Dγ0*. So $M, \omega, \tau \Vdash A \wedge B$, by CL.

Hence not-($R\omega'\omega\tau \wedge M, \omega, \tau \Vdash A \wedge B \wedge \forall\omega''(R\omega'\omega''\tau \wedge M, \omega'', \tau \Vdash A \wedge B \rightarrow \omega \geq_{\omega'} \omega'')$), by *C-Dγ0*. So $\exists\omega''(R\omega'\omega''\tau \wedge M, \omega'', \tau \Vdash A \wedge B \wedge \neg(\omega \geq_{\omega'} \omega''))$; $R\omega'\omega''\tau \wedge M, \omega'', \tau \Vdash A \wedge B \wedge \neg(\omega \geq_{\omega'} \omega'')$; and $R\omega'\omega''\tau \wedge M, \omega'', \tau \Vdash A$. Thus, by *C-Dγ0*, we obtain $R\omega'\omega''\tau \wedge M, \omega'', \tau \Vdash A \rightarrow \omega \geq_{\omega'} \omega''$. So $\omega \geq_{\omega'} \omega''$. Accordingly $\perp$. So $(S_A\omega'\omega\tau \wedge M, \omega, \tau \Vdash B) \rightarrow S_{A\wedge B}\omega'\omega\tau$. Hence $\forall\tau\forall\omega\forall\omega'((S_A\omega\omega'\tau \wedge M, \omega', \tau \Vdash B) \rightarrow S_{A\wedge B}\omega\omega'\tau)$, by CL.

Ad 2. For *α3*: Assume that $\exists\omega'(R\omega\omega'\tau \wedge M, \omega', \tau \Vdash A)$. Then $\|A\|^{M,\omega,\tau} \neq \emptyset$, by the definition of $\|A\|^{M,\omega,\tau}$. So $\{\omega' \in \|A\|^{M,\omega,\tau} : (\forall\omega'' \in \|A\|^{M,\omega,\tau})\ \omega' \geq_{\omega} \omega''\} \neq \emptyset$, by *C-Dδ3* and CL. Hence $\exists\omega'(R\omega\omega'\tau$ and $M, \omega', \tau \Vdash A$ and $\forall\omega''((R\omega\omega''\tau$ and $M, \omega'', \tau \Vdash A) \rightarrow \omega' \geq_{\omega} \omega'')$. Thus, by *C-Dγ0* and CL, $\exists\omega'' S_A\omega\omega''\tau$; $\exists\omega'(R\omega\omega'\tau \wedge M, \omega', \tau \Vdash A) \rightarrow \exists\omega'' S_A\omega\omega''\tau$; and $\forall\tau\forall\omega(\exists\omega'(R\omega\omega'\tau \wedge M, \omega', \tau \Vdash A) \rightarrow \exists\omega'' S_A\omega\omega''\tau)$.

Ad 3. For *α4*: Note that $\forall\tau\forall\omega\forall\omega'\forall\omega''((S_A\omega\omega'\tau \wedge M, \omega', \tau \Vdash B) \rightarrow (S_{A\wedge B}\omega\omega''\tau \rightarrow (S_A\omega\omega''\tau \wedge M, \omega'', \tau \Vdash B)))$ is equivalent to $\forall\tau\forall\omega\forall\omega'$ $(\exists\omega''(S_A\omega'\omega''\tau \wedge M, \omega'', \tau \Vdash B) \rightarrow (S_{A\wedge B}\omega'\omega\tau \rightarrow (S_A\omega'\omega\tau \wedge M, \omega, \tau \Vdash B)))$. So to prove the former it suffices to prove the latter.

Suppose that $\exists\omega''(S_A\omega'''\omega''\tau \wedge M, \omega'', \tau \Vdash B)$, $S_{A\wedge B}\omega'''\omega\tau$, and not-$(S_A\omega'''\omega\tau \wedge M, \omega, \tau \Vdash B)$. Then $R\omega'''\omega\tau \wedge M, \omega, \tau \Vdash A \wedge B \wedge \forall\omega''(R\omega'''\omega''\tau \wedge M, \omega'', \tau \Vdash A \wedge B \rightarrow \omega \geq_{\omega'''} \omega'')$, by *C-Dγ0* and CL. So $R\omega'''\omega\tau \wedge M, \omega, \tau \Vdash B$, and not-$S_A\omega'''\omega\tau$ or not $M, \omega, \tau \Vdash B$, by CL. Hence not-$S_A\omega'''\omega\tau$. Therefore also not-$(R\omega'''\omega\tau \wedge M, \omega, \tau \Vdash A \wedge \forall\omega''(R\omega'''\omega''\tau \wedge M, \omega'', \tau \Vdash A \rightarrow \omega \geq_{\omega'''} \omega''))$, by *C-Dγ0* and CL. Moreover, $R\omega'''\omega\tau \wedge M, \omega, \tau \Vdash A$. So not-$\forall\omega''(R\omega'''\omega''\tau \wedge M, \omega'', \tau \Vdash A \rightarrow \omega \geq_{\omega'''} \omega'')$ and $\exists\omega''(R\omega'''\omega''\tau \wedge M, \omega'', \tau \Vdash A \wedge \neg(\omega \geq_{\omega'''} \omega''))$. Hence $R\omega'''\omega'\tau \wedge M, \omega', \tau \Vdash A \wedge \neg(\omega \geq_{\omega'''} \omega')$. Thus, we obtain $S_A\omega'''\omega''''\tau \wedge M, \omega'''', \tau \Vdash B$. Consequently $R\omega'''\omega''''\tau \wedge M, \omega'''', \tau \Vdash A \wedge B$, by *C-Dγ0* and CL. So $\omega \geq_{\omega'''} \omega''''$. Moreover, $\forall\omega'''''(R\omega'''\omega'''''\tau \wedge M, \omega''''', \tau \Vdash A \rightarrow \omega'''' \geq_{\omega'''} \omega''''')$, by *C-Dγ0* and CL. So $\omega'''' \geq_{\omega'''} \omega'$. Hence $\omega \geq_{\omega'''} \omega'$, by *C-Dδ4*. So we obtain $\perp$. Thus, $\forall\tau\forall\omega\forall\omega'(\exists\omega''(S_A\omega'\omega''\tau \wedge M, \omega'', \tau \Vdash B) \rightarrow (S_{A\wedge B}\omega'\omega\tau \rightarrow (S_A\omega'\omega\tau \wedge M, \omega, \tau \Vdash B)))$.

Ad 4. For *C-Dad4*: Assume that $R\omega\omega'\tau$, and $S_A\omega'\omega''\tau$, and not-$S_A\omega\omega''\tau$. Then $R\omega'\omega''\tau \wedge M, \omega'', \tau \Vdash A \wedge \forall\omega'''(R\omega'\omega'''\tau \wedge M, \omega''', \tau \Vdash A \rightarrow \omega'' \geq_{\omega'} \omega''')$, by *C-Dγ0*. Hence $R\omega\omega''\tau$, by *C-a4*. So $R\omega\omega''\tau$ and $M, \omega'', \tau \Vdash A$. Moreover, $\forall\omega'''(R\omega'\omega'''\tau \wedge M, \omega''', \tau \Vdash A \rightarrow \omega'' \geq_{\omega'} \omega''')$ and not-$(R\omega\omega''\tau \wedge M, \omega'', \tau \Vdash A \wedge \forall\omega'''(R\omega\omega'''\tau \wedge M, \omega''', \tau \Vdash A \rightarrow \omega'' \geq_{\omega} \omega'''))$, by assumption, the definition of $S_A\omega\omega''\tau$, and CL. So $\exists\omega'''(R\omega\omega'''\tau \wedge M, \omega''', \tau \Vdash A \wedge \neg(\omega'' \geq_{\omega} \omega'''))$. Hence $R\omega\omega'''\tau \wedge M, \omega''', \tau \Vdash A \wedge \neg(\omega'' \geq_{\omega} \omega''')$. Moreover, $R\omega'\omega'''\tau \wedge M, \omega''', \tau \Vdash A \rightarrow \omega'' \geq_{\omega'} \omega'''$. But $(R\omega\omega'\tau \wedge R\omega\omega'''\tau) \rightarrow R\omega'\omega'''\tau$, by *C-a5* and CL. So $R\omega'\omega'''\tau$,

$R\omega'\omega'''\tau$ and $M, \omega''', \tau \Vdash A$. Hence $\omega'' \geq_{\omega'} \omega'''$. Accordingly, $\omega'' \geq_{\omega'} \omega''' \to \omega'' \geq_{\omega} \omega'''$, by *C-Dδ6* and CL. Hence $\omega'' \geq_{\omega} \omega'''$. So we obtain $\bot$. Thus, $S_A\omega\omega''\tau$. In conclusion, $(R\omega\omega'\tau \wedge S_A\omega'\omega''\tau) \to S_A\omega\omega''\tau$ and $\forall\tau\forall\omega\forall\omega'\forall\omega''((R\omega\omega'\tau \wedge S_A\omega'\omega''\tau) \to S_A\omega\omega''\tau)$.

For *C-Dad5*: Similarly.

Ad 5. For *C-Dd7*: Suppose that $S_A\omega\omega'\tau$ and $S_A\omega\omega''\tau$. By *C-Dγ0*, $S_A\omega\omega'\tau$ iff $R\omega\omega'\tau$, $M, \omega', \tau \Vdash A$ and $\forall\omega''(R\omega\omega''\tau$ and $M, \omega'', \tau \Vdash A \to \omega' \geq_{\omega} \omega'')$. Moreover, by *C-Dγ0*, $S_A\omega\omega''\tau$ iff $R\omega\omega''\tau$ and $M, \omega'', \tau \Vdash A$ and $\forall\omega'''(R\omega\omega'''\tau \wedge M, \omega''', \tau \Vdash A \to \omega'' \geq_{\omega} \omega''')$. So $R\omega\omega'\tau$ and $M, \omega', \tau \Vdash A$, and $\forall\omega''(R\omega\omega''\tau \wedge M, \omega'', \tau \Vdash A \to \omega' \geq_{\omega} \omega'')$. Hence $R\omega\omega''\tau$ and $M, \omega'', \tau \Vdash A$, and $\forall\omega'''(R\omega\omega'''\tau \wedge M, \omega''', \tau \Vdash A \to \omega'' \geq_{\omega} \omega''')$. Therefore, $R\omega\omega'\tau \wedge M, \omega', \tau \Vdash A$. Moreover, $\forall\omega''(R\omega\omega''\tau \wedge M, \omega'', \tau \Vdash A \to \omega' \geq_{\omega} \omega'')$. Hence $R\omega\omega''\tau \wedge M, \omega'', \tau \Vdash A$. But $\forall\omega'''(R\omega\omega'''\tau \wedge M, \omega''', \tau \Vdash A \to \omega'' \geq_{\omega} \omega''')$ and $R\omega\omega''\tau \wedge M, \omega'', \tau \Vdash A \to \omega' \geq_{\omega} \omega''$. So $R\omega\omega'\tau \wedge M, \omega', \tau \Vdash A \to \omega'' \geq_{\omega} \omega'$, $\omega' \geq_{\omega} \omega''$, and $\omega'' \geq_{\omega} \omega'$. But, by *C-Dδ7*, we have $(\omega' \geq_{\omega} \omega'' \wedge \omega'' \geq_{\omega} \omega') \to \omega' = \omega''$. So $\omega' = \omega''$. In conclusion, $\forall\tau\forall\omega\forall\omega'\forall\omega''((S_A\omega\omega'\tau \wedge S_A\omega\omega''\tau) \to \omega' = \omega'')$. ⊣

4. Proof theory

4.1. Semantic tableaux

In this section, we describe a set of tableau systems. The propositional part is similar to systems introduced by Raymond Smullyan [47] and Richard Jeffrey [31]. The alethic modal part is inspired by e.g. Melvin Fitting and Graham Priest [24, 25, 38]. The concepts of semantic tableau, branch, open and closed branch, etc., are essentially defined as in [44, 46, 38]. For more information on semantic tableaux, see D'Agostino, Gabbay, Hähnle and Posegga [19].

4.2. Tableau rules

Many rules that we use in temporal alethic dyadic deontic logic are described by [44] and [46]. In addition, we will consider some new tableau rules for the dyadic deontic operators and some new rules that "correspond to" the conditions imposed on the dyadic deontic accessibility relations that were introduced in Section 3.2.

We use the same propositional rules, basic alethic and temporal rules, and alethic and temporal accessibility rules as in [44] and [46].

D O-pos (**O**)	D P-pos (**P**)	D F-pos (**F**)
$\mathbf{O}[A]B, w_i t_j$ $s_A w_i w_k t_j$ $\downarrow$ $B, w_k t_j$	$\mathbf{P}[A]B, w_i t_j$ $\downarrow$ $s_A w_i w_k t_j$ $B, w_k t_j$ where w_k is new	$\mathbf{F}[A]B, w_i t_j$ $\downarrow$ $\mathbf{O}[A]\neg B, w_i t_j$
D O-neg ($\neg$**O**)	D P-neg ($\neg$**P**)	D F-neg ($\neg$**F**)
$\neg\mathbf{O}[A]B, w_i t_j$ $\downarrow$ $\mathbf{P}[A]\neg B, w_i t_j$	$\neg\mathbf{P}[A]B, w_i t_j$ $\downarrow$ $\mathbf{O}[A]\neg B, w_i t_j$	$\neg\mathbf{F}[A]B, w_i t_j$ $\downarrow$ $\mathbf{P}[A]B, w_i t_j$

Table 6. Basic dyadic deontic rules (b dd-rules)

CUT	TId(I)	TId(II)	AId(I)	AId(II)
$*$ $\swarrow \quad \searrow$ $\neg A, w_i t_j \qquad A, w_i t_j$	$A(t_i)$ $t_i = t_j$ $\downarrow$ $A(t_j)$	$A(t_i)$ $t_j = t_i$ $\downarrow$ $A(t_j)$	$A(w_i)$ $w_i = w_j$ $\downarrow$ $A(w_j)$	$A(w_i)$ $w_j = w_i$ $\downarrow$ $A(w_j)$

Table 7. CUT, TId(I), TId(II), AId(I) and AId(II)

We will include CUT and all the identity rules, TId(I), TId(II), AId(I), and AId(II), in every tableau system. However, in many systems these rules are redundant. "TId(I)" stands for "Temporal Identity I", "AId(I)" stands for "Alethic Identity I", etc. Intuitively "$t_i = t_j$" says that the temporal point t_i is identical to the temporal point t_j; and "$w_i = w_j$" says that the possible world w_i is identical to the possible world w_j. $A(t_j)$ is exactly as $A(t_i)$ except that "t_i" has been replaced by "t_j", etc.[5]

4.3. Tableau systems

A tableau system is a set of tableau rules. A temporal alethic dyadic deontic tableau system includes all propositional rules, all basic alethic rules, all basic dyadic deontic rules, all basic temporal rules (including the rules for A and S; see [46]), CUT and all identity rules (tables 6–7). The minimal temporal alethic dyadic deontic tableau system is called "T". By adding any subset of the rules introduced in tables 8–11, or various accessibility rules introduced by [44] or [46],

[5] We need TId(I) and TId(II) if we add some temporal accessibility rules, for example T-FC, introduced by [44] or [46], to our systems.

T-Dd4	*T-Dd4′*	*T-Dd5*	*T-Dd5′*
$s_A w_i w_j t_l$ $s_A w_j w_k t_l$ $\downarrow$ $s_A w_i w_k t_l$	$s_A w_i w_j t_l$ $s_B w_j w_k t_l$ $\downarrow$ $s_B w_i w_k t_l$	$s_A w_i w_j t_l$ $s_A w_i w_k t_l$ $\downarrow$ $s_A w_j w_k t_l$	$s_A w_i w_j t_l$ $s_B w_i w_k t_l$ $\downarrow$ $s_B w_j w_k t_l$

T-DdT′	*T-DdB′*	*T-Dd7*
$s_A w_i w_j t_l$ $\downarrow$ $s_A w_j w_j t_l$	$s_A w_i w_j t_l$ $s_A w_j w_k t_l$ $\downarrow$ $s_A w_k w_j t_l$	$s_A w_i w_j t_l$ $s_A w_i w_k t_l$ $\downarrow$ $w_j = w_k$

Table 8. Dyadic deontic accessibility rules (dd-rules)

T-DMO	*T-Dad4*	*T-Dad5*
$s_A w_i w_j t_k$ $\downarrow$ $r w_i w_j t_k$	$r w_i w_j t_l$ $s_A w_j w_k t_l$ $\downarrow$ $s_A w_i w_k t_l$	$r w_i w_j t_l$ $s_A w_i w_k t_l$ $\downarrow$ $s_A w_j w_k t_l$

Table 9. Alethic dyadic deontic accessibility rules (add-rules)

T-Dα0	*T-Dα1*	*T-Dα2*	*T-Dα3*	*T-Dα4*
If D is of the form $\Box(A \leftrightarrow B) \rightarrow$ $(\mathbf{O}[A]C \leftrightarrow \mathbf{O}[B]C)$, $D, w_i t_j$ can be added to any open branch on which w_i and t_j occur	$s_A w_i w_j t_k$ $\downarrow$ $A, w_j t_k$	$s_A w_i w_j t_k$ $B, w_j t_k$ $\downarrow$ $s_{A \wedge B} w_i w_j t_k$	$r w_i w_j t_l$ $A, w_j t_l$ $\downarrow$ $s_A w_i w_k t_l$ where w_k is new	$s_A w_i w_j t_l$ $B, w_j t_l$ $s_{A \wedge B} w_i w_k t_l$ $\downarrow$ $s_A w_i w_k t_l$ $B, w_k t_l$

Table 10. Rules concerning R, S_A and V

T-FT	*T-BT*	*T-SP*
$A, w_i t_k$ $r w_i w_j t_k$ $\downarrow$ $A, w_j t_k$ where A is atomic	$A, w_j t_k$ $r w_i w_j t_k$ $\downarrow$ $A, w_i t_k$ where A is atomic	$r w_i w_j t_l$ $t_k < t_l$ $\downarrow$ $r w_i w_j t_k$

Table 11. Rules concerning R, $<$ and V (adt-rules)

we obtain an extension of T (note that some of these are deductively equivalent). We use the following conventions for naming systems. We write "$aA_1 \ldots A_i dB_1 \ldots B_j \alpha C_1 \ldots C_k adD_1 \ldots D_l tE_1 \ldots E_m adtF_1 \ldots F_n$", where $A_1 \ldots A_i$ is a list (possibly empty) of a-rules (see [44]), $B_1 \ldots B_j$ is a list (possibly empty) of dd-rules, $C_1 \ldots C_k$ is a list (possibly empty) of α-rules, $D_1 \ldots D_l$ is a list (possibly empty) of add-rules, $E_1 \ldots E_m$ is a list (possibly empty) of t-rules (see [44, 46]), and $F_1 \ldots F_n$ is a list (possibly empty) of adt-rules. We sometimes abbreviate by omitting "redundant" letters in a name, if it does not lead to any ambiguity. E.g. *aTd4′5′α12adDMOt4adtSP* is the temporal alethic dyadic deontic system that includes the rules *T-aT*, *T-Dd4′*, *T-Dd5′*, *T-Dα1*, *T-Dα2*, *T-DMO*, *T-t4*, and *T-SP*. Let us call the system that includes all basic rules, all a-rules, and all rules in tables 8–11 except *T-Dd7* Strong temporal alethic dyadic deontic logic or **STADDL**. We shall call the system that includes all rules in tables 6–11, all alethic rules that correspond to the fact that the alethic accessibility relation is an equivalence relation and the temporal rules that correspond to the fact that the temporal accessibility relation is transitive and comparable and does not branch towards the future or the past **T**. **T** is sound with respect to the class of all **T**-models (see Section 3.3 and Soundness Theorem II).[6]

4.4. Some proof-theoretical concepts

The concepts of proof, theorem, derivation, consistency, inconsistency in a system, etc., are defined as in [44] and [46]. Let S be a tableau system. Then the logic (or the logical system) of S, $\mathcal{L}(S)$, is the set of all sentences (in L) that are provable in S, in symbols $\mathcal{L}(S) = \{A \in L : \vdash_S A\}$. E.g. $\mathcal{L}$(*aTdT′α123t4*) is the set of all sentences that are provable in the system *aTdT′α123t4*, i.e., in the system that includes all the basic rules and the (non-basic) rules *T-aT*, *T-DdT′*, *T-Dα1*, *T-Dα2*, *T-Dα3* and *T-t4* [44].

5. Examples of theorems

In this section, we will consider some examples of theorems in some systems. The proofs are usually straightforward and are left to the reader. We will say that a schema is a theorem (in a system S) iff every instance of this schema is a theorem (in S).

[6] **STADDL** includes some redundant rules and there are several systems with fewer primitive rules that are deductively equivalent. The same is true of **T**. However, we will not say anything more about this in the present paper.

Name	Theorem	System
DK	$\mathbf{O}[r](p \to q) \to (\mathbf{O}[r]p \to \mathbf{O}[r]q)$	*T*
TDd4	$\mathbf{O}[r]p \to \mathbf{O}[r]\mathbf{O}[r]p$	*Dd4*
TDd4′	$\mathbf{O}[r]p \to \mathbf{O}[s]\mathbf{O}[r]p$	*Dd4′*
TDd5	$\mathbf{P}[r]p \to \mathbf{O}[r]\mathbf{P}[r]p$	*Dd5*
TDd5′	$\mathbf{P}[r]p \to \mathbf{O}[s]\mathbf{P}[r]p$	*Dd5′*
TDdT′	$\mathbf{O}[r](\mathbf{O}[r]p \to p)$	*DdT′*
TDdB′	$\mathbf{O}[r](p \to \mathbf{O}[r]\mathbf{P}[r]p)$	*DdB′*
—	$\mathbf{O}[r](\mathbf{P}[r]\mathbf{O}[r]p \to p)$	*DdB′*
TDd7	$\mathbf{P}[q]p \to \mathbf{O}[q]p$	*Dd7*
—	$\mathbf{O}[q]p \vee \mathbf{F}[q]p$	*Dd7*
—	$\mathbf{O}[q]p \vee \mathbf{O}[q]\neg p$	*Dd7*
—	$\mathbf{O}[p](q \vee r) \to (\mathbf{O}[p]q \vee \mathbf{O}[p]r)$	*Dd7*

Table 12. Examples of theorems

Name	Theorem	System
TDMO	$\Box q \to \mathbf{O}[p]q$	*DMO*
—	$\mathbf{P}[p]q \to \Diamond q$	*DMO*
TDad4	$\mathbf{O}[p]q \to \Box\mathbf{O}[p]q$	*Dad4*
—	$\mathbf{F}[p]q \to \Box\mathbf{F}[p]q$	*Dad4*
TDad5	$\mathbf{P}[p]q \to \Box\mathbf{P}[p]q$	*Dad5*
Tα0	$\Box(p \leftrightarrow q) \to (\mathbf{O}[p]r \leftrightarrow \mathbf{O}[q]r)$	*Dα0*
—	$\Box(p \leftrightarrow q) \to (\mathbf{P}[p]r \leftrightarrow \mathbf{P}[q]r)$	*Dα0*
—	$\Box(p \leftrightarrow q) \to (\mathbf{F}[p]r \leftrightarrow \mathbf{F}[q]r)$	*Dα0*
TDα1	$\mathbf{O}[p]p$	*Dα1*
—	$\Box(p \to q) \to \mathbf{O}[p]q$	*Dα1DMO*
TDα2	$\mathbf{O}[p \wedge q]r \to \mathbf{O}[p](q \to r)$	*Dα2*
TDα3	$\Diamond p \to (\mathbf{O}[p]q \to \mathbf{P}[p]q)$	*Dα3*
TDα4	$\mathbf{P}[p]q \to (\mathbf{O}[p](q \to r) \to \mathbf{O}[p \wedge q]r)$	*Dα4*

Table 13. Examples of theorems

Theorem 2. *The sentences in tables 12 and 13 are theorems in the indicated systems. E.g. TDMO is a theorem in every system that includes T-DMO, TDd4′ in every system that includes T-Dd4′ and TDα1 in every system that includes T-Dα1, etc.*

Several theorems that can be proved in dyadic deontic systems are mentioned by [43]. Many of these can be proved in the "corresponding" temporal alethic dyadic deontic systems in this paper. However, since our systems do not have any counterpart of the tableau rule *Ta6*, and since *T-Dα3* is somewhat different from *Tα3*, the systems in this paper do not "match" the systems in [43] perfectly. Furthermore, note that

Theorems
$\Box p \to (\mathbf{O}[p]q \leftrightarrow \mathbf{O}(p \to q))$
$\Box p \to (\mathbf{O}(p \to q) \leftrightarrow (p \to \mathbf{O}q))$
$\Box p \to (\mathbf{O}[p]q \leftrightarrow (p \to \mathbf{O}q))$
$\Box \neg p \to (\mathbf{O}[p]q \leftrightarrow \mathbf{O}(p \to q))$
$\Box \neg p \to (\mathbf{O}(p \to q) \leftrightarrow (p \to \mathbf{O}q))$
$\Box \neg p \to (\mathbf{O}[p]q \leftrightarrow (p \to \mathbf{O}q))$
$\triangle\, p \to (\mathbf{O}[p]q \leftrightarrow \mathbf{O}(p \to q))$
$\triangle\, p \to (\mathbf{O}(p \to q) \leftrightarrow (p \to \mathbf{O}q))$
$\triangle\, p \to (\mathbf{O}[p]q \leftrightarrow (p \to \mathbf{O}q))$

Table 14. Examples of theorems

the modal operators $\Box$, $\Diamond$, and $\diamondminus$ are interpreted as so-called universal modalities in the dyadic deontic systems introduced by [43], while this is not the case in our systems in this essay. In this paper, **U** and **M** represent absolute necessity and possibility, respectively, while $\Box$, $\Diamond$, and $\diamondminus$ stand for historical necessity, possibility, and impossibility, respectively.

Theorem 3. 1. *If a system S includes T-a4, T-a5* (see [44]), *T-DMO, T-Dad4, T-Dad5, T-Dα1, and T-Dα3, then the following schemas are theorems in S:* $\mathbf{O}'[B]A \to \Box\mathbf{O}'[B]A$, $\mathbf{P}'[B]A \to \Box\mathbf{P}'[B]A$, *and* $\mathbf{F}'[B]A \to \Box\mathbf{F}'[B]A$.
2. *Every theorem mentioned by* [43] *is a theorem in* ***STADDL***.

Note that not all instances of the following schemas are theorems (not even in our strongest system): $\mathbf{O}[A]B \leftrightarrow \mathbf{O}(A \to B)$, $\mathbf{O}(A \to B) \leftrightarrow (A \to \mathbf{O}B)$, $\mathbf{O}[A]B \leftrightarrow (A \to \mathbf{O}B)$. However, we do have the following interesting result.

Theorem 4. *Let S be a dyadic deontic tableau system that includes the rules T-DMO, T-Dα0, T-Dα2, T-Dα3, T-Dα4, T-FT, and T-BT. Then all sentences in Table 14 are theorems in S.*

So, when p is non-contingent, i.e., when either p or $\neg p$ is historically necessary, all the conditional obligations $\mathbf{O}[p]q$, $\mathbf{O}(p \to q)$ and $p \to \mathbf{O}q$ are equivalent. Furthermore, note that not every instance of $(A \wedge \mathbf{O}[A]B) \to \mathbf{O}B$ $[(A \wedge \mathbf{O}(A \to B)) \to \mathbf{O}B]$ is valid. Hence, we cannot always detach an unconditional obligation from the conditional obligation $\mathbf{O}[A]B$ $[\mathbf{O}(A \to B)]$ and the condition A. In other words, it is not true that if it ought to be the case that B given A and A, then it ought to be that B, for every A and B. However, if the condition is historically necessary, we can detach an unconditional obligation, at least in some systems. The following schemas can be proved in **STADDL** (and also in some weaker systems): $(\Box A \wedge \mathbf{O}[A]B) \to \mathbf{O}B$ and $(\Box A \wedge \mathbf{O}(A \to B)) \to \mathbf{O}B$. $(A \wedge (A \to \mathbf{O}B)) \to \mathbf{O}B$ is provable in every system, and $(\Box A \wedge (A \to \mathbf{O}B)) \to \mathbf{O}B$ is provable in every system that includes the tableau rule T-aT [44].

Let us say that A is "non-future" iff A does not contain any operator of the form G, F, S, A, or $\mathsf{R}t$ (at least if $v(t)$ is a time later than the time of the valuation of the sentence). Now let A be non-future. Then $\Box A$ and $\triangle A$ are theorems in **STADDL** (and every extension of this system). Furthermore, if A is non-future, then all of the following conditional obligations are equivalent in **STADDL**: $\mathbf{O}[A]B$, $\mathbf{O}(A \to B)$ and $A \to \mathbf{O}B$. Since $(\Box A \wedge \mathbf{O}[A]B) \to \mathbf{O}B$, it follows that we can detach the unconditional obligation $\mathbf{O}B$ in **STADDL** from $\mathbf{O}[A]B$ and A, if A is non-future. The same is also true for conditional obligations of the forms $\mathbf{O}(A \to B)$ and $A \to \mathbf{O}B$.

These facts may shed some light on the so-called dilemma of commitment and detachment (see [51, chapters II and IV] and Reason 12 in Section 6.3 below).

6. Contrary-to-duty obligations and the contrary-to-duty (obligation) paradox

In this section, we will briefly describe the so-called contrary-to-duty (obligation) paradox (Section 6.1). Then we will show how we can use our formal systems to solve this puzzle (Section 6.2). Finally, we consider some reasons why we think this solution is attractive (Section 6.3).

A contrary-to-duty obligation is an obligation telling us what ought to be the case if something forbidden is true. Here are some examples of contrary-to-duty obligations (or sentences that express such obligations):

- If you are guilty, you should confess.
- If he has hurt his friend, he should apologise to her.
- If she will not keep her promise to him, she ought to call him.

We might also say that a contrary-to-duty obligation is a conditional obligation where the condition (in the obligation) is forbidden, or where the condition is fulfilled only if a primary obligation is violated. You should not be guilty; but if you are, you should confess. He should not have hurt his friend; but if he has, he should apologise. She should keep her promise to him; but if she will not, she ought to call him.

Contrary-to-duty obligations turn up in discussions concerning guilt, blame, confession, restoration, reparation, punishment, repentance, retributive justice, etc., and hence they form an important part of our moral thinking. Consequently, we want to know how to symbolise them adequately in some deontic system. The rationale of a contrary-to-duty obligation is the fact that most of us do neglect our primary duties from time to time and yet it is reasonable to believe that we should make the best of a bad situation, or at least that it matters what we do when this is the case.

However, it is difficult to find a satisfactory symbolisation of such sentences in many deontic systems. This is shown by the so-called contrary-to-duty (obligation) paradox, sometimes called the contrary-to-duty imperative paradox. Roderick Chisholm [16] was one of the first philosophers to address this puzzle. The contrary-to-duty paradox arises when we try to formalise certain intuitively consistent sets of ordinary language sentences, sets that include at least one contrary-to-duty obligation sentence, by means of ordinary counterparts available in various monadic deontic systems, such as for instance so-called Standard Deontic Logic and similar systems. In many of these systems the resulting sets are inconsistent in the sense that it is possible to deduce contradictions from them, or else they violate some other intuitively plausible condition, e.g. that the members of the sets should be independent of each other.[7]

6.1. Description of the contrary-to-duty paradox

Scenario I: contrary-to-duty obligations concerning promises

Consider the following scenario. It is Monday and you promise a friend to meet her on Friday to help her with some task. Suppose further that you always meet your friend on Saturdays. In this example the following sentences all seem to be true:

N-CTD

N1. (On Monday it is true that) You ought to keep your promise (and see your friend on Friday).

N2. (On Monday it is true that) It ought to be that if you keep your promise, you do not apologise (when you meet your friend on Saturday).

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[7] For a general introduction to many contrary-to-duty paradoxes and an overview of several different solutions, see Carmo and Jones [13]. See also [45, pp. 60–118].

N3. (On Monday it is true that) If you do not keep your promise (i.e., if you do not see your friend on Friday and help her out), you ought to apologise (when you meet her on Saturday).

N4. (On Monday it is true that) You do not keep your promise (on Friday).

Let N-CTD := {N1, N2, N3, N4}. N3 is a contrary-to-duty obligation (or expresses a contrary-to-duty obligation). If the condition is true, the primary obligation that you should keep your promise (expressed by N1) is violated. The set N-CTD seems to be consistent: it does not seem to entail any contradiction. However, if we try to formalise N-CTD in so-called Standard Deontic Logic, for instance, we run into problems. Consider the following symbolisation:

SDL-CTD

SDL1 $\mathbf{O}k$

SDL2 $\mathbf{O}(k \to \neg a)$

SDL3 $\neg k \to \mathbf{O}a$

SDL4 $\neg k$

where k formalises "You keep your promise (meet your friend on Friday and help her with her task)" and a formalises "You apologise (to your friend for not keeping your promise)". In this symbolisation SDL1 is supposed to express a primary obligation and SDL3 a contrary-to-duty obligation telling us what ought to be the case if the primary obligation is violated.

As is well known, the set SDL-CTD := {SDL1, SDL2, SDL3, SDL4} is not consistent in Standard Deontic Logic. $\mathbf{O}\neg a$ follows from SDL1 and SDL2, and from SDL3 and SDL4 we can derive $\mathbf{O}a$. Together these sentences entail the following formula: $\mathbf{O}a \wedge \mathbf{O}\neg a$ ("It is obligatory that you apologise and it is obligatory that you do not apologise"), which directly contradicts the so-called axiom D, i.e., the schema $\neg(\mathbf{O}A \wedge \mathbf{O}\neg A)$, which rules out explicit moral dilemmas and is included in Standard Deontic Logic. Since N-CTD seems to be consistent, while SDL-CTD is inconsistent, something must be wrong with our formalisation, with Standard Deontic Logic or with our intuitions. In a nutshell, this puzzle is the contrary-to-duty (obligation) paradox.[8]

[8] This is, of course, just one example of a contrary-to-duty paradox. But many other similar puzzles have the same or a very similar structure. So, much of what we say about this example can be generalised in a more or less obvious way.

6.2. Solution to the contrary-to-duty paradox in temporal alethic dyadic deontic logic

Many different solutions to this paradox have been suggested in the literature. We can try to find some alternative formalisation of N-CTD, we can try to develop some other kind of deontic logic or we can tinker with our intuitions about N-CTD. This is not the place to consider and evaluate all such attempts[9], even though we will briefly mention some possible solutions in Section 6.3. We will instead describe how the contrary-to-duty paradox can be solved in temporal alethic dyadic deontic logic. Then, we will consider some reasons why this solution is attractive (Section 6.3).[10]

In temporal alethic dyadic deontic logic N-CTD can be symbolised in the following way:

F-CTD

F1. $\mathsf{R}t_1\mathbf{O}[\top]\mathsf{R}t_2k$
F2. $\mathsf{R}t_1\mathbf{O}[\mathsf{R}t_2k]\mathsf{R}t_3\neg a$
F3. $\mathsf{R}t_1\mathbf{O}[\mathsf{R}t_2\neg k]\mathsf{R}t_3a$
F4. $\mathsf{R}t_1\mathsf{R}t_2\neg k$ $[\Leftrightarrow \mathsf{R}t_2\neg k]$

where k and a are interpreted as in SDL-CTD, t_1 refers to the moment on Monday when you make your promise, t_2 refers to the moment on Friday when you should keep your promise and t_3 refers to the moment on Saturday when you should apologise if you do not keep your promise on Friday. So, F1 is read as "It is true on Monday that you ought to keep your promise on Friday". F2 is read as "It is true on Monday that it ought to be the case that you do not apologise on Saturday given that you keep your promise on Friday". F3 is read as "It is true on Monday that it ought to be the case that you apologise on Saturday given that you do not keep your promise on Friday". F4 is read as "It is true on Monday that it is true on Friday that you do not keep your promise". This seems to be a plausible rendering of N-CTD.

Some sentences in temporal alethic dyadic deontic logic are temporally settled. This means that if they are true (in a possible world), they are true at every moment of time (in this possible world), and if they are false (in a possible world), they are false at every moment of time

[9] For a discussion of some possible solutions, see e.g. [45, pp. 60–118].

[10] For some solutions that are similar to the one presented in this paper, see e.g. van Eck [51, 52, 53], Feldman [22, 23], Loewer and Belzer [36], and Åqvist [66].

(in this possible world). All the sentences F1–F4 are temporally settled. $\mathsf{O}[\top]\mathsf{R}t_2k$, $\mathsf{O}[\mathsf{R}t_2k]\mathsf{R}t_3\neg a$ and $\mathsf{O}[\mathsf{R}t_2\neg k]\mathsf{R}t_3a$ are examples of sentences that are not temporally settled, as their truth values may vary from one moment of time to another (in one and the same possible world).

It is true on Monday that it is true on Friday that you do not keep your promise iff it is true on Friday that you do not keep your promise. $\mathsf{R}t_1\mathsf{R}t_2\neg k$ is equivalent to $\mathsf{R}t_2\neg k$. So, from now on, we will use $\mathsf{R}t_2\neg k$ as a symbolisation of N4. (We mention $\mathsf{R}t_1\mathsf{R}t_2\neg k$ because we want to know what is true on Monday, and according to our scenario, it is true on Monday that you will not keep your promise on Friday.) Note that it might be true on Monday that you will not keep your promise on Friday (in some possible world) even though this is not a settled fact, i.e., even though it is not historically necessary. In some possible worlds you will keep your promise on Friday and in some possible worlds you will not. F4 is true at t_1 (i.e. on Monday) in the possible worlds where you do not keep your promise at t_2 (i.e. on Friday).

Let F-CTD := $\{F1, F2, F3, F4\}$. We will show that F-CTD is consistent in all systems weaker than or deductively equivalent to the system **T** (see Section 4.3). Most philosophically interesting temporal alethic dyadic deontic systems (of the kind used in this paper) are included in this set.[11] (In Section 6.3 we will show that F-CTD has many other intuitively plausible properties, for instance that it is non-redundant.) To show this, it is enough to establish that this is true in the system **T**. Then it follows that the result holds in all weaker systems too. Hence, we can solve the contrary-to-duty paradox in temporal alethic dyadic deontic logic.

Model I. Consider the following supplemented temporal alethic dyadic deontic model. $W = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. The model satisfies all conditions in tables 4 and 5 (and thus all conditions in tables 1–3), i.e., $>$ (better than) is transitive, etc. Hence, the ranking is the same in every possible world in the model: $\omega_1 > \omega_2 > \omega_3 > \omega_4$. This means, for instance, that ω_1 is better than ω_2 in every possible world, i.e., $\omega_1 > \omega_2$ is an

[11] It is possible to construct several systems that are stronger than **T** by adding temporal rules to this system, e.g. we can add rules that correspond to the fact that time is dense or that there is no first or last point in time. As far as we can see, the results in this section can be extended to all such systems that are philosophically interesting. However, we will not explicitly consider these examples since we want to keep our models as simple as possible, e.g. we want to avoid models with an infinite number of temporal points.

abbreviation of the following conditions: $\omega_1 >_{\omega_1} \omega_2$, $\omega_1 >_{\omega_2} \omega_2$, $\omega_1 >_{\omega_3} \omega_2$, $\omega_1 >_{\omega_4} \omega_2$, etc. Since ω_1 is better than ω_2 and ω_2 is better than ω_3, ω_1 is better than ω_3, etc. In other words, ω_1 is the best member of W, ω_2 the second best, etc. The alethic accessibility relation is an equivalence relation (i.e., it is reflexive, symmetric, transitive (at every moment in time) etc.). $T = \{\tau_1, \tau_2, \tau_3\}$, where $\tau_1 < \tau_2 < \tau_3$. The temporal relation is transitive, comparable, and it does not branch towards the future or the past. At τ_1 all possible worlds are alethically accessible from all possible worlds. At τ_2, ω_2 is alethically accessible from ω_1 and ω_1 is alethically accessible from ω_2. At τ_2, ω_3 can see ω_4 alethically and ω_4 can see ω_3 alethically. At all times every possible world is alethically accessible to itself. The deontic accessibility relations are defined by $C - D\gamma 0$. $v(t_1) = \tau_1$, $v(t_2) = \tau_2$, and $v(t_3) = \tau_3$. k is true in ω_1 and ω_2 at τ_2 and k is false in ω_3 and ω_4 at τ_2. a is false in ω_1 and ω_4 at τ_3 and a is true in ω_2 and ω_3 at τ_3. For our purposes, we do not need any further information about this model. Model I is a so-called **T**-model (see Section 3.3).

We are now in a position to prove the following theorem:

THEOREM 5. *F-CTD is consistent in all systems weaker than or deductively equivalent to the system* **T**.

PROOF. Every sentence in F-CTD is true in ω_3 at τ_1 in Model I. Hence, F-CTD is satisfiable in this model. By the definition of **T** and the soundness results in Section 7.1, it follows that F-CTD is consistent in **T**. Let us verify that every sentence in F-CTD is true in ω_3 at τ_1 in Model I.

F1. $\mathsf{R}t_1 O[\top]\mathsf{R}t_2 k$. $\mathsf{R}t_1 O[\top]\mathsf{R}t_2 k$ is true in ω_3 at τ_1 iff $\mathbf{O}[\top]\mathsf{R}t_2 k$ is true in ω_3 at τ_1. $\mathbf{O}[\top]\mathsf{R}t_2 k$ is true in ω_3 at τ_1 iff $\mathsf{R}t_2 k$ is true in all the best worlds that are alethically accessible from ω_3 at τ_1 where $\top$ is true at τ_1. Since $\top$ is true in every possible world at every moment in time, ω_1 is the best world that is alethically accessible from ω_3 at τ_1 in which $\top$ is true. Since k is true in ω_1 at τ_2, it follows that $\mathsf{R}t_2 k$ is true in ω_1 at τ_1. So, $\mathsf{R}t_2 k$ is true in all the best worlds that are alethically accessible from ω_3 at τ_1 where $\top$ is true. Consequently, $\mathbf{O}[\top]\mathsf{R}t_2 k$ is true in ω_3 at τ_1. In conclusion, $\mathsf{R}t_1 \mathbf{O}[\top]\mathsf{R}t_2 k$ is true in ω_3 at τ_1.

F2. $\mathsf{R}t_1 \mathbf{O}[\mathsf{R}t_2 k]\mathsf{R}t_3 \neg a$. $\mathsf{R}t_1 \mathbf{O}[\mathsf{R}t_2 k]\mathsf{R}t_3 \neg a$ is true in ω_3 at τ_1 iff $\mathbf{O}[\mathsf{R}t_2 k]\mathsf{R}t_3 \neg a$ is true in ω_3 at τ_1. $\mathbf{O}[\mathsf{R}t_2 k]\mathsf{R}t_3 \neg a$ is true in ω_3 at τ_1 iff $\mathsf{R}t_3 \neg a$ is true in all the best worlds that are alethically accessible from

ω_3 at τ_1 in which $\mathsf{R}t_2k$ is true at τ_1. $\mathsf{R}t_2k$ is true at τ_1 in ω_1 and ω_2. The best of these worlds is ω_1. It follows that $\mathbf{O}[\mathsf{R}t_2k]\mathsf{R}t_3\neg a$ is true in ω_3 at τ_1 iff $\mathsf{R}t_3\neg a$ is true in ω_1 at τ_1. $\mathsf{R}t_3\neg a$ is true in ω_1 at τ_1 iff $\neg a$ is true in ω_1 at τ_3. Since a is false in ω_1 at τ_3, $\neg a$ is true in ω_1 at τ_3. Hence, $Rt_3\neg a$ is true in ω_1 at τ_1. Consequently, $\mathbf{O}[\mathsf{R}t_2k]\mathsf{R}t_3\neg a$ is true in ω_3 at τ_1. It follows that $\mathsf{R}t_1\mathbf{O}[\mathsf{R}t_2k]\mathsf{R}t_3\neg a$ is true in ω_3 at τ_1.

F3. $\mathsf{R}t_1\mathbf{O}[\mathsf{R}t_2\neg k]\mathsf{R}t_3a$. $\mathsf{R}t_1\mathbf{O}[\mathsf{R}t_2\neg k]\mathsf{R}t_3a$ is true in ω_3 at τ_1 iff $\mathbf{O}[\mathsf{R}t_2\neg k]\mathsf{R}t_3a$ is true in ω_3 at τ_1. $\mathbf{O}[\mathsf{R}t_2\neg k]\mathsf{R}t_3a$ is true in ω_3 at τ_1 iff $\mathsf{R}t_3a$ is true in all the best worlds that are alethically accessible from ω_3 at τ_1 in which $\mathsf{R}t_2\neg k$ is true at τ_1. $\mathsf{R}t_2\neg k$ is true at τ_1 in ω_3 and ω_4. The best of these worlds is ω_3. Hence, $\mathbf{O}[\mathsf{R}t_2\neg k]\mathsf{R}t_3a$ is true in ω_3 at τ_1 iff $\mathsf{R}t_3a$ is true in ω_3 at τ_1. $\mathsf{R}t_3a$ is true in ω_3 at τ_1 iff a is true in ω_3 at τ_3. But a is true in ω_3 at τ_3. Accordingly, $\mathsf{R}t_3a$ is true in ω_3 at τ_1. Therefore, $\mathbf{O}[\mathsf{R}t_2\neg k]\mathsf{R}t_3a$ is true in ω_3 at τ_1. It follows that $\mathsf{R}t_1\mathbf{O}[\mathsf{R}t_2\neg k]\mathsf{R}t_3a$ is true in ω_3 at τ_1.

F4. $\mathsf{R}t_2\neg k$. $\mathsf{R}t_2\neg k$ is true in ω_3 at τ_1 iff $\neg k$ is true in ω_3 at τ_2. $\neg k$ is true in ω_3 at τ_2 iff k is false in ω_3 at τ_2. Since k is false in ω_3 at τ_2, it follows that $\mathsf{R}t_2\neg k$ is true in ω_3 at τ_1. $\dashv$

6.3. Reasons why the solution to the contrary-to-duty paradox in temporal alethic dyadic deontic logic is attractive

In this section, we will consider 12 reasons why the solution to the contrary-to-duty paradox suggested in Section 6.2 is attractive. Taken individually, each reason might not seem that impressive, but together they really show how powerful this solution is. Without further ado, let us turn to our reasons.

Reason 1 (F-CTD is consistent). N-CTD seems to be consistent, i.e., it does not seem to be the case that we can derive a contradiction from this set. Hence, we want our symbolisation of N-CTD to be consistent. We have already shown that this is the case (Theorem 5).

Reason 2 (F-CTD is non-redundant). N-CTD seems to be non-redundant, i.e., it seems to be the case that no member of this set is derivable from the others. Therefore, we want our symbolisation of N-CTD to be non-redundant. In monomodal deontic logic, for instance Standard Deontic Logic, we can solve the contrary-to-duty paradox by finding some other formalisation of the sentences in N-CTD. Instead of SDL2 we can use $k \to \mathbf{O}\neg a$ and instead of SDL3 we can use $\mathbf{O}(\neg k \to a)$. Then we obtain three consistent alternative symbolisations of N-CTD. However, these alternatives are not non-redundant. For $\mathbf{O}(\neg k \to a)$ follows from $\mathbf{O}k$ in every so-called normal deontic logic, including Standard Deontic Logic, and $k \to \mathbf{O}\neg a$ follows from $\neg k$ by propositional logic. But intuitively, N3 does not seem to follow from N1, and N2 does not seem to follow from N4. In temporal alethic dyadic deontic logic, we can avoid this problem since we can prove the following theorem:

Theorem 6. *F-CTD is non-redundant in all systems weaker than or deductively equivalent to the system* **T***.*

To prove this theorem, we must establish propositions 1–4 below.

Proposition 1. F1 *is not derivable from* $\{\text{F2}, \text{F3}, \text{F4}\}$ *in* **T***. To prove that* F1 *is not derivable from* $\{\text{F2}, \text{F3}, \text{F4}\}$ *it is sufficient to come up with a* **T***-model* M*, a world* ω *in* M *and a time* τ *in* M *such that all members of* $\{\text{F2}, \text{F3}, \text{F4}\}$ *are true in* ω *at* τ *and* F1 *false in this world at this time. For* **T** *is strongly sound with respect to the class of all* **T***-models. Consider the following model.*

Model II. This model is exactly like Model I except that we use the following ranking of possible worlds instead: $\omega_3 > \omega_1 > \omega_2 > \omega_4$. In world ω_4 at time τ_1 all members of $\{\text{F2}, \text{F3}, \text{F4}\}$ are true and F1 false. Hence, F1 is not derivable from $\{\text{F2}, \text{F3}, \text{F4}\}$ in **T** (or any weaker system).

Proposition 2. F2 *is not derivable from* $\{\text{F1}, \text{F3}, \text{F4}\}$ *in* **T***. The following model proves this proposition.*

Model III. This model is exactly like Model I except that a is true in ω_1 at τ_3 and that we use the following ranking for the possible worlds: $\omega_1 > \omega_3 > \omega_2 > \omega_4$. In ω_4 at τ_1 all members of $\{\text{F1}, \text{F3}, \text{F4}\}$ are true and F2 false. Consequently, F2 is not derivable from $\{\text{F1}, \text{F3}, \text{F4}\}$ in **T**.

Proposition 3. F3 *is not derivable from* $\{\text{F1}, \text{F2}, \text{F4}\}$ *in* **T***. To prove this claim we use the following model.*

Model IV. This model is exactly like Model I except that we use the same ranking as in Model III and that a is false in ω_3 at time τ_3. In world ω_4 at time τ_1 all members of $\{\text{F1}, \text{F2}, \text{F4}\}$ are true and F3 false. Accordingly, F3 is not derivable from $\{\text{F1}, \text{F2}, \text{F4}\}$ in **T**.

Proposition 4. F4 *is not derivable from* $\{\text{F1}, \text{F2}, \text{F3}\}$ *in* **T***. To prove this proposition, we can use Model I. In world* ω_2 *at time* τ_1 *all members of* $\{\text{F1}, \text{F2}, \text{F3}\}$ *are true and* F4 *false.*

Reason 3 (F-CTD is dilemma free). One way of avoiding the contrary-to-duty paradox in monomodal deontic systems is to give up the axiom D, $\neg(\mathbf{O}A \wedge \mathbf{O}\neg A)$ ("It is not obligatory that A and obligatory that not-A"). Without this axiom (or something equivalent), it is no longer possible to derive a contradiction from SDL1–SDL4. In the so-called smallest normal deontic system K, for instance, SDL-CTD is consistent. Some might think that there are independent reasons for rejecting D since they think there are, or could be, genuine moral dilemmas. But even if this were true (which is debatable), rejecting D does not seem to be a good solution to the contrary-to-duty paradox for several reasons. Firstly, even if we reject axiom D, it is problematic to assume that a dilemma follows from N-CTD. We can still derive the sentence $\mathbf{O}a \wedge \mathbf{O}\neg a$ from SDL-CTD in every normal deontic system, which says that it is obligatory that you apologise and it is obligatory that you do not apologise. And this proposition does not seem to follow from N-CTD. Secondly, if there are any moral dilemmas of this kind, we can derive the claim that everything is both obligatory and forbidden in every normal deontic system, which is absurd (see Reason 4 below). Thirdly, such a solution might still have problems with the so-called pragmatic oddity (see Reason 5 below).

Our solution in temporal alethic dyadic deontic logic avoids this problem. From F1 and F2 we can derive the sentence $\mathsf{R}t_1\mathbf{O}[\top]\mathsf{R}t_3\neg a$ (in some systems) (see Reason 10 below) and from F3b and F4 we can derive the sentence $\mathsf{R}t_2\mathbf{O}[\top]\mathsf{R}t_3 a$ (in some systems under some circumstances) (see Reason 11 below). And from this we can derive the following formula: $\mathsf{R}t_1\mathbf{O}[\top]\mathsf{R}t_3\neg a \wedge \mathsf{R}t_2\mathbf{O}[\top]\mathsf{R}t_3 a$, from $\{F1, F2, F3b, F4\}$ (in certain systems). But this is not a moral dilemma. $\mathsf{R}t_1\mathbf{O}[\top]\mathsf{R}t_3\neg a$ says "On Monday [when you have not yet broken your promise] it ought to be the case that you do not apologise on Saturday", and $\mathsf{R}t_2\mathbf{O}[\top]\mathsf{R}t_3 a$ says "On Friday [when you have broken your promise] it ought to be the case that you apologise on Saturday". But $\mathbf{O}[\top]\mathsf{R}t_3 a$ and $\mathbf{O}[\top]\mathsf{R}t_3\neg a$ are not true at the same time. Neither $\mathsf{R}t_1\mathbf{O}[\top]\mathsf{R}t_3\neg a \wedge \mathsf{R}t_1\mathbf{O}[\top]\mathsf{R}t_3 a$ nor $\mathsf{R}t_2\mathbf{O}[\top]\mathsf{R}t_3\neg a \wedge \mathsf{R}t_2\mathbf{O}[\top]\mathsf{R}t_3 a$ is derivable from F-CTD in **T** or any weaker temporal alethic dyadic deontic system. N-CTD seems to be dilemma free. So, we want our formalisation of N-CTD to be dilemma free too. The following theorem shows that F-CTD is dilemma free in **T** and any weaker temporal alethic dyadic deontic system:

Theorem 7. *F-CTD is dilemma free in* **T** *and any weaker temporal alethic dyadic deontic system.*

Proof. Consider Model I. Every sentence in F-CTD is true in ω_3 at τ_1 in this model. However, $\mathsf{R}t_1\mathbf{O}[\top]\mathsf{R}t_3a$ is false in this world at this time. Hence, $\mathsf{R}t_1\mathbf{O}[\top]\mathsf{R}t_3\neg a \wedge \mathsf{R}t_1\mathbf{O}[\top]\mathsf{R}t_3a$ is also false in this world at this time. Since Model I is a **T**-model and **T** is sound with respect to the class of all **T**-models, it follows that $\mathsf{R}t_1\mathbf{O}[\top]\mathsf{R}t_3\neg a \wedge \mathsf{R}t_1\mathbf{O}[\top]\mathsf{R}t_3a$ is not derivable from F-CTD in **T** or any weaker temporal alethic dyadic deontic system. $\mathsf{R}t_2\mathbf{O}[\top]\mathsf{R}t_3\neg a$ is also false in ω_3 at τ_1. Hence, $\mathsf{R}t_2\mathbf{O}[\top]\mathsf{R}t_3\neg a \wedge \mathsf{R}t_2\mathbf{O}[\top]\mathsf{R}t_3a$ is false in this world at this time. It follows that $\mathsf{R}t_2\mathbf{O}[\top]\mathsf{R}t_3\neg a \wedge \mathsf{R}t_2\mathbf{O}[\top]\mathsf{R}t_3a$ is not derivable from F-CTD in **T**. ⊣

Reason 4 (It is not possible to derive the proposition that everything is both obligatory and forbidden from F-CTD). In every so-called normal deontic logic (even without the axiom D), we can derive the conclusion that everything is both obligatory and forbidden if there is at least one moral dilemma. This follows from the equivalence $\mathbf{F}A$ iff $\mathbf{O}\neg A$ and the fact that $\mathbf{O}a \wedge \mathbf{O}\neg a$ entails $\mathbf{O}r$ for any r. This is clearly absurd. N-CTD does not seem to entail that everything is both obligatory and forbidden. Hence, we do not want our symbolisation to entail this. Our solution in temporal alethic dyadic deontic logic has no such consequences. We have already seen that F-CTD is dilemma free (Reason 3 above). The following theorem shows that our solution avoids this problem:

Theorem 8. *F-CTD does not entail that for every A it is both obligatory that A and obligatory that not-A in **T** or any weaker temporal alethic dyadic deontic system.*

Proof. Again, we can use Model I to prove this. Every sentence in F-CTD is true in ω_3 at time τ_1 in this model. However, $\mathsf{R}t_1\mathbf{O}[\top]\mathsf{R}t_3\neg a$ is true while $\mathsf{R}t_1\mathbf{O}[\top]\neg\mathsf{R}t_3\neg a$ and $\mathsf{R}t_1F[\top]\mathsf{R}t_3\neg a$ are false in this world at this time. From this our theorem follows easily. ⊣

Reason 5 (F-CTD avoids the so-called pragmatic oddity). Pragmatic oddity is a problem for many possible solutions to the contrary-to-duty paradox. In every so-called normal deontic logic (with or without the axiom D) it is possible to derive the following sentence from SDL-CTD: $\mathbf{O}(k \wedge a)$, which says that it is obligatory that you keep your promise *and* apologise (for not keeping your promise). Several solutions that use bimodal alethic-deontic logic or counterfactual deontic logic, for instance, also have this problem. The sentence $\mathbf{O}(k \wedge a)$ is not inconsistent, but it is certainly very odd and it does not seem to follow from N-CTD that you should keep your promise *and* apologise. Hence, we do not want our formalisation of N-CTD to entail this counterintuitive conclusion or anything similar to it. The following theorem shows that neither $\mathsf{R}t_1\mathbf{O}[\top](\mathsf{R}t_2k \wedge \mathsf{R}t_3a)$ nor $\mathsf{R}t_2\mathbf{O}[\top](\mathsf{R}t_2k \wedge \mathsf{R}t_3a)$ is derivable from F-CTD in **T** or any weaker system:

Theorem 9. *Neither* $\mathsf{R}t_1\mathbf{O}[\top](\mathsf{R}t_2k \wedge \mathsf{R}t_3a)$ *nor* $\mathsf{R}t_2\mathbf{O}[\top](\mathsf{R}t_2k \wedge \mathsf{R}t_3a)$ *is derivable from F-CTD in* **T**.

Proof. We have seen that all sentences in F-CTD are true in ω_3 at τ_1 in Model I. However, neither $\mathsf{R}t_1\mathbf{O}[\top](\mathsf{R}t_2k \wedge \mathsf{R}t_3a)$ nor $\mathsf{R}t_2\mathbf{O}[\top](\mathsf{R}t_2k \wedge \mathsf{R}t_3a)$ is true in this world at this time (in this model). Now our theorem follows easily from this fact. ⊣

Reason 6 (The solution in temporal alethic dyadic deontic logic is applicable to (at least apparently) actionless contrary-to-duty examples). It might be possible to solve some contrary-to-duty paradoxes by combining deontic logic with some kind of action logic, for instance some kind of Stit (“Seeing to it”) logic, or dynamic logic. However, there also seem to be examples of contrary-to-duty paradoxes that involve actionless contrary-to-duty obligations. And it is difficult to see how to solve these paradoxes in such systems.

Scenario II: Contrary-to-duty paradoxes involving (apparently) actionless contrary-to-duty obligations

Consider the following scenario. At t_1, you are about to get into your car and drive somewhere. Then at t_1 it ought to be the case that the doors are closed at t_2, when you are in your car. If the doors are not closed, then a warning light ought to appear on the car instrument panel (at t_3, a point in time as soon as possible after t_2). It ought to be that if the doors are closed (at t_2), then it is not the case that a warning light appears on the car instrument panel (at t_3). Furthermore, the doors are not closed (at t_2 when you are in the car). In this example, all of the following sentences seem to be true:

N2-CTD

AN1. (At t_1) The doors ought to be closed (at t_2).

AN2. (At t_1) It ought to be that if the doors are closed (at t_2), then it is not the case that a warning light appears on the car instrument panel (at t_3).

AN3. (At t_1) If the doors are not closed (at t_2) then a warning light ought to appear on the car instrument panel (at t_3).

AN4. (At t_1 it is the case that at t_2) The doors are not closed.

N2-CTD is similar to N-CTD. In this set, AN1 expresses a primary obligation (or ought), and AN3 expresses a contrary-to-duty obligation. The condition in AN3 is satisfied only if the primary obligation expressed by AN1 is violated. But AN3 does not seem to tell us anything about what you or someone else ought to do. AN3 seems to be an actionless contrary-to-duty obligation. It tells us something about what ought to be the case if the world is not as it ought to be according to AN1.

In temporal alethic dyadic deontic logic, we have no trouble symbolising such (apparently) actionless contrary-to-duty obligations. The logical form of the sentences in N2-CTD exactly parallels the logical form of the sentences in N-CTD. Contrary-to-duty paradoxes of this kind can therefore be solved in exactly the same way as we solved our original paradox.

Reason 7 (We can assign formal sentences with analogous structures to all conditional obligations in N-CTD in temporal alethic dyadic deontic logic). Some deontic logicians have suggested that a formalisation of N-CTD is adequate only if the formal sentences assigned to N2 and N3 have the same (or analogous) logical form (see e.g. [13]). Our solution in temporal alethic dyadic deontic logic satisfies this requirement, in contrast to many other solutions. F2 and F3 have the "same" logical form; both are formalised using dyadic obligation.

Reason 8 (We can express the idea that an obligation has been violated in temporal alethic dyadic deontic logic). It might be possible to solve some contrary-to-duty paradoxes by applying ordinary concepts of defeasibility from so-called non-monotonic logic. However, it is not obvious that such solutions can explain the difference between violation and defeat. If you will not see your friend and help her, the obligation to keep your promise will be violated. It is not the case that this obligation is defeated, overridden or cancelled. It is not the case that one of the conditional norms in N-CTD defeat or override the other. Nor is it the case that they cancel each other out.

In temporal alethic dyadic deontic logic, we can express the idea that an obligation has been violated. Nevertheless, we must be careful when we describe the facts. At τ_2, when it is already settled that you do not keep your promise, it is no longer obligatory that you keep your promise, since by then it is no longer possible to keep it (in the strong models, for instance the **T**-models, we are considering). However, the following sentence is still true (in ω_1 and ω_2 at τ_2): $\mathsf{R}t_1\mathbf{O}[\top]\mathsf{R}t_2k$, i.e., it is still true at τ_2 (in the worlds where you do not keep your promise) that you should have kept your promise. So, when you do not keep your promise, you violate this earlier duty.

Reason 9 (We can symbolise higher order contrary-to-duty obligations in temporal alethic dyadic deontic logic). There are contrary-to-duty obligations of a higher order or degree. Consider the following variation of Scenario I:

Scenario III

One could claim that what you ought to do on Monday if you will not help your friend on Friday is call her on Wednesday, tell her that you will not keep your promise and apologise. If you neither keep your promise on Friday nor call your friend on Wednesday, then you ought to apologise (when you meet your friend on Saturday). If you keep your promise, then you ought not to apologise (when you meet your friend on Saturday) and you ought not to call her (on Wednesday). In this scenario all of the following sentences seem to be true:

N-HCTD

HN1. (On Monday it is true that) You ought to keep your promise (and see your friend on Friday).

HN2. (On Monday it is true that) It ought to be that if you keep your promise, you do not apologise (when you meet your friend on Saturday).

HN3. (On Monday it is true that) If you do not keep your promise (i.e., if you will not see your friend on Friday and help her out), you ought to call her (tell her that you will not keep your promise and apologise on Wednesday).

HN4. (On Monday it is true that) You do not keep your promise (on Friday).

HN5. (On Monday it is true that) If you do not keep your promise (on Friday) and you do not call your friend (on Wednesday), you ought to apologise (when you meet your friend on Saturday).

HN6. (On Monday it is true that) You do not call your friend (on Wednesday).

HN7. (On Monday it is true that) It ought to be that if you keep your promise, you do not call your friend (on Wednesday).

Let N-HCTD := {HN1, ..., HN7}. Here HN3 is an ordinary, first-order or first-degree contrary-to-duty obligation that tells us what ought to be the case if the primary obligation expressed by HN1 is violated. HN5 expresses a contrary-to-contrary to duty obligation, a second-order or second-degree contrary-to-duty obligation. The condition in this obligation is fulfilled only if the primary obligation expressed by HN1 is violated and the first-order contrary-to-duty obligation to call your friend is violated.

A reasonable solution to the contrary-to-duty paradox should be able to deal with higher-order contrary-to-duty obligations as well as ordinary first-degree contrary-to-duty obligations. In our temporal alethic dyadic deontic systems, we do not seem to have any trouble symbolising such higher-order contrary-to-duty obligations. N-HCTD can, for instance, be symbolised in the following way in temporal alethic dyadic deontic logic:

F-HCTD

HF1. $\mathsf{R}t_1\mathbf{O}[\top]\mathsf{R}t_3k$
HF2. $\mathsf{R}t_1\mathbf{O}[\mathsf{R}t_3k]\mathsf{R}t_4\neg a$
HF3. $\mathsf{R}t_1\mathbf{O}[\mathsf{R}t_3\neg k]\mathsf{R}t_2c$
HF4. $\mathsf{R}t_1\mathsf{R}t_3\neg k \quad [\Leftrightarrow \mathsf{R}t_3\neg k]$
HF5. $\mathsf{R}t_1\mathbf{O}[\mathsf{R}t_3\neg k \wedge \mathsf{R}t_2\neg c]\mathsf{R}t_4a$
HF6. $\mathsf{R}t_1\mathsf{R}t_2\neg c \quad [\Leftrightarrow \mathsf{R}t_2\neg c]$
HF7. $\mathsf{R}t_1\mathbf{O}[\mathsf{R}t_3k]\mathsf{R}t_2\neg c$

Let F-HCTD := {HF1, ..., HF7}. F-HCTD is also consistent, non-redundant, etc. Not all solutions to the contrary-to-duty paradox seem to accommodate such examples.

Reason 10 (In temporal alethic dyadic deontic logic we can derive "ideal" obligations). It seems to follow that you ought not to apologise from N1 and N2. Ideally you ought to keep your promise, and ideally it ought to be that if you keep your promise, then you do not apologise (for not keeping your promise). Hence, ideally you ought not to apologise. We want our formalisation of N-CTD to reflect this intuition. In every tableau system that includes T-$D\alpha0$ and T-$D\alpha2$, $\mathsf{R}t_1\mathbf{O}[\top]\mathsf{R}t_3\neg a$ is derivable from F1 ($\mathsf{R}t_1\mathbf{O}[\top]\mathsf{R}t_2k$) and F2 ($\mathsf{R}t_1\mathbf{O}[\mathsf{R}t_2k]\mathsf{R}t_3\neg a$). To illustrate how to use our tableau systems we will now show this.

In our derivation below (and in the derivation in Reason 11), we will use two derived rules. According to the first derived rule, DR1, we may add $\neg A$, w_it to any open branch in a tree that includes $\neg\mathsf{R}tA$, w_it_j.

This rule holds in every system. According to the second derived rule, DR2, we may add $\mathbf{O}[\top](A \to B)$, $w_i t_j$ to any open branch in a tree that contains $\mathbf{O}[A]B$, $w_i t_j$. This derived rule holds in every system that includes the rules *T-Dα0* and *T-Dα2*.

$$
\begin{array}{c}
(1)\ \mathsf{R}t_1\mathbf{O}[\top]\mathsf{R}t_2k, w_0t_0 \\
(2)\ \mathsf{R}t_1\mathbf{O}[\mathsf{R}t_2k]\mathsf{R}t_3\neg a, w_0t_0 \\
(3)\ \neg\mathsf{R}t_1\mathbf{O}[\top]\mathsf{R}t_3\neg a, w_0t_0 \\
(4)\ \neg\mathbf{O}[\top]\mathsf{R}t_3\neg a, w_0t_1\ [3, \text{DR1}] \\
(5)\ \mathbf{P}[\top]\neg\mathsf{R}t_3\neg a, w_0t_1\ [4, \neg\mathbf{O}] \\
(6)\ s_\top w_0w_1t_1\ [5, \mathbf{P}] \\
(7)\ \neg\mathsf{R}t_3\neg a, w_1t_1\ [5, \mathbf{P}] \\
(8)\ \neg\neg a, w_1t_3\ [7, \text{DR1}] \\
(9)\ \mathbf{O}[\top]\mathsf{R}t_2k, w_0t_1\ [1, \mathsf{R}t] \\
(10)\ \mathsf{R}t_2k, w_1t_1\ [9, 6, \mathbf{O}] \\
(11)\ k, w_1t_2\ [10, \mathsf{R}t] \\
(12)\ \mathbf{O}[\mathsf{R}t_2k]\mathsf{R}t_3\neg a, w_0t_1\ [2, \mathsf{R}t] \\
(13)\ \mathbf{O}[\top](\mathsf{R}t_2k \to \mathsf{R}t_3\neg a), w_0t_1\ [12, \text{DR2}] \\
(14)\ \mathsf{R}t_2k \to \mathsf{R}t_3\neg a, w_1t_1\ [13, 6, \mathbf{O}] \\
\swarrow \qquad \searrow \\
\begin{array}{cc}
(15)\ \neg\mathsf{R}t_2k, w_1t_1\ [14, \to] & (16)\ \mathsf{R}t_3\neg a, w_1t_1\ [14, \to] \\
(17)\ \neg k, w_1t_2\ [15, \text{DR1}] & (18)\ \neg a, w_1t_3\ [16, \mathsf{R}t] \\
(19)\ *\ [11, 17] & (20)\ *\ [8, 18]
\end{array}
\end{array}
$$

According to $\mathsf{R}t_1\mathbf{O}[\top]\mathsf{R}t_3\neg a$, it is true at t_1, i.e., on Monday, that it ought to be the case that you will not apologise on Saturday when you meet your friend. For, ideally, you keep your promise on Friday. Note, however, that $\mathsf{R}t_2\mathbf{O}[\top]\mathsf{R}t_3\neg a$ does not follow from F1 and F2 (see Reason 3 above). On Friday, when you have broken your promise, and when it is no longer historically possible for you to keep your promise, then it is not obligatory that you do not apologise on Saturday. In fact, then it is obligatory that you apologise when you meet your friend on Saturday (see Reason 11). But on Monday it is not a settled fact that you will not keep your promise. Hence, it is reasonable to claim that it is true on Monday that it ought to be the case that you do not apologise on Saturday. For on Monday it is still possible for you to keep your promise on Friday, which you ought to do. All of these conclusions that follow from F-CTD correspond well with our intuitions about Scenario I.

REASON 11 (In temporal alethic dyadic deontic logic we can derive "actual" obligations in certain circumstances). It seems to follow that you ought to apologise from N3 and N4. Ideally you ought to keep your promise. But if you do not keep your promise, you ought to apologise. And in fact, you do not keep your promise. Hence, you should apologise. We want our formalisation of N-CTD to reflect this intuition. Accordingly, we will assume that the conditional (contrary-to-duty) obligation expressed by N3 is still in force at time t_2 (τ_2), i.e., we assume that the following sentence is true:

F3b. $\mathsf{R}t_2\mathbf{O}[\mathsf{R}t_2\neg k]\mathsf{R}t_3a$.

According to F3b it is true at t_2 (i.e., on Friday) that if you do not keep your promise on Friday, then you ought to apologise on Saturday. In every tableau system that includes *T-Dα0*, *T-Dα2*, *T-DMO* (the dyadic must-ought principle), and *T-BT* (backward transfer), $\mathsf{R}t_2\mathbf{O}[\top]\mathsf{R}t_3a$ is derivable from F4 ($\mathsf{R}t_2\neg k$) and F3b. According to $\mathsf{R}t_2\mathbf{O}[\top]\mathsf{R}t_3a$, it is true at t_2, i.e., on Friday when you have broken your promise to your friend, that it ought to be the case that you apologise to your friend on Saturday when you meet her.

Note that $\mathsf{R}t_1\mathbf{O}[\top]\mathsf{R}t_3a$ is not derivable from F3 (or F3b or F3 and F3b) and F4 (see Reason 3). $\mathsf{R}t_1\mathbf{O}[\top]\mathsf{R}t_3a$ says that it is true at t_1, i.e., on Monday, that you should apologise to you friend on Saturday when you meet her. But on Monday it is not yet a settled fact that you will not keep your promise to your friend. Hence, it is not true on Monday that you should apologise on Saturday. On Monday it is still open to you to keep your promise on Friday. And since you ought to keep your promise, and it ought to be that if you keep your promise then you do not apologise, it follows that it is true on Monday that it ought to be the case that you do not apologise on Saturday (see Reason 10). All of these facts correspond well with our intuitions about Scenario I.

Here is our derivation of $\mathsf{R}t_2\mathbf{O}[\top]\mathsf{R}t_3a$ from F3b and F4; this derivation also illustrates how to use our tableau systems:

(1) $\mathsf{R}t_2\neg k, w_0t_0$
(2) $\mathsf{R}t_2\mathbf{O}[\mathsf{R}t_2\neg k]\mathsf{R}t_3a, w_0t_0$
(3) $\neg\mathsf{R}t_2\mathbf{O}[\top]\mathsf{R}t_3a, w_0t_0$
(4) $\neg\mathbf{O}[\top]\mathsf{R}t_3a, w_0t_2$ [3, DR1]
(5) $\mathbf{P}[\top]\neg\mathsf{R}t_3a, w_0t_2$ [4, $\neg\mathbf{O}$]
(6) $s_\top w_0w_1t_2$ [5, $\mathbf{P}$]
(7) $\neg\mathsf{R}t_3a, w_1t_2$ [5, $\mathbf{P}$]
(8) $\neg a, w_1t_3$ [7, DR1]
(9) $rw_0w_1t_2$ [6, *T-DMO*]
(10) $\neg k, w_0t_2$ [1, $\mathsf{R}t$]

(11) $\mathbf{O}[\mathsf{R}t_2\neg k]\mathsf{R}t_3a, w_0t_2$ [2, Rt]
(12) $\mathbf{O}[\top](\mathsf{R}t_2\neg k \to \mathsf{R}t_3a), w_0t_2$ [11, DR2]
(13) $\mathsf{R}t_2\neg k \to \mathsf{R}t_3a, w_1t_2$ [6, 12, $\mathbf{O}$]
↙ ↘
(14) $\neg\mathsf{R}t_2\neg k, w_1t_2$ [13, →] | (15) $\mathsf{R}t_3a, w_1t_2$ [13, →]
(16) $\neg\neg k, w_1t_2$ [14, DR1] | (17) a, w_1t_3 [15, Rt]
(18) k, w_1t_2 [16, ¬¬] | (19) * [8, 17]
(20) k, w_0t_2 [9, 18, T-BT]
(21) * [10, 20]

F3 and F3b are independent of each other (in **T** and weaker systems). But if this is true and we assume that the contrary-to-duty obligation to apologise, given that you do not keep your promise, is still in force at t_2, should not N3 be symbolised by a conjunction of F3 and F3b or something similar? It might be interesting to note that we can do this without affecting the main results in this section. $\{F1, F2, F3, F3b, F4\}$ is, for instance, consistent, non-redundant, etc. (in **T** and weaker systems). So, we can use such an alternative formalisation of N3 instead of F3. Furthermore, note that the symbolisation of N2 can be modified in a similar way.

Reason 12 (We can avoid the so-called dilemma of commitment and detachment in temporal alethic dyadic deontic logic). (Factual) Detachment is an inference pattern that allows us to infer or detach an unconditional obligation from a conditional obligation and this conditional obligation's condition. For example, if detachment holds for the conditional (contrary-to-duty) obligation that you should apologise if you do not keep your promise (if detachment is possible), then we can derive the unconditional obligation that you should apologise given that you do not keep your promise.

According to the so-called dilemma of commitment and detachment [52, p. 263]: (1) Detachment should be possible, for we cannot take seriously a conditional obligation if it cannot, by way of detachment, lead to an unconditional obligation. (2) Detachment should not be possible, for if detachment is possible, the following kind of situation would be inconsistent — A, it ought to be the case that B given that A; and C, it ought to be the case that not-B given C. But, such a situation is not necessarily inconsistent.

This dilemma seems to be a problem for solutions to the contrary-to-duty paradox in pure dyadic deontic logic. In pure dyadic deontic logic, we cannot derive the unconditional obligation that it is obligatory that A ($\mathbf{O}A$) from the dyadic obligation that it is obligatory that A given B ($\mathbf{O}[B]A$) and B. But how can we then take such conditional obligations seriously? Be that as it may, in temporal alethic dyadic deontic logic we can solve this dilemma. In Section 5, we saw that we cannot always detach an unconditional obligation from a conditional obligation and its condition, but that we can detach the unconditional obligation $\mathbf{O}B$ (in **STADDL**) from $\mathbf{O}[A]B$ and A if A is non-future or historically necessary. This seems to give us exactly the correct answer to our current puzzle. Detachment holds, but it does not hold unrestrictedly. We saw above that $\mathsf{R}t_2\mathbf{O}[\top]\mathsf{R}t_3a$, but not $\mathsf{R}t_1\mathbf{O}[\top]\mathsf{R}t_3a$, is derivable from $\mathsf{R}t_2\neg k$ and $\mathsf{R}t_2\mathbf{O}[\mathsf{R}t_2\neg k]\mathsf{R}t_3a$ in certain systems. In other words, we can detach the former sentence, but not the latter. From this it does not follow that a set of the following kind must be inconsistent: $\{A, \mathbf{O}[A]B, C, \mathbf{O}[C]\neg B\}$; this seems to be exactly what we want.

This concludes our discussion of the reasons for the solution to the contrary-to-duty paradox discussed in the present paper. All other solutions that have been suggested in the literature so far seem to lack at least some of the features mentioned above. This makes the symbolisation of N-CTD in temporal alethic dyadic deontic logic very attractive.

7. Soundness and completeness theorems

The soundness and completeness proofs in this section are modifications and extensions of proofs found in [43] and [44].

Let $S =$ "$aA_1...A_idB_1...B_j\alpha C_1...C_kadD_1...D_ltE_1...E_madtF_1...F_n$" be a temporal alethic dyadic deontic tableau system as defined above. Then we shall say that the class of models, $\mathcal{M}$, corresponds to S just in case $\mathcal{M} = \mathcal{M}(C\text{-}A_1, ..., C\text{-}A_i, C\text{-}B_1, ..., C\text{-}B_j, C\text{-}C_1, ..., C\text{-}C_k, C\text{-}D_1, ..., C\text{-}D_l, C\text{-}E_1, ..., C\text{-}E_m, C\text{-}F_1, ..., C\text{-}F_n)$.

S is strongly sound with respect to $\mathcal{M}$ iff for all Γ and A: $\Gamma \vdash_S A$ (i.e., A is derivable from Γ in S) entails $\mathcal{M}, \Gamma \Vdash A$ (i.e., A follows from Γ in $\mathcal{M}$). S is strongly complete with respect to $\mathcal{M}$ just in case $\mathcal{M}, \Gamma \Vdash A$ entails $\Gamma \vdash_S A$.

7.1. Soundness theorems

Let M be any (ordinary) model and b any branch of a tableau. Then b is satisfiable in M iff there is a function f from $w_0, w_1, w_2, \ldots$ to

W and a function g from $t_0, t_1, t_2, \ldots$ to T such that (i) A is true in $f(w_i)$ at $g(t_j)$ in M, for every node $A, w_i t_j$ on b, (ii) if $rw_i w_j t_k$ is on b, then $Rf(w_i)f(w_j)g(t_k)$ in M, (iii) if $s_A w_i w_j t_k$ is on b, then $S_A f(w_i)f(w_j)g(t_k)$ in M, (iv) if $t_i < t_j$ is on b, then $g(t_i) < g(t_j)$ in M, (v) if $w_i = w_j$ is on b, then $f(w_i) = f(w_j)$ in M, (vi) if $t_i = t_j$ is on b, then $g(t_i) = g(t_j)$ in M. If these conditions are fulfilled, we say that f and g show that b is satisfiable in M.

LEMMA 10 (Soundness Lemma). *Let b be any branch of a tableau and M be any temporal alethic dyadic deontic model. If b is satisfiable in M and a tableau rule is applied to it, then it produces at least one extension, b', of b such that b' is satisfiable in M.*

PROOF. The proof proceeds by going through all the tableau rules. Here are some steps to illustrate the method.

For *T-DdT′*: Assume that $s_A w_i w_j t_l$ is on b, and that we apply $T-DdT'$ to give an extended branch of b including $s_A w_j w_j t_l$. Since b is satisfiable in M, $S_A f(w_i)f(w_j)g(t_l)$. Accordingly, $S_A f(w_j)f(w_j)g(t_l)$, since M satisfies the condition *C-DdT′*.

For *T-Dα2*: Suppose that $s_A w_i w_j t_l$ and $B, w_j t_l$ are on b, and that we apply *T-Dα2* to give an extended branch of b containing $s_{A \wedge B} w_i w_j t_l$. Since b is satisfiable in M, $S_A f(w_i)f(w_j)g(t_l)$ and B is true in $f(w_j)$ at $g(t_l)$. Accordingly, $S_{A \wedge B} f(w_i)f(w_j)g(t_l)$, since M satisfies *C-Dα2*.

For *T-Dα4*: Suppose that $s_A w_i w_j t_l$, $B, w_j t_l$, and $s_{A \wedge B} w_i w_k t_l$ are on b, and that we apply *T-Dα4* to give an extended branch of b containing $s_A w_i w_k t_l$ and $B, w_k t_l$. Since b is satisfiable in M, $S_A f(w_i)f(w_j)g(t_l)$, $S_{A \wedge B} f(w_i)f(w_k)g(t_l)$ and B is true in $f(w_j)$ at $g(t_l)$. Accordingly, $S_A f(w_i)f(w_k)g(t_l)$ and B is true in $f(w_k)$ at $g(t_l)$, since M satisfies the condition *C-Dα4*. ⊣

THEOREM 11 (Soundness Theorem I). *Let S be any of the tableau systems discussed in this essay and let $\mathcal{M}$ be the class of models that corresponds to S. Then S is strongly sound with respect to $\mathcal{M}$.*

PROOF. Once Soundness Lemma is established, the proof is an easy modification of similar proofs found e.g. in [38, 43, 44, 46]. ⊣

From the above theorem and Theorem 1 we obtain:

THEOREM 12 (Soundness Theorem II). 1. *α012 is sound with respect to the class of all supplemented models that satisfies C-Dγ0.*

2. *$\alpha 0123$ is sound with respect to the class of all supplemented models that satisfies C-D$\gamma 0$ and C-D$\delta 3$.*
3. *$\alpha 01234$ is sound with respect to the class of all supplemented models that satisfies C-D$\gamma 0$, C-D$\delta 3$, and C-D$\delta 4$.*
4. *$a45d4'5'\alpha 012ad45$ is sound with respect to the class of all supplemented models that satisfies C-D$\gamma 0$, C-a4, C-a5 and C-D$\delta 6$.*
5. *$d7\alpha 012$ is sound with respect to the class of all supplemented models that satisfies C-D$\gamma 0$, and C-D$\delta 7$.* (Soundness results for other combinations of these conditions are easily obtained.)
6. *The tableau system* **T** (see Section 4.3) *is sound with respect to the class of all* **T***-models* (see Section 3.3).

7.2. Completeness theorems

Let b be an open complete branch of a tableau and let I be the set of numbers on b immediately preceded by a "t". We shall say that $i \rightleftharpoons j$ just in case $i = j$, or "$t_i = t_j$" or "$t_j = t_i$" occurs on b. $\rightleftharpoons$ is an equivalence relation and $[i]$ is the equivalence class of i. Furthermore, let K be the set of numbers on b immediately preceded by a "w". We shall say that $k \approx l$ just in case $k = l$, or "$w_k = w_l$" or "$w_l = w_k$" occurs on b. $\approx$ is an equivalence relation and $[k]$ is the equivalence class of k.

Definition 13 (Induced model). The temporal alethic dyadic deontic model, $M = \langle W, T, < R, \{S_A : A \in L\}, V, v\rangle$, induced by b is defined as follows. $W = \{\omega_{[k]} : k \in K\}$, $T = \{\tau_{[i]} : i \in I\}$, $\tau_{[i]} < \tau_{[j]}$ iff $t_i < t_j$ occurs on b, $R\omega_{[i]}\omega_{[j]}\tau_{[k]}$ iff $rw_iw_jt_k$ occurs on b, $S_A\omega_{[i]}\omega_{[j]}\tau_{[k]}$ iff $s_Aw_iw_jt_k$ occurs on b. If p, w_it_j occurs on b, then p is true in $\omega_{[i]}$ at $\tau_{[j]}$ (i.e., then $\langle\omega_{[i]}, \tau_{[j]}\rangle \in V(p)$); if $\neg p, w_it_j$ occurs on b, then p is false in $\omega_{[i]}$ at $\tau_{[j]}$ (i.e., then it is not the case that $\langle\omega_{[i]}, \tau_{[j]}\rangle \in V(p)$). If t_i occurs on b, then $v(t_i) = \tau_{[i]}$.

If our tableau system neither includes *T-FC*, *T-PC* (see [44]) nor *T-C* (see [46]), $\rightleftharpoons$ is reduced to identity and $[i] = i$. Hence, in such systems, we may take T to be $\{\tau_i : t_i$ occurs on $b\}$ and dispense with the equivalence classes. Likewise, if our tableau system does not include *T-Dd7*, $\approx$ is reduced to identity and $[k] = k$. Accordingly, in such systems, we may take W to be $\{\omega_i : w_i$ occurs on $b\}$ and dispense with the equivalence classes.

LEMMA 14 (Completeness Lemma). *Let b be an open branch in a complete tableau and let M be a temporal alethic dyadic deontic model induced by b. Then*:

(i) *A is true in $\omega_{[i]}$ at $\tau_{[j]}$, if $A, w_i t_j$ is on b,*
(ii) *A is false in $\omega_{[i]}$ at $\tau_{[j]}$, if $\neg A, w_i t_j$ is on b.*

PROOF. The proof is by induction on the complexity of A.

We will only go through one example to illustrate the method.

$A = \mathbf{O}[B]C$. Suppose $A, w_i t_k$ occurs on b, i.e., $\mathbf{O}[B]C, w_i t_k$ is on b. Since b is complete ($\mathbf{O}$) has been applied to $\mathbf{O}[B]C, w_i t_k$. Thus, for all w_j on b such that $s_B w_i w_j t_k$, $C, w_j t_k$ is on b. By the induction hypothesis, for all $\omega_{[j]}$ such that $S_B \omega_{[i]} \omega_{[j]} \tau_{[k]}$, C is true in $\omega_{[j]}$ at $\tau_{[k]}$. Hence, $\mathbf{O}[B]C$ is true in $\omega_{[i]}$ at $\tau_{[k]}$. Suppose that $\neg A, w_i t_k$ occurs on b, i.e., $\neg\mathbf{O}[B]C, w_i t_k$ is on b. Then $\mathbf{P}[B]\neg C, w_i t_k$ is on b (by $\neg\mathbf{O}$). For b is complete. Furthermore, since b is complete ($\mathbf{P}$) has been applied to $\mathbf{P}[B]\neg C, w_i t_k$. Thus, for some w_j, $s_B w_i w_j t_k$ and $\neg C, w_j t_k$ are on b. By the induction hypothesis, $S_B \omega_{[i]} \omega_{[j]} \tau_{[k]}$ and C is false in $\omega_{[j]}$ at $\tau_{[k]}$. Hence, $\mathbf{O}[B]C$ is false in $\omega_{[i]}$ at $\tau_{[k]}$. ⊣

THEOREM 15 (Completeness Theorem). *Let S be any of the tableau systems discussed in this essay, not including T-Dα0, and let $\mathcal{M}$ be the class of models that corresponds to S. Then S is strongly complete with respect to $\mathcal{M}$.*[12]

PROOF. The proof is a modification of similar proofs in [43, 44] (see also [38, 46]).

First we show that the weakest system is complete. Then we have to check that the model induced by the open branch, b, is of the right kind in every case. We only consider some cases to illustrate the method.

For *C-DdT′*: Suppose that $S_A \omega_{[i]} \omega_{[j]} \tau_{[k]}$. Then $s_A w_i w_j t_k$ occurs on b [by the definition of an induced model]. Since the tableau is complete, *T-DdT′* has been applied and $s_A w_j w_j t_k$ occurs on b. Hence, $S_A \omega_{[j]} \omega_{[j]} \tau_{[k]}$, as required (by the definition of an induced model).

For *C-DMO*: Suppose that $S_A \omega_{[i]} \omega_{[j]} \tau_{[k]}$. Then $s_A w_i w_j t_k$ occurs on b (by the definition of an induced model). Since b is complete, ($T-DMO$)

[12] We have excluded systems that contain *T-Dα0* because we have not been able to prove that these systems are complete. It should be noted that this does not mean that we have proven that they are incomplete. Our conjecture is that *all* systems in this paper are complete with respect to their (ordinary) semantics (including all systems that contain *T-Dα0*). Hopefully, someone will be able to prove this in the future or refute this conjecture.

has been applied and $rw_iw_jt_k$ occurs on b. Accordingly, $R\omega_{[i]}\omega_{[j]}\tau_{[k]}$, as required (by the definition of an induced model).

For *C-Dα3*: Assume that $R\omega_{[i]}\omega_{[j]}\tau_{[l]}$ and that A is true in $\omega_{[j]}$ at $\tau_{[l]}$. Then $rw_iw_jt_l$ [by the definition of an induced model]. Since the tableau is complete *CUT* has been applied and either A, w_jt_l or $\neg A, w_jt_l$ is on b. Suppose that $\neg A, w_jt_l$ is on b. Then A is false in $\omega_{[j]}$ at $\tau_{[l]}$ (by Completeness Lemma). But this is absurd. Hence, A, w_jt_l is on b. Since the tableau is complete *T-Dα3* has been applied. So, $s_Aw_iw_kt_l$ is on b. Accordingly, $S_A\omega_{[i]}\omega_{[k]}\tau_{[l]}$, as required (by the definition of an induced model).

For *C-Dα4*: Suppose that $S_A\omega_{[i]}\omega_{[j]}\tau_{[l]}$, $S_{A\wedge B}\omega_{[i]}\omega_{[k]}\tau_{[l]}$ and that B is true in $\omega_{[j]}$ at $\tau_{[l]}$. Then $s_Aw_iw_jt_l$ and $s_{A\wedge B}w_iw_kt_l$ occur on b (by the definition of an induced model). Since the tableau is complete *CUT* has been applied and either B, w_jt_l or $\neg B, w_jt_l$ is on b. Assume that $\neg B, w_jt_l$ is on b. Then B is false in $\omega_{[j]}$ at $\tau_{[l]}$ (by Completeness Lemma). But this is absurd. So, B, w_jt_l is on b. Since the tableau is complete *T-Dα4* has been applied and $s_Aw_iw_kt_l$ and B, w_kt_l occur on b. It follows that $S_A\omega_{[i]}\omega_{[k]}\tau_{[l]}$ and that B is true in $\omega_{[k]}$ at $\tau_{[l]}$, as required (by the definition of an induced model and Completeness Lemma). ⊣

Problem. Whether any other system discussed in this paper is complete is left as an open question.

Acknowledgement. We would like to thank an anonymous reviewer for some valuable comments on an earlier version of this paper.

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