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A SIMULATION OF NATURAL DEDUCTION AND GENTZEN SEQUENT CALCULUS

Abstract. We consider four natural deduction systems: Fitch-style systems, Gentzen-style systems (in the form of dags), general deduction Frege systems and nested deduction Frege systems, as well as dag-like Gentzen-style sequent calculi. All these calculi soundly and completely formalise classical propositional logic.

We show that general deduction Frege systems and Gentzen-style natural calculi provide at most quadratic speedup over nested deduction Frege systems and Fitch-style natural calculi and at most cubic speedup over Gentzen-style sequent calculi.

Keywords: Speedup, natural deduction; Gentzen-style calculi; simulation; proof system

1. Introduction

1.1. Preliminaries

We use a propositional language over $\{\neg, \wedge, \vee, \supset\}$ with falsum — $\curlywedge$. We denote propositional variables with p_i $(i \in \{1, 2, \ldots\})$.

Convention 1. Following Pelletier [12] and Reckhow [13] we consider an adequate (sound and complete) calculus to be a natural one if it allows the use of arbitrary assumptions in proofs of theorems and incorporates the deduction theorem as a rule.

In our paper we consider a simulation of various proof systems for classical propositional logic. These systems are as follows: general deduction Frege systems ($d\mathscr{F}$) and nested deduction Frege systems ($nd\mathscr{F}$)

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from [1], systems of subordinate proofs by Fitch [7][1] (**F**), the natural deduction calculus of Gentzen (ND) from [8] and two versions of the sequent calculus (PKT and PKT*[2]) formulated by Buss in [2].

We don't need precise formulations of the axiom schemas used in $d\mathscr{F}$ and $nd\mathscr{F}$ since as Buss and Bonet pointed out in [1], all Frege systems linearly simulate one another.

We provide simulation procedures between them as well as speedups of one over the other.

Convention 2. We assume that both the Gentzen-style natural deduction calculus and the Gentzen-style sequent calculus PKT and its version PKT* use not tree-like but dag-like proofs. We follow Cook and Reckhow [13, 4] here and this convention will be crucial for our results shown in the theorems concerning a simulation of ND, PKT and PKT*.

We designate the modus ponens rule and the deduction rule used in $nd\mathscr{F}$ as mp_n and dr_n while mp_g and dr_g stand for modus ponens and the deduction rule in $d\mathscr{F}$. The reader is directed to [1] for their precise formulations.

Note that deduction rule is actually a rule-realization of deduction theorem

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}$$

In case of $nd\mathscr{F}$ the use of the deduction rule is restricted in such a fashion that it is possible to discharge only the last open assumption.

In case of the other calculi we use the definitions of their respective proofs as provided in the sources cited above.

Definition 1. We will, following [11], write

$$f(n) = O(g(n))$$

with f and g being functions mapping $\mathbb{N} = \{0, 1, 2, \ldots\}$ into itself if there are such $c, n_0 \in \mathbb{N}$ that $\forall n > n_0 : f(n) \leqslant c \cdot g(n)$.

Definition 2. We will, following [1], say that S simulates T with an increase in size $f(x)$, if for any T-derivation of length n there exists an S-derivation of the same (or equivalent) formula from the same assumptions of length $O(f(n))$. We will further say that T provides at most an $f(x)$ speedup if S can simulate T with an increase in size $f(x)$. We will

[1] One could also consider the system of suppositions devised by Jaśkowski in [9].

[2] PKT* (cf. [1]) does not count steps inferred from weak structural rules.

also, following Reckhow [13] and Urquhart [18], name two proof systems polynomially equivalent (p-equivalent), if they polynomially simulate one another.

1.2. Survey of previous results

We will now give a short overview of previous results on a simulation of various proof systems. It is noteworthy, however, to say that a simulation of various kinds of natural deduction and, more importantly, of speedups is not very well researched, since most papers only consider such systems as analytical tableaux (cf. [14]), resolution (cf. [15]), Frege systems and their various extensions.

It is noteworthy to mention that all results concerning simulations of various proof systems can be, according to D'Agostino [5], divided into two kinds: it is either proven that one system p-simulates another (in this case a simulation procedure as well as speedup is provided), or it is proven that f (see Definition 2) has super-polynomial lower bound (in this case lower bound is presented).

The main result in the field is due to Reckhow [13]: he showed that all natural calculi, Gentzen-style calculi with cut rule and Frege systems p-simulate each other independently of the connectives they use, provided they are sound and complete. This means that we don't have to prove that our systems can simulate one another polynomially.

Another result we are going to use extensively in the present paper is due to S. R. Buss and M. L. Bonet [1]. They investigate speedups provided by different proof systems, including $nd\mathscr{F}$, $d\mathscr{F}$, PKT, PKT* and ND. Their results are presented in Figure 1.

A simulation of Gentzen systems has been much better investigated. The main results are due to Urquhart.

1. It is shown in [16] that Gentzen systems with cut p-simulate cut-free Gentzen systems but the reverse p-simulation is impossible.
2. It is shown in [17] that, although tree-like resolution (cf. [15]) p-simulates tree-like[3] Gentzen systems without cut, the reverse simulation is impossible. On the other hand, dag-like Gentzen systems without cut are p-equivalent to resolution.
3. It is, furthermore, shown in [18] that tree-like Gentzen systems without cut are p-equivalent to analytic tableaux described in [14] but

[3] A Gentzen system is called tree-like if all sequents are counted.

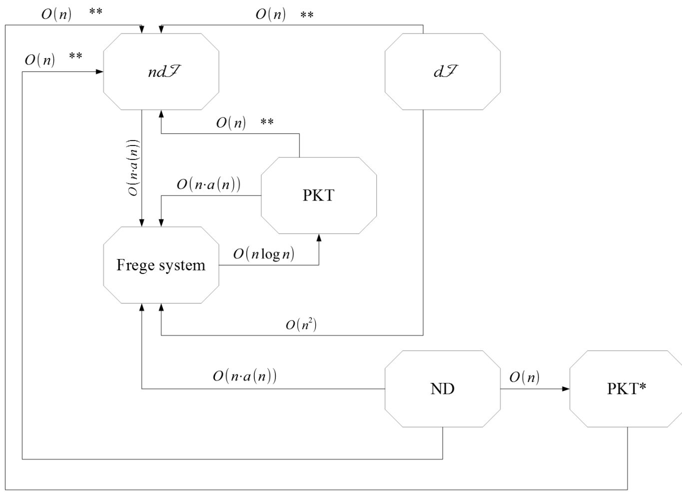

Figure 1. ** — for proofs wherein no line is used as a hypothesis of a rule of inference more than once; a is the inverse Ackermann function

cannot p-simulate truth tables and the reverse simulation is impossible too. Another important result is that tree-like Gentzen systems without cut cannot p-simulate dag-like Gentzen systems without cut.

It is also important to mention the paper by Finger [6] wherein it was proved that dag-like Gentzen systems without cut but with substitution rule linearly simulate tree-like Gentzen systems with cut.

Another important result is mentioned by Cook and Nguyen in [3] and attributed to Krajìček [10]. He shows that tree-like Gentzen systems with cut p-simulate dag-like Gentzen systems with cut.

2. A simulation of natural deduction

In this section we prove some theorems that provide speedups of one natural deduction system over another.

THEOREM 1. $\Gamma|\frac{nd\mathscr{F}}{n} A \Rightarrow \Gamma|\frac{d\mathscr{F}}{O(n)} A$

PROOF. The proof is straightforward, since every instance of axiom schema and every hypothesis in $nd\mathscr{F}$-derivation remain w.l.o.g. an axiom and hypothesis respectively in $d\mathscr{F}$-derivation, every instance of mp_n and dr_n becomes an instance of mp_g and dr_g respectively. □

THEOREM 2. $\Gamma|\frac{d\mathscr{F}}{n} D \Rightarrow \Gamma|\frac{nd\mathscr{F}}{O(n^2)} D$

PROOF. We prove this theorem with the method used by Buss and Bonet in [1] in their proof of Theorem 4.

Recall that every line in a $d\mathscr{F}$-derivation is a sequent of the form $\Gamma \vDash D$. Suppose we have a $d\mathscr{F}$-proof π_g of a sequent $\Gamma \vDash D$ of length n. This means that the $d\mathscr{F}$-derivation is actually the following sequence:

$$\Gamma_1 \vDash A_1, \dots, \Gamma_n \vDash A_n = \Gamma \vDash D$$

Our simulation goes as follows. We first substitute each sequent $\Gamma_i \vDash A_i$ $(1 \leqslant i \leqslant n)$ for the formula $\bigwedge \Gamma_i \supset A_i$ with $\bigwedge \Gamma_i$ denoting the conjunction of all formulas in Γ_i ordered and associated arbitrarily. This gives us the following sequence π':

$$\bigwedge \Gamma_1 \supset A_1, \dots, \bigwedge \Gamma_n \supset A_n \tag{1}$$

It is important to note that since $d\mathscr{F}$ is complete, all formulas in (1) are tautologies. However, π' is not a valid $nd\mathscr{F}$-proof, which means that we have to show by induction on n that we can fill in every gap in no more than $O(n)$ steps. The proof splits into four cases depending on how $\Gamma_n \vDash A_n$ was inferred.

CASE 2.1. $\Gamma_n \vDash A_n$ *is an assumption or hypothesis.*

In this case $\Gamma_n \vDash A_n$ has the form $A_n \vDash A_n$ and becomes $A_n \supset A_n$ in π'. One can prove $A_n \supset A_n$ in $nd\mathscr{F}$ in a constant number of steps.

CASE 2.2. $\Gamma_n \vDash A_n$ *is an instance of an axiom schema.*

$\Gamma_n \vDash A_n$ has the form $\vDash A_n$ and becomes A_n in an $nd\mathscr{F}$-derivation with A_n being an axiom schema. Since $d\mathscr{F}$ and $nd\mathscr{F}$ use the same axiom schemata, we prove A_n in $nd\mathscr{F}$ in one step by simply writing it down.

CASE 2.3. $\Gamma_n \vDash A_n$ *is inferred by* mp_g.

This means that $\Gamma_n \vDash A_n$ has the form $\Gamma_n \vDash B$ and there are two such sequents, namely $\Gamma_{n_1} \vDash A$ and $\Gamma_{n_2} \vDash A \supset B$, that $n_1, n_2 < n$ and $\Gamma_{n_1} \cup \Gamma_{n_2} = \Gamma_n$. These sequents become $\bigwedge \Gamma_n \supset B$, $\bigwedge \Gamma_{n_1} \supset A$, and $\bigwedge \Gamma_{n_2} \supset (A \supset B)$ in π' respectively. We fill in the gap as follows.

$$\begin{array}{l}
\vdots \\
\bigwedge \Gamma_{n_1} \supset A \\
\vdots \\
\bigwedge \Gamma_{n_2} \supset (A \supset B) \\
\left[\begin{array}{l}
\bigwedge \Gamma_n \text{ — assumption} \\
\vdots (*) \\
\bigwedge \Gamma_{n_1} \\
A \text{ — } mp_n \\
\vdots (*) \\
\bigwedge \Gamma_{n_2} \\
A \supset B \text{ — } mp_n \\
B \text{ — } mp_n
\end{array}\right. \\
\bigwedge \Gamma_n \supset B \text{ — } dr_n
\end{array}$$

Let Γ_n contain m formulas, Γ_{n_1} — m_1 formulas, and Γ_{n_2} — m_2 formulas. It is clear that neither of these numbers is greater than n. One can show by induction on m that both $\bigwedge \Gamma_{n_1}$ and $\bigwedge \Gamma_{n_2}$ can be inferred from $\bigwedge \Gamma_n$ in $O(m)$ steps.

CASE 2.4. *$\Gamma_n \vDash A_n$ is inferred by dr_g.*

$\Gamma_n \vDash A_n$ has the form $\Gamma_n \vDash A \supset B$ which becomes $\bigwedge \Gamma_n \supset (A \supset B)$ in π'. Moreover, we have $\Gamma_{n_1} \vDash B$ in π_g such that $n_1 < n$ and $\Gamma_n = \Gamma_{n_1} \setminus A$. $\Gamma_{n_1} \vDash B$ becomes $\bigwedge \Gamma_{n_1} \supset B$. We have two cases depending on whether A is in Γ_{n_1}.

CASE 2.4.1. $A \notin \Gamma_{n_1}$.

In this case $\Gamma_n = \Gamma_{n_1}$. We proceed as follows.

$$\begin{array}{l}
\vdots \\
\bigwedge \Gamma_{n_1} \supset B \\
\left[\begin{array}{l}
\bigwedge \Gamma_{n_1} \text{ — assumption} \\
B \text{ — } mp_n \\
\left[\; A \text{ — assumption} \right. \\
A \supset B \text{ — } dr_n
\end{array}\right. \\
\bigwedge \Gamma_n \supset (A \supset B) \text{ — } dr_n
\end{array}$$

CASE 2.4.2. $A \in \Gamma_{n_1}$

The simulation goes as follows.

$$
\begin{array}{l}
\vdots \\
\bigwedge \Gamma_{n_1} \supset B \\
\left[\begin{array}{l}
\bigwedge \Gamma_n - \text{assumption} \\
\left[\begin{array}{l}
A - \text{assumption} \\
\vdots (*) \\
\bigwedge \Gamma_{n_1} \\
B - mp_n
\end{array}\right. \\
A \supset B - dr_n
\end{array}\right. \\
\bigwedge \Gamma_n \supset (A \supset B) - dr_n \text{ because } \Gamma_n = \Gamma_{n_1}
\end{array}
$$

Let Γ_n contain m formulas and $\Gamma_{n_1} - m_1$ formulas. It is clear that neither of these numbers is greater than n. One can show by induction on m that $\bigwedge \Gamma_{n_1}$ can be inferred from $\bigwedge \Gamma_n$ and A in $O(m)$ steps. □

2.1. Fitch-style systems

THEOREM 3. $\Gamma \big|\frac{nd\mathscr{F}}{n} D \Rightarrow \Gamma \big|\frac{\mathbf{F}}{O(n)} D$

PROOF. We prove this theorem by induction on n. Our induction hypothesis is as follows. *For any $m < n$ **F** linearly simulates $nd\mathscr{F}$ so that all formulas occurring in an $nd\mathscr{F}$ derivation also occur in the **F** derivation.*

The proof splits into four cases depending on how D was derived.

CASE 3.1. *D is either an assumption or a member of Γ.*

In this case D becomes an assumption or a member of Γ in the **F**-derivation respectively.

CASE 3.2. *D is an instance of an axiom schema.*

In this case D can be proved in **F** in a constant number of steps.

CASE 3.3. *D is derived by mp_n.*

In this case $D = B$ and an $nd\mathscr{F}$-derivation has one of the two following forms:

$$
\left[\begin{array}{l}
\vdots \\
A \supset B \\
\vdots \\
A \\
B - mp_n
\end{array}\right.
\qquad\qquad
\begin{array}{l}
\vdots \\
A \supset B \\
\left[\begin{array}{l}
\vdots \\
A \\
B - mp_n
\end{array}\right.
\end{array}
$$

We proceed respectively as follows.

$$
\begin{array}{l|ll}
 & \vdots & \\
 & A \supset B & \text{IH} \\
 & \vdots & \\
 & A & \text{IH} \\ \cline{2-2}
 & B & \supset\text{E}
\end{array}
\qquad
\begin{array}{l|l|ll}
 & \vdots & & \\
 & A \supset B & & \text{IH} \\ \cline{2-3}
 & & \vdots & \\
 & & A & \text{IH} \\
 & & A \supset B & \text{R} \\ \cline{3-3}
 & & B & \supset\text{E}
\end{array}
$$

In both cases we derive A and $A \supset B$ by induction hypothesis and B by $\supset$E in a constant number of steps. In the right-hand case, however, we have to reiterate one of the hypotheses of $\supset$E which takes us exactly one step.

Case 3.4. *D was derived by dr_n.*

The proof of this case is straightforward since in both $nd\mathscr{F}$ and **F** when dr_n (respectively $\supset$I) is applied, the last open assumption has to be closed. Hence, we simply substitute an instance of dr_n with an instance of $\supset$I. □

Theorem 4. $\Gamma |\frac{\mathbf{F}}{n} D \Rightarrow \Gamma |\frac{nd\mathscr{F}}{O(n)} D$.

Proof. We prove the theorem by induction on n. The proof splits into cases, depending on how D was derived.

If D is a hypothesis or an assumption in **F**-derivation, it becomes a hypothesis (resp. assumption) in an $nd\mathscr{F}$-derivation.

If D was reiterated, this simply means that it had already been derived earlier. Hence we obtain it by the induction hypothesis.

Finally, D could be derived by one of the rules of inference. The only interesting case here would be rule $\vee$E since we substitute all instances of $\supset$I with dr_n and in all other cases we simply derive conclusion of a rule from its premises. This can be done in a constant number of steps because both $nd\mathscr{F}$ and **F** are sound and complete. If D was derived by $\vee$E, **F**-derivation has the following form.

$$
\begin{array}{|l}
\vdots \\
A \vee B \\
\hline
\quad \begin{array}{|l} A \\ \hline \vdots \\ D \end{array} \\
\quad \begin{array}{|l} B \\ \hline \vdots \\ D \end{array} \\
D \qquad\qquad \vee\mathrm{E}
\end{array}
$$

We proceed as follows.

$$
\begin{array}{l}
\vdots \\
A \vee B \\
\left[\begin{array}{l} A - \text{assumption} \\ \vdots \\ D - \text{by IH} \end{array}\right. \\
A \supset D - dr_n \\
\left[\begin{array}{l} B - \text{assumption} \\ \vdots \\ D - \text{by IH} \end{array}\right. \\
B \supset D - dr_n \\
\vdots \\
D - \text{in a constant number of steps}
\end{array}
$$

As one can see, we have derived D in a constant number of steps. □

2.2. Gentzen's natural deduction ND

THEOREM 5. $\Gamma|\frac{d\mathscr{F}}{n} D \Rightarrow \Gamma|\frac{\mathrm{ND}}{O(n)} D$.

PROOF. qThe proof is straightforward since each step in $d\mathscr{F}$-derivation is either an axiom which has an ND-proof of constant length, an assumption or member of Γ which remains an assumption or member of Γ in ND-derivation, or was derived by either mp_g or dr_g. We substitute an instance of mp_g for FB — $\frac{A \quad A \supset B}{B}$ — and dr_g for FE — $\dfrac{\begin{matrix}[A]\\ B\end{matrix}}{A \supset B}$ [4]. We can do this since we aren't required to close the last open assumption in both $d\mathscr{F}$ and ND. □

THEOREM 6. $\Gamma|\frac{\text{ND}}{n} D \Rightarrow \Gamma|\frac{d\mathscr{F}}{O(n)} D$.

PROOF. Recall that we assumed that ND has dag-like proofs. This means that we can, following Reckhow [13] designate each step of the derivation of C from Γ as $\Delta \vDash A$ with Δ being the set of all open assumptions and formulas from Γ.

We need this convention because $d\mathscr{F}$ proofs are sequences of sequents (cf. [1, P.691]), not trees of them so our simulation results won't be affected by the necessity of deriving one formula multiple times from same assumptions.

We prove the theorem by induction on n.

CASE 6.1 (Base case, $n = 1$).

If $n = 1$, D is either an assumption or a member of Γ. In both cases we have a $d\mathscr{F}$-derivation of exactly one line, namely, $D \vDash D$.

CASE 6.2 (Induction step).

We suppose that for all $m < n$ if all occurrences of falsum are substituted for the fixed contradictory formula — $p_1 \wedge \neg p_1$ $d\mathscr{F}$ simulates ND linearly so that every step occuring in the ND derivation, except for occurrences of falsum is present in the $d\mathscr{F}$ derivation. Occurrences of $\curlywedge$ are substituted with $p_1 \wedge \neg p_1$. We now need to show this for n. We have different cases depending on how D was derived.

The cases of all rules, except for

$$\text{OB: } \frac{A \vee B \qquad \begin{matrix}[A]\\ C\end{matrix} \qquad \begin{matrix}[B]\\ C\end{matrix}}{C}$$

and

$$\text{NE: } \frac{\begin{matrix}[A]\\ \curlywedge\end{matrix}}{\neg A}$$

are straightforward and will be omitted here.

CASE 6.2.1. *D was derived by OB.*

[4] $[A]$ means that assumption A is discharged.

In this case $D = C$ and the ND-derivation has the following form.

$$\text{OB: } \dfrac{\begin{array}{c}\vdots\\ A \vee B\end{array} \qquad \begin{array}{c}[A]\\ \vdots\\ C\end{array} \qquad \begin{array}{c}[B]\\ \vdots\\ C\end{array}}{C}$$

Since ND derivations are dag-like, we can rewrite this derivation and get the following one.

$$\begin{array}{rcl}
 & \vdots & \\
A & \models & A \\
 & \vdots & \\
\Gamma_1, A & \models & C \\
B & \models & B \\
 & \vdots & \\
\Gamma_2, B & \models & C \\
 & \vdots & \\
\Gamma_3 & \models & A \vee B \\
 & \vdots & \\
\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 & \models & C \qquad\qquad \text{OB}
\end{array}$$

$$\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \Gamma$$

Our simulation goes as follows.

$$\begin{array}{rcll}
A & \models & A & \\
 & \vdots & & \\
\Gamma_1, A & \models & C & \text{by IH} \\
B & \models & B & \\
 & \vdots & & \\
\Gamma_2, B & \models & C & \text{by IH} \\
 & \vdots & & \\
\Gamma_3 & \models & A \vee B & \text{by IH} \\
\Gamma_1 & \models & A \supset C & dr_g \\
\Gamma_2 & \models & B \supset C & dr_g \\
 & \vdots & & \\
\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 & \models & C & \text{in a constant number of steps}
\end{array}$$

Case 6.2.2. *D was derived by NE.*

In this case $D \rightleftharpoons \neg A$ and the ND-derivation has the following form.

$$\text{NE: } \frac{\begin{array}{c}[A]\\ \vdots \\ \curlywedge\end{array}}{\neg A}$$

Our simulation goes as follows.

$$\begin{array}{rcll}
A & \vDash & A & \\
 & \vdots & & \\
\Gamma, A & \vDash & p_1 \wedge \neg p_1 & \text{by IH} \\
 & \vdots & & \\
 & \vDash & (p_1 \wedge \neg p_1) \supset p_1 & \text{in a constant number of steps} \\
 & \vDash & (p_1 \wedge \neg p_1) \supset \neg p_1 & \text{in a constant number of steps} \\
\Gamma, A & \vDash & p_1 & mp_g \\
\Gamma, A & \vDash & \neg p_1 & mp_g \\
\Gamma & \vDash & A \supset p_1 & dr_n \\
\Gamma & \vDash & A \supset \neg p_1 & dr_n \\
 & \vdots & & \\
 & \vDash & (A \supset p_1) \supset ((A \supset \neg p_1) \supset \neg A) & \text{in a constant number of steps} \\
\Gamma & \vDash & (A \supset \neg p_1) \supset \neg A & mp_g \\
\Gamma & \vDash & \neg A & mp_g
\end{array}$$

In both cases we have derived D in a constant number of steps. □

3. A simulation of Gentzen-style systems

In this section we will prove theorems considering pairwise simulation of $d\mathscr{F}$, PKT and PKT*. We start with a theorem about the simulation of PKT by $d\mathscr{F}$.

Once again, recall that we assume our PKT and PKT* proofs to be dag-like, not tree-like. This lets us use one sequent many times without need to derive it multiple times. Note, also, that the additive version of the cut rule is used.

Lemma 1. *Assume, we have a sequent $A_1, \ldots, A_n \rightarrow C$. Then it takes no more than $O(n^2)$ steps of* PKT *proof to remove all repeated formulas from the antecedent.*

Proof. This statement can be easily proved since we need no more than $2n$ permutations and then one contraction for any repeated formula to be removed. □

Lemma 2. *Assume, we have a sequent $A_1, \ldots, A_n \to C$. Then we need no more than $O(m \cdot n)$ steps of* PKT *proof to augment its antecedent with m formulas in any order.*

Proof. The proof is straightforward since we only need to add m formulas and put them on their places which won't take more than n permutations for each formula. □

Theorem 7. $\Gamma |\frac{d\mathscr{F}}{n} D \Rightarrow \Gamma |\frac{\text{PKT}}{O(n^3)} D$.

Proof. We will now prove the theorem by induction on n. The simulation goes as follows: every sequent $\Gamma_i \vDash A_i$ from $d\mathscr{F}$-derivation becomes a sequent $\Gamma_i \to A_i$ with formulas in the antecedent being in an arbitrary order. We need to show that the gaps can be filled in in $O(n^2)$ steps.

An assumption or a member of Γ, i.e., the line of the form $A \vDash A$ becomes $A \to A$ which is an initial sequent. An axiom $\vDash A$ becomes $\to A$ which has a constant-length PKT-proof.

Sequents constituting mp_g $\left(\frac{\Gamma_1 \vDash A \supset B \quad \Gamma_2 \vDash A}{\Gamma_1 \cup \Gamma_2 \vDash B}\right)$ rule of inference become $\Gamma_1 \to A \supset B$, $\Gamma_2 \to A$ and $\Gamma_1 \cup \Gamma_2 \to B$ in PKT derivation. We show that third sequent can be derived in PKT from first and second ones in $O(n^2)$ steps.

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma_1 \to A \supset B$}
\LeftLabel{$O(n^2)$ W_l and E_l}
\doubleLine
\UnaryInfC{$\Gamma_1, \Gamma_2 \to A \supset B$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma_2 \to A$}
\LeftLabel{W_l}
\UnaryInfC{$\Gamma_2 \to A, B$}
\LeftLabel{E_l}
\UnaryInfC{$\Gamma_2 \to B, A$}
\AxiomC{$B \to B$}
\LeftLabel{W_l}
\doubleLine
\UnaryInfC{$\Gamma_2, B \to B$}
\LeftLabel{$\supset_l$}
\BinaryInfC{$A \supset B, \Gamma_2 \to B$}
\LeftLabel{W_l and E_l}
\doubleLine
\UnaryInfC{$A \supset B, \Gamma_1, \Gamma_2 \to B$}
\LeftLabel{Cut}
\BinaryInfC{$\Gamma_1, \Gamma_2 \to B$}
\LeftLabel{$O(n^2)$ E_l and C_l}
\doubleLine
\UnaryInfC{$\Gamma \to B$}
\end{prooftree}
$$

We use lemmas 1 and 2 to derive $\Gamma \to B$ from $\Gamma_1, \Gamma_2 \to B$ and $\Gamma_1, \Gamma_2 \to A \supset B$ from $\Gamma_1 \to A \supset B$ respectively. It is also clear that we need $O(n)$ steps to derive $\Gamma_2, B \to B$ from $B \to B$ and $A \supset B, \Gamma_1, \Gamma_2 \to B$ from $A \supset B, \Gamma_2 \to B$.

Finally consider the case of dr_g $\left(\frac{\Gamma \vDash B}{\Gamma \setminus \{A\} \vDash A \supset B}\right)$. The sequents become $\Gamma \to B$ and $\Gamma \setminus \{A\} \to A \supset B$ in the PKT derivation. We have two cases: $A \in \Gamma$ (left) and $A \notin \Gamma$ (right).

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \to B}}{A, \Gamma \setminus \{A\} \to B}\leqslant n\ E_l}{\Gamma \setminus \{A\} \to A \supset B}\supset_r \qquad \dfrac{\dfrac{\dfrac{\vdots}{\Gamma \to B}}{A, \Gamma \to B}W_l}{\Gamma \to A \supset B}\supset_r$$

□

Theorem 8. $\Gamma|\frac{d\mathscr{F}}{n} D \Rightarrow \Gamma|\frac{\mathrm{PKT}^*}{O(n)} D$.

The proof is straightforward since we can see from the proof of Theorem 7 that if we don't count steps made by weakening, permutation and contraction rules (which in the case of PKT* we don't), then we can fill every gap in a constant number of steps. □

We will finally prove that $d\mathscr{F}$ linearly simulates PKT*. Since it is clear that PKT* linearly simulates PKT, it will also entail the linear simulation of PKT by $d\mathscr{F}$.

Theorem 9. *If* $\Gamma|\frac{\mathrm{PKT}^*}{n} D$, *then there is such a subset* $\Gamma' \subseteq \Gamma$ *that* $\Gamma'|\frac{d\mathscr{F}}{O(n)} D$.

We prove this theorem the same way as Buss and Bonet proved Theorem 11 in [1]. We need the following lemma.

Lemma 3. *If there is a PKT*-proof* π_{PKT^*} *of* $\Gamma \to \Delta$ *of length* n*, then there is such a subset* $\Xi \subseteq (\Gamma \cup \neg\Delta)$ ($\neg\Delta$ *denotes that every formula in* Δ *is negated), that there exists a* $d\mathscr{F}$*-proof* $\pi_{d\mathscr{F}}$ *of a sequent* $\Xi \vDash p_1 \wedge \neg p_1$ *of length* $O(n)$.

Case 3.1. *Base case.* $n = 1$. If the length of π_{PKT^*} is 1, then it consists only of an initial sequent, say, $A \to A$. It takes a constant number of steps to prove $A, \neg A \vDash p_1 \wedge \neg p_1$ in $d\mathscr{F}$.

Case 3.2. *Induction step.* We assume that for all $m < n$ there exists such a constant c that $|\frac{d\mathscr{F}}{c \cdot m} \Xi \vDash p_1 \wedge \neg p_1$ and prove the lemma for n. The proof splits depending on how the last line of π_{PKT^*} was inferred. We will prove the most representative cases.

Case 3.2.1. $\neg_r$. The end of π_{PKT^*} has the following form.

$$\neg_r\,\dfrac{\dfrac{\vdots}{A, \Gamma \to \Delta}}{\Gamma \to \Delta, \neg A}$$

By the induction hypothesis there exist a $d\mathscr{F}$-proof $\pi'_{d\mathscr{F}}$ of $\Xi' \vDash p_1 \wedge \neg p_1$ ($\Xi' \subseteq (\{A\} \cup \Gamma \cup \neg\Delta)$) of length $c \cdot m$. We need to construct a

$d\mathscr{F}$-proof $\pi_{d\mathscr{F}}$ of $\Xi \models p_1 \wedge \neg p_1$ ($\Xi \subseteq (\{\neg\neg A\} \cup \Gamma \cup \neg\Delta)$) of length $O(n)$.

$$\begin{array}{rcll} \neg\neg A & \models & \neg\neg A & \text{assumption} \\ \Xi & \models & \Xi & \text{assumptions} \\ & \vdots & & \\ \neg\neg A & \models & A & \text{in a constant number of steps} \\ & \vdots & & \\ \Xi & \models & p_1 \wedge \neg p_1 & \text{from } \pi'_{d\mathscr{F}} \end{array}$$

Case 3.2.2. $\wedge_r$. The end of π_{PKT^*} has the following form.

$$\wedge_r \frac{\begin{array}{cc} \vdots & \vdots \\ \Gamma \to \Delta, A & \Gamma \to \Delta, B \end{array}}{\Gamma \to \Delta, A \wedge B}$$

By the induction hypothesis there exists a proof $\pi'_{d\mathscr{F}}$ of length $c \cdot m$ containing both $\Xi_1 \models p_1 \wedge \neg p_1$ ($\Xi_1 \subseteq (\{\neg A\} \cup \Gamma \cup \neg\Delta)$) and $\Xi_2 \models p_1 \wedge \neg p_1$ ($\Xi_2 \subseteq (\{\neg B\} \cup \Gamma \cup \neg\Delta)$). We construct the $d\mathscr{F}$-proof $\pi_{d\mathscr{F}}$ of $\Xi \models p_1 \wedge \neg p_1$ ($\Xi \subseteq (\{\neg(A \wedge B)\} \cup \Gamma \cup \neg\Delta)$) as follows.

$$\begin{array}{rcll} \neg(A \wedge B) & \models & \neg(A \wedge B) & \text{assumption} \\ \Xi & \models & \Xi & \text{assumptions} \\ & \vdots & & \\ \neg(A \wedge B) & \models & \neg A \vee \neg B & \text{in a constant number of steps} \\ & \vdots & & \\ \Xi_1 & \models & p_1 \wedge \neg p_1 & \text{from } \pi'_{d\mathscr{F}} \\ \Xi_2 & \models & p_1 \wedge \neg p_1 & \text{from } \pi'_{d\mathscr{F}} \\ \Xi_1 \setminus \{A\} & \models & \neg A \supset (p_1 \wedge \neg p_1) & dr_g \\ \Xi_2 \setminus \{B\} & \models & \neg B \supset (p_1 \wedge \neg p_1) & dr_g \\ & \vdots & & \\ \Xi \setminus \{\neg(A \wedge B)\} & \models & (\neg A \vee \neg B) \supset (p_1 \wedge \neg p_1) & \text{in a constant number of steps} \\ \Xi & \models & (p_1 \wedge \neg p_1) & mp_g \end{array}$$

Case 3.2.3. *Cut.* The end of π_{PKT^*} has the following form:

$$\mathrm{Cut} \frac{\begin{array}{cc} \vdots & \vdots \\ \Gamma \to \Delta, A & A, \Gamma \to \Delta \end{array}}{\Gamma \to \Delta}$$

By the induction hypothesis there exists a $d\mathscr{F}$-proof $\pi'_{d\mathscr{F}}$ of length $c \cdot m$ containing both $\Xi_1 \vDash p_1 \wedge \neg p_1$ $(\Xi_1 \subseteq (\Gamma \cup \neg\Delta \cup \{\neg A\}))$ and $\Xi_2 \vDash p_1 \wedge \neg p_1$ $(\Xi_2 \subseteq (\Gamma \cup \neg\Delta \cup \{A\}))$. We construct the proof of $\Xi \vDash p_1 \wedge \neg p_1$ $(\Xi \subseteq (\Gamma \cup \neg\Delta))$ as follows.

$$
\begin{array}{rcll}
\neg A & \vDash & \neg A & \text{assumption} \\
A & \vDash & A & \text{assumption} \\
\Xi \setminus \{A, \neg A\} & \vDash & \Xi \setminus \{A, \neg A\} & \text{assumptions} \\
 & \vdots & & \\
\Xi_1 & \vDash & p_1 \wedge \neg p_1 & \text{from } \pi'_{d\mathscr{F}} \\
\Xi_2 & \vDash & p_1 \wedge \neg p_1 & \text{from } \pi'_{d\mathscr{F}} \\
\Xi & \vDash & A \supset (p_1 \wedge \neg p_1) & dr_g \\
\Xi & \vDash & \neg A \supset (p_1 \wedge \neg p_1) & dr_g \\
 & \vdots & & \\
 & \vDash & A \vee \neg A & \text{in a constant number of steps} \\
 & \vdots & & \\
\Xi & \vDash & p_1 \wedge \neg p_1 & \text{in a constant number of steps}
\end{array}
$$

The result follows taking c to be not less than any constant number of steps in any case. ⊣

Theorem 9 follows by the application of Lemma 13 from [1] and Theorem 1 to Lemma 3. □

4. Concluding remarks

The results from this paper allow us to include **F** in the scheme depicted on Figure 1. Combining our results with those from [1], we get the scheme depicted in Figure 2.

It is worth mentioning that the quadratic speedup of $d\mathscr{F}$ over $nd\mathscr{F}$ is due to the deduction rule that allowed for the exclusion of arbitrary assumption instead of the last open one. It turns out that the necessity of reiterating formulas does not lead to a non-linear speedup of one calculus over another since we do not need to infer a formula once more.

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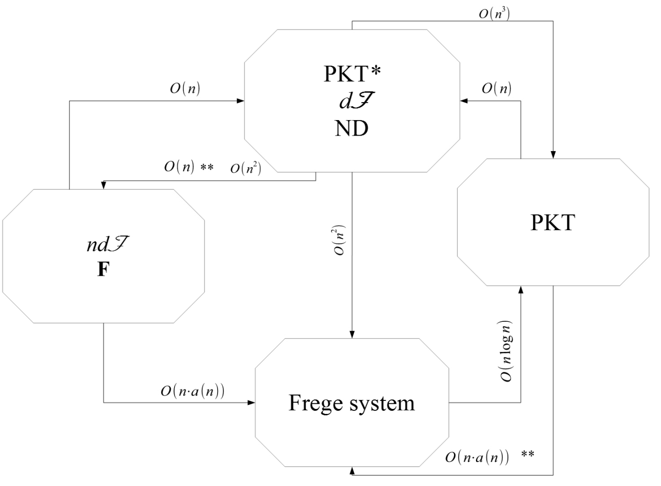

Figure 2. ** — for proofs wherein no line is used a hypothesis of a rule of inference more than once; a is inverse Ackermann function

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