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A CERTAIN VERSION OF PRESERVATIONISM

Abstract. A certain approach to paraconsistency was initiated by works of R. Jennings and P. Schotch. In their "Inference and necessity" [4] they proposed a notion of a *level of inconsistency* (incoherence) of a given set of premises. This level is a measure that assigns to a given set of premises X, the least number of elements of covers of X that consist of consistent subsets of X. The idea of the level of inconsistency allows to formulate a paraconsistent inference relation called by the authors *forcing*, while the obtained approach — preservationism. Similarly as classical inference relation is truth-preserving, the obtained inference relation is preserving the level of inconsistency.

We will discuss some examples of inferences that are valid in the sense of Jennings-Schotch inference relation and rise some questions on them. Based on that we formulate an inference relation as an answer to the mentioned doubts.

As regards *forcing* inference relation, the set of premises needed to derive a given conclusion can vary when changing covers from one to another. Our proposal is to stipulate to have some common set of relevant premises.

Keywords: preservationism; inconsistency; Jennings; Schotch; Canadian school of paraconsistency; incoherence; level of inconsistency

1. Introduction

We consider a certain approach to paraconsistency that was initiated by works of Raymond Jennings and Peter Schotch. In [4] they proposed a notion of level of inconsistency (incoherence) of a given set of premises. This level is a measure that assigns to a set of premises X, the least number of elements of covers of X that consist of consistent subsets of X. The idea of using subsets of set of premises matches inferences known

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from everyday life — we do not use all of our beliefs simultaneously (see [4, p. 329]). The notion of the level of inconsistency allows to formulate a paraconsistent inference relation called by the two Canadian scholars *forcing*. Similarly as classical inference relation is *truth-preserving*, the obtained inference relation is *level of inconsistency-preserving*.

In the present paper we will propose a certain variant of Jennings-Schotch inference relation.

2. Basic notions and Canadian forcing

First, we recall the elementary notions needed to define the *forcing* and to discuss some examples of its application.

Let the following symbols be the only components of the alphabet of a language $\mathcal{L}$: p, $'$, $\neg$, $\wedge$, $\to$, (,). Symbols 'p' and '$'$' will be used to build a propositional variables of the language $\mathcal{L}$. For clarity of writing, first six propositional variables will be denoted by 'p', 'q', 'r', 's', 't', 'w'. The set of all formulas of the language $\mathcal{L}$ will be denoted by For.

A *logic* $\boldsymbol{L}$ over the language $\mathcal{L}$ is any set of pairs $\langle \Gamma, \alpha\rangle \in 2^{\mathrm{For}} \times \mathrm{For}$. For each pair $\langle \Gamma, \alpha\rangle \in \boldsymbol{L}$, we write $\Gamma \vdash_{\boldsymbol{L}} \alpha$. We say that a logic $\boldsymbol{L}$ is *consistent* iff there is $\beta \in \mathrm{For}$ such that $\Gamma \nvdash_{\boldsymbol{L}} \beta$. Of course, a logic $\boldsymbol{L}$ is *inconsistent* iff $\boldsymbol{L}$ is not consistent. We will refer to some fixed consistent logic $\boldsymbol{L}$. We assume that the relation $\vdash_{\boldsymbol{L}}$ fulfills the properties of *reflexivity*, *monotonicity*, and *cut*.

A set Γ of formulas is *consistent relatively to a logic* $\boldsymbol{L}$ (in short: Γ is $\boldsymbol{L}$-*consistent*) iff there is $\beta \in \mathrm{For}$ such that $\Gamma \nvdash_{\boldsymbol{L}} \beta$. By monotonicity:

LEMMA 2.1. 1. *$\boldsymbol{L}$ is consistent iff the empty set $\varnothing$ is $\boldsymbol{L}$-consistent.*
2. *For any $\Gamma \subseteq \mathrm{For}$: if Γ is $\boldsymbol{L}$-consistent, then $\boldsymbol{L}$ is consistent.*

Of course, a set Γ of formulas is *inconsistent relatively to a logic* $\boldsymbol{L}$ (in short: Γ is $\boldsymbol{L}$-*inconsistent*) iff Γ is not $\boldsymbol{L}$-consistent. A formula α is *self-$\boldsymbol{L}$-inconsistent* iff the set $\{\alpha\}$ is $\boldsymbol{L}$-inconsistent. By monotonicity:

LEMMA 2.2. *For any $\Gamma \subseteq \mathrm{For}$: if Γ contains some self-$\boldsymbol{L}$-inconsistent formula, then Γ is $\boldsymbol{L}$-inconsistent.*

DEFINITION 2.1. Let Γ be any finite set of formulas. A family $\mathcal{C}$ of sets of formulas is *logical cover of Γ* (relatively to a logic $\boldsymbol{L}$) iff the following three conditions are fulfilled: (i) $\varnothing \in \mathcal{C}$; (ii) for any $\Omega \in \mathcal{C}$, Ω is $\boldsymbol{L}$-consistent; (iii) $\Gamma = \bigcup \mathcal{C}$.

The empty set $\varnothing$ is a member of all logical covers. It is needed to fulfil conditions imposed on the *forcing* relation in the case of the empty set of premises (see Definition 2.2).

Let $\mathrm{Cov}_{\boldsymbol{L}}(\Gamma)$ be the family of all logical covers of Γ (relatively to $\boldsymbol{L}$).

LEMMA 2.3. *For any* $\Gamma \subseteq \mathrm{For}$:

1. *If* $\mathrm{Cov}_{\boldsymbol{L}}(\Gamma) \neq \varnothing$, *then* $\boldsymbol{L}$ *is consistent.*
2. Γ *is* $\boldsymbol{L}$*-consistent iff* $\{\varnothing, \Gamma\} \in \mathrm{Cov}_{\boldsymbol{L}}(\Gamma)$.
3. *If* $\{\varnothing\} \in \mathrm{Cov}_{\boldsymbol{L}}(\Gamma)$, *then both* $\boldsymbol{L}$ *is consistent and* $\Gamma = \varnothing$.
4. *If* $\boldsymbol{L}$ *is consistent, then* $\mathrm{Cov}_{\boldsymbol{L}}(\varnothing) = \{\{\varnothing\}\}$.
5. *If* $\boldsymbol{L}$ *is consistent and* Γ *contains no self-*$\boldsymbol{L}$*-inconsistent formula, then the family* $\{\varnothing\} \cup \{\{\alpha\} : \alpha \in \Gamma\}$ *belongs to* $\mathrm{Cov}_{\boldsymbol{L}}(\Gamma)$.
6. $\mathrm{Cov}_{\boldsymbol{L}}(\Gamma) \neq \varnothing$ *iff* $\boldsymbol{L}$ *is consistent and* Γ *contains no self-*$\boldsymbol{L}$*-inconsistent formula.*

PROOF. *Ad* 1. If $\mathcal{C} \in \mathrm{Cov}_{\boldsymbol{L}}(\Gamma)$, then $\varnothing \in \mathcal{C}$, by (i) in Definition 2.1. So $\varnothing$ is $\boldsymbol{L}$-consistent, by (ii).

Ad 2. By definitions.

Ad 3. Let $\{\varnothing\} \in \mathrm{Cov}_{\boldsymbol{L}}(\Gamma)$. Then $\varnothing$ is $\boldsymbol{L}$-consistent (see also 1). Moreover, by (iii), $\Gamma = \bigcup\{\varnothing\} = \varnothing$.

Ad 4. Let $\boldsymbol{L}$ be consistent. Then $\varnothing$ is $\boldsymbol{L}$-consistent. Hence, by 2, $\{\varnothing\} \in \mathrm{Cov}_{\boldsymbol{L}}(\varnothing)$. Moreover, by (i) and (iii), for any $\mathcal{C} \in \mathrm{Cov}_{\boldsymbol{L}}(\Gamma)$ we have $\varnothing = \bigcup\mathcal{C}$; so $\mathcal{C} = \{\varnothing\}$.

Ad 5. Let $\boldsymbol{L}$ be consistent and Γ contain no self-$\boldsymbol{L}$-inconsistent formula. Then $\{\varnothing\} \cup \{\{\alpha\} : \alpha \in \Gamma\}$ satisfies three conditions (i)–(iii) from Definition 2.1.

Ad 6. "$\Rightarrow$" Let $\mathrm{Cov}_{\boldsymbol{L}}(\Gamma) \neq \varnothing$. Then, by 1, $\boldsymbol{L}$ is consistent. Now suppose towards contradiction that $\mathcal{C} \in \mathrm{Cov}_{\boldsymbol{L}}(\Gamma)$ and Γ contains some self-$\boldsymbol{L}$-inconsistent formula α. Then $\alpha \in \Omega$, for some $\Omega \in \mathcal{C}$, by (iii). So, by monotonicity, Ω is $\boldsymbol{L}$-inconsistent — a contradiction, by (ii).

"$\Leftarrow$" By 5. $\square$

For any set S let $\mathrm{Card}(S)$ denote its *cardinality*. For any $\mathcal{C}$ from $\mathrm{Cov}_{\boldsymbol{L}}(\Gamma)$, the *width of* $\mathcal{C}$ is the number $\mathrm{w}(\mathcal{C}) := \mathrm{Card}(\mathcal{C}) - 1$, i.e., $\mathrm{w}(\mathcal{C})$ is equal to the number of all non-empty subsets of $\mathcal{C}$.

Example 2.1. If $\boldsymbol{L}$ is Propositional Classical Logic (**CL**) and $\Gamma := \{p, \neg p, q, p \to q, p \wedge \neg q\}$, then the family $\{\varnothing, \{p, p \to q\}, \{\neg p, q\}, \{p \wedge \neg q\}\}$ is a logical cover of Γ. The width of this cover equals 3. $\square$

An important role in considerations on preservationism plays the following function ℓ defined on the family of all finite subsets of For:

$$\ell^{\boldsymbol{L}}(\Gamma) := \begin{cases} \min\{\mathrm{w}(\mathcal{C}) : \mathcal{C} \in \mathrm{Cov}_{\boldsymbol{L}}(\Gamma)\} & \text{if } \mathrm{Cov}_{\boldsymbol{L}}(\Gamma) \neq \varnothing \\ \infty & \text{if } \mathrm{Cov}_{\boldsymbol{L}}(\Gamma) = \varnothing \end{cases}$$

The number $\ell^{\boldsymbol{L}}(\Gamma)$ is called the *level of inconsistency* of Γ (relatively to the logic $\boldsymbol{L}$). The function $\ell^{\boldsymbol{L}}$ differentiates two ways in which a set can be inconsistent: cases where inconsistencies of a set of premises are arising from an occurrence of a self-inconsistent formula and those that are caused by a subset (whose elements are consistent) of mutually contradictory premises (see [5, p. 308]).

Fact 2.1. *Let* $\boldsymbol{L}$ *be a consistent logic and* Γ *be a finite subset of* For. *Then:*

1. $\ell^{\boldsymbol{L}}(\Gamma) = 0$ *iff* $\Gamma = \varnothing$.
2. *For any* $\Gamma \neq \varnothing$: Γ *is* $\boldsymbol{L}$*-consistent iff* $\ell^{\boldsymbol{L}}(\Gamma) = 1$.
3. $\ell^{\boldsymbol{L}}(\Gamma) = \infty$ *iff* Γ *contains some self-*$\boldsymbol{L}$*-inconsistent formula.*
4. $\infty \neq \ell^{\boldsymbol{L}}(\Gamma) > 1$ *iff* Γ *is* $\boldsymbol{L}$*-inconsistent and* Γ *contains no self-*$\boldsymbol{L}$*-inconsistent formula.*

Proof. *Ad* 1. “⇒” If $\ell^{\boldsymbol{L}}(\Gamma) = 0$, then $\mathrm{Cov}_{\boldsymbol{L}}(\Gamma)$ contains some singletons S. By (i) in Definition 2.1, $S = \{\varnothing\}$. So we use Lemma 2.3(3). “⇐” By Lemma 2.3(4).

Ad 2. Let $\Gamma \neq \varnothing$. “⇒” If Γ is $\boldsymbol{L}$-consistent, then $\ell^{\boldsymbol{L}}(\Gamma) = 1$, by Lemma 2.3(2). “⇐” Let $\ell^{\boldsymbol{L}}(\Gamma) = 1$. Then for some $\Omega \subseteq \mathrm{For} \setminus \{\varnothing\}$ we have $\{\varnothing, \Omega\} \in \mathrm{Cov}_{\boldsymbol{L}}(\Gamma)$. So $\Gamma = \bigcup\{\varnothing, \Omega\}$ and Ω is $\boldsymbol{L}$-consistent. Hence $\Gamma = \Omega$.

Ad 3. By Lemma 2.3(6).

Ad 4. “⇒” Let $\infty \neq \ell^{\boldsymbol{L}}(\Gamma) > 1$. Then Γ is $\boldsymbol{L}$-inconsistent, by 1 and 2. Moreover, we use 3. “⇐” Suppose that Γ is $\boldsymbol{L}$-inconsistent and Γ contains no self-$\boldsymbol{L}$-inconsistent formula. Then $\Gamma \neq \varnothing$ and $\mathrm{Cov}_{\boldsymbol{L}}(\Gamma) \neq \varnothing$, by Lemma 2.3(6). Hence $\infty \neq \ell^{\boldsymbol{L}}(\Gamma) > 1$, by 2, 3, and definitions. □

Fact 2.1(4) occurs when either there are formulas $\alpha, \beta \in \Gamma$ such that $\{\alpha, \beta\}$ is $\boldsymbol{L}$-inconsistent or $\mathrm{Card}(\Gamma) \geqslant 3$ and there are no $\alpha, \beta \in \Gamma$ such that $\{\alpha, \beta\}$ is $\boldsymbol{L}$-inconsistent. The next two examples correspond to the mentioned two cases.

Example 2.2. Let $\boldsymbol{L} := \mathbf{CL}$ and $\Gamma := \{p, q, p \wedge \neg q, r \wedge s, t \wedge \neg r, w, q \wedge t\}$. Since Γ is $\mathbf{CL}$-inconsistent, so $\ell^{\boldsymbol{L}}(\Gamma) > 1$, by Fact 2.1(2). Moreover, the family $\mathcal{C} := \{\{t \wedge \neg r, q \wedge t, p, q\}, \{r \wedge s, p \wedge \neg q, w, p\}\}$ belongs to $\mathrm{Cov}_{\boldsymbol{L}}(\Gamma)$ and $\mathrm{w}(\mathcal{C}) = 2$. Thus, $\ell^{\boldsymbol{L}}(\Gamma) = 2$. □

Example 2.3. Let $\boldsymbol{L} := \mathbf{CL}$ and $\Gamma := \{p, p \to q, \neg q\}$. By Fact 2.1(4), $\infty \neq \ell^{\boldsymbol{L}}(\Gamma) > 1$, since Γ is **CL**-inconsistent and Γ contains no self-**CL**-inconsistent formula. Moreover, the family $\mathcal{C} := \{\{\neg q\}, \{p, p \to q\}\}$ belongs to $\mathrm{Cov}_{\boldsymbol{L}}(\Gamma)$ and $\mathrm{w}(\mathcal{C}) = 2$. Thus, $\ell^{\boldsymbol{L}}(\Gamma) = 2$. □

By definition, any $\mathcal{C}$ from $\mathrm{Cov}_{\boldsymbol{L}}(\Gamma)$ such that $\mathrm{Card}(\mathcal{C}) = \ell^{\boldsymbol{L}}(\Gamma) + 1$ is called a *minimal logical cover* of Γ. Let $\mathbb{MLC}_{\boldsymbol{L}}(\Gamma)$ be the set of all such covers. So we put:

$$\mathbb{MLC}_{\boldsymbol{L}}(\Gamma) := \{\mathcal{C} \in \mathrm{Cov}_{\boldsymbol{L}}(\Gamma) : \mathrm{Card}(\mathcal{C}) = \ell^{\boldsymbol{L}}(\Gamma) + 1\}.$$

By definitions, Lemma 2.3, and Fact 2.1 we obtain:

Fact 2.2. *Let* $\boldsymbol{L}$ *be any consistent logic and* Γ *be any finite subset of* For*. Then*:

1. *If* $\ell^{\boldsymbol{L}}(\Gamma) \neq \infty$ *(i.e.* $\mathrm{Cov}_{\boldsymbol{L}}(\Gamma) \neq \varnothing$*), then* $\mathbb{MLC}_{\boldsymbol{L}}(\Gamma) \neq \varnothing$*.*
2. *If either* $\ell^{\boldsymbol{L}}(\Gamma) = 0$ *or* $\ell^{\boldsymbol{L}}(\Gamma) = 1$*, then* $\mathrm{Card}(\mathbb{MLC}_{\boldsymbol{L}}(\Gamma)) = 1$*.*

Proof. *Ad* 1. By definitions.

Ad 2. First, if $\ell^{\boldsymbol{L}}(\Gamma) = 0$, then $\Gamma = \varnothing$ and $\mathrm{Cov}_{\boldsymbol{L}}(\varnothing) = \{\{\varnothing\}\}$, by Fact 2.1(1) and Lemma 2.3(4). Second, if $\ell^{\boldsymbol{L}}(\Gamma) = 1$, then $\Gamma \neq \varnothing$ and Γ is $\boldsymbol{L}$-consistent, by Fact 2.1(1,2). Moreover, $\{\varnothing, \Gamma\} \in \mathrm{Cov}_{\boldsymbol{L}}(\Gamma)$, by Lemma 2.3(2). Of course, if $\{\varnothing, \Omega\} \in \mathrm{Cov}_{\boldsymbol{L}}(\Gamma)$, then $\Gamma = \Omega$. □

Definition 2.2. In the following product

$$\{\Gamma \in 2^{\mathrm{For}} : \mathrm{Card}(\Gamma) < \aleph_0 \text{ and } \ell^{\boldsymbol{L}}(\Gamma) \neq \infty\} \times \mathrm{For}$$

by the following conditions

$$\Gamma \Vdash_{\boldsymbol{L}} \alpha \text{ iff for each } \mathcal{C} \in \mathbb{MLC}_{\boldsymbol{L}}(\Gamma) \text{ there is } \Omega \in \mathcal{C} \text{ such that } \Omega \vdash_{\boldsymbol{L}} \alpha$$

we define the *forcing inference relation.*

Notice that if either $\ell^{\boldsymbol{L}}(\Gamma) = 0$ or $\ell^{\boldsymbol{L}}(\Gamma) = 1$ (i.e., when we conclude from the empty set of premises or from a non-empty consistent set), then *forcing* behaves like $\vdash_{\boldsymbol{L}}$. The following fact states that the forcing relation preserves the level of inconsistency of a given set of premises (see [3, p. 96]).

Fact 2.3. *For any finite subset* Γ *of* For *such that* $\infty \neq \ell^{\boldsymbol{L}}(\Gamma) > 0$ *and for any* $\alpha \in$ For:

$$\text{if } \Gamma \Vdash_{\boldsymbol{L}} \alpha, \text{ then } \ell^{\boldsymbol{L}}(\Gamma) = \ell^{\boldsymbol{L}}(\Gamma \cup \{\alpha\})$$

3. A discussion of some examples

Using the forcing inference relation we transform consequences of inconsistent set of premises Γ into consequences of its certain consistent subsets defined by the value of the function $\ell^{\boldsymbol{L}}$. Let us look at the Jennings and Schotch method from the point of view of the isolation of premises, which are the basis of the inferences.[1]

The Definition 2.2 states that a formula is called the conclusion if in each minimal logical cover we can find premises supporting that formula.

Example 3.1. Let $\boldsymbol{L} := \mathbf{CL}$ and $\Gamma := \{r \wedge p, \neg r \wedge \neg s, s \wedge (p \to q), t\}$. Then $\mathbb{MLC}_{\boldsymbol{L}}(\Gamma)$ consists of three covers:

$$\begin{aligned}
\mathcal{C}_1 &= \{\{r \wedge p, s \wedge (p \to q), t\}, \{\neg r \wedge \neg s\}\},\\
\mathcal{C}_2 &= \{\{r \wedge p, s \wedge (p \to q)\}, \{\neg r \wedge \neg s, t\}\},\\
\mathcal{C}_3 &= \{\{r \wedge p, s \wedge (p \to q), t\}, \{\neg r \wedge \neg s, t\}\}.
\end{aligned}$$

Thus $\Gamma \Vdash_{\boldsymbol{L}} q$, since for any $\mathcal{C}$ from $\mathbb{MLC}_{\boldsymbol{L}}(\Gamma)$ there is a set $\Omega \in \mathcal{C}$ such that $\Omega \vdash q$. □

Let us observe that in the above example we not only find the right premises, but moreover, these premises are the same in the case of each minimal cover. Definition 2.2 forces only that for each logical cover there is a consistent element from which a given conclusion can be derived, while it is not necessary that this set of premises is contained in some element of each logical cover.

Example 3.2. Let $\boldsymbol{L} := \mathbf{CL}$ and $\Gamma := \{\neg r \wedge p, r \wedge p, p \to q\}$. Then $\mathbb{MLC}_{\boldsymbol{L}}(\Gamma)$ consists of two covers:

$$\begin{aligned}
\mathcal{C}_1 &= \{\{r \wedge p, p \to q\}, \{\neg r \wedge p\}\},\\
\mathcal{C}_2 &= \{\{r \wedge p\}, \{\neg r \wedge p, p \to q\}\}.
\end{aligned}$$

Having those covers we can conclude that $\Gamma \Vdash_{\boldsymbol{L}} q$. □

In the example above we have two subsets of the set of premises that allow us to obtain the conclusion q: $\{p \wedge r, p \to q\}$ and $\{p \wedge \neg r, p \to q\}$. In contrast to Example 3.1, where the set of relevant premises is included in some element of each minimal logical cover, here we have to use different and even mutually contradictory configurations of premises to reach the conclusion q. It seems counterintuitive for us that while trying to obtain some conclusion α and changing covers from $\mathcal{C}_1$ to $\mathcal{C}_2$ we have to use

[1] We omit the case of receiving conclusions from singletons of a set of premises.

some X_1 and X_2, such that there are $\Omega_1 \in \mathcal{C}_1$ and $\Omega_2 \in \mathcal{C}_2$ for which $X_1 \subseteq \Omega_1$, $X_2 \subseteq \Omega_2$, and X_1 and X_2 are mutually contradictory.

But what about the case where using contradictory premises we can reach the same conclusions? Our answer is that they should be accessible in each minimal cover. More formally we say about a situation when for each $\mathcal{C} \in \mathbb{MLC}_{\boldsymbol{L}}(\Gamma)$ there are $\Omega_1, \Omega_2 \in \mathcal{C}$ such that our mutually contradictory sets X_1 and X_2 are contained respectively in Ω_1 and Ω_2, and moreover $X_1 \vdash_{\boldsymbol{L}} \alpha$ $X_2 \vdash_{\boldsymbol{L}} \alpha$?

Example 3.3. Let $\boldsymbol{L} := \mathbf{CL}$ and $\Gamma := \{\neg p \wedge \neg s, s \wedge (\neg q \to r), p \wedge \neg q, q \wedge (\neg s \to r), t\}$. Then $\mathbb{MLC}_{\boldsymbol{L}}(\Gamma)$ consists of three covers:

$$\mathcal{C}_1 = \{\{s \wedge (\neg q \to r), p \wedge \neg q, t\}, \{\neg p \wedge \neg s, q \wedge (\neg s \to r)\}\},$$
$$\mathcal{C}_2 = \{\{s \wedge (\neg q \to r), p \wedge \neg q\}, \{\neg p \wedge \neg s, q \wedge (\neg s \to r), t\}\},$$
$$\mathcal{C}_3 = \{\{s \wedge (\neg q \to r), p \wedge \neg q, t\}, \{\neg p \wedge \neg s, q \wedge (\neg s \to r), t\}\}.$$

As we see, $\Gamma \Vdash_{\boldsymbol{L}} r$, since for each logical cover there is at least one of its elements that support this conclusion. In fact, for each cover two sets of premises are available: $\{s \wedge (\neg q \to r), p \wedge \neg q\}$ and $\{\neg p \wedge \neg s, q \wedge (\neg s \to r)\}$. Although these sets are mutually inconsistent we can choose one of them to drawn the inference from — it doesn't bother us which one we chose, since both are at our disposal for each minimal cover. □

Such a situation is acceptable for us, since a given conclusion can be obtained from each mutually inconsistent elements of a given cover of the set of premises.

As a remedy to the above doubts we let ourselves to propose a modified version of the preservationism inference relation.

4. A new version of preserving forcing

Due to the lucidity of the following deliberation we assume that no self-inconsistent formula belongs to considered sets of premises. It is easy to observe that there are infinitely many sets Γ for which $\mathbb{MLC}_{\boldsymbol{L}}(\Gamma)$ is non-empty — we can take pairs of a propositional variable and its negation.

Let Γ be a finite subset of For such that $\mathbb{MLC}_{\boldsymbol{L}}(\Gamma) \neq \varnothing$.[2] Then we put

$$\mathrm{Const}_{\boldsymbol{L}}(\Gamma) := \{\Delta \in 2^{\Gamma} : \forall_{\mathcal{C} \in \mathbb{MLC}_{\boldsymbol{L}}(\Gamma)} \exists_{\Omega \in \mathcal{C}}\ \Delta \subseteq \Omega\}.$$

Elements of $\mathrm{Const}_{\boldsymbol{L}}(\Gamma)$ are called *fixed elements of logical covers* of Γ.

[2] $\mathbb{MLC}_{\boldsymbol{L}}(\Gamma) \neq \varnothing$ guarantees that a set Γ contains no self-$\boldsymbol{L}$-inconsistent formula.

The set $\mathrm{Const}_{\boldsymbol{L}}(\Gamma)$ is non-empty, since singletons of the set Γ are elements of $\mathrm{Const}_{\boldsymbol{L}}(\Gamma)$, by the condition (ii) in Definition 2.1. Moreover, if $\ell^{\boldsymbol{L}}(\Gamma) = 1$ then $\mathrm{Const}_{\boldsymbol{L}}(\Gamma) = 2^{\Gamma}$.

Example 4.1. Let $\boldsymbol{L} := \mathbf{CL}$ and $\Gamma := \{p, \neg p, q \wedge r, q \wedge \neg r\}$. Then we see that $\ell^{\boldsymbol{L}}(\Gamma) = 2$ and $\mathbb{MLC}_{\boldsymbol{L}}(\Gamma)$ contains only the following covers:

$$\begin{aligned}\mathcal{C}_1 &= \{\{p, q \wedge r\}, \{\neg p, q \wedge \neg r\}\},\\ \mathcal{C}_2 &= \{\{p, q \wedge \neg r\}, \{\neg p, q \wedge r\}\}.\end{aligned}$$

We observe that singletons are the only subsets of the set Γ that are contained in at least one element of $\mathcal{C}$, for each cover $\mathcal{C}$ of Γ: $\{p\}$, $\{\neg p\}$, $\{q \wedge r\}$, $\{q \wedge \neg r\}$. Thus, these sets are the only elements of the set $\mathrm{Const}_{\boldsymbol{L}}(\Gamma)$. □

In the next example we will show that the singletons are not the only elements of $\mathrm{Const}_{\boldsymbol{L}}(\Gamma)$.

Example 4.2. $\boldsymbol{L} := \mathbf{CL}$ and $\Gamma := \{\neg p \wedge q, \neg t \wedge p \wedge \neg q, \neg p \wedge \neg r \wedge \neg q, \neg p \wedge r, t \wedge w\}$. Then $\ell^{\boldsymbol{L}}(\Gamma) = 3$ and elements of $\mathbb{MLC}_{\boldsymbol{L}}(\Gamma)$ are the following covers:

$$\begin{aligned}\mathcal{C}_1 &= \{\{\neg t \wedge p \wedge \neg q\}, \{\neg p \wedge r, \neg p \wedge q, t \wedge w\}, \{\neg p \wedge \neg r \wedge \neg q\}\},\\ \mathcal{C}_2 &= \{\{\neg t \wedge p \wedge \neg q\}, \{\neg p \wedge r, \neg p \wedge q\}, \{\neg p \wedge \neg r \wedge \neg q, t \wedge w\}\},\\ \mathcal{C}_3 &= \{\{\neg t \wedge p \wedge \neg q\}, \{\neg p \wedge r, \neg p \wedge q, t \wedge w\}, \{\neg p \wedge \neg r \wedge \neg q, t \wedge w\}\}.\end{aligned}$$

So $\mathrm{Const}_{\boldsymbol{L}}(\Gamma) = \{\{\neg t \wedge p \wedge \neg q\}, \{\neg p \wedge r\}, \{\neg p \wedge r, \neg p \wedge q\}, \{\neg p \wedge q\}, \{\neg p \wedge \neg r \wedge \neg q\}, \{t \wedge w\}\}$. □

Notice that if $\Delta \in \mathrm{Const}_{\boldsymbol{L}}(\Gamma)$ is not a singleton, then $\mathrm{Const}_{\boldsymbol{L}}(\Gamma)$ also contains all proper subsets of Δ. Because we are interested in the largest fixed elements with respect to the inclusion we propose the following definition of a set of *maximal fixed elements* of logical covers.

Let Γ be a finite subset of For such that $\mathbb{MLC}_{\boldsymbol{L}}(\Gamma) \neq \varnothing$. We put

$$\mathrm{Const}^{\max}_{\boldsymbol{L}}(\Gamma) := \{\Delta \in \mathrm{Const}_{\boldsymbol{L}}(\Gamma) : \neg\exists_{\Delta' \in \mathrm{Const}_{\boldsymbol{L}}(\Gamma)}\ \Delta \subsetneq \Delta'\}.$$

Elements of $\mathrm{Const}^{\max}_{\boldsymbol{L}}(\Gamma)$ are called *maximal fixed elements of logical covers* of Γ.

Remark 4.1. Let us observe that if $\ell^{\boldsymbol{L}}(\Gamma) = 1$, then $\mathrm{Const}^{\max}_{\boldsymbol{L}}(\Gamma) = \{\Gamma\}$. Similarly, one can see that if there is a self-inconsistent formula in Γ, i.e. $\ell^{\boldsymbol{L}}(\Gamma) = \infty$, then $\mathbb{MLC}_{\boldsymbol{L}}(\Gamma) = \varnothing$, so $\mathrm{Const}^{\max}_{\boldsymbol{L}}(\Gamma) = \{\Gamma\}$. □

We easily see that:

Fact 4.1. *For any* $\Delta \in \mathrm{Const}_{\boldsymbol{L}}(\Gamma)$ *there is* $\Delta' \in \mathrm{Const}^{\max}_{\boldsymbol{L}}(\Gamma)$ *such that* $\Delta \subseteq \Delta'$.

Lemma 4.1. *Let* Γ *be a finite subset of* For *such that* $\ell^{\boldsymbol{L}}(\Gamma) > 1$ *and* $\alpha \in \Gamma$. *If for some* $\mathcal{C} \in \mathbb{MLC}_{\boldsymbol{L}}(\Gamma)$ *there is a set* $\Omega \in \mathcal{C}$ *such that* $\alpha \in \Omega$, *then there is* $\mathcal{C}' \in \mathbb{MLC}_{\boldsymbol{L}}(\Gamma)$ *such that* $\mathcal{C}'$ *differs from* $\mathcal{C}$ *in that the formula* α *has been removed from all elements of cover* $\mathcal{C}$ *but* Ω.

Proof. Let us assume that $\Gamma \subset_{\mathrm{fin}}$ For is a set which satisfies the condition $\ell^{\boldsymbol{L}}(\Gamma) > 1$. Moreover, let $\mathcal{C} \in \mathbb{MLC}_{\boldsymbol{L}}(\Gamma)$ be a cover of the set Γ and let $\Omega \in \mathcal{C}$ be an element of that cover such that $\alpha \in \Omega$. If the only element of $\mathcal{C}$ that contains the formula α is Ω, then thesis holds in a trivial way, since $\mathcal{C}' = \mathcal{C}$.

More generally, a family $\mathcal{C}'$ is obtained from $\mathcal{C}$ in the following way: for any $\Omega' \in \mathcal{C}$ if $\alpha \notin \Omega'$ or $\Omega = \Omega'$, then $\Omega' \in \mathcal{C}'$, otherwise $\Omega' \setminus \{\alpha\} \in \mathcal{C}'$.

By Definition 2.1 all members of $\mathcal{C}$ are $\boldsymbol{L}$-consistent and $\Gamma = \bigcup \mathcal{C}$. Hence also all members of $\mathcal{C}'$ are $\boldsymbol{L}$-consistent. Furthermore, $\bigcup \mathcal{C} = \bigcup \mathcal{C}' = \Gamma$. Therefore $\mathcal{C}' \in \mathrm{Cov}_{\boldsymbol{L}}(\Gamma)$. Now notice that the method of construction guarantees that the width of the obtained cover is not greater then the width of $\mathcal{C}$. Since $\mathcal{C} \in \mathbb{MLC}_{\boldsymbol{L}}(\Gamma)$, i.e. $\mathrm{w}(\mathcal{C}) = \ell^{\boldsymbol{L}}(\Gamma)$, so also $\mathcal{C}' \in \mathbb{MLC}_{\boldsymbol{L}}(\Gamma)$.[3] So $\mathrm{w}(\mathcal{C}') = \mathrm{w}(\mathcal{C})$. Hence $\mathcal{C}' \in \mathbb{MLC}_{\boldsymbol{L}}(\Gamma)$ and there is only one element $\Omega' \in \mathcal{C}'$ such that $\alpha \in \Omega'$ — the set Ω. $\square$

Lemma 4.2. *Let* Γ *be any finite subset of* For. *Then for all* $\Delta_1, \Delta_2 \in \mathrm{Const}^{\max}_{\boldsymbol{L}}(\Gamma)$ *such that* $\Delta_1 \neq \Delta_2$, *it holds that* $\Delta_1 \cap \Delta_2 = \varnothing$.

Proof. Suppose that Δ_1 and Δ_2 are arbitrarily chosen elements of the set $\mathrm{Const}^{\max}_{\boldsymbol{L}}(\Gamma)$ and $\Delta_1 \cap \Delta_2 \neq \varnothing$, i.e. there is $\mathrm{D} \subsetneq \Gamma$, $\Delta_1 \cap \Delta_2 = \mathrm{D}$.

Assume that for any cover $\mathcal{C} \in \mathbb{MLC}_{\boldsymbol{L}}(\Gamma)$ there is $\Omega \in \mathcal{C}$ such that $\Delta_1, \Delta_2 \subseteq \Omega$. But then, for any $\mathcal{C} \in \mathbb{MLC}_{\boldsymbol{L}}(\Gamma)$ there is $\Omega \in \mathcal{C}$ such that $\Delta_1 \cup \Delta_2 \subseteq \Omega$. Hence, it is not the case that $\Delta_1, \Delta_2 \in \mathrm{Const}^{\max}_{\boldsymbol{L}}(\Gamma)$, contrary to the assumption. Therefore, we may assume that for some logical cover $\mathcal{C} \in \mathbb{MLC}_{\boldsymbol{L}}(\Gamma)$ does not exist $\Omega \in \mathcal{C}$ such that $\Delta_1, \Delta_2 \subseteq \Omega$. Then, there are $\Omega, \Omega' \in \mathcal{C}$, such that $\Omega \neq \Omega'$, $\Delta_1 \subseteq \Omega$ and $\Delta_2 \subseteq \Omega'$. Let $\alpha \in \mathrm{D}$ be an arbitrarily chosen formula. Since $\alpha \in \Delta_1$, and $\alpha \in \Delta_2$, we know that $\alpha \in \Omega$ and $\alpha \in \Omega'$ (but we do not prejudge if the formula is contained in other elements of the cover). By Lemma 4.1 there is

[3] Notice that any element of $\mathcal{C}$ that has been reduced is non-empty — otherwise $\mathrm{w}(\mathcal{C}') < \ell^{\boldsymbol{L}}(\Gamma)$.

$\mathcal{C}' \in \mathbb{MLC}_{\boldsymbol{L}}(\Gamma)$ that results from $\mathcal{C}$ by removing α from all elements of the cover $\mathcal{C}$ but Ω. Hence there is no $\overline{\Omega} \in \mathcal{C}'$ such that $\Delta_2 \subseteq \overline{\Omega}$ — notice that $\Delta_2 \nsubseteq \Omega$. Therefore, $\Delta_2 \notin \mathrm{Const}_{\boldsymbol{L}}(\Gamma)$, which is in contrary to the assumption. □

Now we introduce some variant of the *forcing* relation.

Definition 4.1. For any finite subset Γ of For and any $\alpha \in$ For we put:

$$\Gamma \Vdash^{c}_{\boldsymbol{L}} \alpha \text{ iff for some } \Delta \in \mathrm{Const}^{\max}_{\boldsymbol{L}}(\Gamma),\ \Delta \vdash_{\boldsymbol{L}} \alpha$$

The inference relation $\Vdash^{c}_{\boldsymbol{L}}$ we called a *c-forcing*.

By Remark 4.1 we see that:

Fact 4.2. *For any finite subset Γ of* For *and any $\alpha \in$* For*: if $\ell^{\boldsymbol{L}}(\Gamma) = \infty$, then $\Gamma \Vdash^{c}_{\boldsymbol{L}} \alpha$.*

The above fact can be treated as another reason for excluding from our consideration any set Γ for which $\ell^{\boldsymbol{L}}(\Gamma) = \infty$.

Lemma 4.3. *For any finite subset Γ of* For *such that $\ell^{\boldsymbol{L}}(\Gamma) \neq \infty$ and any $\alpha \in$* For*: if $\Gamma \Vdash^{c}_{\boldsymbol{L}} \alpha$, then $\Gamma \Vdash_{\boldsymbol{L}} \alpha$.*

Proof. Let us assume that $\Gamma \Vdash^{c}_{\boldsymbol{L}} \alpha$. Hence, for some $\Delta \in \mathrm{Const}^{\max}_{\boldsymbol{L}}(\Gamma)$ we have $\Delta \vdash_{\boldsymbol{L}} \alpha$. Let $\mathcal{C} \in \mathbb{MLC}_{\boldsymbol{L}}(\Gamma)$. Since $\mathrm{Const}^{\max}_{\boldsymbol{L}}(\Gamma) \subseteq \mathrm{Const}_{\boldsymbol{L}}(\Gamma)$, by definition, there is $\Omega \in \mathcal{C}$ such that $\Delta \subseteq \Omega$. The relation $\vdash_{\boldsymbol{L}}$ satisfies the condition of monotonicity, hence $\Omega \vdash_{\boldsymbol{L}} \alpha$. Therefore, $\Gamma \Vdash_{\boldsymbol{L}} \alpha$. □

Using Example 3.2 we conclude that reverse implication does not hold. The set of *fixed elements* of logical covers of Γ given in Example 3.2 has the form:

$$\mathrm{Const}_{\boldsymbol{L}}(\Gamma) = \{\{p \to q\}, \{r \wedge p\}, \{\neg r \wedge p\}\}$$

Let us recall that $\Gamma \Vdash_{\boldsymbol{L}} q$ but the formula q is not a classical consequence of any of the *fixed elements* of logical covers of Γ. Thus,

Fact 4.3. *For some finite subset Γ of* For *and $\alpha \in$* For *we have $\Gamma \Vdash_{\boldsymbol{L}} \alpha$ and $\Gamma \nVdash^{c}_{\boldsymbol{L}} \alpha$.*

We show that *c-forcing* also preserve the *level of inconsistency* of a set of premises.

Lemma 4.4. *For any finite subset* Γ *of* For *such that* $\infty \neq \ell^{\boldsymbol{L}}(\Gamma) > 0$ *and for any* $\alpha \in$ For*:*

$$\textit{if } \Gamma \Vdash^{\mathrm{c}}_{\boldsymbol{L}} \alpha, \textit{ then } \ell^{\boldsymbol{L}}(\Gamma) = \ell^{\boldsymbol{L}}(\Gamma \cup \{\alpha\}).$$

Proof. Assume that $\Gamma \Vdash^{\mathrm{c}}_{\boldsymbol{L}} \alpha$. By Lemma 4.3, $\Gamma \Vdash_{\boldsymbol{L}} \alpha$. So we use Fact 2.3. □

Now we formulate and prove some facts on the introduced inference relation. First, we prove that $\Vdash^{\mathrm{c}}_{\boldsymbol{L}}$ satisfies the condition of reflexivity.

Theorem 4.1. *For any finite subset* Γ *of* For *such that* $\ell^{\boldsymbol{L}}(\Gamma) > 0$ *and any* $\alpha \in \Gamma$ *we have* $\Gamma \Vdash^{\mathrm{c}}_{\boldsymbol{L}} \alpha$*.*

Proof. Let Γ be a finite subset of For such that $\ell^{\boldsymbol{L}}(\Gamma) > 0$, $\alpha \in \Gamma$. By Fact 4.2, the thesis holds if $\ell^{\boldsymbol{L}}(\Gamma) = \infty$. So we can assume that $\ell^{\boldsymbol{L}}(\Gamma) \neq \infty$, i. e. $\mathbb{MLC}_{\boldsymbol{L}}(\Gamma) \neq \varnothing$, by Fact 2.2. Let $\Delta = \{\alpha\}$. For any $\mathcal{C} \in \mathbb{MLC}_{\boldsymbol{L}}(\Gamma)$, we have $\Gamma = \bigcup \mathcal{C}$ (by Definition 2.1), so there is $\Omega \in \mathcal{C}$ such that $\alpha \in \Omega$. This means that $\Delta \in \mathrm{Const}_{\boldsymbol{L}}(\Gamma)$. Thus, by Fact 4.1, there is $\Delta' \in \mathrm{Const}^{\max}_{\boldsymbol{L}}(\Gamma)$ such that $\Delta \subseteq \Delta'$. By *reflexivity* of $\vdash_{\boldsymbol{L}}$ we have $\Delta' \vdash_{\boldsymbol{L}} \alpha$. So $\Gamma \Vdash^{\mathrm{c}}_{\boldsymbol{L}} \alpha$. □

Similarly as the *forcing* inference relation $\Vdash_{\boldsymbol{L}}$, the relation $\Vdash^{\mathrm{c}}_{\boldsymbol{L}}$ satisfies the condition of monotonicity only in the following restricted version.

Theorem 4.2. *Let* Γ *be a finite subset of* For *such that* $\ell^{\boldsymbol{L}}(\Gamma) > 0$, $\beta \in$ For*, and* $\ell^{\boldsymbol{L}}(\Gamma) = \ell^{\boldsymbol{L}}(\Gamma \cup \{\beta\})$*. Then for any* $\alpha \in$ For*: if* $\Gamma \Vdash^{\mathrm{c}}_{\boldsymbol{L}} \alpha$ *then* $\Gamma \cup \{\beta\} \Vdash^{\mathrm{c}}_{\boldsymbol{L}} \alpha$*.*

Before showing the proof of this theorem we prove:

Lemma 4.5. *Let* Γ *be a finite subset of* For*,* $\Delta \in \mathrm{Const}_{\boldsymbol{L}}(\Gamma)$*,* $\alpha \in$ For*, and* $\ell^{\boldsymbol{L}}(\Gamma) = \ell^{\boldsymbol{L}}(\Gamma \cup \{\alpha\})$*. If* $\alpha \notin \Gamma$*, then* $\Delta \in \mathrm{Const}_{\boldsymbol{L}}(\Gamma \cup \{\alpha\})$*.*

Proof. Under the adopted assumptions, let us suppose that both $\alpha \notin \Gamma$ and $\Delta \notin \mathrm{Const}_{\boldsymbol{L}}(\Gamma \cup \{\alpha\})$. Hence for some $\mathcal{C} \in \mathbb{MLC}_{\boldsymbol{L}}(\Gamma \cup \{\alpha\})$ there is no $\Omega \in \mathcal{C}$ such that $\Delta \subseteq \Omega$. Let $\mathcal{C}_\alpha$ be a family of sets obtained from $\mathcal{C}$ by removing the formula α from each $\Omega \in \mathcal{C}$, i.e., we put $\mathcal{C}_\alpha := \{\Omega \backslash \{\alpha\} : \Omega \in \mathcal{C}\}$.

We obtain $\bigcup \mathcal{C}_\alpha = \bigcup \{\Omega \backslash \{\alpha\} : \Omega \in \mathcal{C}\} = \bigcup \mathcal{C} \backslash \{\alpha\} = (\Gamma \cup \{\alpha\}) \backslash \{\alpha\} = \Gamma$. Moreover, all members of $\mathcal{C}_\alpha$ are $\boldsymbol{L}$-consistent. Hence $\mathcal{C}_\alpha \in \mathrm{Cov}_{\boldsymbol{L}}(\Gamma)$. The way in which we received $\mathcal{C}_\alpha$ from $\mathcal{C}$ guarantees that $\mathrm{w}(\mathcal{C}_\alpha) \leqslant \mathrm{w}(\mathcal{C})$. But, since $\ell^{\boldsymbol{L}}(\Gamma) = \ell^{\boldsymbol{L}}(\Gamma \cup \{\alpha\})$, so width of the cover $\mathcal{C}_\alpha$ cannot be smaller then $\ell^{\boldsymbol{L}}(\mathit{\Gamma}\cup\{\alpha\})$; so $\mathrm{w}(\mathcal{C}_\alpha)=\mathrm{w}(\mathcal{C})$. Thus, $\mathcal{C}_\alpha\in\mathbb{MLC}_{\boldsymbol{L}}(\mathit{\Gamma})$. But there is no $\mathit{\Omega}\in\mathcal{C}_\alpha$ such that $\mathit{\Delta}\subseteq\mathit{\Omega}$, contrary to the assumption that $\mathit{\Delta}\in\mathrm{Const}_{\boldsymbol{L}}(\mathit{\Gamma})$. ☐

Proof of Theorem 4.2. Under the adopted assumptions, let $\mathit{\Gamma}\Vdash^{\mathrm{c}}_{\boldsymbol{L}}\alpha$. Then, by Definition 4.1, there is $\mathit{\Delta}\in\mathrm{Const}^{\max}_{\boldsymbol{L}}(\mathit{\Gamma})$ such that $\mathit{\Delta}\vdash_{\boldsymbol{L}}\alpha$. If $\alpha\in\mathit{\Gamma}$, then the thesis holds by Theorem 4.1 in a trivial way. Therefore we may assume that $\alpha\notin\mathit{\Gamma}$. Then $\mathit{\Delta}\in\mathrm{Const}_{\boldsymbol{L}}(\mathit{\Gamma}\cup\{\alpha\})$, by Lemma 4.5. Moreover, by definitions and Fact 4.1, there is a set $\mathit{\Delta}'\in\mathrm{Const}^{\max}_{\boldsymbol{L}}(\mathit{\Gamma}\cup\{\alpha\})$ such that $\mathit{\Delta}\subseteq\mathit{\Delta}'$. Hence $\mathit{\Delta}'\vdash_{\boldsymbol{L}}\alpha$, by the monotonicity of $\vdash_{\boldsymbol{L}}$. Thereby $\mathit{\Gamma}\cup\{\beta\}\Vdash^{\mathrm{c}}_{\boldsymbol{L}}\alpha$. ☐

Another property of $\Vdash^{\mathrm{c}}_{\boldsymbol{L}}$, which we consider below, is *the cut rule*.

Theorem 4.3. *For any finite subset $\mathit{\Gamma}$ of* For *such that $\ell^{\boldsymbol{L}}(\mathit{\Gamma})>0$ and for all $\alpha,\beta\in$* For, *if $\mathit{\Gamma}\cup\{\alpha\}\Vdash^{\mathrm{c}}_{\boldsymbol{L}}\beta$ and $\mathit{\Gamma}\Vdash^{\mathrm{c}}_{\boldsymbol{L}}\alpha$, then $\mathit{\Gamma}\Vdash^{\mathrm{c}}_{\boldsymbol{L}}\beta$.*

First we prove two auxiliary facts.

Lemma 4.6. *Let $\mathit{\Gamma}$ be finite subset of* For, *$\alpha\in$* For, *$\mathit{\Delta}\in\mathrm{Const}_{\boldsymbol{L}}(\mathit{\Gamma}\cup\{\alpha\})$, and $\mathit{\Delta}\subseteq\mathit{\Gamma}$. If $\alpha\notin\mathit{\Gamma}$ and $\mathit{\Gamma}\Vdash^{\mathrm{c}}_{\boldsymbol{L}}\alpha$, then $\mathit{\Delta}\in\mathrm{Const}_{\boldsymbol{L}}(\mathit{\Gamma})$.*

Proof. Under the adopted assumptions, let $\alpha\notin\mathit{\Gamma}$ and $\mathit{\Gamma}\Vdash^{\mathrm{c}}_{\boldsymbol{L}}\alpha$. Suppose towards contradiction that $\mathit{\Delta}\notin\mathrm{Const}_{\boldsymbol{L}}(\mathit{\Gamma})$. Hence, for some $\mathcal{C}\in\mathbb{MLC}_{\boldsymbol{L}}(\mathit{\Gamma})$ there is no element $\mathit{\Omega}\in\mathcal{C}$ such that (†): $\mathit{\Delta}\subseteq\mathit{\Omega}$. Let $\mathcal{C}'$ be such logical cover. Since $\mathit{\Gamma}\Vdash^{\mathrm{c}}_{\boldsymbol{L}}\alpha$, so for some $\mathit{\Delta}_\alpha\in\mathrm{Const}^{\max}_{\boldsymbol{L}}(\mathit{\Gamma})$ we have $\mathit{\Delta}_\alpha\vdash_{\boldsymbol{L}}\alpha$. Therefore, by definitions of the sets $\mathrm{Const}_{\boldsymbol{L}}$ and $\mathrm{Const}^{\max}_{\boldsymbol{L}}$, for some $\mathit{\Omega}_\alpha\in\mathcal{C}'$ we have $\mathit{\Delta}_\alpha\subseteq\mathit{\Omega}_\alpha$. Since $\mathit{\Delta}_\alpha\vdash_{\boldsymbol{L}}\alpha$ and $\mathit{\Delta}_\alpha\subseteq\mathit{\Omega}_\alpha$, so $\mathit{\Omega}_\alpha\cup\{\alpha\}$ is $\boldsymbol{L}$-consistent (indeed, as an element of a cover, $\mathit{\Omega}_\alpha$ is a consistent set and by monotonicity $\mathit{\Omega}_\alpha\vdash_{\boldsymbol{L}}\alpha$, so $\mathit{\Omega}_\alpha\cup\{\alpha\}$ is $\boldsymbol{L}$-consistent, by the cut rule). Let $\mathcal{C}'_\alpha$ be a logical cover of $\mathit{\Gamma}\cup\{\alpha\}$ obtained from $\mathcal{C}'$ by adding the formula α to $\mathit{\Omega}_\alpha$ and leaving all the other elements of $\mathcal{C}'$ unchanged. By Lemma 4.4, $\ell^{\boldsymbol{L}}(\mathit{\Gamma})=\ell^{\boldsymbol{L}}(\mathit{\Gamma}\cup\{\alpha\})=\mathrm{w}(\mathcal{C}'_\alpha)$. Thus, $\mathcal{C}'_\alpha\in\mathbb{MLC}_{\boldsymbol{L}}(\mathit{\Gamma}\cup\{\alpha\})$.

If for some $\mathit{\Omega}\in\mathcal{C}'_\alpha$ and $\mathit{\Delta}\subseteq\mathit{\Omega}$, then either $\mathit{\Omega}\in\mathcal{C}'$ or $\mathit{\Omega}=\mathit{\Omega}_\alpha\cup\{\alpha\}$. The first case is obviously not possible due to (†). In the second case, we would have $\mathit{\Delta}\subseteq\mathit{\Omega}_\alpha$, since $\mathit{\Delta}\subseteq\mathit{\Gamma}$ and $\alpha\notin\mathit{\Gamma}$ — again contrary to (†). Hence $\mathcal{C}'_\alpha\in\mathrm{Cov}_{\boldsymbol{L}}(\mathit{\Gamma}\cup\{\alpha\})$ and there is no $\mathit{\Omega}\in\mathcal{C}'_\alpha$ such that $\mathit{\Delta}\subseteq\mathit{\Omega}$. So $\mathit{\Delta}\notin\mathrm{Const}_{\boldsymbol{L}}(\mathit{\Gamma}\cup\{\alpha\})$ — a contradiction. ☐

Lemma 4.7. *Let $\mathit{\Gamma}$ be a finite subset of* For *such that $\ell^{\boldsymbol{L}}(\mathit{\Gamma})>0$, $\alpha\in$* For, *and $\alpha\notin\mathit{\Gamma}$. Let $\mathit{\Delta}\vdash_{\boldsymbol{L}}\alpha$, for some $\mathit{\Delta}\in\mathrm{Const}^{\max}_{\boldsymbol{L}}(\mathit{\Gamma})$. Let $\alpha\in\mathit{\Delta}'$, for* *some* $\Delta' \in \mathrm{Const}^{\max}_{\boldsymbol{L}}(\Gamma \cup \{\alpha\})$ *such that* $\mathrm{Card}(\Delta') > 1$. *Then* $\Delta = \Delta' \setminus \{\alpha\}$.

Proof. Under the adopted assumptions, we put $\Delta'' := \Delta' \setminus \{\alpha\}$. Then $\Delta'' \in \mathrm{Const}_{\boldsymbol{L}}(\Gamma \cup \{\alpha\})$. Since $\Delta'' \in \mathrm{Const}_{\boldsymbol{L}}(\Gamma \cup \{\alpha\})$ and $\alpha \notin \Gamma$, so $\Delta'' \subseteq \Gamma$; additionally $\Gamma \Vdash^{c}_{\boldsymbol{L}} \alpha$ (because $\Delta \vdash_{\boldsymbol{L}} \alpha$ and $\Delta \in \mathrm{Const}^{\max}_{\boldsymbol{L}}(\Gamma)$). Thus, by Lemma 4.6, we have $\Delta'' \in \mathrm{Const}_{\boldsymbol{L}}(\Gamma)$.

We know that for any $\mathcal{C} \in \mathbb{MLC}_{\boldsymbol{L}}(\Gamma)$ there are $\Omega \in \mathcal{C}$, $\Omega' \in \mathcal{C}$ such that $\Delta \subseteq \Omega$ and $\Delta'' \subseteq \Omega'$. Now, we have to consider two cases. First, for some cover $\mathcal{C}$, Δ and Δ'' are included in different elements of the cover. Second, for each cover $\mathcal{C}$, Δ and Δ'' are included in the same element of the cover. Below we prove that the first case is not possible.

Let us assume that for some $\mathcal{C}_0 \in \mathbb{MLC}_{\boldsymbol{L}}(\Gamma)$ there is no $\Omega \in \mathcal{C}_0$ such that $\Delta \subseteq \Omega$ and $\Delta'' \subseteq \Omega$. Although there are $\Omega, \Omega' \in \mathcal{C}_0$ such that $\Delta \subseteq \Omega$ and $\Delta'' \subseteq \Omega'$. Therefore $\Omega \neq \Omega'$. Since $\Delta \vdash_{\boldsymbol{L}} \alpha$, we can add consistently the formula α to the set Ω. Therefore let $\mathcal{C}_\alpha$ be a logical cover of Γ that is obtained from $\mathcal{C}_0$ by adding the formula α to Ω (we put $\Omega_\alpha := \Omega \cup \{\alpha\}$), and leaving unchanged the rest of elements of $\mathcal{C}_0$ — in particular $\Omega' \in \mathcal{C}_\alpha$. Notice that the proposed modification does not change the width of the cover, i.e. $\ell^{\boldsymbol{L}}(\Gamma) = \mathrm{w}(\mathcal{C}_0) = \mathrm{w}(\mathcal{C}_\alpha)$. By Lemma 4.4 we have that $\mathrm{w}(\mathcal{C}_\alpha) = \ell^{\boldsymbol{L}}(\Gamma \cup \{\alpha\})$. Furthermore, every element of $\mathcal{C}_\alpha$ is $\boldsymbol{L}$-consistent and $\bigcup \mathcal{C}_\alpha = \Gamma \cup \{\alpha\}$. Therefore $\mathcal{C}_\alpha \in \mathbb{MLC}_{\boldsymbol{L}}(\Gamma \cup \{\alpha\})$. By the assumption that $\Delta' \in \mathrm{Const}^{\max}_{\boldsymbol{L}}(\Gamma \cup \{\alpha\})$, for some $\hat{\Omega} \in \mathcal{C}_\alpha$ it holds that $\Delta' \subseteq \hat{\Omega}$. But the only element of $\mathcal{C}_\alpha$ that contains the formula α is Ω_α, thus $\hat{\Omega} = \Omega_\alpha$. Then, $\Delta'' \subseteq \Omega_\alpha$, since $\Delta'' \subsetneq \Delta'$. However, $\Delta'' \subseteq \Omega'$ — as it was mentioned, in the proposed modification of the cover $\mathcal{C}_0$ the element Ω' remained unchanged, thereby $\Omega' \in \mathcal{C}_\alpha$. Thus, $\Delta'' \subsetneq \Omega_\alpha$ and $\Delta'' \subseteq \Omega'$. Since $\Delta'' \neq \varnothing$, $\delta \in \Omega_\alpha$ and $\delta \in \Omega'$, for some $\delta \in \Delta''$. By Lemma 4.1 applied to $\mathcal{C}_\alpha$, $\Omega' \in \mathcal{C}_\alpha$ and δ there is $\mathcal{C}'_\alpha \in \mathrm{Cov}_{\boldsymbol{L}}(\Gamma \cup \{\alpha\})$ such that δ has been removed from all elements of $\mathcal{C}_\alpha$ but Ω'. Since there is no $\Omega \in \mathcal{C}'_\alpha$ such that $\Delta' \subseteq \Omega$ and $\mathcal{C}'_\alpha \in \mathbb{MLC}_{\boldsymbol{L}}(\Gamma \cup \{\alpha\})$, so $\Delta' \notin \mathrm{Const}_{\boldsymbol{L}}(\Gamma \cup \{\alpha\})$ — a contradiction.

By the last paragraph we have shown that for any $\mathcal{C} \in \mathbb{MLC}_{\boldsymbol{L}}(\Gamma)$ there is $\Omega \in \mathcal{C}$ such that $\Delta \subseteq \Omega$ and $\Delta'' \subseteq \Omega$. Therefore for $\hat{\Delta} := \Delta \cup \Delta''$ we have $\hat{\Delta} \in \mathrm{Const}_{\boldsymbol{L}}(\Gamma)$. Since $\Delta \subseteq \hat{\Delta}$ and $\Delta \in \mathrm{Const}^{\max}_{\boldsymbol{L}}(\Gamma)$, so $\hat{\Delta} = \Delta$. Hence $\Delta'' \subseteq \Delta$. By lemmas 4.4 and 4.5 we know that $\Delta \in \mathrm{Const}_{\boldsymbol{L}}(\Gamma \cup \{\alpha\})$. Notice that if for some $\Delta''' \in \mathrm{Const}^{\max}_{\boldsymbol{L}}(\Gamma \cup \{\alpha\})$, $\Delta''' \neq \Delta'$ and $\Delta'' \subseteq \Delta'''$ then since $\Delta'' \subsetneq \Delta'$ we would have $\Delta' \cap \Delta''' \neq \varnothing$ — a contradiction with Lemma 4.2. So Δ' is the unique element of

$\mathrm{Const}^{\max}_{\boldsymbol{L}}(\Gamma \cup \{\alpha\})$ that contains Δ''. Therefore $\Delta \subsetneq \Delta'$ (because $\alpha \in \Delta'$ and $\alpha \notin \Delta$). Since $\Delta'' \subseteq \Delta \subsetneq \Delta'$, so $\Delta'' \cup \{\alpha\} \subseteq \Delta \cup \{\alpha\} \subseteq \Delta' \cup \{\alpha\}$. Thus, $\Delta' \subseteq \Delta \cup \{\alpha\} \subseteq \Delta'$. Therefore $\Delta = \Delta' \setminus \{\alpha\}$. $\square$

Proof of Theorem 4.3. Under the adopted assumptions, suppose that $\Gamma \cup \{\alpha\} \Vdash^{c}_{\boldsymbol{L}} \beta$ and $\Gamma \Vdash^{c}_{\boldsymbol{L}} \alpha$. If α or β belong to the set Γ, then the thesis trivially holds (for the second case see Theorem 4.1). Therefore we may assume that $\alpha \notin \Gamma$ and $\beta \notin \Gamma$. By Definition 4.1 there are sets $\Delta \in \mathrm{Const}^{\max}_{\boldsymbol{L}}(\Gamma)$ and $\Delta' \in \mathrm{Const}^{\max}_{\boldsymbol{L}}(\Gamma \cup \{\alpha\})$ such that $\Delta \vdash_{\boldsymbol{L}} \alpha$ and $\Delta' \vdash_{\boldsymbol{L}} \beta$. (Furthermore, by Lemma 4.4, $\ell^{\boldsymbol{L}}(\Gamma) = \ell^{\boldsymbol{L}}(\Gamma \cup \{\alpha\})$.)

First assume that $\alpha \in \Delta'$ and $\mathrm{Card}(\Delta') > 1$. Then, by Lemma 4.7, $\Delta' = \Delta \cup \{\alpha\}$, since $\Delta' \in \mathrm{Const}^{\max}_{\boldsymbol{L}}(\Gamma \cup \{\alpha\})$, $\Delta \in \mathrm{Const}^{\max}_{\boldsymbol{L}}(\Gamma)$, $\alpha \notin \Gamma$, $\Delta \vdash_{\boldsymbol{L}} \alpha$. Hence, by the cut rule for $\boldsymbol{L}$, $\Delta \vdash_{\boldsymbol{L}} \beta$, since $\Delta \vdash_{\boldsymbol{L}} \alpha$ and $\Delta \cup \{\alpha\} \vdash_{\boldsymbol{L}} \beta$. So for some $\hat{\Delta} \in \mathrm{Const}^{\max}_{\boldsymbol{L}}(\Gamma)$ we have $\hat{\Delta} \vdash_{\boldsymbol{L}} \beta$. Therefore $\Gamma \Vdash^{c}_{\boldsymbol{L}} \beta$.

Second assume that $\alpha \in \Delta'$ and $\mathrm{Card}(\Delta') = 1$. So $\{\alpha\} \vdash_{\boldsymbol{L}} \beta$. Then, by *the monotonicity*, $\Delta \cup \{\alpha\} \vdash_{\boldsymbol{L}} \beta$ and since $\Delta \vdash_{\boldsymbol{L}} \alpha$, we have $\Delta \vdash_{\boldsymbol{L}} \beta$, by *the cut rule*. Finally $\Gamma \Vdash^{c}_{\boldsymbol{L}} \beta$, because $\Delta \in \mathrm{Const}^{\max}_{\boldsymbol{L}}(\Gamma)$.

Thirdly assume that $\alpha \notin \Delta'$. Then $\Delta' \subseteq \Gamma \cup \{\alpha\}$, since $\Delta' \in \mathrm{Const}^{\max}_{\boldsymbol{L}}(\Gamma \cup \{\alpha\})$. Thus, $\Delta' \subseteq \Gamma$. Furthermore, $\Delta' \in \mathrm{Const}_{\boldsymbol{L}}(\Gamma)$, by Lemma 4.6 applied for Δ' and also by the assumptions: $\alpha \notin \Gamma$ and $\Gamma \Vdash^{c}_{\boldsymbol{L}} \alpha$. Hence there is $\Delta'' \in \mathrm{Const}^{\max}_{\boldsymbol{L}}(\Gamma)$ such that $\Delta' \subseteq \Delta''$. Then, by *the monotonicity*, $\Delta'' \vdash_{\boldsymbol{L}} \beta$. Hence $\Gamma \Vdash^{c}_{\boldsymbol{L}} \beta$. $\square$

We demonstrated that *c-forcing* is weaker than *the forcing* inference relation. While *the forcing* guarantees only the same set of conclusions derived from elements of each minimal logical cover, the *c-forcing* in addition requires fixed sets of premises. The existence of such constant elements for inconsistent sets of premises seems to be worth of the attention. Crucial for disclosure of these elements is the concept of the level of inconsistency of the set of premises — sets of premises ought to satisfy the condition that a number of consistent subsets into which we divide an inconsistent set of premises is possibly the smallest (see [4]).

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