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## COMMENTARY AND ILLOCUTIONARY EXPRESSIONS IN LINEAR CALCULI OF NATURAL DEDUCTION

**Abstract.** We argue that the need for commentary in commonly used linear calculi of natural deduction is connected to the “deletion” of illocutionary expressions that express the role of propositions as reasons, assumptions, or inferred propositions. We first analyze the formalization of an informal proof in some common calculi which do not formalize natural language illocutionary expressions, and show that in these calculi the formalizations of the example proof rely on commentary devices that have no counterpart in the original proof. We then present a linear natural deduction calculus that makes use of formal illocutionary expressions in such a way that unique readability for derivations is guaranteed — thus showing that formalizing illocutionary expressions can eliminate the need for commentary.

**Keywords:** Logic; natural deduction; application of logic; formalization

### 1. Aim, scope and outline

Natural deduction as developed by Gentzen and Jaśkowski was connected with the aim of modelling actual (mathematical) reasoning more closely (see [16, p. 5], [10, pp. 176, 183–184, 186, 188, 190], and [11, pp. 499, 511–512]). Moreover, calculi of natural deduction have often been associated with a guiding function for actual reasoning in other domains as well.<sup>1</sup> With respect to this guiding task and to the application of calculi to (the formalizations of) natural language arguments,

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<sup>1</sup> We are here concerned with reasoning as an overt, linguistic activity. With MacFarlane, we assume that “it is relatively uncontroversial that logic provides norms

the naturalness of calculi is often seen as an important property which may outweigh considerations of simplicity and elegance.<sup>2</sup>

Mostly, linear calculi of natural deduction seem to be viewed as (the most) promising candidates for a natural regulation.<sup>3</sup> Among these, one can distinguish two main families, namely *dependency calculi* that follow what Indrzejczak calls a “*recorded assumption approach*” and *subproof calculi* that follow what he calls an “*ordered assumption approach*” ([15,

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for inferring (in the narrow sense of drawing out consequences)” [20, p. 4]. We are not concerned, however, with the role logic might play in inference and reasoning when these are conceived as “psychological processes” [12, p. 171]. For the question of the normativity of logic for the performance of certain speech acts see [26, 6]. As ‘calculus’ is used with some variance in the literature, a note on our terminology may be in order. We use ‘calculus’ as follows: In a narrower sense, a calculus consists of a (formal) language, a set of inference rules for this language and a possibly empty set of axioms. In a wider sense, a calculus consists of a set of inference rules and (possibly) axioms or axiom schemata which can be combined with languages of a certain type to form a calculus in the narrower sense. Logistic systems in the sense of Church [2, pp. 48–51] and logical calculi in the sense of Prawitz [24, p. 13] are calculi in the narrower sense, while Prawitz’s systems of natural deduction [24, pp. 13, 23–24, 23: n. 1] are calculi in the wider sense.

<sup>2</sup> The demand for naturalness with respect to application oriented calculi is reflected in the choice of calculi in introductory textbooks. For example, in the preface to the first edition of their textbook, Kalish and Montague claim for their calculus to be “a closer approximation to everyday and mathematical reasoning than has previously been achieved by any formal system” [17, p. xv]. Link [19, p. 300] distinguishes a calculus of natural deduction for “*applications of logic*, be they within the realm of mathematics, of natural science or even of philosophical theories” with respect to its naturalness: “the calculus of natural deduction formalizes in a certain sense the natural course of argumentation in these applications [our translation].” Suppes opines that his calculus “has been designed to correspond as closely as possible to the author’s conception of the most natural techniques of informal proof” [30, p. iv]. Barwise and Etchemendy justify the addition of a redundant repetition rule by pointing out that its application “will make some proofs look more natural” [1, p. 56]. The demand of naturalness also arises in the project of proof theory, insofar as one considers proof theory, with Gentzen, as a formal theory of the actual proofs constructed by mathematicians [11, p. 499].

<sup>3</sup> Gentzen regards the tree format as a deviation with regard to actual, linear reasoning ([10, p. 184]; see also [15, pp. 38–39]). That linear calculi are considered to be more natural can be seen from the fact that although calculi of natural deduction are a standard for introductory textbooks (see [22, p. 1], [23, p. 105]), tree calculi of natural deduction are rarely ever used (see [22, p. 8] and [23, p. 113]). Prawitz observes: “If one replaces the tree-form in Gentzen’s system by a linear arrangement [...], one seems indeed to have an eminently useful system also from a pedagogical point of view” [24, p. 105].

p. 44], see also [14]). In dependency calculi, formulas in a derivation depend on certain (sets of) assumptions and inferences are legitimate with respect to the assumptions on which a formula depends — regardless of their order. By contrast, the discharge of an assumption in a derivation in a subproof calculus requires the construction of an appropriate subproof which starts with this assumption and contains no further undischarged assumptions.<sup>4</sup> Virtually all commonly used linear calculi of natural deduction — whether they are subproof or dependency calculi — rely on commentary devices to indicate the scope of assumptions. Such commentary devices may have counterparts in natural language proofs, but there are easily understood informal proofs that do not need them. In usual linear calculi of natural deduction, however, commentary devices have to be employed if one wants to determine a specific reading of a derivation.

We hold that this need for commentary in commonly used calculi is connected to the “deletion” of illocutionary expressions, i.e., expressions that indicate the role of propositions as reasons, assumptions or inferred propositions. To support our thesis, we will argue for two sub-theses:

- (i) Formalizations of certain (relatively) “illocutionarily explicit” and easy to understand informal proofs in typical and widely-used calculi from both families have to employ commentary devices which have no direct natural language counterpart in such proofs.
- (ii) One can construct linear calculi of natural deduction in which illocutionary expressions are formalized and in which there is unique readability for derivations and thus no need to make use of commentary devices.

The first part of our paper is devoted to the first sub-thesis. We will first present an informal proof and identify the speech acts that are being performed by uttering the derivation (2). Subsequently we will discuss formalizations of the proof in two dependency calculi (in a sequent and a quasi-sequent variation) and in two Jaśkowskian subproof calculi (in

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<sup>4</sup> For the distinction between what we call dependency calculi and subproof calculi see also [24, pp. 101–102] and [15, pp. 40–45]. A review of different variants can be found in [22, 23]. Indrzejczak mentions mixed forms of calculi [15, pp. 44–45]. The historical origin of the dependency calculi seems to be [11], while the subproof calculi can be traced back to [16]. Examples for dependency calculi are presented in [30] and [18]. Typical subproof calculi are the Fitch calculi, tracing back to [8], and the Kalish-Montague calculus [17].

a box and a numeral-string variant).<sup>5</sup> It will emerge that derivations in these calculi only partially formalize the expressions of the original proof, more precisely, only the propositions receive a formalization while the illocutionary expressions are deleted. As we will argue in the following sections, it is precisely because of this deletion of illocutionary expressions that the formalizations have to rely on additional commentary elements that have no counterpart in the original proof. We will also point out that the need for commentary elements may introduce some measure of ambiguity (3). In the second part of the paper, we will turn to the second sub-thesis. We combine Jaśkowski's subproof approach with the speech-act theoretic approach to calculi developed by Hinst and Siegart [13, 21, 27, 28]. Following Hinst and Siegart, we introduce illocutionary operators which correspond to the illocutionary expressions used in the original proof. We will argue that a subproof calculus whose language contains such illocutionary operators allows a natural formalization that closely mirrors the original proof and does not rely on any commentary (4). Then, we give a short presentation of this calculus. The use of illocutionary operators will allow us to define derivations in such a way that the scope and discharge of assumptions can be "read off" a given derivation — there being no need for boxes, numeral strings or other "bookkeeping devices" [15, p. 40] (5).

Our approach assumes a speech-act theoretic perspective according to which informal proofs are made up of (possibly elliptical) sentences which are used to perform speech acts, and not merely of propositions which have to be arranged in some way, so that illocutionary expressions are part of the proof and not merely another form of commentary. In the last section, we will submit that, from such a perspective, a calculus that allows the formalization of proofs as sentence sequences for which unique readability with respect to the type of sentences and the discharge of assumptions is given seems more natural than the formalization in calculi which rely on commentary devices. Commentary is often useful, and nothing speaks against employing commentary devices in order to make formal proofs easier to read. However, if commentary devices have to be employed in order to determine a certain reading of a derivation, this may be seen as problematic not just from a speech-act-theoretic perspective (6).

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<sup>5</sup> Our talk of quasi-sequents follows Indrzejczak, who speaks of "Suppes' quasi-sequents" [15, p. 45].

## 2. The example proof

In a first step we present and explain an example derivation, a simple arithmetic proof:

### 2.1. *John's proof*

First it holds that 0 is not identical with  $0+1$ . Now, assume  $k$  is not identical with  $k+1$ . Suppose  $k+1$  was identical with  $(k+1)+1$ . It holds for all  $i, j$  that if  $i+1 = j+1$ , then  $i = j$ . Hence  $k$  would be identical with  $k+1$ . On the other hand, we have that  $k$  is not identical with  $k+1$ . So,  $k+1$  can't be identical with  $(k+1)+1$ . Thus we have that if  $k$  is not identical with  $k+1$ , then also  $k+1$  is not identical with  $(k+1)+1$ . Obviously this then holds for all  $k$ . But we know: If 0 is not identical with  $0+1$  and if furthermore it is true for all  $k$  that given  $k$  is not identical with  $k+1$ , then also  $k+1$  is not identical with  $(k+1)+1$ , then for all  $k$ :  $k$  is not identical with  $k+1$ . Therefore, it holds for all  $k$  that  $k$  is not identical with  $k+1$ .

Given a suitable context, it seems fair to go on the assumption that some author, say John, would be carrying out a proof by uttering above text. Let us assume such a context of utterance for what follows. First of all, we note that the text has a linear structure in the sense that the natural language sentences it consists of are ordered in succession. For reasons of clarity we can emphasize this feature by numbering the sentences and arranging them vertically according to their order:

### 2.2. *John's proof in vertical form*

- 0 First it holds that 0 is not identical with  $0+1$ .
- 1 Now, assume  $k$  is not identical with  $k+1$ .
- 2 Suppose  $k+1$  was identical with  $(k+1)+1$ .
- 3 It holds for all  $i, j$  that if  $i+1 = j+1$ , then  $i = j$ .
- 4 Hence  $k$  would be identical with  $k+1$ .
- 5 On the other hand, we have that  $k$  is not identical with  $k+1$ .
- 6 So,  $k+1$  can't be identical with  $(k+1)+1$ .
- 7 Thus we have that if  $k$  is not identical with  $k+1$ , then also  $k+1$  is not identical with  $(k+1)+1$ .
- 8 Obviously this then holds for all  $k$ .
- 9 But we know: If 0 is not identical with  $0+1$  and if furthermore it is true for all  $k$  that given  $k$  is not identical with  $k+1$ , then also  $k+1$  is not identical with  $(k+1)+1$ , then for all  $k$ :  $k$  is not identical with  $k+1$ .
- 10 Therefore, it holds for all  $k$  that  $k$  is not identical with  $k+1$ .

What is John doing when he produces his proof? One possible answer: He utters sentences and thereby performs certain speech acts, namely assumptions, inferences and adductions of reasons. In the following presentation, the proof is commented on accordingly (ADD – adduction, ASS – assumption, INF – inference):

### 2.3. *John's proof in vertical form with commentary*

- |    |   |       |
|----|---|-------|
| 0  | First it holds that 0 is not identical with $0+1$ .   | (ADD) |
| 1  | Now, assume $k$ is not identical with $k+1$ .   | (ASS) |
| 2  | Suppose $k+1$ was identical with $(k+1)+1$ .  | (ASS) |
| 3  | It holds for all $i, j$ that if $i+1 = j+1$ , then $i = j$ .  | (ADD) |
| 4  | Hence $k$ would be identical with $k+1$ .   | (INF) |
| 5  | On the other hand, we have that $k$ is not identical with $k+1$ .   | (INF) |
| 6  | So, $k+1$ can't be identical with $(k+1)+1$ .   | (INF) |
| 7  | Thus we have that if $k$ is not identical with $k+1$ , then also $k+1$ is not identical with $(k+1)+1$ .  | (INF) |
| 8  | Obviously this then holds for all $k$ .   | (INF) |
| 9  | But we know: If 0 is not identical with $0+1$ and if furthermore it is true for all $k$ that given $k$ is not identical with $k+1$ , then also $k+1$ is not identical with $(k+1)+1$ , then for all $k$ : $k$ is not identical with $k+1$ . | (ADD) |
| 10 | Therefore, it holds for all $k$ that $k$ is not identical with $k+1$ .  | (INF) |

John commences his proof by adducing a reason – more specifically a consequence of a PA (Peano arithmetic) axiom – namely that 0 is not identical to  $0+1$ .<sup>6</sup> He adduces this reason by uttering the sentence ‘First it holds that 0 is not identical with  $0+1$ ’. In lines 3 and 9 John adduces additional reasons (this time PA axioms) by uttering suitable sentences. In lines 1 and 2 John assumes propositions by uttering sentences that start with ‘Now, assume’ and ‘Suppose’ respectively. In lines 4 through 8 and in line 10 John performs inferences, which are legitimated by speech acts he made beforehand. He performs these inferences by uttering inference sentences marked by expressions such as ‘hence’, ‘so’, ‘thus’ and ‘therefore’. In line 4, for example, John infers from the propositions in lines 2 and 3 that  $k$  is identical with  $k+1$ . He does this by uttering the sentence ‘Hence  $k$  would be identical with  $k+1$ ’. To give another example: In line 6 John utters the sentence ‘So,  $k+1$  can't be identical

<sup>6</sup> For a first-order version of the axioms of PA see, for example, [7, pp. 173–174].

with  $(k+1)+1$ ' and thus infers the negation of a proposition which he assumed earlier (in line 2) and which he led to a contradiction (in lines 4 and 5). This step discharges the assumption made in line 2, while the next inference discharges the assumption made in line 1.

To put it briefly: Propositions are adduced, assumed and inferred. Which of these three acts is performed by the utterance of a sentence is (in this case) clearly shown by illocutionary expressions ('it holds that', 'assume', 'hence' etc.). Given these expressions and the order in which the sentences are uttered, one can also grasp the relationship between the adductions, assumptions and inferences performed by John. This holds in general for successful natural language derivations: At least in the context of utterance and for direct addressees, the speech acts and their relationships are identifiable. Obviously this is neither true in general for all kinds of verbal behavior nor for all contexts or all recipients of a sequence of speech acts. As is well known, natural languages are limited with regard to the precision and context-independency that can be achieved. This is one of the reasons why one makes use of formal languages.<sup>7</sup> We neither claim that John's proof does not pose interpretative problems that would not arise for a formal proof, for example with regard to the propositional side, nor do we want to present John's proof as a particularly representative specimen of informal proofs. It is simply taken as a representative for natural language proofs in which the speech acts and their relationships are easily identifiable.<sup>8</sup> How could John's proof be represented formally?

### 3. Commentary in some typical linear calculi of natural deduction

We will now formalize the propositions of the sentences in John's proof. Note that we use 'proposition' in a somewhat unusual way, namely to refer to the expressions that are assumed, adduced, inferred — or uttered with any other kind of illocutionary force. The results of the formalization are formulas of a first-order language (of arithmetic). We would

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<sup>7</sup> As already pointed out by Frege (see, for example, [9, p. iv]).

<sup>8</sup> Concerning the view of deductive reasoning as an activity that consists in the performance of certain speech acts, we follow Hinst [13]; Meggle and Siegart [21], and Siegart [27, 28]. Dutilh Novaes [6] also provides an account of (the dialogical origins of) deduction that locates it in the sphere of speech acts.

ask the reader to regard the following formalization of the propositions, which will be used in all following sections, as unproblematic:<sup>9</sup>

3.1. *Formalization of the propositions in John's proof*

- 0  $\neg 0 = 0 + 1$
- 1  $\neg x = x + 1$
- 2  $x + 1 = (x + 1) + 1$
- 3  $\forall x \forall y (x + 1 = y + 1 \rightarrow x = y)$
- 4  $x = x + 1$
- 5  $\neg x = x + 1$
- 6  $\neg x + 1 = (x + 1) + 1$
- 7  $\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1$
- 8  $\forall x (\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1)$
- 9  $\neg 0 = 0 + 1 \wedge \forall x (\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1)$   
 $\rightarrow \forall x \neg x = x + 1$
- 10  $\forall x \neg x = x + 1$

In this simple sequence of formulas, the role of the propositions is not immediately identifiable anymore. In the case of Gentzen's 1936 calculus, derivations are sequences of sequents and derivations in the other calculi considered in the following can be formally defined as structures that are not merely sequences of formulas but, for example, sequences of ordered pairs of sets of formulas and formulas (dependency calculi) or sequences of formulas and subproofs (subproof calculi). However, as will be illustrated in the following, within the presentations of derivations in these calculi commentary elements which indicate dependencies or subproofs can be identified. One question is if these commentary devices are considered as integral parts of derivations in the calculi or as inessential metalinguistic addenda. Concerning this question, one can note that in the case of Gentzen's sequent format and Jaśkowski's (original) numeral-string format, the commentary devices form part of the proof, while in Suppes' quasi-sequent format they do not. Another question, and the one we want to focus on, is which status these commentary devices have with respect to the original proof: Which role do they play in formalizations of the original proof? Here, one can note in advance that only the formalizations of the propositions as listed above are direct counterparts of expressions occurring in John's proof. We will argue that the commentary devices employed by the different calculi — independent of

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<sup>9</sup> Note that with the exception of the identity predicate all logical operators apply to formulas only.



their status within the calculi — constitute an “essentially metalinguistic” comment on a propositionally reduced version of John’s proof.

What is given in John’s proof is a sequence of sentences in which the role of the propositions is expressed by illocutionary expressions. The order of these sentences allows one to identify the scopes of assumptions and their discharge. However, with respect to the calculi considered in the following, only the propositions receive a direct formalization, while the role of the propositions, their status as assumed, inferred, or adduced propositions, and the scope of assumptions and their discharge are indicated by commentary devices. These commentary devices consist of a rule commentary which records if a proposition is assumed, adduced or inferred in accordance with some rule and “bookkeeping devices for separating the parts of proof which are in the scope of discharged assumption” [15, p. 40]. The rule commentaries are not essential with regard to the question of whether or not a sequence is a derivation of a certain proposition from a certain set of propositions, though they may not just be omitted if one wants to determine if, for example, a proposition is used as reason or if it is inferred. The “bookkeeping devices”, on the other hand, are essential and will constitute the focus of our analysis. In the following, when we speak of commentary (devices), we refer to this essential part of the overall commentary.

If these commentary devices are considered as external to the formalized proof (e.g. Suppes), the formalized proof will just be a sequence of propositions such as the one above (3.1). If one wants to assess whether such a sequence of formulas is a proof or non-proof, one has to try whether it is possible to ascribe roles to the propositions and scopes to the assumptions in such a way that every line of the sequence is in accordance with the rules of the calculus. This “trying out” amounts essentially to finding some commentary that makes clear which formulas are assumed, which formulas are adduced and which formulas are inferred, and also indicates the scope of those formulas that are marked as assumptions by the commentary, and according to which every line of the sequence is in accordance with the rules of the calculus.<sup>10</sup> If, on

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<sup>10</sup> Of course, the proof-predicate is still decidable (with respect to a decidable set of axioms) if one treats derivations as sequences of formulas: If there is a commentary according to which a sequence of formulas is a proof, then it is a proof. However, our claim is simply that determining proofhood requires one to check whether there exists a suitable commentary, a commentary which cannot simply be read off the sequence of formulas. This raises problems with respect to the formalization of derivations and

the other hand, the commentary devices are considered as an integral part of the formalized proof (e.g. Gentzen), the process of “trying out” has to occur as part of the formalization and may require substantial interpretive efforts.

In the following, these remarks will be substantiated by an analysis of formalizations of John’s proof in different types of linear calculi of natural deduction. Note that we use streamlined classical versions of these calculi, in particular with regard to the use of parameters instead of free variables,<sup>11</sup> the rules for negation and the quantifier rules, and the distinction between assumed and adduced propositions. This streamlining facilitates exposition and comparison without having any substantial impact on the relevant properties of the calculi under consideration. In particular, we will always suppose that the operators for conditional, conjunction, disjunction, biconditional, negation, the existential and universal quantifier and the identity predicate are the only logical operators and that each of these is regulated by an introduction and an elimination rule.

First we will consider dependency calculi ( $\uparrow 1$ ), which can be further divided into sequent and quasi-sequent variants. Gentzen’s calculus of natural deduction in sequent format serves as a typical example for the first group.<sup>12</sup> The rightmost column contains a rule commentary that names the rules and marks the adduction of reasons:

### 3.2. *John’s proof: Gentzen(1936)-style*

0	$\Rightarrow \neg 0 = 0 + 1$	(ADD)
1	$\neg x = x + 1 \Rightarrow \neg x = x + 1$	(ASS)

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failed proof attempts: If we want to formalize a derivation from propositions which are also axioms of the system at hand or a proof attempt which fails because an axiom is assumed and this assumption is not discharged, the formalization can also only be identified as a derivation or a non-proof relative to a commentary.

<sup>11</sup> As parameters we use ‘ $x$ ’, ‘ $y$ ’, ‘ $z$ ’,  $\dots$ , while ‘ $x$ ’, ‘ $y$ ’, ‘ $z$ ’,  $\dots$  are used as variables. Closed terms and formulas are terms and formulas, respectively, in which no variables occur free (but which may contain parameters).  $A, B, \Gamma, \Delta, \dots$  are used as metavariables for formulas,  $\vartheta, \vartheta^*, \vartheta', \dots$  for terms,  $\zeta, \xi, \omega, \dots$  for variables, and  $\beta, \beta^*, \beta', \dots$  for parameters. ‘ $[\dots, \dots, \dots]$ ’ is used as a metalinguistic substitution operator, where  $[\vartheta, \xi, \Delta]$  designates the result of substituting the term  $\vartheta$  for the variable  $\xi$  in the formula  $\Delta$ . On the formal side, propositions are always closed formulas.

<sup>12</sup> Gentzen [11] develops this calculus for the formalization of mathematical proofs. In order to facilitate the exposition and comparison of different calculi, the induction rule (see [11, p. 515]) is eliminated and the relevant instance of the induction schema is adduced in line 10.

2	$x + 1 = (x + 1) + 1 \Rightarrow x + 1 = (x + 1) + 1$	(ASS)
3	$\Rightarrow \forall x \forall y (x + 1 = y + 1 \rightarrow x = y)$	(ADD)
4	$\Rightarrow \forall y (x + 1 = y + 1 \rightarrow x = y)$	(UE; 3)
5	$\Rightarrow x + 1 = (x + 1) + 1 \rightarrow x = x + 1$	(UE; 4)
6	$x + 1 = (x + 1) + 1 \Rightarrow x = x + 1$	(CdE; 2, 5)
7	$\neg x = x + 1 \Rightarrow \neg x + 1 = (x + 1) + 1$	(NI; 1, 6)
8	$\Rightarrow \neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1$	(CdI; 7)
9	$\Rightarrow \forall x (\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1)$	(UI; 8)
10	$\Rightarrow \neg 0 = 0 + 1 \wedge \forall x (\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1)$	(ADD)
	$\rightarrow \forall x \neg x = x + 1$	
11	$\Rightarrow \neg 0 = 0 + 1 \wedge \forall x (\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1)$	(CI; 0, 9)
12	$\Rightarrow \forall x \neg x = x + 1$	(CdE; 10, 11)

Assumptions correspond to “logische Grundsequenzen” (“logical basic sequents”) of the form  $\ulcorner \Delta \Rightarrow \Delta \urcorner$  and adductions of axioms (and here: of reasons in general) to “mathematische Grundsequenzen” (“mathematical basic sequents”) of the form  $\ulcorner \Rightarrow \Delta \urcorner$  [11, p. 513]. The adductions in lines 0, 3, and 10 are accordingly rendered as sequents with empty antecedents, while the assumptions in lines 1 and 2 are represented by sequents in which the assumed formula is both the antecedent and the succedent. Inferences normally inherit the formulas to the left of ‘ $\Rightarrow$ ’ from their premise lines (for example in line 6). Exempt are inferences by negation introduction (NI), conditional introduction (CdI) and existential elimination (EE), where the discharged assumption is eliminated from the formulas that appear to the left of ‘ $\Rightarrow$ ’ (e.g. in lines 7 and 8).

In this calculus, derivations are sequences of sequents. Although the formalization of the propositions in John’s proof yields the succedents of these sequents ( $\uparrow$ 3.1), the sequents as a whole have no direct counterpart. The interpretation intended by Gentzen insofar as 3.2 is regarded as a formalization of 2.1 is that the propositions to the right of ‘ $\Rightarrow$ ’ depend on the propositions to the left of ‘ $\Rightarrow$ ’ [11, pp. 511–513]. Gentzen holds that (the formalizations of) the propositions on which (the formalization of) a proposition depends should be recorded as antecedent formulas of a sequent whose succedent formula is (the formalization of) this proposition [11, p. 512]. With respect to the propositions that are actually assumed, adduced and inferred in a proof, the sequent format thus serves to keep record of the assumptions on which these propositions depend. The sequent format thus serves to provide a dependency commentary for the formulas that correspond directly to the propositions as they are used in

a proof. The requirement that the propositions on which a proposition depends at a certain position in a proof have to be recorded forces one to deliberate on dependency relations between propositions in order to determine the antecedents of the sequents even if one formalizes simple and clearly structured proofs.

Insofar as only the succedents of the sequents have direct counterparts in the natural language proof, quasi-sequent variants in the style of Suppes [30] might seem closer to natural language proofs, because the antecedents of sequents are “moved” to a metalinguistic commentary.<sup>13</sup> Under this approach, dependencies are recorded by sets of line numbers in a separate column:

### 3.3. *John’ s proof: Suppes-style*

$\emptyset$	0	$\neg 0 = 0 + 1$	(ADD)
{1}	1	$\neg x = x + 1$	(ASS)
{2}	2	$x + 1 = (x + 1) + 1$	(ASS)
$\emptyset$	3	$\forall x \forall y (x + 1 = y + 1 \rightarrow x = y)$	(ADD)
$\emptyset$	4	$\forall y (x + 1 = y + 1 \rightarrow x = y)$	(UE; 3)
$\emptyset$	5	$x + 1 = (x + 1) + 1 \rightarrow x = x + 1$	(UE; 4)
{2}	6	$x = x + 1$	(CdE; 2, 5)
{1}	7	$\neg x + 1 = (x + 1) + 1$	(NI; 1, 6)
$\emptyset$	8	$\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1$	(CdI; 7)
$\emptyset$	9	$\forall x (\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1)$	(UI; 8)
$\emptyset$	10	$\neg 0 = 0 + 1 \wedge \forall x (\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1) \rightarrow \forall x \neg x = x + 1$	(ADD)
$\emptyset$	11	$\neg 0 = 0 + 1 \wedge \forall x (\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1)$	(CI; 0, 9)
$\emptyset$	12	$\forall x \neg x = x + 1$	(CdE; 10, 11)

The entries in the leftmost column record the set of propositions on which the proposition in a given line depends, where line numbers point to the lines in which the assumptions (if any) were made. Analogously to Gentzen’s format, adduced propositions depend on the empty set, while assumed propositions depend on their unit set.<sup>14</sup> Accordingly, the entries for lines 0, 3 and 10 show the empty set, while the entries for

<sup>13</sup> On the relationship between Gentzen’s sequent format and its further developments in the style of Suppes see [15, pp. 43–45] and [24, pp. 101–102].

<sup>14</sup> As remarked above, we work with streamlined versions of the calculi, where adductions are distinguished from assumptions. This is not the case for Suppes’s and (with some qualification) Jaśkowski’s original calculi.

the assumptions in lines 1 and 2 contain the unit set of the respective line numbers. The propositions inferred in lines 6 and 7 depend on the propositions assumed in lines 2 and 1 respectively, which is recorded in the respective entries.

Note that the dependency commentary in the left column has no direct counterpart in John's proof but is added in its formalization. So, as in Gentzen's case, formalizing a proof requires one to find a fitting dependency commentary. As remarked above, in contrast to Gentzen's case, Suppes' dependency commentary is not normally viewed as part of the object language, but as metalinguistic like the rule commentary on the right. However, if one omits the commentary, this results in a simple sequence of propositions in which, as in 3.1, the role of the respective propositions is not uniquely determined. Sequences of propositions can be commented on in different ways in the Suppes-variant of the quasi-sequent format, so that, for example, one and the same sequence of propositions can be commented on as a proof / derivation (from a set of formulas) or as a non-proof / non-derivation (from this set of formulas). Thus the sequence of formulas in the Suppes formalization of John's proof can be "transformed" into a derivation of the last formula from the set of propositions *adduced as reasons* by John. Concerning this problem, one might also look at the following, very short, example proof (or non-proof):

#### 3.4. *Proof and non-proof with identical sequence of formulas*

	proof				non-proof			
$\emptyset$	0	$x = x$	(II)	$\{0\}$	0	$x = x$	(ASS)	
$\emptyset$	1	$\forall x(x = x)$	(UI)	$\{0\}$	1	$\forall x(x = x)$	(UI?)	

Comment: In the non-proof, the "UI" is not correct, since the parameter of the UI must not occur in any assumption on which the premise of the UI depends. Yet even if the inference were legitimate, the left sequence would be a derivation from the empty set, and thus a proof, while the sequence to the right would be one in which the formula in the last line depended on a non-empty set, and thus not a proof. Thus, in Suppes approach, the sequences that form the object-language core of derivations, i.e., sequences of propositions without the dependency commentary, are ambiguous and in need of interpretation.<sup>15</sup>

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<sup>15</sup> Of course, the provisions of n. 10 still hold.

An alternative to the dependency calculi is provided by the subproof calculi, which go back to Jaśkowski. In such calculi certain (segments of) sequences are characterized as subproofs whose closure discharges the assumption they start with. What matters is not whether or not a proposition depends on (a set of) other propositions but whether or not it occurs at a certain position in a proof with a certain structure. Jaśkowski [16] proposed two variants: subproofs are either demarcated by boxes (or other graphic means) or by strings of numerals. A formalization of John's proof according to the first method would look something like this:

3.5. *John's proof: Jaśkowski box style*

0	$\neg 0 = 0 + 1$	(ADD)
1	$\neg x = x + 1$	(ASS)
2	$x + 1 = (x + 1) + 1$	(ASS)
3	$\forall x \forall y (x + 1 = y + 1 \rightarrow x = y)$	(ADD)
4	$\forall y (x + 1 = y + 1 \rightarrow x = y)$	(UE; 3)
5	$x + 1 = (x + 1) + 1 \rightarrow x = x + 1$	(UE; 4)
6	$x = x + 1$	(CdE; 2, 5)
7	$\neg x = x + 1$	(R; 1)
8	$\neg x + 1 = (x + 1) + 1$	(NI; 2-7)
9	$\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1$	(CdI; 1-8)
10	$\forall x (\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1)$	(UI; 9)
11	$\neg 0 = 0 + 1 \wedge \forall x (\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1)$	(ADD)
	$\rightarrow \forall x \neg x = x + 1$	
12	$\neg 0 = 0 + 1 \wedge \forall x (\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1)$	(CI; 0, 10)
13	$\forall x \neg x = x + 1$	(CdE; 11, 12)

In line 2, an assumption is made, which is indicated by the fact that the proposition appears at the beginning of a box. The proposition assumed in line 2 is then led to a contradiction. This legitimates the inference of the negation of the assumed proposition in line 8, which discharges the assumption and closes the subproof, which is indicated by closing the box. Similarly, the subproof that starts with the assumption in line 1 is closed by the inference of the conditional in line 9, which is legitimate, because the subproof led from the assumption of the conditional's antecedent to the inference of its consequent in line 8, the last line of the subproof. The boxes serve to demarcate the subproofs associated with assumptions: The assumption that starts a subproof and

the propositions that are inferred from this assumption are only available as premises in this subproof and in embedded subproofs. This ensures that assumptions and propositions that are inferred from them cannot be used as premises once the respective assumption has been discharged.<sup>16</sup>

The boxes amount to start and end markers for subproofs and thereby serve as commentary devices to separate these subproofs. Like the dependency commentary, boxes have to be added in the formalization. In contrast to the dependency commentary, however, the boxes serve to delineate structures, namely subproofs that can also be identified on the surface of natural language derivations such as the example proof—only that they are not necessarily commented on there. The dependency commentary, by contrast, states that certain (sets of) formulas stand in a certain relation to certain formulas, where this dependency relation is often a subclass of the consequence relation for the calculus in question.<sup>17</sup> However, whether a proposition stands in the dependency relation to itself or to other propositions found in a natural language derivation can normally not be determined by just inspecting the surface of a natural language derivation but demands further interpretative efforts which require, amongst other things, knowledge of the respective dependency relation. Thus, the subproof commentary by boxes or by numeral-strings (see below), seems less external to John’s original proof than the dependency commentary. On the other hand, John does neither use start and end markers for subproofs nor a numeral system to keep track of proof structure. Ordinary language expressions that express illocutionary force can, by contrast, easily be identified and subproofs in the example proof can simply be identified by inspecting the sentences and their order.

The status of the boxes with respect to their object-linguistic or metalinguistic nature is left open by Jaśkowski.<sup>18</sup> If one considers it as a metalinguistic commentary that consists in “drawing boxes or rectangles

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<sup>16</sup> This function is also fulfilled by boxes or other graphical means when they do not have to start with an assumption as in Jaśkowski’s calculus or the variants of it tracing back to Fitch [8], but are only applied once a subproof is closed as in the Kalish-Montague calculus [17].

<sup>17</sup> The “metalogical nature” often attributed to sequent calculi (see e.g. [24, p. 90] seems also to apply to natural deduction calculi in sequent and quasi-sequent format. Thus Suppes writes: “The intuitive significance of the numerals at the left should be emphasized: each line is a *logical consequence* of the set of premises corresponding to the numerals at the left” [30, p. 27].

<sup>18</sup> In formal definitions of derivations in the box format, the boxes may be “represented” by special expressions (e.g. in [15, pp. 50–51]). However, this does not settle

around portions of a proof” (see [22, p. 3] and [23, p. 108]), where the proof is just the sequence of formulas,<sup>19</sup> one again encounters the problem of ambiguity that is faced by Suppes’ metalinguistic dependency commentary. Obviously a given sequence of formulas can then be boxed in different ways and thus there exists again the possibility of boxing one and the same sequence of formulas in different ways, such that, for example, it may be commented on as a proof / derivation of a formula (from a certain set of propositions) or as a non-proof / non-derivation of a formula (from a certain set of propositions). For example, if one lets the outer box in the box formalization of John’s proof end two lines later, a non-proof results, because the inference in line 9 then becomes incorrect.<sup>20</sup>

While the status of the boxes with respect to their object-linguistic or metalinguistic nature is left open by Jaśkowski, he clearly considers the numeral-strings of his alternative format as part of the “significant expressions” of his system. Jaśkowski uses ‘S’ to mark assumptions and calls concatenations of numeral strings, (and possibly ‘S’) and formulas “new expressions which must be considered as significant ones” [16, p. 7]. This suggests that numeral strings (and ‘S’) are supposed to belong to the object language. Jaśkowski’s use of ‘S’ can be seen as constituting the first use of an assumption operator in a formal system and thus provides a starting point for the approach presented below in sections 4 and 5. However, while such an assumption operator has a natural language counterpart, there is no natural language counterpart to the numeral strings.

Jaśkowski’s numeral method allows one to split subproofs and makes a rule of repetition superfluous [16, p. 8]. Thus, for example, the members of a contradiction that legitimate an NI do not have to appear under

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the question of whether or not these special metalinguistic expressions refer to object language expressions.

<sup>19</sup> With respect to the box variant, it seems unclear whether Pelletier considers the proof to consist just in the sequence of formulas.

<sup>20</sup> As remarked above, we here suppose calculi with primitive rules, i.e., one introduction and elimination rule for each logical operator. In the case at hand, we suppose that a conditional introduction has to be legitimated by a subproof that leads from the assumption of the conditional’s antecedent to its consequent. If one supposes a more liberal calculus in which one could infer a conditional simply from its consequent, not the CdI in line 9, but the UI in line 10 would be incorrect (because the parameter of the supposed UI would still be part of the undischarged assumption in line 1).



the assumption that is to be negated. Numerals show in which subproof a line is embedded. A formalization could look something like this:<sup>21</sup>

3.6. *John's proof: Jaśkowski numeral-string style*

0	$\neg 0 = 0 + 1$	(ADD)
1	1 S $\neg x = x + 1$	(ASS)
2	1, 1 $x + 1 = (x + 1) + 1$	(ASS)
	S	
3	$\forall x \forall y (x + 1 = y + 1 \rightarrow x = y)$	(ADD)
4	$\forall y (x + 1 = y + 1 \rightarrow x = y)$	(UE; 3)
5	$x + 1 = (x + 1) + 1 \rightarrow x = x + 1$	(UE; 4)
6	1, 1 $x = x + 1$	(CdE; 2, 5)
7	1 $\neg x + 1 = (x + 1) + 1$	(NI; 1, 2, 6)
8	$\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1$	(CdI; 1, 7)
9	$\forall x (\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1)$	(UI; 8)
10	$\neg 0 = 0 + 1 \wedge \forall x (\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1) \rightarrow \forall x \neg x = x + 1$	(ADD)
11	$\neg 0 = 0 + 1 \wedge \forall x (\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1)$	(CI; 0, 9)
12	$\forall x \neg x = x + 1$	(CdE; 10, 11)

As in the case of the box format, it holds that ambiguity is introduced at the formal level if one treats Jaśkowski's numeral strings and 'S' as a metalinguistic "book-keeping annotation alongside the sequences of formulas that constitutes a proof" (see [22, p. 3] and [23, p. 8]). If one then omits the commentary (and thus the numeral strings), a simple sequence of propositions results — as in the case of Suppes. As a consequence, one and the same sequence of propositions can then be given numeral commentaries that present it as a proof and numeral commentaries that present it as a non-proof. Thus, for example, the following commentary presents John's proof as a non-proof:

3.7. *John's proof with "wrong" commentary*

0	$\neg 0 = 0 + 1$	(ADD)
1	1 S $\neg x = x + 1$	(ASS)

<sup>21</sup> We loosely follow the presentation in Prawitz [24, pp. 99–101]. In particular, we work with classical quantifier rules, while Jaśkowski's original calculus only allows the proof of propositions which are also valid in the empty domain, as terms have to be "supposed" as well [24, pp. 99–101]. In his original system for quantifier logic, Jaśkowski used 'T' "analogous to the symbol of supposition 'S'" [16, p. 29] to mark the "supposition" of terms.

2	2 S	$x + 1 = (x + 1) + 1$	(ASS)
3		$\forall x \forall y (x + 1 = y + 1 \rightarrow x = y)$	(ADD)
4		$\forall y (x + 1 = y + 1 \rightarrow x = y)$	(UE; 3)
5		$x + 1 = (x + 1) + 1 \rightarrow x = x + 1$	(UE; 4)
6	2	$x = x + 1$	(CdE; 2, 5)
7	1	$\neg x + 1 = (x + 1) + 1$	(NI; 1, 2, 6?)
8		$\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1$	(CdI; 1, 7)
9		$\forall x (\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1)$	(UI; 8)
10		$\neg 0 = 0 + 1 \wedge \forall x (\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1) \rightarrow \forall x \neg x = x + 1$	(ADD)
11		$\neg 0 = 0 + 1 \wedge \forall x (\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1)$	(CI; 0, 9)
12		$\forall x \neg x = x + 1$	(CdE; 10, 11)

One of the reasons why this presents John’s proof as a non-proof is that the “NI” in line 7 is incorrect: NI requires that (i) the numeral string of the line in which the negation is inferred consists exactly of the numeral string of the line in which the proposition to be negated has been assumed without the last member of that string, and that (ii) the numeral string of the lines that contain the members of the contradiction have to be initial segments of the numeral string of the line in which the proposition to be negated has been assumed. Casually put: Two subproofs have been inadmissibly mixed – at least according to the commentary. Of course, one could always try to come up with an alternative commentary, like the one in 3.6, according to which John’s proof is presented as a proof.

#### 4. Sequences of sentences without commentary

According to our analysis, the calculi discussed so far rely on commenting devices which have no counterpart in John’s proof. Moreover, it seems as if John’s proof would work quite well without a commentary. But why? One possible answer: John does not have to provide a commentary for his proof, because he does not simply utter a sequence of *propositions*, but performs a sequence of speech acts – adductions, assumptions, and inferences – by uttering a sequence of *sentences*.<sup>22</sup> Sentences in this sense are those expressions through whose utterances (in normal cir-

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<sup>22</sup> Here and in the following, we draw heavily on the work of Hinst and Siegart [13, 21, 27, 28].

cumstances) speech acts are performed. More generally, putting forward a natural language proof can be conceived as performing a sequence of speech acts. Of course, not every natural language proof is illocutionarily explicit to the degree that John's proof is. But at least some natural language proofs work without dependency or subproof commentaries because they are not sequences of propositions, but sequences of sentences whose illocutionary status is clearly expressed.

Sentences can be assigned a standard format  $\lceil \Xi \Gamma \rceil$ , where  $\Xi$  is a performator, i.e., an atomic object-language illocutionary operator, and  $\Gamma$  is a proposition. The performators determine which speech act is performed by the utterance of a sentence. So, for example, John uses natural language performators such as 'hence', 'thus' and 'therefore' to perform inferences. He uses 'it holds' and 'we know' to perform adductions and 'assume' and 'suppose' to perform assumptions. The list of natural language performators may easily be extended. Note that — similar to natural language expressions that would be assigned to other atomic logical categories — natural language performators may consist of more than one word. It seems rather natural to use performators for different speech acts, and thus sentences, in a formal representation as well. So, if one chooses 'SINCE' as adduction performatore, 'SUPPOSE' as assumption performatore and 'THUS' as inference performatore and if  $\Gamma$  is a proposition, then  $\lceil \text{SINCE } \Gamma \rceil$  is the adduction sentence for  $\Gamma$ , by whose utterance  $\Gamma$  is adduced as a reason,  $\lceil \text{SUPPOSE } \Gamma \rceil$  is the assumption sentence for  $\Gamma$ , by whose utterance  $\Gamma$  is assumed, and  $\lceil \text{THUS } \Gamma \rceil$  is the inference sentence for  $\Gamma$ , by whose utterance  $\Gamma$  is inferred.<sup>23</sup> Thus equipped, John's proof can be formalized as a sequence of sentences as follows:

#### 4.1. *John's proof as a sentence sequence*

0	SINCE	$\neg 0 = 0 + 1$	(ADD)
1	SUPPOSE	$\neg x = x + 1$	(ASS)
2	SUPPOSE	$x + 1 = (x + 1) + 1$	(ASS)
3	SINCE	$\forall x \forall y (x + 1 = y + 1 \rightarrow x = y)$	(ADD)
4	THUS	$\forall y (x + 1 = y + 1 \rightarrow x = y)$	(UE; 3)
5	THUS	$x + 1 = (x + 1) + 1 \rightarrow x = x + 1$	(UE; 4)

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<sup>23</sup> The idea of formal, object language expressions which show the illocutionary status of an utterance is not new. Frege's judgment stroke [9] and Jaśkowski's 'S'-operator [16] can be seen as precursors to an assertion and an assumption performatore respectively (see also [3]). However, the systematic integration of performators into the vocabulary of a formal language is due to Hinst (e.g. [13]).

6	THUS	$x = x + 1$	(CdE; 2, 5)
7	THUS	$\neg x = x + 1$	(R; 1)
8	THUS	$\neg x + 1 = (x + 1) + 1$	(NI; 2-7)
9	THUS	$\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1$	(CdI; 1-8)
10	THUS	$\forall x(\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1)$	(UI; 9)
11	SINCE	$\neg 0 = 0 + 1 \wedge \forall x(\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1) \rightarrow \forall x \neg x = x + 1$	(ADD)
12	THUS	$\neg 0 = 0 + 1 \wedge \forall x(\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1)$	(CI; 0, 10)
13	THUS	$\forall x \neg x = x + 1$	(CdE; 11, 12)

In line 0, the proof is opened by an adduction sentence. Through the utterance of the adduction sentence ‘SINCE  $\neg 0 = 0+1$ ’ the proposition ‘ $\neg 0 = 0+1$ ’ is adduced as a reason. As one can easily read off the object-language surface, more precisely the performators, propositions are adduced in lines 3 and 11 as well. In lines 1 and 2, two propositions are assumed by a written utterance of two assumption sentences. In the remaining lines, successive inferences are performed by utterances of inference sentences. So, for example, in line 6 the sentence ‘THUS  $x = x+1$ ’ is uttered in accordance with CdE and thereby the proposition ‘ $x = x+1$ ’ is inferred. In line 7 the negation of this proposition is inferred in accordance with the repetition rule. By thus inferring a proposition and its negation, a contradiction has been inferred on the basis of the assumption in line 2. This allows NI in line 8, whereby the assumed proposition is negated. In this way, a reductio subproof is closed. The last assumption and all following inferences up to and including the one in line 7 are thereby made unavailable. The segment from line 2 through line 7 corresponds to a subproof in a Jaśkowski box calculus. However, in such a calculus, the subproof would have to be boxed to show which lines are available and which are not ([↑3.5](#)).

Because the assumption in line 2 has been discharged, the proposition in line 8 has now been reached on the basis of the assumption in line 1, which is now the last available assumption. By CdI, this assumption is discharged as well and the conditional ‘ $\neg x = x+1 \rightarrow \neg x+1 = (x+1)+1$ ’ is inferred in line 9. This again closes a subproof, which is marked by the outer box in the Jaśkowski box-style proof under [3.5](#). Because all assumptions containing the parameter ‘x’ have been discharged, the proposition in line 9 can serve as a premise for the UI carried out in line 10. The remaining part of the proof proceeds as outlined above.

With the proof under 4.1, one has a purely object-language proof which does not rely on any commentary: It is clearly identifiable as a proof in which three reasons are adduced — just as John’s natural language proof is recognizable as such by any competent recipient (of course, in the case of the formalization, we can also *show* that it is such a proof). The rule commentary facilitates the understanding, but is not necessary to identify the proof. In particular, the rule commentary is not needed to distinguish inferred and adduced propositions. That the structure of John’s original natural language proof is preserved, is illustrated by the following comparison:

4.2. *John’s proof and its formalization as a sentence sequence*

- 0 First it holds that 0 is not identical with  $0+1$ .  
     SINCE  $\neg 0 = 0 + 1$
- 1 Now, assume  $k$  is not identical with  $k+1$ .  
     SUPPOSE  $\neg x = x + 1$
- 2 Suppose  $k+1$  was identical with  $(k+1)+1$ .  
     SUPPOSE  $x + 1 = (x + 1) + 1$
- 3 It holds for all  $i, j$  that if  $i+1 = j+1$ , then  $i = j$ .  
     SINCE  $\forall x \forall y (x + 1 = y + 1 \rightarrow x = y)$
- 4 Hence  $k$  would be identical with  $k+1$ .  
     THUS  $\forall y (x + 1 = y + 1 \rightarrow x = y)$   
     THUS  $x + 1 = (x + 1) + 1 \rightarrow x = x + 1$   
     THUS  $x = x + 1$
- 5 On the other hand we have that  $k$  is not identical with  $k+1$ .  
     THUS  $\neg x = x + 1$
- 6 So,  $k+1$  can’t be identical with  $(k+1)+1$ .  
     THUS  $\neg x + 1 = (x + 1) + 1$
- 7 Thus we have that if  $k$  is not identical with  $k+1$ , then also  $k+1$  is not identical with  $(k+1)+1$ .  
     THUS  $\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1$
- 8 Obviously this then holds for all  $k$ .  
     THUS  $\forall x (\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1)$
- 9 But we know: If 0 is not identical with  $0+1$  and if furthermore it is true for all  $k$  that given  $k$  is not identical with  $k+1$ , then also  $k+1$  is not identical with  $(k+1)+1$ , then for all  $k$ :  $k$  is not identical with  $k+1$ .  
     SINCE  $\neg 0 = 0 + 1 \wedge \forall x (\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1)$   
          $\rightarrow \forall x \neg x = x + 1$

10 Therefore it holds for all  $k$  that  $k$  is not identical with  $k+1$ .

$$\begin{array}{l} \text{THUS} \quad \neg 0 = 0 + 1 \wedge \forall x(\neg x = x + 1 \rightarrow \neg x + 1 = (x + 1) + 1) \\ \text{THUS} \quad \forall x \neg x = x + 1 \end{array}$$

The calculus used here ( $\downarrow 5$ ), regulates assumptions, adductions and inferences in such a way that it is always defined which of the propositions reached before (if any) remain available as premises and which propositions are made newly available by their assumption, adduction or correct inference. Thus it can be shown that the formalization of John's proof under 4.1 is not, for example, a derivation from the assumptions in lines 1 and 2, but a PA-derivation from the empty set, i.e., a PA-proof. Which propositions are available in which lines of a given sentence sequence depends solely on this sentence sequence. For any given sentence sequence there is thus just one correct commentary with respect to the availability of propositions.

## 5. A speech act calculus

In this section, we provide some technical background to the preceding section.<sup>24</sup> We present a characterization of the derivation concept of the speech act calculus we used to formalize John's proof. This calculus is a subproof calculus. However, subproofs do not have to be demarcated by commentary devices but can be read off a given derivation, as illustrated in Section 4. The following inductive definition defines under what circumstances  $\mathfrak{H}$  is a derivation of length  $k$  with the set  $X$  of available sentence lines and the set  $Y$  of available assumption lines and the set  $Z$  of adduction lines. A derivation will be a sentence sequence of some finite length  $k$ , i.e., a function on the natural numbers  $i < k$  which assigns to each  $i < k$  a sentence, i.e., an expression of the form  $\lceil \Xi \Gamma \rceil$ , where  $\Xi$  is a performatore and  $\Gamma$  is a proposition ( $\uparrow 4$ ). The proposition of the sentence in the last line of a non-empty sentence sequence  $\mathfrak{H}$  will be called *the conclusion of  $\mathfrak{H}$* . In preparation, we define *the extension of a sentence sequence  $\mathfrak{H}$  of length  $k$  by a sentence  $\Sigma$*  to be  $\mathfrak{H} \cup \{(k, \Sigma)\}$ .  $\mathfrak{H}^*$  is an extension of the sentence sequence  $\mathfrak{H}$  iff  $\mathfrak{H}^*$  is a sentence sequence and  $\mathfrak{H}$  is an initial segment of  $\mathfrak{H}^*$ .

The rule-compliant extensions of sentence sequences and the corresponding adjustments in the availability sets will be characterized simul-

<sup>24</sup> For notation and terminology, see also n. 11.

taneously.<sup>25</sup> We will set the empty sequence (i.e.  $\emptyset$ ) as a base, for which the set  $X$  of available sentence lines and the set  $Y$  of available assumption lines and the set  $Z$  of adduction lines are all empty (clause (i)). The inductive clauses then specify how a given derivation (so initially the empty one)  $\mathfrak{H}$  of length  $k$  with the set  $X$  of available sentence lines and the set  $Y$  of available assumption lines and the set  $Z$  of adduction lines may be extended. This can be achieved by extending  $\mathfrak{H}$  by an adduction sentence for a parameter-free proposition  $\Gamma$  (clause (ii-i)). Such an extension will result in a derivation  $\mathfrak{H}^*$  of the form

$$\begin{array}{l} \mathfrak{H} \\ k \text{ SINCE } \Gamma \end{array}$$

Here,  $\mathfrak{H}$  appears as an initial segment of  $\mathfrak{H}^*$ .  $\mathfrak{H}^*$  will be of length  $k+1$  and will have the set  $X \cup \{k\}$  of available sentence lines, the set  $Y$  of available assumption lines and the set  $Z \cup \{k\}$  of adduction lines. Note that the set of available assumption lines, the set of adduction lines, and the set of available inference lines are always disjoint subsets of the set of available sentence lines.

An extension of  $\mathfrak{H}$  may also be achieved by extending  $\mathfrak{H}$  by an assumption sentence for a proposition  $\Gamma$  (clause (ii-ii)). Such an extension will result in a derivation  $\mathfrak{H}^\bullet$  of the form

$$\begin{array}{l} \mathfrak{H} \\ k \text{ SUPPOSE } \Gamma \end{array}$$

$\mathfrak{H}^\bullet$  will be of length  $k+1$  and will have the set  $X \cup \{k\}$  of available sentence lines, the set  $Y \cup \{k\}$  of available assumption lines and the set  $Z$  of adduction lines.

Under certain circumstances,  $\mathfrak{H}$  may also be extended by an inference sentence. So for example, if  $\mathfrak{H}$  is of the form

$$\begin{array}{l} \dots \\ \max(Y) \text{ SUPPOSE } \Delta \\ \dots \\ k-1 \quad \text{THUS} \quad \Gamma \end{array}$$


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<sup>25</sup> This follows the approach in [5], where a definition for derivations without adduction lines is given. A more precise development of this kind of calculus can be found in [4], where we followed the different and somehow more unwieldy approach of defining certain segments in sequences which correspond to closed subproofs (including the closing inference). For a presentation of the speech act calculus suitable for introductory logic courses taught in philosophy departments see [29, chap. 4].

where  $Y$ , the set of available assumption lines, is not empty, and  $\Delta$  is an available assumption which is not followed by another available assumption, i.e.,  $\Delta$  is the assumption in the line  $\max(Y)$  of  $\mathfrak{H}$ , where  $\max(Y)$  is the maximal member of  $Y$ , then one may extend  $\mathfrak{H}$  by an inference of  $\ulcorner \Delta \rightarrow \Gamma \urcorner$  to a derivation  $\mathfrak{H}^\circ$  of the form

$$\begin{array}{l} \dots \\ \max(Y) \text{ SUPPOSE } \Delta \\ \dots \\ k-1 \quad \text{THUS} \quad \Gamma \\ k \quad \text{THUS} \quad \Delta \rightarrow \Gamma \end{array}$$

$\mathfrak{H}^\circ$  will be a derivation of length  $k+1$  (clause (ii-iii)). The set of available assumption lines of  $\mathfrak{H}^\circ$  will be  $Y \setminus \max(Y)$ . So, the assumption will be discharged by this conditional introduction and the subproof from line  $\max(Y)$  to line  $k-1$  will be closed. Accordingly, inference lines between  $\max(Y)$  and  $k-1$ , including the latter, will also become unavailable. Adduction lines, however, remain available. Also, we will have a new available line, namely  $k$ . So, the set of available sentence lines of  $\mathfrak{H}^\circ$  will be  $(X \setminus (\{j \mid \max(Y) \leq j \leq k-1\} \setminus Z)) \cup \{k\}$ , while the set of adduction lines of  $\mathfrak{H}^\circ$  will be just  $Z$ . The two other discharging inferences, negation introduction and existential quantifier elimination, work similarly (clauses (ii-iv) and (ii-v)). In short: With each discharging inference the subproof that legitimizes the inference is closed and all lines in this subproof, with the exception of adduction lines, become unavailable.

Of course,  $\mathfrak{H}$  may under the right circumstances also be extended by a non-discharging inference. So, for example, if there is an  $i \in X$ , i.e., if there is an available line in  $\mathfrak{H}$ , whose sentence has the proposition  $\ulcorner \Delta \wedge B \urcorner$  or the proposition  $\ulcorner B \wedge \Delta \urcorner$ ,  $\mathfrak{H}$  may be extended by an inference of  $B$ . Now, such an inference will always be legitimate, but it may not always be treated as a simple, non-discharging conjunction elimination. For example, it could coincide with a discharging conditional introduction. In order to remove any ambiguity, such cases will always be treated as discharging inferences. In order to achieve this, the clauses for the non-discharging inferences (clauses (ii-vi)-a) through (ii-vi)-n) are under an exclusion clause, which demands that none of the conditions for a discharging inference is met, i.e., that the same inference is not also covered by conditional introduction, negation introduction, or existential quantifier elimination. If this clause is not fulfilled, the inference will still be legitimate, but the available (assumption) lines will change in accordance



with the clauses for the discharging inferences. If the exclusion clause is fulfilled, the set of available sentence lines will also be extended by  $k$ , while the set of available assumption lines and, of course, the set of adduction lines, remain unchanged. Now for the formal definition:

- (i) (*Base clause*) The empty sequence is a derivation of length 0 with the set  $\emptyset$  of available sentence lines and set  $\emptyset$  of available assumption lines and the set  $\emptyset$  of adduction lines; and
- (ii) (*Inductive Clause*) If  $\mathfrak{H}$  is a derivation of length  $k$  with the set  $X$  of available sentence lines and the set  $Y$  of available assumption lines and the set  $Z$  of adduction lines, then:
  - (ii-i) (*Adduction clause*) If  $\Gamma$  is a parameter-free proposition, then the extension of  $\mathfrak{H}$  by  $\ulcorner \text{SINCE } \Gamma \urcorner$  is a derivation of length  $k+1$  with the set  $X \cup \{k\}$  of available sentence lines and the set  $Y$  of available assumption lines and the set  $Z \cup \{k\}$  of adduction lines; and
  - (ii-ii) (*Assumption clause*) If  $\Gamma$  is a proposition, then the extension of  $\mathfrak{H}$  by  $\ulcorner \text{SUPPOSE } \Gamma \urcorner$  is a derivation of length  $k+1$  with the set  $X \cup \{k\}$  of available sentence lines and the set  $Y \cup \{k\}$  of available assumption lines and the set  $Z$  of adduction lines; and
  - (ii-iii) (*CdI clause*) If  $Y \neq \emptyset$  and  $\Delta$  is the proposition of the sentence in line  $\max(Y)$  of  $\mathfrak{H}$  and  $\Gamma$  is the conclusion of  $\mathfrak{H}$ , then the extension of  $\mathfrak{H}$  by  $\ulcorner \text{THUS } \Delta \rightarrow \Gamma \urcorner$  is a derivation of length  $k+1$  with the set  $(X \setminus (\{j \mid \max(Y) \leq j \leq k-1\} \setminus Z)) \cup \{k\}$  of available sentence lines and the set  $Y \setminus \max(Y)$  of available assumption lines and the set  $Z$  of adduction lines; and
  - (ii-iv) (*NI clause*) If  $Y \neq \emptyset$  and  $i \in X$  and  $\max(Y) \leq i$  and  $\Delta$  is the proposition of the sentence in line  $\max(Y)$  of  $\mathfrak{H}$  and  $\Gamma$  is the proposition of the sentence in line  $i$  of  $\mathfrak{H}$  and  $\ulcorner \neg \Gamma \urcorner$  is the conclusion of  $\mathfrak{H}$  or  $\ulcorner \neg \Gamma \urcorner$  is the proposition of the sentence in line  $i$  of  $\mathfrak{H}$  and  $\Gamma$  is the conclusion of  $\mathfrak{H}$ , then the extension of  $\mathfrak{H}$  by  $\ulcorner \text{THUS } \neg \Delta \urcorner$  is a derivation of length  $k+1$  with the set  $(X \setminus (\{j \mid \max(Y) \leq j \leq k-1\} \setminus Z)) \cup \{k\}$  of available sentence lines and the set  $Y \setminus \max(Y)$  of available assumption lines and the set  $Z$  of adduction lines; and
  - (ii-v) (*EE clause*) If  $Y \neq \emptyset$  and  $i \in X$  and  $\max(Y) = i + 1$  and  $\Gamma$  is the conclusion of  $\mathfrak{H}$  and  $\Delta$  is a formula and  $\beta$  is a parameter which neither appears in any line  $m$  of  $\mathfrak{H}$  with  $m \leq i$  nor in  $\Gamma$  and  $\ulcorner \exists \xi \Delta \urcorner$  is the proposition of the sentence in line  $i$  of  $\mathfrak{H}$  and  $[\beta, \xi, \Delta]$  is the proposition of the sentence in line  $\max(Y)$  of  $\mathfrak{H}$ , then the extension of  $\mathfrak{H}$  by  $\ulcorner \text{THUS } \Gamma \urcorner$  is a derivation of length  $k+1$  with the set  $(X \setminus (\{j \mid \max(Y) \leq j \leq k-1\} \setminus Z)) \cup \{k\}$  of available sentence lines and the set  $Y \setminus \max(Y)$  of available assumption lines and the set  $Z$  of adduction lines; and

- (ii-vi) (*Clauses for non-discharging inferences*) If  $Y = \emptyset$  or  $[Y \neq \emptyset$  and  $[B$  is not the conditional of the proposition of the sentence in line  $\max(Y)$  of  $\mathfrak{H}$  and the conclusion of  $\mathfrak{H}]$  and  $[[$ there is no  $i \in X$  with  $\max(Y) \leq i$  and no  $\Gamma$  such that  $\Gamma$  is the proposition of the sentence in line  $i$  of  $\mathfrak{H}$  and  $\lceil \neg\Gamma \rceil$  is the conclusion of  $\mathfrak{H}$  or  $\lceil \neg\Gamma \rceil$  is the proposition of the sentence in line  $i$  of  $\mathfrak{H}$  and  $\Gamma$  is the conclusion of  $\mathfrak{H}$ ] or  $[B$  is not the negation of the proposition of the sentence in line  $\max(Y)$  of  $\mathfrak{H}]$  and  $[[$ there is no  $i \in X$  with  $\max(Y) = i + 1$  and a formula  $\Delta$  and a parameter  $\beta$  which appears neither in any line  $m$  of  $\mathfrak{H}$  with  $m \leq i$  nor in the conclusion of  $\mathfrak{H}$  such that  $\lceil \exists \xi \Delta \rceil$  is the proposition of the sentence in line  $i$  of  $\mathfrak{H}$  and  $[\beta, \xi, \Delta]$  is the proposition of the sentence in line  $\max(Y)$  of  $\mathfrak{H}]$  or  $[B$  is not the conclusion of  $\mathfrak{H}]]$ , and
- a) (*CdE clause*) If  $i, j \in X$  and  $\Delta$  is the proposition of the sentence in line  $i$  of  $\mathfrak{H}$  and  $\lceil \Delta \rightarrow B \rceil$  is the proposition of the sentence in line  $j$  of  $\mathfrak{H}$ ; or
  - b) (*CI clause*) If  $i, j \in X$  and  $\Delta$  is the proposition of the sentence in line  $i$  of  $\mathfrak{H}$  and  $\Gamma$  is the proposition of the sentence in line  $j$  of  $\mathfrak{H}$  and  $B$  is identical with  $\lceil \Delta \wedge \Gamma \rceil$ ; or
  - c) (*CE clause*) If  $i \in X$  and the sentence in line  $i$  of  $\mathfrak{H}$  is identical with  $\lceil \Delta \wedge B \rceil$  or with  $\lceil B \wedge \Delta \rceil$ ; or
  - d) (*BI clause*) If  $i, j \in X$  and  $\lceil \Delta \rightarrow \Gamma \rceil$  is the proposition of the sentence in line  $i$  of  $\mathfrak{H}$  and  $\lceil \Gamma \rightarrow \Delta \rceil$  is the proposition of the sentence in line  $j$  of  $\mathfrak{H}$  and  $B$  is identical with  $\lceil \Delta \leftrightarrow \Gamma \rceil$ ; or
  - e) (*BE clause*) If  $i, j \in X$  and  $\Delta$  is the proposition of the sentence in line  $i$  of  $\mathfrak{H}$  and  $\lceil \Delta \leftrightarrow B \rceil$  or  $\lceil B \leftrightarrow \Delta \rceil$  is the proposition of the sentence in line  $j$  of  $\mathfrak{H}$ ; or
  - f) (*DI clause*) If  $i \in X$  and  $\Delta$  is the proposition of the sentence in line  $i$  of  $\mathfrak{H}$  and  $\Gamma$  is a proposition and  $B$  is identical with  $\lceil \Delta \vee \Gamma \rceil$  or with  $\lceil \Gamma \vee \Delta \rceil$ ; or
  - g) (*DE clause*) If  $i, j, l \in X$  and  $\lceil \Delta \vee \Gamma \rceil$  is the proposition of the sentence in line  $i$  of  $\mathfrak{H}$  and  $\lceil \Delta \rightarrow B \rceil$  is the proposition of the sentence in line  $j$  of  $\mathfrak{H}$  and  $\lceil \Gamma \rightarrow B \rceil$  is the proposition of the sentence in line  $l$  of  $\mathfrak{H}$ ; or
  - h) (*NE clause*) If  $i \in X$  and  $\lceil \neg\neg B \rceil$  is the proposition of the sentence in line  $i$  of  $\mathfrak{H}$ ; or
  - i) (*UI clause*) If  $i \in X$  and  $\Delta$  is a formula and  $\xi$  is a variable and  $\beta$  is a parameter which appears neither in any line  $l$  of  $\mathfrak{H}$  with  $l \in Y$  nor in  $\Delta$  and  $[\beta, \xi, \Delta]$  is the proposition of the sentence in line  $i$  of  $\mathfrak{H}$  and  $B$  is identical with  $\lceil \forall \xi \Delta \rceil$ ; or

- j) (*UE clause*) If  $i \in X$  and  $\vartheta$  is a closed term and  $\ulcorner \forall \xi \Delta \urcorner$  is the proposition in line  $i$  of  $\mathfrak{H}$  and  $B$  is identical with  $[\vartheta, \xi, \Delta]$ ; or
- k) (*EI clause*) If  $i \in X$  and  $\Delta$  is a formula and  $\xi$  is a variable and  $\vartheta$  is closed term and  $[\vartheta, \xi, \Delta]$  is the proposition of the sentence in line  $i$  of  $\mathfrak{H}$  and  $B$  is identical with  $\ulcorner \exists \xi \Delta \urcorner$ ; or
- l) (*II clause*) If  $\vartheta$  is a closed term and  $B$  is identical with  $\ulcorner \vartheta = \vartheta \urcorner$ ; or
- m) (*IE clause*) If  $i, j \in X$  and  $\Delta$  is a formula and  $\xi$  is a variable and  $\vartheta_0, \vartheta_1$  are closed terms and  $\ulcorner \vartheta_0 = \vartheta_1 \urcorner$  is the proposition of the sentence in line  $i$  of  $\mathfrak{H}$  and  $[\vartheta_0, \xi, \Delta]$  is the proposition of the sentence in line  $j$  of  $\mathfrak{H}$  and  $B$  is identical with  $[\vartheta_1, \xi, \Delta]$ ; or
- n) (*R clause*) If  $i \in X$  and  $B$  is the proposition of the sentence in line  $i$  of  $\mathfrak{H}$ ;

then the extension of  $\mathfrak{H}$  by  $\ulcorner \text{THUS } B \urcorner$  is a derivation of length  $k+1$  with the set  $X \cup \{k\}$  of available sentence lines and the set  $Y$  of available assumption lines and the set  $Z$  of adduction lines; and

- (iii) (*Closure clause*) Else no  $\mathfrak{H}$  is a derivation of length  $k$  with the set  $X$  of available sentence lines and the set  $Y$  of available assumption lines and the set  $Z$  of adduction lines for any  $k, X, Y, Z$ .

To (partly) repeat; Clause (i), the base clause, stipulates that the empty set is a derivation. In clause (ii), which contains the inductive clauses, the rule-compliant extensions of a given derivation by a sentence that result in a change of the set of available assumption lines or the set of adduction lines are treated first: Extensions by (ii-i) and (ii-ii) make a reason ((ii-i)) or an assumption ((ii-ii)) available; extensions by (ii-iii) through (ii-v) result in the discharge of an assumption and close the respective subproofs. The first part of (ii-vi) excludes extensions by CdI, NI, and EE. The sub-clauses a) through n) then list all other rule-compliant extensions by an inference sentence. These extensions are allowed even if the exclusion condition is not satisfied. But if this is the case, the respective extension is also an extension by (ii-iii), (ii-iv) or (ii-v) and the set of available sentence lines and the set of available assumption lines change accordingly.

Now we define:  $\mathfrak{H}$  is a *T-derivation* of  $\Gamma$  from  $M$  iff  $T$  is a set of propositions and there are  $k \neq 0$  and  $X, Y, Z$  such that  $\mathfrak{H}$  is a derivation of length  $k$  with the set  $X$  of available sentence lines and the set  $Y$  of available assumption lines and the set  $Z$  of adduction lines and  $M = \{\Delta \mid \text{there is an } i \in Y \text{ and } \Delta \text{ is the proposition of the sentence in line } i \text{ of } \mathfrak{H}\}$  and  $\{\Delta \mid \text{there is an } i \in Z \text{ and } \Delta \text{ is the proposition of the sentence in line } i \text{ of } \mathfrak{H}\}$ .

$i$  of  $\mathfrak{H}\}$   $\subseteq T$  and  $\Gamma$  is the conclusion of  $\mathfrak{H}$ . With these definitions we have unique readability for derivations: If  $\mathfrak{H}$  is a  $T$ -derivation of  $\Gamma$  from  $M$  for any  $T$ ,  $\Gamma$ ,  $M$ , then, as usual, the conclusion of  $\mathfrak{H}$  is uniquely determined, but, in addition, the set of available sentence lines, the set of available assumption lines, and the set of adduction lines are uniquely determined as well.<sup>26</sup> Thus, it is always uniquely determined for any derivation which propositions are available and which of those are available as assumed or adduced or inferred propositions.<sup>27</sup>

Moreover, we define:  $\mathfrak{H}$  is a  $T$ -proof for  $\Gamma$  iff  $\mathfrak{H}$  is a  $T$ -derivation of  $\Gamma$  from  $\emptyset$ .  $\mathfrak{H}$  is a (purely logical) derivation of  $\Gamma$  from  $M$  iff  $\mathfrak{H}$  is an  $\emptyset$ -derivation of  $\Gamma$  from  $M$ ; and  $\mathfrak{H}$  is a (purely logical) proof for  $\Gamma$  iff  $\mathfrak{H}$  is an  $\emptyset$ -proof for  $\Gamma$ . The general unique readability for derivations also applies to these special cases: For example, for any derivation it is uniquely determined, for any given (decidable)  $T$ , whether or not it is a  $T$ -proof for its conclusion. Deductive consequence relations can be defined in the usual way.<sup>28</sup>

It is important to see that the preceding definitions and results rely on the full illocutionary explicitness of sentence sequences in the speech act calculus. Each sentence is uniquely determined as an assumption, inference or adduction sentence for a certain proposition. This makes it possible to implement Jaśkowski's subproof approach in such a way that for any given derivation subproofs are uniquely determined. On

<sup>26</sup> This is shown by induction on the length of derivations.

<sup>27</sup> Consider 4.1 as an example: Applying the inductive definition several times, it can be shown that the sequence consisting of lines 0 to 7 is a derivation of length 8 with the set  $\{0, 1, 2, 3, 4, 5, 6, 7\}$  of available sentence lines and the set  $\{1, 2\}$  of available assumption lines and the set  $\{0, 3\}$  of adduction lines. Thus it is a PA-derivation of ' $\neg x = x + 1$ ' from ' $\neg x = x + 1$ ', ' $x + 1 = (x + 1) + 1$ '. The way the inductive definition works can be illustrated a bit more detailed: Building on the preceding results clause (ii-iv) guarantees that the sequence consisting of lines 0 to 8 is a derivation of length 9 with the set  $\{0, 1, 2, 3, 8\}$  of available sentence lines and the set  $\{1\}$  of available assumption lines and the set  $\{0, 3\}$  of adduction lines. Thus it is a PA-derivation of ' $\neg x + 1 = (x + 1) + 1$ ' from ' $\neg x = x + 1$ '. All this is determined without recurring to the rule commentary. Proceeding this way, one vindicates the earlier claim that the full sentence sequence under 4.1 (which does not include the rule commentary to the right) is a PA-derivation of ' $\forall x \neg x = x + 1$ ' from  $\emptyset$ .

<sup>28</sup>  $\Gamma$  is a deductive consequence of  $M$  (short:  $M \vdash \Gamma$ ) iff  $M$  is a set of propositions and there is a derivation  $\mathfrak{H}$  of  $\Gamma$  from some  $N \subseteq M$ .  $\Gamma$  is a deductive consequence of  $M$  under  $T$  (short:  $M \vdash_T \Gamma$ ) iff  $M$  is a set of propositions and there is an  $\mathfrak{H}$  such that  $\mathfrak{H}$  is a  $T$ -derivation of  $\Gamma$  from some  $N \subseteq M$ . We then have for parameter-free  $T$ :  $M \vdash_T \Gamma$  iff  $M \cup T \vdash \Gamma$ .

the other hand, the potential of the performators is only fully exploited under the subproof approach. The use of performators alone does not guarantee unique readability: The dependency approach still requires a record of the premises used for inferences. Thus, the quasi-sequent calculi developed by Hinst (e.g. [13]) and Siegart (e.g. [27]) rely on a commentary apparatus to keep track of assumptions. In contrast, under the subproof approach, the order of sentences and subproofs determines availability. It is only by combining the Hinst-Siegwart-approach with that of Jaśkowski that the need for “bookkeeping” is eliminated.

## 6. Speech acts and commentary

In the speech act calculus, derivations are sequences of object-language expressions, namely sentences. The purely object-language nature of derivations alone, however, does not guarantee that the formalization of John’s proof does not contain added commentary devices for “bookkeeping”. The calculi discussed in Section 3 can also be characterized formally in such a way that no metalinguistic (or graphical) devices are needed and indeed some of them are so characterized. This basically requires that object languages are extended by new expressions and derivations are defined not as sequences of formulas, but as sequences of more complex entities. Thus, in Gentzen’s sequent calculus, derivations are sequences of object-language sequents, with ‘ $\Rightarrow$ ’ as an (auxiliary) object-language symbol (see [10, pp. 179–180], [11, pp. 512–513], and [24, p. 88]).

Obviously, our approach also follows this strategy: the usual formal languages are extended by expressions of a new atomic category (performators) and derivations are then not defined as sequences of formulas but of sentences, i.e., of expressions which consist of an expression from the new category followed by a formula. However, the analysis of the calculi in Section 3 does not essentially depend on whether or not the commentary devices are incorporated into the object language: derivations in the resulting calculi (would) still contain expressions which – in contrast to the performators used in the speech act calculus – have no counterpart in John’s proof.

With the provisions made in the preceding paragraph one may say that the calculi considered in Section 3 rely on commentary where John’s proof does not while discarding illocutionary expressions present in the

original proof. Under the speech-act theoretic perspective adopted in Section 4, John performs a sequence of speech acts when he puts forward his proof, where each speech act consists in his utterance of one of the sentences of the proof. According to this view, the illocutionary expressions are not commentary on a sequence of propositions but an essential part of the sentences that constitute the proof. For someone who shares this perspective, the formal performatives thus represent not some commentary devices but essential constituents of the sentences that make up the original proof. In this respect then, the formalization of John's proof in the speech act calculus is more natural than its formalizations in the other calculi. Moreover, from such a speech-act theoretic perspective, putting forward a natural language proof always consists in performing a sequence of speech acts. Accordingly, under such a perspective, informal proofs (and attempted proofs) are conceived of as sequences of – possibly elliptical – sentences. Adding illocutionary operators to a formalization of a proof in which the illocutionary status is not as clearly marked as in John's proof will then be an interpretative exercise that aims at making something explicit that is needed for a full understanding of the proof in any case, namely the illocutionary status of the sentences it consists of. For someone who is not opposed to such a perspective, the speech act calculus offers a format for a natural and uniform formalization of (attempted) natural language proofs.<sup>29</sup>

Of course, natural language derivations often contain commenting devices that resemble those used in the dependency and subproof calculi from Section 3. However, such derivations can also be formalized as sequences of sentences in the speech act calculus. Under the assumption that utterances of proofs consist in the performance of certain sequences of speech acts, the general naturalness of the speech act calculus lies in the fact that natural language derivations that contain metalinguistic phrases as well as those that do not can be uniformly formalized without additional commentary devices. This is not true of the calculi presented in Section 3. In the speech act calculus, subproof structures and available assumptions in derivations are uniquely determined by the sequences of object language sentences.<sup>30</sup>

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<sup>29</sup> Of course, calculi of this type can be constructed not only for classical regulations of inference but also for alternative ones. An extension to classical second-order logic with a definite and an indefinite description operator and a “thinkability” operator can be found in [25, Appendix II.ii].

<sup>30</sup> Moreover, the speech act calculus may be used to represent failed proof at-

We acknowledge that metalinguistic commentaries that indicate, for example, the scope of assumptions or the rules applied in an inference, can be very helpful. We do not want to suggest that there is something inherently wrong with them as long as they are not obligatory. But even independently from a speech act theoretic point of view, obligatory metalinguistic commentaries are problematic if one wants “to express proofs [...] in a fully formalized object language” [2, p. 53: n. 121] and holds with Church that

as long as any part of the proof remains in an unformalized meta-language the logical analysis must be held to be incomplete. [...] Though we use a meta-language to set up the object language, we require that, once set up, the object language shall be an independent language capable, without continuous support and supplementation from the meta-language, of expressing those things for which it was designed. [2, p. 53: n. 121]

Church’s demand – which is easily fulfilled for axiomatic calculi – can also be fulfilled for linear calculi of natural deduction. The speech act calculus shows that it is possible to do this in a natural way – by incorporating illocutionary expressions into our formal languages.

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tempts. Here, it seems to fare definitely better than the other approaches, since a failed proof attempt can be represented as such by its formalization, without relying on any additional commentary. In contrast, formalizations of failed proof attempts in the calculi from Section 3 can sometimes also be commented on as a proof; in such cases, the formalization without the commentary does not unambiguously represent the fact that it is a formalization of a failed proof attempt and not of a proof.

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