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NATURAL DEDUCTION FOR FOUR-VALUED BOTH REGULAR AND MONOTONIC LOGICS

Abstract. The development of recursion theory motivated Kleene to create regular three-valued logics. Taking his inspiration from the computer science, Fitting later continued to investigate regular three-valued logics and defined them as monotonic ones. Afterwards, Komendantskaya proved that there are four regular three-valued logics and in the three-valued case the set of regular logics coincides with the set of monotonic logics. Next, Tomova showed that in the four-valued case regularity and monotonicity do not coincide. She counted that there are 6400 four-valued regular logics, but only six of them are monotonic. The purpose of this paper is to create natural deduction systems for them. We also describe some functional properties of these logics.

Keywords: natural deduction; four-valued logic; regular logic; monotonic logic; Kleene's logics; Belnap–Dunn's logic

1. Introduction

1.1. Preliminaries

All logics described in this paper are built in a propositional language $\mathcal{L}$ which we define in Backus–Naur form as follows:

$$A := p \mid \neg A \mid A \wedge A \mid A \vee A.$$

Let Prop and Form abbreviate, respectively, the set of all propositional variables and the set of all formulae of $\mathcal{L}$. Let V_3 and V_4 be, respectively, the set $\{1, u, 0\}$ of truth values "true", "undefined", and "false" and the set $\{1, b, n, 0\}$ of truth values "true", "both true and

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false", "neither true no false", and "false". In all t-valued ($t \in \{3, 4\}$) logics described in this paper, a valuation is a function v from Prop to V_t. Moreover, let us denote a truth-table f for a connective c by f_c.

1.2. Three-valued both regular and monotonic logics

Let us call *regular logics* those systems in which all connectives are regular in the sense specified below. The investigation of them began in Kleene's paper [15] where two regular logics were introduced: Kleene's strong logic $\mathbf{K_3}$ and Kleene's weak logic $\mathbf{K_3^w}$. In [14] Kleene defines regularity and clarifies the motivation behind it as follows:

> We conclude that, in order for the propositional connectives to be partial recursive operations (or at least to produce partial recursive predicates when applied to partial recursive predicates), we must choose tables for them which are *regular*, in the following sense: A given column (row) contains 1 in the u row (column), only if the column (row) consists entirely of 1's; and likewise for 0. [14, p. 334]

In $\mathbf{K_3}$ a valuation v on Prop is extended to a valuation on Form according to the following truth tables:

	$f_\neg$
1	0
u	u
0	1

$f_\wedge$	1	u	0
1	1	u	0
u	u	u	0
0	0	0	0

$f_\vee$	1	u	0
1	1	1	1
u	1	u	u
0	1	u	0

In $\mathbf{K_3}$, an entailment relation is defined via the sole designated value 1. However, Asenjo [1] studied $\mathbf{K_3}$ with two designated values (1 and u) as a logic of antinomies. This logic is well-known as **LP** (Logic of Paradox) due to Priest's [23, 21, 22] continuation of Asenjo's research. Note that $\mathbf{K_3}$ (1938) is a fragment of Łukasiewicz's logic $\mathbf{Ł_3}$ (1920) [18]. Natural deduction systems for $\mathbf{K_3}$ and **LP**, respectively, are presented in [22, 24, 17].

In $\mathbf{K_3^w}$ negation is the same as for $\mathbf{K_3}$; conjunction and disjunction, as was shown in Finn's paper [8], are expressed via $\mathbf{K_3}$'s connectives by equations (1) and (2) (see p. 55), respectively. Notice that $\mathbf{K_3^w}$ (1938) is a fragment of Bochvar's logic $\mathbf{B_3}$ (1938) introduced in [4] independently of [15]. Natural deduction systems for $\mathbf{K_3^w}$ both with one and two designated values are presented in [19].

The next stage in the exploration of regular three-valued logics is Fitting's paper [10] where the intermediate logic $\mathbf{K_3^\rightarrow}$ (**Lisp**) was discovered. In $\mathbf{K_3^\rightarrow}$ negation is the same as for $\mathbf{K_3}$; conjunction and disjunction,

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as was shown in Komendantskaya's paper [16], are expressed via $\mathbf{K_3}$'s connectives by equations (3) and (4) (see p. 55), respectively. Moreover, Komendantskaya [16] described the logic $\mathbf{K_3^{\leftarrow}}$ (**TwinLisp**) which is the dual of $\mathbf{K_3^{\rightarrow}}$. In $\mathbf{K_3^{\leftarrow}}$, negation is the same as for $\mathbf{K_3}$; conjunction and disjunction, as was shown in [16], are defined via $\mathbf{K_3}$'s connectives by equations (5) and (6), respectively. Natural deduction systems for $\mathbf{K_3^{\rightarrow}}$ and $\mathbf{K_3^{\leftarrow}}$ both with one and two designated values are presented in [19].

Note also that $\mathbf{K_3^w}$'s conjunction and disjunction, as was shown in [16], are expressed both via $\mathbf{K_3^{\rightarrow}}$'s and $\mathbf{K_3^{\leftarrow}}$'s connectives (see equations (7)–(10) on p. 55).

Let $\wedge$ and $\vee$ be $\mathbf{K_3}$'s conjunction and disjunction, respectively; let $\cap$ and $\cup$ be $\mathbf{K_3^w}$'s conjunction and disjunction, respectively; let $\wedge^{\rightarrow}$ and $\vee^{\rightarrow}$ be $\mathbf{K_3^{\rightarrow}}$'s conjunction and disjunction, respectively; let $\wedge^{\leftarrow}$ and $\vee^{\leftarrow}$ be $\mathbf{K_3^{\leftarrow}}$'s conjunction and disjunction, respectively. Then the following equations hold [8, 16]:

$$A \cap B = (A \wedge B) \vee (A \wedge \neg A) \vee (B \wedge \neg B) \tag{1}$$
$$A \cup B = (A \vee B) \wedge (A \vee \neg A) \wedge (B \vee \neg B) \tag{2}$$
$$A \wedge^{\rightarrow} B = (\neg A \vee B) \wedge A \tag{3}$$
$$A \vee^{\rightarrow} B = (\neg A \wedge B) \vee A \tag{4}$$
$$A \wedge^{\leftarrow} B = (A \vee \neg B) \wedge B \tag{5}$$
$$A \vee^{\leftarrow} B = (A \wedge \neg B) \vee B \tag{6}$$
$$A \cap B = (A \wedge^{\rightarrow} B) \vee^{\rightarrow} (B \wedge^{\rightarrow} A) \tag{7}$$
$$A \cup B = (A \vee^{\rightarrow} B) \wedge^{\rightarrow} (B \vee^{\rightarrow} A) \tag{8}$$
$$A \cap B = (A \wedge^{\leftarrow} B) \vee^{\leftarrow} (B \wedge^{\leftarrow} A) \tag{9}$$
$$A \cup B = (A \vee^{\leftarrow} B) \wedge^{\leftarrow} (B \vee^{\leftarrow} A) \tag{10}$$

Monotonic logics are those whose propositional connectives are monotonic functions; a function F is monotonic, if $F(x_1, \ldots, x_z) \leq F(y_1, \ldots, y_k)$, for all truth values $x_1, \ldots, x_z, y_1, \ldots, y_k$ such that $x_1 \leq y_1, \ldots, x_k \leq y_k$. In [9, 10], the set $\{1, u, 0\}$ is ordered as follows: $u \leq 1$, $u \leq 0$, 1 and 0 are incomparable. Using this order, Fitting [10] defined regular logics as monotonic ones. Moreover, as shown in [16], the set of all regular three-valued logics coincides with the set of all normal three-valued[1] monotonic logics.

[1] A many-valued logic is called *normal*, if its connectives are classical on $\{1, 0\}$.

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1.3. Regularity and monotonicity in the four-valued case

In [25] Tomova defined regularity for the *four*-valued case as follows:

> A given column (row) contains 1 in the b or n row (column), only if the column (row) consists entirely of 1's; and likewise for 0. [25, p. 226]

Moreover, Tomova [25] counted that there are 6400 four-valued regular disjunctions (conjunctions are defined in a standard way: $A \wedge B = \neg(\neg A \vee \neg B)$). Furthermore, there are 2^8 $\mathbf{K_3}$-type four-valued disjunctions, 2^{10} $\mathbf{K_3^{\rightarrow}}$-type four-valued disjunctions, 2^{10} $\mathbf{K_3^{\leftarrow}}$-type four-valued disjunctions, and 2^{10} $\mathbf{K_3^{w}}$-type four-valued disjunctions.

In [25], the set $\{1, b, n, 0\}$ is ordered as follows: $n \leq 0 \leq b$, $n \leq 1 \leq b$, 1 and 0 are incomparable. As follows from [25], this order produces 81 monotonic logics; however, only 6 of them are regular. Let us introduce these logics:

- $\mathbf{K_4^{\rightarrow}}$ for the matrix $\langle\{1, b, n, 0\}, f_\neg, f_\wedge, f_\vee, \{1, b\}\rangle$ where

	$f_\neg$
1	0
b	b
n	n
0	1

$f_\wedge$	1	b	n	0
1	1	b	n	0
b	b	b	b	b
n	n	n	n	n
0	0	0	0	0

$f_\vee$	1	b	n	0
1	1	1	1	1
b	b	b	b	b
n	n	n	n	n
0	1	b	n	0

- $\mathbf{K_4^{\leftarrow}}$ for the matrix $\langle\{1, b, n, 0\}, f_\neg, f_\wedge, f_\vee, \{1, b\}\rangle$ where $f_\neg$ is the same as for $\mathbf{K_4^{\rightarrow}}$ and

$f_\wedge$	1	b	n	0
1	1	b	n	0
b	b	b	n	0
n	n	b	n	0
0	0	b	n	0

$f_\vee$	1	b	n	0
1	1	b	n	1
b	1	b	n	b
n	1	b	n	n
0	1	b	n	0

- $\mathbf{K_4^{w}}$ for the matrix $\langle\{1, b, n, 0\}, f_\neg, f_\wedge, f_\vee, \{1, b\}\rangle$ where $f_\neg$ is the same as for $\mathbf{K_4^{\rightarrow}}$ and

$f_\wedge$	1	b	n	0
1	1	b	n	0
b	b	b	b	b
n	n	n	n	n
0	0	b	n	0

$f_\vee$	1	b	n	0
1	1	b	n	1
b	b	b	b	b
n	n	n	n	n
0	1	b	n	0

- $\mathbf{K}^{\mathbf{w}}_{\mathbf{4b}}$ for the matrix $\langle\{1, b, n, 0\}, f_{\neg}, f_{\wedge}, f_{\vee}, \{1, b\}\rangle$ where $f_{\neg}$ is the same as for $\mathbf{K}^{\rightarrow}_{\mathbf{4}}$ and

$f_{\wedge}$	1	b	n	0
1	1	b	n	0
b	b	b	b	b
n	n	b	n	n
0	0	b	n	0

$f_{\vee}$	1	b	n	0
1	1	b	n	1
b	b	b	b	b
n	n	b	n	n
0	1	b	n	0

- $\mathbf{K}^{\mathbf{w}}_{\mathbf{4bn}}$ for the matrix $\langle\{1, b, n, 0\}, f_{\neg}, f_{\wedge}, f_{\vee}, \{1, b\}\rangle$ where $f_{\neg}$ is the same as for $\mathbf{K}^{\rightarrow}_{\mathbf{4}}$ and

$f_{\wedge}$	1	b	n	0
1	1	b	n	0
b	b	b	n	b
n	n	b	n	n
0	0	b	n	0

$f_{\vee}$	1	b	n	0
1	1	b	n	1
b	b	b	n	b
n	n	b	n	n
0	1	b	n	0

- $\mathbf{K}^{\mathbf{w}}_{\mathbf{4n}}$ for the matrix $\langle\{1, b, n, 0\}, f_{\neg}, f_{\wedge}, f_{\vee}, \{1, b\}\rangle$ where $f_{\neg}$ is the same as for $\mathbf{K}^{\rightarrow}_{\mathbf{4}}$ and

$f_{\wedge}$	1	b	n	0
1	1	b	n	0
b	b	b	n	b
n	n	n	n	n
0	0	b	n	0

$f_{\vee}$	1	b	n	0
1	1	b	n	1
b	b	b	n	b
n	n	n	n	n
0	1	b	n	0

1.4. Functional properties of these four-valued logics

We will present here some functional properties of these four-valued logics which were not mentioned in [25].

First of all, let us introduce Belnap–Dunn's logic **FDE** [2, 3, 7][2] for the matrix $\langle\{1, b, n, 0\}, f_{\neg}, f_{\wedge}, f_{\vee}, \{1, b\}\rangle$[3] where $f_{\neg}$ is the same as for $\mathbf{K}^{\rightarrow}_{\mathbf{4}}$ and

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[2] As mentioned in [9, 10, 11], **FDE** is a four-valued generalization of $\mathbf{K_3}$, i.e., with respect to the sets $\{1, n, 0\}$ and $\{1, b, 0\}$ **FDE** is $\mathbf{K_3}$ and **LP**, respectively. A natural deduction system for **FDE** may be found in [22].

[3] Note that Belnap [2, 3] defined an entailment relation in **FDE** via $\leq$. However, Font [12] proved that it is equivalently defined via the set $\{1, b\}$ of designated values. Later Zaitsev and Shramko [26] independently obtained the same result.

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$f_\wedge$	1	b	n	0
1	1	b	n	0
b	b	b	0	0
n	n	0	n	0
0	0	0	0	0

$f_\vee$	1	b	n	0
1	1	1	1	1
b	1	b	1	b
n	1	1	n	n
0	1	b	n	0

If in equations (1) and (2) we replace $\mathbf{K_3}$'s connectives by **FDE**'s connectives, we obtain $\mathbf{K_4^w}$'s connectives. If in equations (3) and (4) we replace $\mathbf{K_3}$'s connectives by **FDE**'s connectives, we obtain $\mathbf{K_4^\rightarrow}$'s connectives. If in equations (5) and (6) we replace $\mathbf{K_3}$'s connectives by **FDE**'s connectives, we obtain $\mathbf{K_4^\leftarrow}$'s connectives. Surprisingly, if in equations (7)–(10) we replace $\mathbf{K_3^\rightarrow}$'s and $\mathbf{K_3^\leftarrow}$'s connectives by $\mathbf{K_4^\rightarrow}$'s and $\mathbf{K_4^\leftarrow}$'s connectives, respectively, we do not obtain $\mathbf{K_4^w}$'s connectives. We will obtain connectives of the logic $\mathbf{K_4^\leftrightarrow}$ for the matrix $\langle\{1, b, n, 0\}, f_\neg, f_\wedge, f_\vee, \{1, b\}\rangle$ where $f_\neg$ is the same as for $\mathbf{K_4^\rightarrow}$ and

$f_\wedge$	1	b	n	0
1	1	b	n	0
b	b	b	1	b
n	n	1	n	n
0	0	b	n	0

$f_\vee$	1	b	n	0
1	1	b	n	1
b	b	b	0	b
n	n	0	n	n
0	1	b	n	0

Although $\mathbf{K_4^\leftrightarrow}$ is not regular, we will consider it on equal terms with both regular and monotonic four-valued logics, since $\mathbf{K_4^\leftrightarrow}$'s connectives are naturally obtained from $\mathbf{K_3^\rightarrow}$'s and $\mathbf{K_3^\leftarrow}$'s ones.

Definition 1.1. Let $\boldsymbol{L} \in \{\mathbf{K_4^\rightarrow}, \mathbf{K_4^\leftarrow}, \mathbf{K_4^w}, \mathbf{K_{4b}^w}, \mathbf{K_{4bn}^w}, \mathbf{K_{4n}^w}, \mathbf{K_4^\leftrightarrow}\}$, $\Gamma \subseteq$ Form, and $A \in$ Form. Then $\Gamma \models_{\boldsymbol{L}} A$ iff for each valuation v, if $v(G) \in \{1, b\}$, for any $G \in \Gamma$, then $v(A) \in \{1, b\}$.

2. Natural deduction systems

We will use the following rules of inference:

$$(\neg\neg\mathrm{I})\ \frac{A}{\neg\neg A} \qquad (\neg\neg\mathrm{E})\ \frac{\neg\neg A}{A}$$

$$(\vee\mathrm{I}_1)\ \frac{A}{A \vee B} \qquad (\vee\mathrm{I}_2)\ \frac{B}{A \vee B} \qquad (\vee\mathrm{I}_3)\ \frac{\neg A \quad B}{A \vee B}$$

$$(\vee\mathrm{I}_4)\ \frac{A \quad \neg B}{A \vee B} \qquad (\vee\mathrm{I}_5)\ \frac{A \quad \neg A}{A \vee B} \qquad (\vee\mathrm{I}_6)\ \frac{B \quad \neg B}{A \vee B} \qquad (\vee\mathrm{I}_7)\ \frac{A \quad B}{A \vee B}$$

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$$(\wedge I_1)\ \frac{A \quad B}{A \wedge B} \qquad (\wedge I_2)\ \frac{A \quad \neg A}{A \wedge B} \qquad (\wedge I_3)\ \frac{B \quad \neg B}{A \wedge B} \qquad (\wedge I_4)\ \frac{A \quad \neg A \quad B}{A \wedge B}$$

$$(\wedge I_5)\ \frac{A \quad \neg A \quad \neg B}{A \wedge B} \qquad (\wedge I_6)\ \frac{A \quad B \quad \neg B}{A \wedge B} \qquad (\wedge I_7)\ \frac{\neg A \quad B \quad \neg B}{A \wedge B}$$

$$(\wedge E_1)\ \frac{A \wedge B}{A} \qquad (\wedge E_2)\ \frac{A \wedge B}{B} \qquad (\wedge E_3)\ \frac{A \wedge B}{\neg A \vee B}$$

$$(\wedge E_4)\ \frac{A \wedge B}{A \vee \neg B} \qquad (\wedge E_5)\ \frac{A \wedge B}{A \vee B}$$

$$(\neg\vee I_1)\ \frac{\neg A \wedge \neg B}{\neg(A \vee B)} \qquad (\neg\vee I_2)\ \frac{A \wedge \neg A}{\neg(A \vee B)} \qquad (\neg\vee I_3)\ \frac{B \wedge \neg B}{\neg(A \vee B)}$$

$$(\neg\vee I_4)\ \frac{A \wedge \neg A \wedge B}{\neg(A \vee B)} \qquad (\neg\vee I_5)\ \frac{A \wedge B \wedge \neg B}{\neg(A \vee B)}$$

$$(\neg\vee E_1)\ \frac{\neg(A \vee B)}{\neg A \wedge \neg B} \qquad (\neg\vee E_2)\ \frac{\neg(A \vee B)}{\neg A} \qquad (\neg\vee E_3)\ \frac{\neg(A \vee B)}{\neg B}$$

$$(\neg\vee E_4)\ \frac{\neg(A \vee B)}{\neg A \vee B} \qquad (\neg\vee E_5)\ \frac{\neg(A \vee B)}{A \vee \neg B}$$

$$(\neg\wedge I)\ \frac{\neg A \vee \neg B}{\neg(A \wedge B)} \qquad (\neg\wedge E)\ \frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

Moreover, we will use the following proof construction rules:

$$(\vee E_1)\ \frac{A \vee B \quad \begin{matrix}[A]\\ C\end{matrix} \quad \begin{matrix}[\neg A][B]\\ C\end{matrix}}{C} \qquad (\vee E_2)\ \frac{A \vee B \quad \begin{matrix}[A][\neg B]\\ C\end{matrix} \quad \begin{matrix}[B]\\ C\end{matrix}}{C}$$

$$(\vee E_3)\ \frac{A \vee B \quad \begin{matrix}[A][B]\\ C\end{matrix} \quad \begin{matrix}[A][\neg B]\\ C\end{matrix} \quad \begin{matrix}[\neg A][B]\\ C\end{matrix}}{C}$$

$$(\vee E_4)\ \frac{A \vee B \quad \begin{matrix}[A][B]\\ C\end{matrix} \quad \begin{matrix}[A][\neg B]\\ C\end{matrix} \quad \begin{matrix}[\neg A][B]\\ C\end{matrix} \quad \begin{matrix}[A][\neg A]\\ C\end{matrix}}{C}$$

$$(\vee E_5)\ \frac{A \vee B \quad \begin{matrix}[A][B]\\ C\end{matrix} \quad \begin{matrix}[A][\neg B]\\ C\end{matrix} \quad \begin{matrix}[\neg A][B]\\ C\end{matrix} \quad \begin{matrix}[B][\neg B]\\ C\end{matrix}}{C}$$

$$(\vee E_6)\ \frac{A \vee B \quad \begin{matrix}[A][B]\\ C\end{matrix} \quad \begin{matrix}[A][\neg B]\\ C\end{matrix} \quad \begin{matrix}[\neg A][B]\\ C\end{matrix} \quad \begin{matrix}[A][\neg A]\\ C\end{matrix} \quad \begin{matrix}[B][\neg B]\\ C\end{matrix}}{C}$$

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$$(\wedge\mathrm{E}_6)\ \frac{A\wedge B \quad \begin{matrix}[A][B]\\ C\end{matrix} \quad \begin{matrix}[A][\neg A]\\ C\end{matrix} \quad \begin{matrix}[B][\neg B]\\ C\end{matrix}}{C}$$

where $\genfrac{}{}{0pt}{}{[X]}{Z}$ means that Z is derivable from the assumption X and this assumption is discharged; and $\genfrac{}{}{0pt}{}{[X][Y]}{Z}$ means that Z is derivable from either the assumption X or the assumption Y and either X or Y is discharged.

It seems that these rules do not exactly meet the standard requirements with respect to natural deduction systems. However, this is a consequence a consequence, on the one hand, of the semantic singularity of the logics and, on the other, the method of axiomatization used.

A set of rules of a natural deduction system for $\mathbf{K_4^{\rightarrow}}$ is as follows: $(\neg\neg\mathrm{I})$, $(\neg\neg\mathrm{E})$, $(\vee\mathrm{I}_1)$, $(\vee\mathrm{I}_3)$, $(\vee\mathrm{E}_1)$, $(\wedge\mathrm{I}_1)$, $(\wedge\mathrm{I}_2)$, $(\wedge\mathrm{E}_1)$, $(\wedge\mathrm{E}_3)$, $(\neg\vee\mathrm{I}_1)$, $(\neg\vee\mathrm{I}_2)$, $(\neg\vee\mathrm{E}_2)$, $(\neg\vee\mathrm{E}_5)$, $(\neg\wedge\mathrm{I})$, $(\neg\wedge\mathrm{E})$.

A set of rules for $\mathbf{K_4^{\leftarrow}}$ is as follows: $(\neg\neg\mathrm{I})$, $(\neg\neg\mathrm{E})$, $(\vee\mathrm{I}_2)$, $(\vee\mathrm{I}_4)$, $(\vee\mathrm{E}_2)$, $(\wedge\mathrm{I}_1)$, $(\wedge\mathrm{I}_3)$, $(\wedge\mathrm{E}_2)$, $(\wedge\mathrm{E}_4)$, $(\neg\vee\mathrm{I}_1)$, $(\neg\vee\mathrm{I}_3)$, $(\neg\vee\mathrm{E}_3)$, $(\neg\vee\mathrm{E}_4)$, $(\neg\wedge\mathrm{I})$, $(\neg\wedge\mathrm{E})$.

A set of rules for $\mathbf{K_4^w}$ is as follows: $(\neg\neg\mathrm{I})$, $(\neg\neg\mathrm{E})$, $(\vee\mathrm{I}_3)$, $(\vee\mathrm{I}_4)$, $(\vee\mathrm{I}_5)$, $(\vee\mathrm{E}_4)$, $(\wedge\mathrm{I}_1)$, $(\wedge\mathrm{I}_2)$, $(\wedge\mathrm{I}_7)$, $(\wedge\mathrm{E}_3)$, $(\wedge\mathrm{E}_4)$, $(\wedge\mathrm{E}_5)$, $(\neg\vee\mathrm{I}_1)$, $(\neg\vee\mathrm{I}_2)$, $(\neg\vee\mathrm{I}_5)$, $(\neg\vee\mathrm{E}_1)$, $(\neg\wedge\mathrm{I})$, $(\neg\wedge\mathrm{E})$.

A set of rules for $\mathbf{K_{4b}^w}$ is as follows: $(\neg\neg\mathrm{I})$, $(\neg\neg\mathrm{E})$, $(\vee\mathrm{I}_3)$, $(\vee\mathrm{I}_4)$, $(\vee\mathrm{I}_5)$, $(\vee\mathrm{I}_6)$, $(\vee\mathrm{E}_6)$, $(\wedge\mathrm{I}_1)$, $(\wedge\mathrm{I}_2)$, $(\wedge\mathrm{I}_3)$, $(\wedge\mathrm{E}_3)$, $(\wedge\mathrm{E}_4)$, $(\wedge\mathrm{E}_5)$, $(\neg\vee\mathrm{I}_1)$, $(\neg\vee\mathrm{I}_2)$, $(\neg\vee\mathrm{I}_3)$, $(\neg\vee\mathrm{E}_1)$, $(\neg\wedge\mathrm{I})$, $(\neg\wedge\mathrm{E})$.

A set of rules for $\mathbf{K_{4bn}^w}$ is as follows: $(\neg\neg\mathrm{I})$, $(\neg\neg\mathrm{E})$, $(\vee\mathrm{I}_3)$, $(\vee\mathrm{I}_4)$, $(\vee\mathrm{I}_6)$, $(\vee\mathrm{E}_5)$, $(\wedge\mathrm{I}_1)$, $(\wedge\mathrm{I}_3)$, $(\wedge\mathrm{I}_4)$, $(\wedge\mathrm{I}_5)$, $(\wedge\mathrm{E}_3)$, $(\wedge\mathrm{E}_4)$, $(\wedge\mathrm{E}_5)$, $(\neg\vee\mathrm{I}_1)$, $(\neg\vee\mathrm{I}_3)$, $(\neg\vee\mathrm{I}_4)$, $(\neg\vee\mathrm{E}_1)$, $(\neg\wedge\mathrm{I})$, $(\neg\wedge\mathrm{E})$.

A set of rules for $\mathbf{K_{4n}^w}$ is as follows: $(\neg\neg\mathrm{I})$, $(\neg\neg\mathrm{E})$, $(\vee\mathrm{I}_3)$, $(\vee\mathrm{I}_4)$, $(\vee\mathrm{E}_3)$, $(\wedge\mathrm{I}_1)$, $(\wedge\mathrm{I}_4)$, $(\wedge\mathrm{I}_7)$, $(\wedge\mathrm{E}_3)$, $(\wedge\mathrm{E}_4)$, $(\wedge\mathrm{E}_5)$, $(\neg\vee\mathrm{I}_1)$, $(\neg\vee\mathrm{I}_4)$, $(\neg\vee\mathrm{I}_5)$, $(\neg\vee\mathrm{E}_1)$, $(\neg\wedge\mathrm{I})$, $(\neg\wedge\mathrm{E})$.

A set of rules for $\mathbf{K_4^{\leftrightarrow}}$ is as follows: $(\neg\neg\mathrm{I})$, $(\neg\neg\mathrm{E})$, $(\vee\mathrm{I}_3)$, $(\vee\mathrm{I}_4)$, $(\vee\mathrm{I}_7)$, $(\vee\mathrm{E}_3)$, $(\wedge\mathrm{I}_1)$, $(\wedge\mathrm{I}_2)$, $(\wedge\mathrm{I}_3)$, $(\wedge\mathrm{E}_6)$, $(\neg\vee\mathrm{I}_1)$, $(\neg\vee\mathrm{I}_2)$, $(\neg\vee\mathrm{I}_3)$, $(\neg\vee\mathrm{E}_1)$, $(\neg\wedge\mathrm{I})$, $(\neg\wedge\mathrm{E})$.

Definition 2.1. $\Gamma \vdash_{\mathbf{K_4^{\rightarrow}}} A$ iff there is a derivation in the natural deduction system for $\mathbf{K_4^{\rightarrow}}$ of a formula A from a set of assumptions Γ, i.e., there is a finite non-empty sequence of formulae with the following conditions: (i) each formula is an assumption or follows from the previous formulae via $\mathbf{K_4^{\rightarrow}}$'s rule of inference and (ii) by applying $(\vee\mathrm{E}_1)$ each formula starting from the assumption A until a formula C, inclusively,

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as well as each formula starting either from the assumption $\neg A$ until a formula C, inclusively, or from the assumption B until a formula C, inclusively, is discarded from the derivation.[4] Note that the notion of a derivation in the natural deduction system for $\mathbf{K}_\mathbf{4}^{\rightarrow}$ of A from Γ may be defined in an alternative way as a finite tree labeled with formulae such that conditions (i) and (ii) hold. The notion of $\Gamma \vdash_{\boldsymbol{L}} A$ (for $\boldsymbol{L} \in \{\mathbf{K}_\mathbf{4}^{\leftarrow}, \mathbf{K}_\mathbf{4}^\mathbf{w}, \mathbf{K}_\mathbf{4b}^\mathbf{w}, \mathbf{K}_\mathbf{4bn}^\mathbf{w}, \mathbf{K}_\mathbf{4n}^\mathbf{w}, \mathbf{K}_\mathbf{4}^{\leftrightarrow}\}$) is defined similarly.

Recall that the definition of $\Gamma \models_{\boldsymbol{L}} A$ (for $\boldsymbol{L} \in \{\mathbf{K}_\mathbf{4}^{\rightarrow}, \mathbf{K}_\mathbf{4}^{\leftarrow}, \mathbf{K}_\mathbf{4}^\mathbf{w}, \mathbf{K}_\mathbf{4b}^\mathbf{w}, \mathbf{K}_\mathbf{4bn}^\mathbf{w}, \mathbf{K}_\mathbf{4n}^\mathbf{w}, \mathbf{K}_\mathbf{4}^{\leftrightarrow}\}$) is given in Definition 1.1.

Now we are ready to formulate the main result of this paper:

THEOREM 2.1. *Let* $\boldsymbol{L} \in \{\mathbf{K}_\mathbf{4}^{\rightarrow}, \mathbf{K}_\mathbf{4}^{\leftarrow}, \mathbf{K}_\mathbf{4}^\mathbf{w}, \mathbf{K}_\mathbf{4b}^\mathbf{w}, \mathbf{K}_\mathbf{4bn}^\mathbf{w}, \mathbf{K}_\mathbf{4n}^\mathbf{w}, \mathbf{K}_\mathbf{4}^{\leftrightarrow}\}$. *Then for all* $\Gamma \subseteq \mathrm{Form}$ *and* $A \in \mathrm{Form}$*:*

$$\Gamma \vdash_{\boldsymbol{L}} A \quad \textit{iff} \quad \Gamma \models_{\boldsymbol{L}} A.$$

3. Proof of Theorem 2.1

As an example, we will prove Theorem 2.1 for the logic $\mathbf{K}_\mathbf{4}^{\rightarrow}$. For other logics this theorem is proved similarly. So let us write $\Gamma \vdash A$ for $\Gamma \vdash_{\mathbf{K}_\mathbf{4}^{\rightarrow}} A$ and $\Gamma \models A$ for $\Gamma \models_{\mathbf{K}_\mathbf{4}^{\rightarrow}} A$. The soundness proof is by a routine check.

PROPOSITION 3.1 (Soundness). *For all* $\Gamma \subseteq \mathrm{Form}$ *and* $A \in \mathrm{Form}$*:*

$$\textit{if } \Gamma \vdash A \textit{ then } \Gamma \models A.$$

For the completeness proof we use Henkin's method and adopt the notational conventions of [17, 24]. A set of formulae Γ is a *nontrivial prime theory* iff the following conditions are met:

- (Γ1) $\Gamma \neq \mathrm{Form}$ (non-triviality);
- (Γ2) $\Gamma \vdash A$ iff $A \in \Gamma$ (closure of $\vdash$);
- (Γ3) if $A \vee B \in \Gamma$ then either $A \in \Gamma$ or both $\neg A \in \Gamma$ and $B \in \Gamma$ (primeness).

For all $\Gamma \subseteq A$ and $A \in \mathrm{Form}$, $e(A, \Gamma)$ is a *canonic valuation* iff the following conditions are met:

[4] This definition is an adaptation for our case of Copi, Cohen, and McMahon's one [6, p. 366].

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$$e(A,\Gamma) = \begin{cases} 1 & \text{iff } A \in \Gamma, \neg A \notin \Gamma \\ b & \text{iff } A \in \Gamma, \neg A \in \Gamma \\ n & \text{iff } A \notin \Gamma, \neg A \notin \Gamma \\ 0 & \text{iff } A \notin \Gamma, \neg A \in \Gamma \end{cases}$$

Lemma 3.1. *For any nontrivial prime theory* Γ *and for all* $A, B \in \text{Form}$*:*

(1) $f_{\neg}(e(A,\Gamma)) = e(\neg A,\Gamma)$;
(2) $f_{\vee}(e(A,\Gamma), e(B,\Gamma)) = e(A \vee B,\Gamma)$;
(3) $f_{\wedge}(e(A,\Gamma), e(B,\Gamma)) = e(A \wedge B,\Gamma)$.

Proof. (1.1) Let $e(A,\Gamma) = 0$. Then $A \notin \Gamma$, $\neg A \in \Gamma$. Suppose $\neg\neg A \in \Gamma$. By the rule $(\neg\neg\text{E})$, $A \in \Gamma$. Contradiction. Hence, $\neg\neg A \notin \Gamma$. Therefore, $e(\neg A,\Gamma) = 1 = f_{\neg}(0) = f_{\neg}(e(A,\Gamma))$.

(1.2) Let $e(A,\Gamma) = b$. Then $A \in \Gamma$, $\neg A \in \Gamma$. By the rule $(\neg\neg\text{I})$, $\neg\neg A \in \Gamma$. Therefore, $e(\neg A,\Gamma) = b = f_{\neg}(b) = f_{\neg}(e(A,\Gamma))$. The other cases are proved similarly.

(2.1) Let $e(A,\Gamma) = b$ and $e(B,\Gamma) = 1$. Then $A \in \Gamma$, $\neg A \in \Gamma$, $B \in \Gamma$, and $\neg B \notin \Gamma$. By the rule $(\vee\text{I}_1)$, $A \vee B \in \Gamma$. By the rules $(\wedge\text{I}_1)$ and $(\neg\vee\text{I}_2)$, $\neg(A \vee B) \in \Gamma$. Hence, $e(A \vee B,\Gamma) = b = f_{\vee}(b, 1) = f_{\vee}(e(A,\Gamma), e(B,\Gamma))$.

(2.2) Let $e(A,\Gamma) = n$ and $e(B,\Gamma) = 1$. Then $A \notin \Gamma$, $\neg A \notin \Gamma$, $B \in \Gamma$, and $\neg B \notin \Gamma$. Suppose $A \vee B \in \Gamma$. Then, by $(\Gamma 3)$, either $A \in \Gamma$ or both $\neg A \in \Gamma$ and $B \in \Gamma$. Contradiction. Hence, $A \vee B \notin \Gamma$. Suppose $\neg(A \vee B) \in \Gamma$. By the rule $(\neg\vee\text{E}_2)$ $\neg A \in \Gamma$. Contradiction. Hence, $\neg(A \vee B) \notin \Gamma$. So $e(A \vee B,\Gamma) = n = f_{\vee}(n, 1) = f_{\vee}(e(A,\Gamma), e(B,\Gamma))$. The other cases are proved similarly.

(3.1) Let $e(A,\Gamma) = 1$ and $e(B,\Gamma) = 0$. Then $A \in \Gamma$, $\neg A \notin \Gamma$, $B \notin \Gamma$, and $\neg B \in \Gamma$. Suppose $A \wedge B \in \Gamma$. By the rule $(\wedge\text{E}_3)$, $\neg A \vee B \in \Gamma$. By $(\Gamma 3)$, either $\neg A \in \Gamma$ or both $\neg\neg A \in \Gamma$ and $B \in \Gamma$. Contradiction. So $A \wedge B \notin \Gamma$. By the rule $(\neg\neg\text{I})$, $\neg\neg A \in \Gamma$. Then by the rule $(\vee\text{I}_3)$, $\neg A \vee \neg B \in \Gamma$. By the rule $(\neg\wedge\text{I})$, $\neg(A \wedge B) \in \Gamma$. Hence, $e(A \wedge B,\Gamma) = 0 = f_{\vee}(1, 0) = f_{\vee}(e(A,\Gamma), e(B,\Gamma))$.

(3.2) Let $e(A,\Gamma) = b$ and $e(B,\Gamma) = n$. Then $A \in \Gamma$, $\neg A \in \Gamma$, $B \notin \Gamma$, and $\neg B \notin \Gamma$. By the rule $(\wedge\text{I}_2)$, $A \wedge B \in \Gamma$. By the rules $(\vee\text{I}_1)$ and $(\neg\wedge\text{I})$, $\neg(A \wedge B) \in \Gamma$. Hence, $e(A \wedge B,\Gamma) = b = f_{\vee}(b, n) = f_{\vee}(e(A,\Gamma), e(B,\Gamma))$. The other cases are proved similarly. □

By a structural induction on formulae, using Lemma 3.1 we obtain:

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LEMMA 3.2. *Let* Γ *be any nontrivial prime theory and* v_Γ *be an arbitrary valuation such that* $v_\Gamma(p) = e(p, \Gamma)$, *for any* $p \in \mathrm{Prop}$. *Then we have* $v_\Gamma(A) = e(A, \Gamma)$, *for any* $A \in \mathrm{Form}$.

LEMMA 3.3 (Lindenbaum). *For all* $\Gamma \subseteq \mathrm{Form}$, $A \in \mathrm{Form}$, *if* $\Gamma \nvdash A$ *then there is* $\Gamma^* \subseteq \mathrm{Form}$ *such that* (1) $\Gamma \subseteq \Gamma^*$, (2) $\Gamma^* \nvdash A$, *and* (3) Γ^* *is a nontrivial prime theory.*

PROOF. Suppose $\Gamma \nvdash A$. Let B_1, B_2, ... be an enumeration of Form. Let Γ_0, Γ_1, ... be a sequence of sets of formulae defined as follows:

$$\Gamma_0 = \Gamma$$

$$\Gamma_{i+1} = \begin{cases} \Gamma_i \cup \{B_{i+1}\}, & \text{if } \Gamma_i \cup \{B_{i+1}\} \nvdash A; \\ \Gamma_i, & \text{otherwise.} \end{cases}$$

We take $\Gamma^* = \bigcup_{i=1}^{\infty} \Gamma_i$. Then:

(1) Follows from the definition of Γ^*.
(2) By straightforward induction on i.
(3) We prove only the case (Γ3) as it is the most complicated one.

(Γ3) Suppose $B \vee C \in \Gamma^*$, but $B \notin \Gamma^*$ and either $\neg B \notin \Gamma^*$ or $C \notin \Gamma^*$. Since $B \vee C \in \Gamma^*$, so $\Gamma^* \vdash B \vee C$ (cf.(Γ2)). Moreover, for some i, j, and k we have: $B = B_i$, $\neg B = B_j$, and $C = B_k$. Furthermore, $\Gamma_{i-1} \cup \{B_i\} \vdash A$ and either $\Gamma_{j-1} \cup \{B_j\} \vdash A$ or $\Gamma_{k-1} \cup \{B_k\} \vdash A$. Since $\Gamma_{i-1} \subseteq \Gamma^*$, $\Gamma_{j-1} \subseteq \Gamma^*$, and $\Gamma_{k-1} \subseteq \Gamma^*$, so $\Gamma^* \cup \{B_i\} \vdash A$ and either $\Gamma^* \cup \{B_j\} \vdash A$ or $\Gamma^* \cup \{B_k\} \vdash A$. From the latter and the fact that $\Gamma \vdash B \vee C$, by the rule ($\vee$E$_1$), we obtain $\Gamma^* \vdash A$. This contradicts (2). The statement (Γ3) is proved. □

PROPOSITION 3.2 (Completeness). *For all* $\Gamma \subseteq \mathrm{Form}$ *and* $A \in \mathrm{Form}$*:*

$$\textit{if } \Gamma \models A \textit{ then } \Gamma \vdash A.$$

PROOF. Suppose $\Gamma \nvdash A$. Then, by Lemma 3.3, there is $\Gamma^* \subseteq \mathrm{Form}$ such that (1) $\Gamma \subseteq \Gamma^*$, (2) $\Gamma^* \nvdash A$, and (3) Γ^* is a nontrivial prime theory. By Lemma 3.2, there is a valuation v_{Γ^*} such that: $v_{\Gamma^*}(B) \in \{1, b\}$, for any $B \in \Gamma$, and $v_{\Gamma^*}(A) \notin \{1, b\}$. Then $\Gamma \not\models A$. So if $\Gamma \nvdash A$ then $\Gamma \not\models A$. By contraposition we obtain that if $\Gamma \models A$ then $\Gamma \vdash A$. □

Theorem 2.1 immediately follows from propositions 3.1 and 3.2 for the case of $\mathbf{K}_4^{\rightarrow}$. Recall that for other logics Theorem 2.1 is proved similarly.

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4. Conclusion

In this paper, we have constructed natural deduction systems for regular and monotonic four-valued logics that is a continuation of [17, 19, 22, 24] where regular three-valued logics are formalized via natural deduction systems.

The future work concerns, firstly, exploring the other possible generalizations for the four-valued case of regular three-valued logics; secondly, the development of proof-search algorithms in the spirit of [5] for the calculi described in this paper; and thirdly, an investigation of the logics studied here with other sets of designed values; for example, with the sole designated value 1.[5]

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References

[1] Asenjo, F. G., "A calculus of antinomies", *Notre Dame Journal of Formal Logic* 7 (1966): 103-105. DOI: 10.1305/ndjfl/1093958482

[2] Belnap, N. D., "A useful four-valued logic", pages 7–37 in J. M. Dunn and G. Epstein, *Modern Uses of Multiple-Valued Logic*, Boston: Reidel Publishing Company, 1977. DOI: 10.1007/978-94-010-1161-7_2

[3] Belnap, N. D., "How a computer should think", pages 30–56 in G. Rule (ed.), *Contemporary Aspects of Philosophy*, Stocksfield: Oriel Press, 1977.

[4] Bochvar, D. A., "On a three-valued logical calculus and its application to the analysis of the paradoxes of the classical extended functional calculus", *History and Philosophy of Logic* 2 (1981): 87–112. English translation of Bochvar's paper of 1938. DOI: 10.1080/01445348108837023

[5] Bolotov, A., and V. Shangin, "Natural deduction system in paraconsistent setting: Proof search for PCont", *Journal of Intelligent Systems* 21 (2012): 1–24. DOI: 10.1515/jisys-2011-0021

[6] Copi, I. M., C. Cohen, and K. McMahon, *Introduction to Logic*, Fourteenth Edition, Routledge, New York, 2011.

[5] Note that for **FDE** such an investigation was completed in Pietz and Rivieccio's paper [20].

CC BY ND

[7] Dunn, J. M., "Intuitive semantics for first-degree entailment and coupled trees", *Philosophical Studies* 29 (1976): 149–168. DOI: 10.1007/BF00373152

[8] Finn, V. K., "Axiomatization of some three-valued propositional calculi and their algebras" (in Russian), pages 398–438 in P. Tavanets and V. Smirnov (eds.), *Philosophy in the Contemporary World. Philosophy and Logic*, Moscow: Nauka Publ., 1974.

[9] Fitting, M., "Kleene's logic, generalized", *Journal of Logic and Computation* 1 (1991): 797–810. DOI: 10.1093/logcom/1.6.797

[10] Fitting, M., "Kleene's three valued logics and their children", *Fundamenta Informaticae* 20 (1994): 113–131. DOI: 10.3233/FI-1994-201234

[11] Fitting, M., "Negation as refutation", pages 63–70 in R. Parikh (ed.), *Proceedings of the Fourth Annual Symposium on Logic in Computer Science (1989)*, IEEE, 1989. DOI: 10.1109/LICS.1989.39159

[12] Font, J. M., "Belnap's four-valued logic and De Morgan lattices", *Logic Journal of the IGPL* 5 (1997): 1–29. DOI: 10.1093/jigpal/5.3.1-e

[13] Karpenko, A. S., *The Development of Many-Valued Logic* (in Russian), LKI, 2010.

[14] Kleene, S. C., *Introduction to Metamathematics*, Sixth Reprint, Wolters-Noordhoff Publishing and North-Holland Publishing Company, 1971.

[15] Kleene, S. C., "On a notation for ordinal numbers", *The Journal of Symbolic Logic* 3 (1938): 150–155. DOI: 10.2307/2267778

[16] Komendantskaya, E. Y., "Functional expressibility of regular Kleene's logics" (in Russian), *Logical Investigations* 15 (2009): 116–128.

[17] Kooi, B., and A. Tamminga, "Completeness via correspondence for extensions of the logic of paradox", *The Review of Symbolic Logic* 5 (2012): 720–730. DOI: 10.1017/S1755020312000196

[18] Łukasiewicz, J., "On three-valued logic", pages 87–88 in L. Borkowski (ed.), *Jan Łukasiewicz: Selected Works*, Amsterdam, North-Holland Publishing Company, 1997. English translation of Łukasiewicz's paper of 1920.

[19] Petrukhin, Y., "Natural deduction for three-valued regular logics", *Logic and Logical Philosophy* 26, 2 (2017): 197–206. DOI: 10.12775/LLP.2016.025

[20] Pietz, A., and U. Rivieccio, "Nothing but the truth", *Journal of Philosophical Logic* 42 (2013): 125–135. DOI: 10.1007/s10992-011-9215-1

[21] Priest, G., "Logic of paradox revisited", *Journal of Philosophical Logic* 13 (1984): 153–179. DOI: 10.1007/BF00453020

[22] Priest, G., "Paraconsistent logic", in M. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic*, vol. 6, Second Edition, Dordrecht: Kluwer, 2002. DOI: 10.1007/978-94-017-0460-1_4

cc BY ND

[23] Priest, G., "The logic of paradox", *Journal of Philosophical Logic* 8 (1979): 219–241. DOI: 10.1007/BF00258428

[24] Tamminga, A., "Correspondence analysis for strong three-valued logic", *Logical Investigations* 20 (2014): 255–268.

[25] Tomova, N. E., "About four-valued regular logics" (in Russian), *Logical Investigations* 15 (2009): 223–228.

[26] Zaitsev, D. V., and Y. V. Shramko, "Logical entailment and designated values" (in Russian), *Logical Investigations* 11 (2004): 126–137.

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