Logic and Logical Philosophy
Volume 27 (2018), 193–203
DOI: 10.12775/LLP.2017.008

Anna Maria Karczewska

MAXIMALITY OF THE MINIMAL $\mathcal{R}$-LOGIC

Abstract. The minimal system of the connective of realization — Jarmużek and Pietruszczak's MR — is examined. The single-index rule is defined. Then it is claimed that if a single-index rule non-derivable in MR is derivable in a strengthening of MR, then the strengthening is inconsistent. This property may be called *single-index maximality.*

Keywords: maximality; positional logics; connective of realization

1. Introduction

The system MR that we will be dealing with is a logic of the connective '$\mathcal{R}$' of realization, relating sentences to the contexts they are true in. The first to examine systematically the logical properties of the connective of realization was Jerzy Łoś in his paper [4]. Łoś' objective was to provide a logical tool for formalizing empirical sentences such as 'it is raining in Lublin on 11^{th} May 2015'. According to him the logical schemata of such sentences should reflect their being composed of a propositional function, referring to some physical phenomenon, and a temporal determination. The expression 'at ... it is the case that ... ' that Łoś studied may be called the connective of temporal realization. Yet realization does not need to be understood temporally: '$\mathcal{R}$' may instead stand for some epistemic, spatial, deontic, or other connective of natural language. Logics containing the connective of realization, irrespective of interpretation, are usually known as topological [6], locative [1] or positional [2] logics.

Tomasz Jarmużek and Andrzej Pietruszczak's system MR has been constructed as the minimal one such that the connective of realization is distributive over all classical connectives. It is claimed here that MR is, in addition, maximal in an interesting sense.

Received September 26, 2016. *Revised* March 10, 2017. *Published online* April 20, 2017

2. Basic positional language

The basic $\mathcal{R}$-language $\mathbb{L}$ is determined by the set $\mathbb{SL}$ of sentence letters and the set $\mathbb{IN}$ of individual constants, both infinite but denumerable, and connectives: '$\neg$" of negation, '$\wedge$' of conjunction, '$\vee$' of disjunction, '$\rightarrow$' of conditional, '$\equiv$' of biconditional, and the positional connective '$\mathcal{R}$' (or the connective of realization), as well as parentheses.

The smallest set containing the set $\mathbb{SL}$ and such that if $\varphi \in \mathbb{QF}$, then $\ulcorner(\neg\varphi)\urcorner \in \mathbb{QF}$, and given that both φ and $\psi \in \mathbb{QF}$ all $\ulcorner(\varphi\wedge\psi)\urcorner$, $\ulcorner(\varphi\vee\psi)\urcorner$, $\ulcorner(\varphi \rightarrow \psi)\urcorner$, $\ulcorner(\varphi \equiv \psi)\urcorner \in \mathbb{QF}$, will be referred to as $\mathbb{QF}$ or the set of quasi–formulas. Clearly, the set $\mathbb{QF}$ is equal to the set of formulas of classical propositional language. Nevertheless, in the language $\mathbb{L}$ quasi–formulas represent propositional functions rather than full–blooded propositions, and may exclusively play the role of arguments (together with individual constants) of the connective '$\mathcal{R}$' in atomic formulas of $\mathbb{L}$.

Accordingly, by an atomic formula we shall mean a string of symbols of the form $\ulcorner\mathcal{R}_\alpha\varphi\urcorner$, where $\varphi \in \mathbb{QF}$, $\alpha \in \mathbb{IN}$. $\ulcorner\mathcal{R}_\alpha\varphi\urcorner$ may be read "φ at α", or similarly, in conformity with the intended interpretation. All atomic formulas are in $\mathbb{L}$.

Compound formulas of $\mathbb{L}$ are built up by means of the propositional connectives in the usual way: if $A \in \mathbb{L}$, then $\ulcorner\neg A\urcorner \in \mathbb{L}$, if both $A, B \in \mathbb{L}$, then $\ulcorner(A \wedge B)\urcorner$, $\ulcorner(A \vee B)\urcorner$, $\ulcorner(A \rightarrow B)\urcorner$, $\ulcorner(A \equiv B)\urcorner \in \mathbb{L}$.

Substitutions relevant to the language $\mathbb{L}$ are determined by the following mappings: l in the set $\mathbb{IN}$ and e from $\mathbb{SL}$ into the set $\mathbb{QF}$. If $\mathfrak{sub}^e$ is an endomorphism in $\mathbb{QF}$ extending e, then a substitution in $\mathbb{L}$ is any mapping $\mathfrak{Sub}^e_l : \mathbb{L} \longrightarrow \mathbb{L}$, such that:

$$\begin{aligned}
\mathfrak{Sub}^e_l(\mathcal{R}_\alpha\varphi) &:= \ulcorner\mathcal{R}_{l(\alpha)}(\mathfrak{sub}^e(\varphi))\urcorner,\\
\mathfrak{Sub}^e_l(\neg A) &:= \neg\mathfrak{Sub}^e_l(A),\\
\mathfrak{Sub}^e_l(A * B) &:= \ulcorner\mathfrak{Sub}^e_l(A) * \mathfrak{Sub}^e_l(B)\urcorner, \quad \text{for } * \in \{\wedge, \vee, \rightarrow, \equiv\}.
\end{aligned}$$

If $X \subseteq \mathbb{L}$, then $\mathfrak{Sub}^e_l(X)$ is an image of X under the function $\mathfrak{Sub}^e_l$. The substitution $\mathfrak{Sub}^e_l$ is adopted, with a slight modification, from [7, pp. 203–204].

A rule r in $\mathbb{L}$ is a set of pairs (X, A), such that $X \subseteq \mathbb{L}$, $A \in \mathbb{L}$. All the rules closed under $\mathfrak{Sub}^e_l$ (and only those), i.e., such that for all $X \subseteq \mathbb{L}$, $A \in \mathbb{L}$, and for all e, l, if $(X, A) \in r$, then $(\mathfrak{Sub}^e_l(X), \mathfrak{Sub}^e_l(A)) \in r$, are said to be *structural* in $\mathbb{L}$.

The simplest extensions of the basic positional language with the above vocabulary involve admitting sentence letters as atomic formulas and allowing for iterations of '$\mathcal{R}$'. Early studies of positional logic concerned even stronger languages containing additional symbols.

3. Maximality

What we are aiming at is to prove a certain kind of maximality result. In the sequel we will make use of some basic proof-theoretic notions. The present section will supply the needed definitions as applied to the basic positional language $\mathbb{L}$. Still more importantly, as we dare to name a new property with an old label, it will serve to situate our approach in the context of the former practice.

A system S in $\mathbb{L}$ is defined by a set $\Phi \subseteq \mathbb{L}$ of axioms and a set R of primitive rules in $\mathbb{L}$.

DEFINITION 1. Given $A \in \mathbb{L}$, $X \subseteq \mathbb{L}$, A is a consequence of X in S (symbolically: $X \vdash_S A$) if and only if there is a finite sequence of formulas each of which is either an axiom of S or a member of X or is obtained from preceding members by means of a primitive rule of S, and whose last member is A. Whenever S is fixed unambiguously, we will forgo the subscript in $\vdash_S$.

If $\varnothing \vdash_S A$ ($\vdash_S A$ for short), then A is said to be a *theorem* of S.

DEFINITION 2. A rule r is derivable in S (or simply: is a rule of S) if and only if $X \vdash_S A$, for every $(X, A) \in r$.

THEOREM 1. *If a rule r is derivable in S, then the set of S's theorems is closed under r, i.e., if $(X, A) \in r$ and for every $B \in X$, $\vdash_S B$, then $\vdash_S A$.*

Theorem 1 may be easily proved using definitions 1 and 2.

DEFINITION 3. $X \subseteq \mathbb{L}$ is inconsistent in S, if $X \vdash_S A$, for every $A \in \mathbb{L}$. Otherwise X is consistent.

DEFINITION 4. A system S is a subsystem of a system S' (symbolically: $S \preccurlyeq S'$) if and only if for all $X \subseteq \mathbb{L}$, $A \in \mathbb{L}$: if $X \vdash_S A$, then $X \vdash_{S'} A$.

DEFINITION 5. A system S is a proper subsystem of a system S' (symbolically: $S \prec S'$) if and only if $S \preccurlyeq S'$, but $S' \not\preccurlyeq S$.

If $S \prec S'$, then S' is said to be *properly stronger* then S (or an extension of S) and S — *weaker* then S'.

DEFINITION 6. A consistent logic S is maximal if and only if any system S' such that $S \prec S'$ is inconsistent.

Definition 6 determines the classical notion of maximality. Several non-classical counterparts of the notion of maximality and of related notion of Post-completeness have been defined, as it has been realized that the classical account does not meet the cases of *invariant* systems, i.e., systems whose all derivable rules are structural. For a general survey see [5]; let us mention here just two examples. First, even if a invariant system cannot be maximal (provided it is consistent), it can still be the strongest among invariant systems. Thus it may be more reasonable to consider only invariant strengthenings instead of considering every strengthening possible. This is how $*$-maximality can be characterized.

Definition 7. A consistent and invariant system S is $*$-maximal if and only if every invariant S', such that $S \prec S'$, is inconsistent.

In an attempt to provide a uniform and generalized treatment, Pogorzelski and Wojtylak introduced the notion of Γ-maximality [5, p. 92]. In short, generalized maximality, or Γ-maximality, is a maximality relativized to some subset Γ of the set of formulas of the apposite language L. Let a rule r be Γ-*structural*, iff for arbitrary formula A, a set of formulas X and a mapping $e\colon At \longrightarrow \Gamma$ (At being the set of atoms), if $(X, A) \in r$, then $(e(X), e(A)) \in r$ [5, p. 29]. A system is Γ-*invariant* if its derivable rules are Γ-structural.

Definition 8. A system S is said to be maximal with respect to Γ (Γ-maximal) if and only if S is Γ-invariant and every Γ-invariant S', such that $S \prec S'$, is inconsistent.

Any system S, if consistent, is $\varnothing$-maximal if and only if S is maximal and S is L–maximal if and only if S is $*$-maximal [5, pp. 98, 120].

4. The system of minimal realization

The system MR in $\mathbb{L}$ has been defined by Jarmużek and Pietruszczak based on classical propositional logic (hereafter: PC): every instance of any PC theorem is an axiom and *modus ponens* is assumed as the unique primitive rule of inference. MR's specific axioms include formulas of the following forms:

$$\mathcal{R}_\alpha\varphi, \text{ for any PC tautology } \varphi, \tag{A1}$$

$$\mathcal{R}_\alpha\neg\varphi \equiv \neg\mathcal{R}_\alpha\varphi, \tag{A2}$$

$$\mathcal{R}_\alpha\varphi \wedge \mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha(\varphi \wedge \psi). \tag{A3}$$

According to (A1) every atomic formula having a classical logic theorem as its argument is an axiom of MR. Schemata (A2) and (A3) determine the relationship between internal (i.e., in the scope of $\mathcal{R}$) and external (outside the scope of $\mathcal{R}$) occurrences of the negation and conjunction connectives.

Formulas falling under (A2) or (A3) are only some examples of distributional laws characteristic for MR. In fact, '$\mathcal{R}$' is distributive over all propositional connectives, that is all equivalences: (A2) and

$$\mathcal{R}_\alpha(\varphi \wedge \psi) \equiv \mathcal{R}_\alpha\varphi \wedge \mathcal{R}_\alpha\psi,$$
$$\mathcal{R}_\alpha(\varphi \vee \psi) \equiv \mathcal{R}_\alpha\varphi \vee \mathcal{R}_\alpha\psi,$$
$$\mathcal{R}_\alpha(\varphi \rightarrow \psi) \equiv \mathcal{R}_\alpha\varphi \rightarrow \mathcal{R}_\alpha\psi,$$
$$\mathcal{R}_\alpha(\varphi \equiv \psi) \equiv (\mathcal{R}_\alpha\varphi \equiv \mathcal{R}_\alpha\psi),$$

are schemata of MR's theorems [2, pp. 151–153]. Actually, MR is intended to be the minimal calculus enjoying this property.

The original semantic interpretation of MR is provided by the structure of the form $\mathfrak{M} = \langle \mathfrak{X}, \mathfrak{d}, \mathfrak{v} \rangle$, where $\mathfrak{X}$ is a non-empty set, $\mathfrak{d}$ is a mapping interpreting members of $\mathbb{N}$, i.e., $\mathfrak{d}\colon \mathbb{N} \longrightarrow \mathfrak{X}$, and $\mathfrak{v}$ is a mapping interpreting members of $\mathbb{QF}$, i.e., $\mathfrak{v}\colon \mathfrak{X} \times \mathbb{QF} \longrightarrow \{1, 0\}$ and for all φ, $\psi \in \mathbb{QF}$ and $x \in \mathfrak{X}$ satisfies the following conditions:

$$\begin{aligned}
\mathfrak{v}(x, \ulcorner\neg\varphi\urcorner) = 1 &\text{ if and only if } \mathfrak{v}(x,\varphi) = 0,\\
\mathfrak{v}(x, \ulcorner\varphi \wedge \psi\urcorner) = 1 &\text{ if and only if } \mathfrak{v}(x,\varphi) = 1 \text{ and } \mathfrak{v}(x,\psi) = 1,\\
\mathfrak{v}(x, \ulcorner\varphi \vee \psi\urcorner) = 1 &\text{ if and only if } \mathfrak{v}(x,\varphi) = 1 \text{ or } \mathfrak{v}(x,\psi) = 1, \qquad (\star)\\
\mathfrak{v}(x, \ulcorner\varphi \rightarrow \psi\urcorner) = 1 &\text{ if and only if } \mathfrak{v}(x,\varphi) = 0 \text{ or } \mathfrak{v}(x,\psi) = 1,\\
\mathfrak{v}(x, \ulcorner\varphi \equiv \psi\urcorner) = 1 &\text{ if and only if } \mathfrak{v}(x,\varphi) = \mathfrak{v}(x,\psi).
\end{aligned}$$

Quasi-formulas that are assigned 1 in a point x of a model are said to be *satisfied* at the point of the model. Quasi-formulas satisfied at every point of every model are said to be *valid*, and quasi-formulas not satisfied at any point of any model are said to be *unsatisfiable*.

An atomic formula $\ulcorner\mathcal{R}_\alpha\varphi\urcorner$ is true in a model $\mathfrak{M}$ if and only if the quasi-formula φ is satisfied at a point x of $\mathfrak{M}$ such that x is $\mathfrak{d}(\alpha)$, i.e.,

$$\mathfrak{M} \models \ulcorner\mathcal{R}_\alpha\varphi\urcorner \text{ if and only if } \mathfrak{v}(\mathfrak{d}(\alpha), \varphi) = 1.$$

The truth-conditions for compound formulas are classical:

$$\mathfrak{M} \models \ulcorner\neg A\urcorner \text{ if and only if } \mathfrak{M} \not\models A,$$
$$\mathfrak{M} \models \ulcorner A \wedge B\urcorner \text{ if and only if } \mathfrak{M} \models A \text{ and } \mathfrak{M} \models B,$$

$$\mathfrak{M} \models \ulcorner A \vee B \urcorner \text{ if and only if } \mathfrak{M} \models A \text{ or } \mathfrak{M} \models B,$$
$$\mathfrak{M} \models \ulcorner A \to B \urcorner \text{ if and only if } \mathfrak{M} \not\models A \text{ or } \mathfrak{M} \models B,$$
$$\mathfrak{M} \models \ulcorner A \equiv B \urcorner \text{ if and only if } \mathfrak{M} \models A, B \text{ or } \mathfrak{M} \not\models A, B.$$

Let $\mathfrak{K}$ be the collection of all models. $A \in \mathbb{L}$ is a *tautology of* $\mathfrak{K}$ if and only if $\mathfrak{M} \models A$, for every $\mathfrak{M} \in \mathfrak{K}$. Call $E(\mathfrak{K})$ the set of all tautologies of $\mathfrak{K}$. $A \in \mathbb{L}$ is an *antilogy* of $\mathfrak{K}$ if and only if $\mathfrak{M} \not\models A$ for every model $\mathfrak{M} \in \mathfrak{K}$. Call $\overline{E(\mathfrak{K})}$ the set of all antilogies of $\mathfrak{K}$. It may be easily seen that the members of $\overline{E(\mathfrak{K})}$ are exactly $\ulcorner \neg A \urcorner$, for every $A \in E(\mathfrak{K})$.

Definition 9. $\mathfrak{M} \models X$, where $X \subseteq \mathbb{L}$, if and only if $\mathfrak{M} \models A$ for every $A \in X$. Given $X \subseteq \mathbb{L}$, $A \in \mathbb{L}$, X entails A, symbolically $X \models A$, if and only if for every model $\mathfrak{M} \in \mathfrak{K}$, if $\mathfrak{M} \models X$, then $\mathfrak{M} \models A$.

Jarmużek and Pietruszczak proved the adequacy (i.e., both soundness: if $X \vdash A$, then $X \models A$, and completeness: if $X \models A$, then $X \vdash A$, for arbitrary $X \subseteq \mathbb{L}$, $A \in \mathbb{L}$) of MR with respect to this semantics.

Theorem 2 ([2, pp. 155–159]). *For all* $X \subseteq \mathbb{L}$ *and* $A \in \mathbb{L}$:

$$X \vdash A \quad \textit{iff} \quad X \models A.$$

In what follows we take for granted this important result, which ensures the interchangeability of the notions of entailment and consequence, and in particular of MR's tautology and theorem.

Alternative semantic structures, as well as different axiomatizations of the system MR have been defined and examined in [8] and [3]. In particular, it has been shown that the semantics presented here may be generalized, without affecting the set of formulas true in a model, to the cases in which the interpretation of quasi-formulas in positions non denoted by constants from $\mathbb{N}$ need not behave classically [3, pp. 51–60].

5. Possible extensions

First, notice that MR is neither maximal nor $*$-maximal. To prove this, consider the formula

$$\mathcal{R}_a p \to \mathcal{R}_b p. \tag{1}$$

Whenever $\mathfrak{d}(a) \neq \mathfrak{d}(b)$, it suffices to put $\mathfrak{v}(\mathfrak{d}(a), p) = 1$ and $\mathfrak{v}(\mathfrak{d}(b), p) = 0$ to determine the countermodel to (1). Thus MR does not include (1).

Nevertheless, not only (1) but all its substitutions may be consistently added as axioms to MR. The schema corresponding to (1), namely

$$\mathcal{R}_\alpha\varphi \to \mathcal{R}_\beta\varphi, \tag{2}$$

defines a set of models with unit set universe. In such models all individual constants have the same referent and, consequently, every example of (2) is true. Thus the system MR^+, being MR strengthened with the axiom (1) or the schema (2) of axioms, is consistent and $\mathsf{MR} \prec \mathsf{MR}^+$.

A similar argument applies to another sample schema:

$$\mathcal{R}_\alpha\varphi \wedge \mathcal{R}_\beta\psi \to \mathcal{R}_\gamma(\varphi \wedge \psi). \tag{3}$$

Observe that schemata of MR's specific axioms (A1)–(A3), unlike (2) and (3), involve only one (albeit possibly repeated) position indicator. Likewise, all MR's rules with the exception of instances of the rules of classical logic determine solely relations among formulas having identical indexes. This may be easily justified, since valuations of quasi-formulas in different points of a model are independent one of another (compare ($\star$)). Consequently, if $\alpha \neq \beta$ and formulas $\ulcorner\mathcal{R}_\alpha\varphi\urcorner$ and $\ulcorner\mathcal{R}_\beta\psi\urcorner$ are neither tautological nor contradictorily by themselves, then there exist models $\mathfrak{M}$ and $\mathfrak{M}'$ such that $\mathfrak{M} \models \ulcorner\mathcal{R}_\alpha\varphi\urcorner$, but $\mathfrak{M} \not\models \ulcorner\mathcal{R}_\beta\psi\urcorner$, and $\mathfrak{M}' \not\models \ulcorner\mathcal{R}_\alpha\varphi\urcorner$, but $\mathfrak{M}' \models \ulcorner\mathcal{R}_\beta\psi\urcorner$. In this respect, atomic formulas with different indexes behave like sentence letters in propositional languages.

We claim that consistent strengthening of the system MR cannot employ a rule non-derivable in MR and defining relations among formulas involving a single individual constant. We call this property *single-index maximality*.

6. Single-index maximality

In order to state our claim more accurately, we introduce two auxiliary notions: that of an *α-formula* and of a *single-index rule*.

Definition 10. The set of α-formulas is the smallest collection containing every atomic formula of the form $\ulcorner\mathcal{R}_\alpha\varphi\urcorner$, for any quasi-formula φ, and such that if A is an α-formula, then $\ulcorner(\neg\varphi)\urcorner$ is an α-formula, and given that both φ and ψ are α-formulas all $\ulcorner\varphi \wedge \psi\urcorner$, $\ulcorner\varphi \vee \psi\urcorner$, $\ulcorner\varphi \to \psi\urcorner$, and $\ulcorner\varphi \equiv \psi\urcorner$ are α-formulas as well.

Definition 11. A rule r is single-index if and only if for any $(X, A) \in r$ there is $\alpha \in \mathbb{N}$ such that each $B \in X$ and A are α-formulas.

We will argue that if a non-derivable α-rule gets derivable in an extension of MR, then the extension is inconsistent. Thus single-index maximality is maximality restricted to the set of single-index rules. In this regard our idea may be considered analogous to Pogorzelski and Wojtylak's idea of generalized maximality.

The scheme of argumentation for our claim is analogous to that used in the case of the classical propositional calculus, but a little more complex. Let us begin with the following lemmas.

Lemma 3. *Let $X \subseteq \mathbb{L}$, $A \in \mathbb{L}$. If $X \nvdash A$, then $X \cup \{\ulcorner\neg A\urcorner\}$ is consistent.*

Lemma 4. *X is consistent if and only if $\mathfrak{M} \models X$, for some $\mathfrak{M} \in \mathfrak{K}$.*

We omit the routine proofs of lemmas 3 and 4.

Lemma 5. *Given an arbitrary model $\mathfrak{M} = \langle \mathfrak{X}, \mathfrak{d}, \mathfrak{v}\rangle$ and $u \in \mathfrak{X}$, let $\mathsf{e}\colon \mathbb{SL} \longrightarrow \mathbb{QF}$ be a mapping relative to u such that:*

$$\text{if } \mathfrak{v}(u,\varphi) = \mathtt{1}, \text{ then } \mathsf{e}(\varphi) = \text{‘}p \equiv p\text{’},$$
$$\text{if } \mathfrak{v}(u,\varphi) = \mathtt{0}, \text{ then } \mathsf{e}(\varphi) = \text{‘}p \equiv \neg p\text{’}.$$

For any $\psi \in \mathbb{QF}$, if $\mathfrak{v}(u,\psi) = \mathtt{1}$, then $\mathfrak{sub}^{\mathsf{e}}(\psi)$ is a valid quasi-formula and if $\mathfrak{v}(u,\psi) = \mathtt{0}$, then $\mathfrak{sub}^{\mathsf{e}}(\psi)$ is unsatisfiable.

Proof. The proof proceeds by induction on the length of a quasi-formula. The only exception is relativization to u, which is not vital for the proof.

The base step is obvious, as ‘$p \equiv p$’ is valid and ‘$p \equiv \neg p$’ is unsatisfiable. Since the connectives ‘$\wedge$’, ‘$\rightarrow$’, ‘$\equiv$’ are definable by means of ‘$\neg$’ and ‘$\vee$’, we will restrict the inductive step to the cases of negation and disjunction. Assume that this lemma is true for quasi-formulas φ and ψ.

If $\mathfrak{v}(u,\ulcorner\neg\varphi\urcorner) = \mathtt{1}$, then $\mathfrak{v}(u,\varphi) = \mathtt{0}$ and $\mathfrak{sub}^{\mathsf{e}}(\varphi)$ is, by the inductive hypothesis, unsatisfiable, which means that $\mathfrak{sub}^{\mathsf{e}}(\ulcorner\neg\varphi\urcorner) = \ulcorner\neg\mathfrak{sub}^{\mathsf{e}}(\varphi)\urcorner$ is valid.

If $\mathfrak{v}(u,\ulcorner\neg\varphi\urcorner) = \mathtt{0}$, then $\mathfrak{v}(u,\varphi) = \mathtt{1}$ and $\mathfrak{sub}^{\mathsf{e}}(\varphi)$ is, by inductive hypothesis, valid, which means that $\mathfrak{sub}^{\mathsf{e}}(\neg\varphi) = \neg\mathfrak{sub}^{\mathsf{e}}(\varphi)$ is unsatisfiable.

If $\mathfrak{v}(u,\ulcorner\varphi \vee \psi\urcorner) = \mathtt{1}$, then either $\mathfrak{v}(u,\varphi) = \mathtt{1}$ or $\mathfrak{v}(u,\psi) = \mathtt{1}$. Hence by inductive hypothesis either $\mathfrak{sub}^{\mathsf{e}}(\varphi)$ or $\mathfrak{sub}^{\mathsf{e}}(\psi)$ is valid. In any case $\ulcorner\mathfrak{sub}^{\mathsf{e}}(\varphi) \vee \mathfrak{sub}^{\mathsf{e}}(\psi)\urcorner$ is also valid and since $\mathfrak{sub}^{\mathsf{e}}(\ulcorner\varphi \vee \psi\urcorner) = \ulcorner\mathfrak{sub}^{\mathsf{e}}(\varphi) \vee \mathfrak{sub}^{\mathsf{e}}(\psi)\urcorner$ we get that $\mathfrak{sub}^{\mathsf{e}}(\ulcorner\varphi \vee \psi\urcorner)$ is valid too.

If $\mathfrak{v}(u,\varphi \vee \psi) = \mathtt{0}$, then $\mathfrak{v}(u,\varphi) = \mathfrak{v}(u,\psi) = \mathtt{0}$ and by inductive hypothesis both $\mathfrak{sub}^{\mathsf{e}}(\varphi)$ and $\mathfrak{sub}^{\mathsf{e}}(\psi)$ are unsatisfiable. Consequently $\mathfrak{sub}^{\mathsf{e}}(\varphi \vee \psi)\urcorner$ is unsatisfiable too. □

LEMMA 6. *Let A be an α-formula and let* $\mathtt{e}$ *be mapping relative to the point* $\mathfrak{d}(\alpha)$ *and* $\mathtt{l}(\alpha) = \alpha$. *For an arbitrary model* $\mathfrak{M}$ *the following holds*:

$$\text{if } \mathfrak{M} \models A, \text{ then } \mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(A) \in E(\mathfrak{K}),$$
$$\text{if } \mathfrak{M} \not\models A, \text{ then } \mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(A) \in \overline{E(\mathfrak{K})}.$$

PROOF. The proof of this lemma is similar to the proof of Lemma 5.

(a) Let A be $\ulcorner \mathcal{R}_\alpha \varphi \urcorner$, for some $\varphi \in \mathbb{QF}$. According to the lemma 5, if $\mathfrak{M}' \models \ulcorner \mathcal{R}_\alpha \varphi \urcorner$, that is $\mathfrak{v}'(\mathfrak{d}(\alpha), \varphi) = \mathtt{1}$, $\mathfrak{sub}^{\mathtt{e}}(\varphi)$ relative to $\mathfrak{d}(\alpha)$ is valid (takes the value $\mathtt{1}$ in all positions of any model $\mathfrak{M}$). Hence $\mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(\ulcorner \mathcal{R}_\alpha \varphi \urcorner) \in E(\mathfrak{K})$.

If on the other hand $\mathfrak{M}' \not\models \ulcorner \mathcal{R}_\alpha \varphi \urcorner$, that is $\mathfrak{v}'(\mathfrak{d}(\alpha), \varphi) = \mathtt{0}$, then $\mathfrak{sub}^{\mathtt{e}}(\varphi)$ relative to $\mathfrak{d}(\alpha)$ is unsatisfiable and for any model $\mathfrak{M}$, $\mathfrak{M} \not\models \mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(\ulcorner \mathcal{R}_\alpha \varphi \urcorner)$, that is $\mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(\ulcorner \mathcal{R}_\alpha \varphi \urcorner) \in \overline{E(\mathfrak{K})}$.

(b) Assume that Lemma 6 is true for some formulas $B, C \in \mathbb{L}$.

If $A = \ulcorner \neg B \urcorner$, then $\mathfrak{M}' \models A$ if and only if $\mathfrak{M}' \not\models B$. Since $\mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(\ulcorner \neg B \urcorner) = \ulcorner \neg \mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(B) \urcorner$, then by inductive hypothesis $\mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(A) \in E(\mathfrak{K})$, provided $\mathfrak{M}' \models A$, and $\mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(A) \in \overline{E(\mathfrak{K})}$, provided $\mathfrak{M}' \not\models A$.

If $A = \ulcorner B \vee C \urcorner$, then $\mathfrak{M}' \models \ulcorner A \urcorner$ if and only if either $\mathfrak{M}' \models B$ or $\mathfrak{M}' \models C$, so by inductive hypothesis in both cases for any model $\mathfrak{M}$, $\mathfrak{M} \models \mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(B)$ or $\mathfrak{M} \models \mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(C)$, then also $\mathfrak{M} \models \ulcorner \mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(B) \vee \mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(C) \urcorner$. Since $\mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(\ulcorner B \vee C \urcorner) = \ulcorner \mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(B) \vee \mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(C) \urcorner$, we get $\mathfrak{M} \models \mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(\ulcorner A \vee B \urcorner)$ for any model $\mathfrak{M}$. Consequently $\mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(A) \in E(\mathfrak{K})$; $\mathfrak{M}' \not\models \ulcorner A \vee B \urcorner$, when both $\mathfrak{M}' \not\models A$ and $\mathfrak{M}' \not\models B$, so by the inductive assumption for any model $\mathfrak{M}$ $\mathfrak{M} \not\models \mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(A)$ and $\mathfrak{M} \not\models \mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(B)$. Consequently $\mathfrak{M} \not\models \ulcorner \mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(A) \vee \mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(B) \urcorner$. Since $\mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(\ulcorner A \vee B \urcorner) = \ulcorner \mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(A) \vee \mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(B) \urcorner$, we get $\mathfrak{M} \not\models \mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(\ulcorner A \vee B \urcorner)$ for any model $\mathfrak{M}$, that means $\mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(\ulcorner A \vee B \urcorner) \in \overline{E(\mathfrak{K})}$. □

Essentially the substitution $\mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}$ transforms formulas true in a given model into tautologies, and formulas false in that model into antilogies. To give an example of such substitution, consider a model $\mathfrak{M}$, such that $\mathfrak{X} = \{u\}$, $\mathfrak{d}(a) = u$, $\mathfrak{v}(u, p) = \mathtt{1}$, $\mathfrak{v}(u, q) = \mathtt{0}$ and the formula

$$\mathcal{R}_a(p \wedge q) \to \mathcal{R}_a q\,.$$

According to the definition of $\mathtt{e}$, for the sentence letters occurring in the formula in question we have $\mathfrak{sub}^{\mathtt{e}}(p) =$ '$p \equiv p$' and $\mathfrak{sub}^{\mathtt{e}}(q) =$ '$p \equiv \neg p$'. Consequently:

$$\mathfrak{Sub}^{\mathtt{e}}_{\mathtt{1}}(\mathcal{R}_a(p \wedge q) \to \mathcal{R}_a q) = \mathcal{R}_a((p \equiv p) \wedge (p \equiv \neg p)) \to \mathcal{R}_a(p \equiv \neg p).$$

THEOREM 7. *Let* MR^+ *be an extension of* MR. *If for some single-index structural rule* r, r *is not derivable in* MR, *but derivable in* MR^+, *then* MR^+ *is inconsistent.*

PROOF. Assume r is a structural single-index rule and is not derivable in MR, then for some $X \subseteq \mathbb{L}$ and $A \in \mathbb{L}$ such that $(X, A) \in r$, $X \nvdash_{\mathsf{MR}} A$. Hence the set $X \cup \{\ulcorner\neg A\urcorner\}$ is consistent and for some model $\mathfrak{M}$, all formulas $B \in X$ and $\ulcorner\neg A\urcorner$ are true, i.e., for any $B \in X$, $\mathfrak{M} \models B$ and $\mathfrak{M} \not\models A$. Then by Lemma 6, substitution $\mathfrak{Sub}^{\mathsf{e}}_{\mathtt{1}}$, determined by the model $\mathfrak{M}$ and $\mathfrak{d}(\alpha)$, makes MR's tautologies out of the elements of X and an antilogy out of A, i.e., $\mathfrak{Sub}^{\mathsf{e}}_{\mathtt{1}}(B) \in E(\mathfrak{K})$ for all $B \in X$, $\mathfrak{Sub}^{\mathsf{e}}_{\mathtt{1}}(A) \in \overline{E(\mathfrak{K})}$. Thus, by Theorem 2, $\vdash_{\mathsf{MR}} \mathfrak{Sub}^{\mathsf{e}}_{\mathtt{1}}(B)$, for any $B \in X$, and $\vdash_{\mathsf{MR}} \neg\mathfrak{Sub}^{\mathsf{e}}_{\mathtt{1}}(A)$ and by assumption $\vdash_{\mathsf{MR}^+} \mathfrak{Sub}^{\mathsf{e}}_{\mathtt{1}}(B)$, for any $B \in X$, and $\vdash_{\mathsf{MR}^+} \neg\mathfrak{Sub}^{\mathsf{e}}_{\mathtt{1}}(A)$.

Since r is structural and $\langle X, A\rangle \in r$, then $(\mathfrak{Sub}^{e}_{l}(X), \mathfrak{Sub}^{e}_{l}(A)) \in r$ for arbitrary e, l, that is also $(\mathfrak{Sub}^{\mathsf{e}}_{\mathtt{1}}(X), \mathfrak{Sub}^{\mathsf{e}}_{\mathtt{1}}(A)) \in r$ and since it is derivable in MR^+, we get by Theorem 1 that $\vdash_{\mathsf{MR}^+} \mathfrak{Sub}^{\mathsf{e}}_{\mathtt{1}}(A)$ as well. Thus, by Duns Scotus rule (PC), we have $A, \neg A \vdash B$. So the system MR^+ is inconsistent. □

Acknowledgements. I benefited from comments by Professor Marcin Tkaczyk. The article is funded by National Science Centre in Poland on grounds of the decision no. DEC-2012/05/E/HS1/03542.

References

[1] Garson, J. W., *Modal Logic for Philosophers*, Cambridge University Press, 2006.

[2] Jarmużek, T., and A. Pietruszczak, "Completeness of Minimal Positional Calculus", *Logic and Logical Philosophy* 13 (2004): 147–162. DOI: 10.12775/LLP.2004.009

[3] Jarmużek, T., and M. Tkaczyk, *Normalne logiki pozycyjne* (Normal Positional Logics), Wydawnictwo KUL, Lublin, 2015.

[4] Łoś J., "Podstawy analizy metodologicznej kanonów Milla" (The foundations of methodological analysis of Mill's canons), *Annales Universitatis Mariae Curie-Skłodowska* 2 (1947): 269–301.

[5] Pogorzelski, W. A., and P. Wojtylak, *Completeness Theory for Propositional Logics*, Birkhäuser, Berlin, 2008.

[6] Rescher, N., and A. Urquhart, *Temporal Logic*, Springer-Verlag, New York and Vienna, 1971.

[7] Tkaczyk, M., *Logika czasu empirycznego* (Logic of Empirical Time), Wydawnictwo KUL, Lublin, 2009.
[8] Tkaczyk, M., "Negation in Weak Positional Calculi", *Logic and Logical Philosophy* 22 (2013): 3–19. DOI: 10.12775/LLP.2013.001

Anna Maria Karczewska
The John Paul II Catholic University of Lublin
Faculty of Philosophy
Lublin, Poland
amkarczewska@kul.pl