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TAUTOLOGY ELIMINATION, CUT ELIMINATION, AND S5

Abstract. Tautology elimination rule was successfully applied in automated deduction and recently considered in the framework of sequent calculi where it is provably equivalent to cut rule. In this paper we focus on the advantages of proving admissibility of tautology elimination rule instead of cut for sequent calculi. It seems that one may find simpler proofs of admissibility for tautology elimination than for cut admissibility. Moreover, one may prove its admissibility for some calculi where constructive proofs of cut admissibility fail. As an illustration we present a cut-free sequent calculus for **S5** based on tableau system of Fitting and prove admissibility of tautology elimination rule for it.

Keywords: sequent calculus; tautology elimination; cut elimination; modal logic S5

1. Introduction

Tautology elimination rule (TE) was first introduced in late 1950s by Davis and Putnam [3] in their automated theorem prover for classical propositional logic **CPL** in the form:

$$\frac{C_1, \ldots, C_{k-1}, C[\top]_k, C_{k+1}, \ldots, C_n}{C_1, \ldots, C_{k-1}, C_{k+1}, \ldots, C_n}$$

where each C_i denotes a clause.

It is well known that Davis and Putnam procedure is one of the most efficient for **CPL** and in particular TE was also applied in many variants of resolution provers as additional technique for improvement

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of performance. However, in this article we are interested in the application of this kind of rule in the setting of sequent calculi (SC) and its possible advantages. In particular, we will show that it may be applied successfully instead of cut rule.

In the framework of sequent calculus TE may be formulated as the following rule:

$$\frac{\top, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad \text{or} \qquad \frac{\varphi \rightarrow \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Although it is a valid schema of rule for elimination of any thesis from the antecedent of any sequent — hence the name tautology elimination — we prefer for our purposes the more specific instance on the right. One may easily prove the following crucial result:

Lemma 1.1 (Equivalence of TE and Cut). TE *and* Cut *are interderivable.*

$$(\mathrm{Cut})\ \frac{\Rightarrow \varphi \rightarrow \varphi \qquad \varphi \rightarrow \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$\dfrac{(\rightarrow\Rightarrow)\ \dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Pi \Rightarrow \Sigma}{\varphi \rightarrow \varphi, \Gamma, \Pi \Rightarrow \Delta, \Sigma}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (\mathrm{TE})$$

In the above proof figures we did not specify which sort of sequent calculus we are using and we have freely applied multiplicative form of cut and of $(\rightarrow\Rightarrow)$. But it should be obvious that it holds also for additive forms in the presence of weakening.

As far as I know in the framework of sequent calculi TE was introduced by Lyaletsky [9] under the name *tautology rule* and it was used rather for building proof-search procedures. As a device for proving (indirectly) cut admissibility it was applied first by Brighton [1] and then by Tourlakis and Gao [7]. In both cases admissibility of TE was proved for some modal logics of provability and provided proofs were slightly complicated because of taking into account not only proof-trees but also proof-search trees. It seems that such complications in some cases may be avoided. Below we provide a proof which is conceptually simpler since we consider only proof-trees and demonstrate things by induction on the complexity (of eliminated formula) and height of the proof only.

One may also notice that rules of this kind are also often applied for obtaining adequate SC formalizations of theories. Thus instead of addition of axioms of the form $\Rightarrow A$ for an axiom A one may equivalently use rules AE of the form:

$$\frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Equivalence of both solutions is obvious and in one direction was shown in the above lemma. In the second it follows by AE from $A \Rightarrow A$. As an example of such an approach one may mention Gallier's [6] formalization of identity obtained by means of the following rules:

$$\frac{x = x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad \frac{\forall x, y(x = y \wedge \varphi \rightarrow \varphi[x/y]), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

In Negri and von Plato [10] such approach is restricted only to axioms being atoms (the leftmost rule), whereas compound axioms are formalized by means of rules. Thus the second Gallier's rule is formulated in the following way:

$$\frac{x = y, \varphi, \varphi[x/y]), \Gamma \Rightarrow \Delta}{x = y, \varphi, \Gamma \Rightarrow \Delta}$$

The advantage of such decomposition of compound axioms is significant; we can prove cut admissibility for theories formalized in such a way.

2. Admissibility of Tautology Elimination

For specific applications we will use a sequent calculus G3 essentially due to Ketonen (see Negri and von Plato [10]) Sequents are composed from two finite (possibly empty) multisets of formulae. There is one schema of axiom: $\varphi, \Gamma \Rightarrow \Delta, \varphi$, where φ is atomic formula. The rules are the following:

$$(\neg\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\neg)\ \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$$

$$(\wedge\Rightarrow)\ \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\wedge)\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$(\Rightarrow\vee)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi} \qquad (\vee\Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow\rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi} \qquad (\rightarrow\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \rightarrow \psi, \Gamma \Rightarrow \Delta}$$

$$(\forall\Rightarrow)\ \frac{\forall x\varphi, \varphi[x/\tau], \Gamma \Rightarrow \Delta}{\forall x\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\forall)^*\ \frac{\Gamma \Rightarrow \Delta, \varphi[x/a]}{\Gamma \Rightarrow \Delta, \forall x\varphi}$$

$$(\exists\Rightarrow)^*\ \frac{\varphi[x/a], \Gamma \Rightarrow \Delta}{\exists x\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\exists)\ \frac{\Gamma \Rightarrow \Delta, \varphi[x/\tau], \exists x\varphi}{\Gamma \Rightarrow \Delta, \exists x\varphi}$$

side conditions: * where a is a free variable (eigenparameter) not occurring in Γ, Δ and φ.

Note that $(\forall\Rightarrow)$ and $(\Rightarrow\exists)$ are contraction-absorbing in the sense that we can always derive the conclusion from the premiss by means of contraction. This makes contraction dispensable (but admissible as we can show) and moreover, both rules are invertible.

Proofs are defined as usual as well-founded trees with leaves labelled by axioms, all other nodes labelled with sequents derived by the application of rules and proven sequent as the root. $\vdash \Gamma \Rightarrow \Delta$ means that the sequent has a proof in our system. The height of the proof a sequent is defined as the length of the longest branch leading to it. Notions of parametric, side and principal formula in a rule application are standard.

This calculus provides adequate characterization of first-order classical logic. Moreover, one may in a standard way (see e.g. Negri and von Plato [10]) prove the following results:

LEMMA 2.1 (General axioms). $\vdash \varphi, \Gamma \Rightarrow \Delta, \varphi$, *for arbitrary* φ.

LEMMA 2.2 (Admissibility of weakening). *If* $\vdash \Gamma \Rightarrow \Delta$, *then* $\vdash \Gamma' \Rightarrow \Delta'$, *for* $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$.

LEMMA 2.3 (Invertibility). *All rules are invertible.*

LEMMA 2.4 (Admissibility of contraction). *If* $\vdash \varphi, \varphi, \Gamma \Rightarrow \Delta$ *then* $\vdash \varphi, \Gamma \Rightarrow \Delta$, *and if* $\vdash \Gamma \Rightarrow \Delta, \varphi, \varphi$ *then* $\vdash \Gamma \Rightarrow \Delta, \varphi$.

The last important auxiliary result is a variant of Substitution lemma for G3. First we must extend the notion of alphabetic variant and proper substitution from formulae to sequents in the following way:

- for (multi)set $\Gamma := \varphi_1, \ldots, \varphi_n$ $\Gamma[x/\tau] := \varphi_1[x/\tau], \ldots, \varphi_n[x/\tau]$.
- $\Gamma \Rightarrow \Delta[x/\tau] := \Gamma[x/\tau] \Rightarrow \Delta[x/\tau]$.

LEMMA 2.5 (Substitution). *If* $\vdash \Gamma \Rightarrow \Delta$ *then* $\vdash \Gamma \Rightarrow \Delta[x/\tau]$.

Now we have enough machinery to prove:

THEOREM 2.6 (Admissibility of TE). *For any* φ, Γ, *and* Δ:

$$\textit{if} \vdash \varphi \to \varphi, \Gamma \Rightarrow \Delta \ \textit{then} \vdash \Gamma \Rightarrow \Delta\,.$$

PROOF. Assume that (a): $\vdash \varphi \to \varphi, \Gamma \Rightarrow \Delta$. Immediately, by invertibility we obtain: (b) $\vdash \Gamma \Rightarrow \Delta, \varphi$; and (c) $\vdash \varphi, \Gamma \Rightarrow \Delta$.

We prove that $\vdash \Gamma \Rightarrow \Delta$ by induction on the complexity of φ.

Basis: φ is atomic. Consider (c), we prove the claim that $\vdash \Gamma \Rightarrow \Delta$, by subsidiary induction on the height of $\vdash \varphi, \Gamma \Rightarrow \Delta$.

If it is an axiom, then either $\Gamma \Rightarrow \Delta$ is an axiom, or $\varphi, \Gamma \Rightarrow \Delta$ is of the form $\varphi, \Gamma \Rightarrow \Delta', \varphi$. Then, by (b), we have $\vdash \Gamma \Rightarrow \Delta', \varphi, \varphi$ which, by contraction, reduces to $\Gamma \Rightarrow \Delta', \varphi$ and we are done.

Induction step: Assume that the claim holds for any proof of $\vdash \varphi, \Gamma \Rightarrow \Delta$ of height $k < n$ and prove it for the height $= n$.

In the case of Boolean rules the proof is trivial since φ may be only a parameter and its deletion by IH in premisses does not affect the application of a rule. Similarly in case of quantifier rules.

Induction step: Assume as IH that the lemma holds for all formulae of lower complexity than φ. The proof goes by cases. For all Boolean formulae it is similar and based on invertibility of respective rules. We consider only one case as an example:

$\varphi := \psi \vee \chi$. From (b) and (c), by invertibility, we obtain $\vdash \Gamma \Rightarrow \Delta, \psi, \chi, \vdash \psi, \Gamma \Rightarrow \Delta, \vdash \chi, \Gamma \Rightarrow \Delta$, and we build the following proof:

$$\dfrac{\text{IH}\;\dfrac{(\to\Rightarrow)\;\dfrac{\Gamma \Rightarrow \Delta, \psi, \chi \qquad \dfrac{\chi, \Gamma \Rightarrow \Delta}{\chi, \Gamma \Rightarrow \Delta, \psi}\,(\Rightarrow W)}{\chi \to \chi, \Gamma \Rightarrow \Delta, \psi}}{\Gamma \Rightarrow \Delta, \psi} \qquad \psi, \Gamma \Rightarrow \Delta}{\dfrac{\psi \to \psi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\text{IH}}\,(\to\Rightarrow)$$

$\varphi := \forall x \psi$. We again prove the claim: If $\vdash \varphi, \Gamma \Rightarrow \Delta$ then $\vdash \Gamma \Rightarrow \Delta$, by subsidiary induction on the height of (c).

Basis: If (c) is an axiom, then $\Gamma \Rightarrow \Delta$ too.

Induction step: In all cases where φ is parametric, we may delete it by IH and perform the rule.

Case φ is principal. Then the premiss is $\forall x \psi, \psi(x/\tau), \Gamma \Rightarrow \Delta$ and, by IH, it reduces to $\psi(x/\tau), \Gamma \Rightarrow \Delta$. From (b), by invertibility, we get $\Gamma \Rightarrow \Delta, \psi(x/\tau)$ and, by $(\to\Rightarrow)$ we get $\psi(x/\tau) \to \psi(x/\tau), \Gamma \Rightarrow \Delta$ from which the result follows by IH (main).

Hence by (c) we obtain the result for this case too.

Similar proof applies for $\varphi := \exists x \psi$ but now we must carry subsidiary induction on the height of (b) since we must refer to invertibility of $(\exists \Rightarrow)$. $\dashv$

One should pay attention to remarkable simplicity of this proof. Subsidiary induction on the height of one of the sequents obtained by invertibility from the premiss of TE is necessary only in atomic case and in case of quantified formulae. In fact, we can get even simpler proof in the sense of required preliminary results. Note that we have used contraction only in the proof of the basis of the main induction. It means that instead of the (rather involved proof of the) admissibility of full contraction rules we need only admissibility of contraction on atomic formulae. Such a result is trivial for G3 since all two-premiss rules are additive and simple observation shows that if in any proof we have two occurrences of the same atom on any side of the root sequent, then these two occurrences are present in any preceding sequent. Therefore we can just delete one occurrence of this atom in all sequents and obtain a proof with only one occurrence of this atom in all sequents including root sequent.

3. Cut-free Sequent Calculus for S5

Simplicity of the proof of the well known result may be seen as not very important achievement. Can we find an application of TE admissibility which works for cases where direct proofs of cut admissibility do not work? It seems that we can find such an example in the field of modal logic. **S5** is one of the most important modal logics having numerous nice syntactic, semantic and algebraic properties and a variety of applications. Let us recall that if we add to the propositional language just one additional constant $\Box$ — unary modal necessity operator, we can axiomatize **S5** by adding to some system for **CPL** the following schemata:

K $\quad \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
T $\quad \Box\varphi \to \varphi$
4 $\quad \Box\varphi \to \Box\Box\varphi$
B $\quad \neg\varphi \to \Box\neg\Box\varphi$

The system is closed under MP (modus ponens) and the rule of necessitation (of every thesis): $\vdash \varphi \ / \ \vdash \Box\varphi$.

Sequent calculi play similar role in the field of modern proof theory as **S5** in the field of modal logics. However, it is a well known fact that formalization of the latter by means of the former leads to serious troubles. It is not hard to find some rules for **S5** but it is hard to obtain such SC which satisfies properties usually required from well-crafted SC. In particular, it is hard to prove cut admissibility/eliminability for SC adequate with respect to **S5**. At least this remark applies to such SC which are standard, in the sense of not going far beyond the construction of the original Gentzen's SC.[1]

Let us recall here the well known SC for **S5** provided by Ohnishi and Matsumoto [11]. They have used as a basis not G3 but Gentzen's original system LK which operates on sequents built from finite lists of formulas so they have used additionally structural rules of weakening, contraction and permutation. For $\Box$ they introduced the following rules:

$$(\Box\Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta} \qquad\qquad (\Rightarrow\Box)\ \frac{\Box\Gamma \Rightarrow \Box\Delta, \varphi}{\Box\Gamma \Rightarrow \Box\Delta, \Box\varphi}$$

instead of our contraction- and weakening-absorbing rules.

The following proof of B shows that cut is indispensable in this formalization of **S5**.

$$(\Rightarrow\to)\ \dfrac{(\neg\Rightarrow)\ \dfrac{(\text{Cut})\ \dfrac{(\Rightarrow\Box)\ \dfrac{(\Rightarrow\neg)\ \dfrac{\Box p \Rightarrow \Box p}{\Rightarrow \neg\Box p, \Box p}}{\Rightarrow \Box\neg\Box p, \Box p} \qquad \dfrac{p \Rightarrow p}{\Box p \Rightarrow p}\ (\Box\Rightarrow)}{\Rightarrow \Box\neg\Box p, p}}{\neg p \Rightarrow \Box\neg\Box p}}{\Rightarrow \neg p \to \Box\neg\Box p}$$

One may obtain suitable counterparts of Ohnishi and Matsumoto rules for G3 in the following way:

$$(\Box\Rightarrow)\ \frac{\Box\varphi, \varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta} \qquad\qquad (\Rightarrow\Box)\ \frac{\Box\Gamma \Rightarrow \Box\Delta, \varphi}{\Pi, \Box\Gamma \Rightarrow \Box\Delta, \Sigma, \Box\varphi}$$

where $(\Box\Rightarrow)$ is contraction-absorbing and $(\Rightarrow\Box)$ is weakening-absorbing. It is easily seen that such modified rules allow for extending the results like admissibility of weakening and contraction as well as invertibility of Boolean rules. Still this is not enough for proving admissibility of cut and our counterexample works as well.

However, a small modification based on the tableau formalization of **S5** due to Fitting [4, 5] provides a solution. In [4] Fitting has shown that the addition of the rule: $\neg\varphi$ / $\neg\Box\varphi$ to incomplete tableau system

[1] A detailed discussion of different approaches to SC formalization of **S5** may be found in [8] and in [2].

for **S5** may replace the applications of cut. Moreover, he remarked that his (weak) completeness proof for this system implies that it is enough to apply such a rule just once, at the beginning, only to an input formula. This observation was later used in [5] to provide a tableau system for **S5** where in order to check if φ is bf S5-valid we just check $\Box\varphi$ and the system satisfies weak completeness result of the form:

Theorem 3.1 (Completeness). *If* $\models \varphi$ *then* $\vdash \Rightarrow \Box\varphi$.

Since in this tableau system rules are so defined that every formula of the shape $\Box\varphi$ or $\neg\Box\varphi$ is allowed to be rewritten in any new world created on some branch it is a simple matter to find their SC counterparts. Fitting's rules (if put in Hintikka's style, i.e., defined on finite sets not on individual formulae as nodes of the tree) are of the form:

$$(\Box E)\ \frac{\Box\varphi, \Gamma}{\varphi, \Box\varphi, \Gamma} \qquad\qquad (\neg\Box E)\ \frac{\Sigma, \Box\Gamma, \neg\Box\Delta, \neg\Box\varphi}{\Box\Gamma, \neg\Box\Delta, \neg\Box\varphi, \neg\varphi}$$

To simulate Fitting's tableau system in SC we must add to G3 the following modal rules:

$$(\Box{\Rightarrow})\ \frac{\varphi, \Box\varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta} \qquad\qquad ({\Rightarrow}\Box)\ \frac{\Box\Gamma \Rightarrow \Box\Delta, \Box\varphi, \varphi}{\Pi, \Box\Gamma \Rightarrow \Sigma, \Box\Delta, \Box\varphi}$$

The only difference with previously formulated rules is in $(\Rightarrow\Box)$ which is also contraction-absorbing. All definitions formulated for G3 are intact. Also all preliminary results concerning invertibility of rules and admissibility of weakening and contraction hold also for this calculus. One may easily check that addition of modal rules in this form does not destroy the proofs of all these results.

As for soundness we may translate sequents in a standard way into formulas of **S5** by treating antecedents as conjunctions, succedents as disjunctions and $\Rightarrow$ as $\rightarrow$. A rule is validity-preserving (in **S5**) iff under translation a conclusion is valid whenever all premises are valid. Thus, from the fact that all rules are validity-preserving in **S5** we obtain:

Theorem 3.2 (Soundness). *If* $\vdash \Gamma \Rightarrow \Delta$ *then* $\models \bigwedge\Gamma \rightarrow \bigvee\Delta$.

As for completeness one may rigorously demonstrate that every tableau proof of $\Box\varphi$ (i.e., a closed tableau starting with $\neg\Box\varphi$) may be translated into a proof of $\Rightarrow \Box\varphi$ in our system. Hence semantic completeness proof of Fitting applies to this sequent system as well. This will show that we obtain a cut-free SC for **S5** but we want more: a constructive syntactical proof of its admissibility. Hence for the moment we assume that cut is admissible and we syntactically prove:

Theorem 3.3. *If $\vdash_{\mathrm{H}} \varphi$ then $\vdash \Rightarrow \Box\varphi$, where $\vdash_{\mathrm{H}}$ denotes provability in the axiomatic system for* **S5**.

Proof. It is easy to provide proofs for all (boxed) axioms, in particular a proof of (boxed) axiom B looks like that:

$$
\begin{array}{rc}
 & p, \Box p \Rightarrow p, \Box(\neg p \to \Box\neg\Box p), \Box\neg\Box p \\ \cline{2-2}
(\neg\Rightarrow) & p, \neg p, \Box p \Rightarrow \Box(\neg p \to \Box\neg\Box p), \Box\neg\Box p, \Box\neg\Box p \\ \cline{2-2}
(\Rightarrow\to) & p, \Box p \Rightarrow \Box(\neg p \to \Box\neg\Box p), \Box\neg\Box p, \neg p \to \Box\neg\Box p \\ \cline{2-2}
(\Box\Rightarrow) & \Box p \Rightarrow \Box(\neg p \to \Box\neg\Box p), \Box\neg\Box p, \neg p \to \Box\neg\Box p \\ \cline{2-2}
(\Rightarrow\Box) & \Box p \Rightarrow \Box(\neg p \to \Box\neg\Box p), \Box\neg\Box p \\ \cline{2-2}
(\Rightarrow\neg) & \Rightarrow \Box(\neg p \to \Box\neg\Box p), \Box\neg\Box p, \neg\Box p \\ \cline{2-2}
(\Rightarrow\Box) & \neg p \Rightarrow \Box(\neg p \to \Box\neg\Box p), \Box\neg\Box p \\ \cline{2-2}
(\Rightarrow\to) & \Rightarrow \Box(\neg p \to \Box\neg\Box p), \neg p \to \Box\neg\Box p \\ \cline{2-2}
(\Rightarrow\Box) & \Rightarrow \Box(\neg p \to \Box\neg\Box p)
\end{array}
$$

We must of course also demonstrate how applications of GR and MP are simulated on (boxed) theses. For GR we have:

$$
\begin{array}{rc}
 & \Rightarrow \Box\varphi \\ \cline{2-2}
(\Rightarrow\mathrm{W}) & \Rightarrow \Box\varphi, \Box\Box\varphi \\ \cline{2-2}
(\Rightarrow\Box) & \Rightarrow \Box\Box\varphi
\end{array}
$$

For MP we have:

$$
\dfrac{\Rightarrow \Box\varphi \qquad
\dfrac{\Rightarrow \Box(\varphi \to \psi) \qquad
\dfrac{\dfrac{\dfrac{\dfrac{\Box(\varphi \to \psi), \Box\varphi, \varphi \Rightarrow \varphi, \Box\psi \qquad \Box(\varphi \to \psi), \Box\varphi, \psi \Rightarrow \psi, \Box\psi}{\varphi \to \psi, \Box(\varphi \to \psi), \Box\varphi, \varphi \Rightarrow \psi, \Box\psi}\,(\to\Rightarrow)}{\Box(\varphi \to \psi), \Box\varphi, \varphi \Rightarrow \psi, \Box\psi}\,(\Box\Rightarrow)}{\Box(\varphi \to \psi), \Box\varphi \Rightarrow \psi, \Box\psi}\,(\Box\Rightarrow)}{\Box(\varphi \to \psi), \Box\varphi \Rightarrow \Box\psi}\,(\Rightarrow\Box)}
{\Box\varphi \Rightarrow \Box\psi}\,(\mathrm{Cut})}
{\Rightarrow \Box\psi}\,(\mathrm{Cut})
$$

Hence the theorem follows by induction on the height of axiomatic proofs. ⊣

Now we consider the SC for G3 formulated in the previous section but without cut. For this system we can prove:

THEOREM 3.4 (Admissibility of TE). *For any* φ, Γ, *and* Δ:

$$\textit{if} \vdash \varphi \to \varphi, \Gamma \Rightarrow \Delta \textit{ then} \vdash \Gamma \Rightarrow \Delta\,.$$

PROOF. The structure of this proof is the same as the proof of Theorem 2.6. Again we prove that $\vdash \Gamma \Rightarrow \Delta$ by induction on the complexity of φ. Also again in the basis we additionally perform a subsidiary induction on the height of $\vdash \varphi, \Gamma \Rightarrow \Delta$. Now additionally, in the induction step we must consider modal rules. For $(\Box\Rightarrow)$ the proof is trivial like for Boolean rules. In case of $(\Rightarrow\Box)$ φ may be introduced only as a part of weakening of the antecedent, i.e., we have $\varphi, \Gamma', \Box\Pi \Rightarrow \Delta', \Box\Sigma, \Box\psi$ (where $\Gamma = \Gamma', \Box\Pi$ and $\Delta = \Delta', \Box\Sigma, \Box\psi$) deduced from $\Box\Pi \Rightarrow \Box\Sigma, \Box\psi, \psi$ and it is enough to deduce $\Gamma', \Box\Pi \Rightarrow \Delta', \Box\Sigma, \Box\psi$ by the same rule.

Induction step: Again we assume as IH that the lemma holds for all formulae of lower complexity than φ. The proof goes by cases. For all Boolean formulae it is the same (i.e., based on the invertibility of respective rules).

The case of $\varphi := \Box\psi$ is similar to cases of quantified formulae and even slightly more complicated. As in the basis we prove the claim: If $\vdash \varphi, \Gamma \Rightarrow \Delta$, then $\vdash \Gamma \Rightarrow \Delta$, by subsidiary induction on the height of (c).

The basis is trivial since, if $\vdash \varphi, \Gamma \Rightarrow \Delta$ is an axiom, then $\vdash \Gamma \Rightarrow \Delta$ is an axiom too.

In the inductive step the cases of parametric φ in the application of Boolean rules and both $(\Rightarrow\Box)$ rules are also trivial — we just delete φ, by IH and apply suitable rule. The only interesting case is when φ is principal in the application of $(\Box\Rightarrow)$. The premiss is $\Box\psi, \psi, \Gamma \Rightarrow \Delta$ and, by IH, we obtain (d) $\psi, \Gamma \Rightarrow \Delta$. In order to deduce $\Gamma \Rightarrow \Delta$ we must additionally consider (b). Now we prove the claim: If $\vdash \Gamma \Rightarrow \Delta, \varphi$, then $\vdash \Gamma \Rightarrow \Delta$, by subsidiary induction on the height of (b).

Again it is straightforward for the case of axiom and φ parametric in Boolean and all modal rules. We must consider the case of φ principal in both $(\Rightarrow\Box)$-rules. We have $\Gamma', \Box\Pi \Rightarrow \Box\Sigma, \Delta', \Box\psi$ (where $\Gamma = \Gamma', \Box\Pi$ and $\Delta = \Box\Sigma, \Delta'$) deduced from $\Box\Pi \Rightarrow \Box\Sigma, \Box\psi, \psi$. We use this premiss and (d) to obtain the following proof:

$$\dfrac{\dfrac{\dfrac{\Box\Pi \Rightarrow \Box\Sigma, \Box\psi, \psi}{\Gamma \Rightarrow \Delta, \psi}\,\text{IH} \qquad \psi, \Gamma \Rightarrow \Delta}{\psi \to \psi, \Gamma \Rightarrow \Delta}\,(\to\Rightarrow)}{\Gamma \Rightarrow \Delta}\,\text{IH}$$

⊣

A specific feature of this proof, when compared with the proof for classical logic, is the fact that we must perform subsidiary induction on the height of both (b) and (c) in modal case. One is not enough because ($\Rightarrow\Box$) is not invertible.

By the extension of TE admissibility to G3 for **S5** we obtain (indirect) proof of cut admissibility of this calculus.

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