Logic and Logical Philosophy
Volume 25 (2016), 309–350
DOI: 10.12775/LLP.2016.023

Karl-Georg Niebergall

MEREOLOGY AND INFINITY

Abstract. This paper deals with the treatment of infinity and finiteness in mereology. After an overview of some first-order mereological theories, finiteness axioms are introduced along with a mereological definition of "x is finite" in terms of which the axioms themselves are derivable in each of those theories. The finiteness axioms also provide the background for definitions of "(mereological theory) T makes an assumption of infinity". In addition, extensions of mereological theories by the axioms are investigated for their own sake. In the final part, a definition of "x is finite" stated in a second-order language is also presented, followed by some concluding remarks on the motivation for the study of the (first-order) extensions of mereological theories dealt with in the paper.

Keywords: mereology; infinity; axioms of finiteness; relative interpretability; second-order language

1. Introduction

This paper is concerned with the mereological treatment of infinity and finiteness. In [Niebergall, 2014] [see also Niebergall, 2011a, 2009a], I offered a general account of the interplay between axioms of finiteness, definitions of finiteness and infinity, and assumptions of infinity. There, the approach was primarily motivated by and applied to set theories as background theories. Here, mereological theories are the main topic of inquiry.

A set-theoretical framework is nonetheless the natural starting point for the study of infinity and finiteness. For the phrases "x is finite" and "x is infinite", if they are introduced explicitly at all, are usually *defined* in *set-theoretic terms*. Put more precisely: let L[$\in$] be the first-order language with the 2-place predicate "$\in$" as its sole non-logical sign, "$x \in y$" being read "x is an element of y"; then there is a formula α (in

Special Issue: Mereology and Beyond (II). Edited by A. C. Varzi and R. Gruszczyński
 Published online August 8, 2016

the free variable "x") from L[$\in$] such that α is a *definiens* of "x is finite" (or, alternatively, of "x is infinite").

Of course, various such definitions have been formulated. To mention just one example, consider:

x is finite $:\longleftrightarrow \exists yu(y \in \omega \wedge u$ is a bijective function from x to $y)$[1],
x is infinite $:\longleftrightarrow \neg\, x$ is finite.

Equivalences such as these are widely employed and accepted as set-theoretic explications of "x is finite" and "x is infinite".[2] To put it differently, it may also be said that it is accepted that, e.g., "x is infinite" *expresses* that x is infinite. In [Niebergall, 2011a, 2014], different kinds of potential reasons for this acceptance are discussed and various explications for "α expresses that x is infinite" are presented. According to the position developed in these papers, in order for a set-theoretic formula α to express that x is infinite, relative to a set theory T, T must prove certain sentences (in which α occurs); sentences which have to be faithful to our linguistic intuitions when $\alpha(x)$ is read "x is infinite".

As a matter of fact, it seems to be easier to find and accept such test-sentences when they are formulated with an $\alpha(x)$ that is supposed to express "x is finite". Certain candidates for them have been suggested quite early [see e.g. Fraenkel, 1927] and have been rediscovered several times, hereby witnessing a remarkable stability of intuitions. These sentences may ultimately be regarded as *axioms of finiteness* which are formulated in L[$\in, \mathcal{F}$], i.e., L[$\in$] extended by the 1-place predicate "$\mathcal{F}$" (where "$\mathcal{F}x$" is supposed to be read "x is finite"). Among them are:

$\mathcal{F}\emptyset$,
$\forall x\, \mathcal{F}(\{x\})$,
$\forall xy(\mathcal{F}x \wedge \mathcal{F}y \to \mathcal{F}(x \cup y))$,

plus an *induction schema*, i.e., the set of all formulas:

$$\psi(\emptyset) \wedge \forall x\, \psi(\{x\}) \wedge \forall xy\, (\psi(x) \wedge \psi(y) \to \psi(x \cup y)) \to \forall x(\mathcal{F}x \to \psi(x))$$

for any ψ from L[$\in, \mathcal{F}$].

[1] "$x \in \omega$" may be defined in different ways. "$\forall z(\emptyset \in z \wedge \forall y(y \in z \to y \cup \{y\} \in z) \to x \in z)$" is a common *definiens*, but there are also others; see [Parsons, 1987] and the appendix of [Niebergall, 2014].

[2] See, for example, [Levy, 1958; Schmidt, 1966; Klaua, 1973] for further definitions.

It can be shown that when "$\mathcal{F}x$" is replaced by "x is finite" (as defined above) in these axioms, and that when ψ is taken from L[$\in$], the resulting L[$\in$]-sentences are in fact provable in set theories such as ZF, i.e., Zermelo-Fraenkel set theory [see e.g. Takeuti, 1982]. In [Niebergall, 2011a, 2014], it is precisely this type of provability which is used as the *criterion of adequacy* for whether a set-theoretic formula $\alpha(x)$ expresses that x is finite (relative to a set theory T). In the present paper, this approach is applied to mereological theories. That is, I take it that for a mereological treatment of infinity and finiteness, plausible axioms of finiteness — which are stated in L[$\circ, \mathcal{F}$] — and a formula α in L[$\circ$], the language of mereological theories, have to be found such that these axioms become provable in mereological theories (when extended by the definition "$\mathcal{F}x :\longleftrightarrow \alpha(x)$").[3]

This programme has already been initiated in [Niebergall, 2009a, 2011a, 2014]. In particular, axioms of finiteness and a suitable *definiens* for "x is finite" stated in mereological terms can be found in those papers. The present study is an elaboration (containing, e.g., proofs missing in the earlier texts) and continuation of this work. After an overview of *mereological theories* in Section 2, *axioms of finiteness* relative to mereological theories are put forward in Section 3. Section 4 then contains a mereological *definiens* for "x is finite" which turns these axioms, when it replaces "$\mathcal{F}x$" in them, into sentences which are derivable in all mereological theories. These axioms also provide the background for definitions of "(mereological theory) T makes an assumption of infinity", which are discussed in Section 5. In addition, extensions of mereological theories by these axioms are investigated for their own sake in Section 6. These are first-order theories. The definition of "x is finite" stated in a second-order language is presented in Section 7. I close with thoughts on the motivation for the study of the (first-order) extensions of mereological theories dealt with in the present paper.

2. Mereological theories

2.1. Basic definitions and axioms

The theories providing the basis for axioms of finiteness studied in the present paper are formulated in the first-order language L[$\circ$], whose vo-

[3] See especially sections 2 and 3 for the unexplained terminology used in this statement.

cabulary consists of the 2-place predicate "$\circ$" and the identity sign "$=$". L[$\circ$] is supplied with classical first-order logic.

L[$\circ$] is the language of the theories called "mereological theories" in this paper. But not every theory in L[$\circ$] should be regarded as a mereological theory. For a theory in L[$\circ$] to be called "mereological", its theorems have to be correct under a specific reading of "$\circ$": namely under the reading of "$x \circ y$" as "x overlaps y". Which sentences of L[$\circ$] *are* correct, given that reading, is not settled. I assume that the following ones are correct:

$$\begin{array}{ll} \text{O} & \forall xy(x \circ y \longleftrightarrow \exists z(z \sqsubseteq x \wedge z \sqsubseteq y)), \\ \text{SUM} & \forall xy \exists z \forall u(u \circ z \longleftrightarrow u \circ x \vee u \circ y), \\ \text{NEG} & \forall x(\neg \forall v\, v \circ x \rightarrow \exists y \forall v(v \sqsubseteq y \longleftrightarrow \neg v \circ x)). \end{array}$$

The 2-place predicate "$\sqsubseteq$", which is read "part of", is defined as follows:

$$x \sqsubseteq y :\longleftrightarrow \forall z(z \circ x \rightarrow z \circ y).$$

Moreover, in all the theories taken into account in this paper certain principles for identity should be provable. Thus, I assume as axioms *the principle of substitutivity*[4] and

$$\text{ANT} \qquad \forall xy\,(x \sqsubseteq y \wedge y \sqsubseteq x \rightarrow x = y).$$

I call the above set of sentences "Ax(CI)" and the theory axiomatized by them "CI".[5]

Often, instances of the so-called *fusion schema* FUS are also included in mereological theories. By employing the common procedure of identifying a schema with the set of "its instances", FUS can be precisely determined as follows. Let ψ be a formula in L[$\circ$]; then:

$$\begin{array}{l} \text{FUS}_\psi\text{: (the universal closure of) } \exists x\, \psi \rightarrow \exists z \forall y(z \circ y \leftrightarrow \exists x(x \circ y \wedge \psi)), \\ \text{FUS} := \{\text{FUS}_\psi \mid \psi \text{ is a } \text{L}^1[\circ]\text{-formula}\}. \end{array}$$

A particular instance of FUS, called "FUS_{At}", will play an exposed role for this investigation. It is obtained by choosing "$At(x)$" ("x is an atom")—which is defined by "$\forall y(y \sqsubseteq x \rightarrow x \sqsubseteq y)$"—for $\psi(x)$ in FUS; i.e.:

$$\text{FUS}_{At}\text{: } \exists x At(x) \rightarrow \exists z \forall y(z \circ y \longleftrightarrow \exists x(x \circ y \wedge At(x))).$$

[4] If L[$\circ$] is extended, the principle of substitutivity is supposed to hold also for the formulas of this extended language.

[5] A theory T in language L is a set of L-formulas such that all L-formulas derivable from T are already elements of T.

CI + $\{\mathrm{FUS}_{At}\}$[6] will be the core of all theories in L[$\circ$] dealt with in this paper. Accordingly, I define:

T is a mereological theory $:\Longleftrightarrow$ CI + $\{\mathrm{FUS}_{At}\} \subseteq T$.[7]

The set of mereological theories can be structured by distinguishing these theories with respect to what they prove about atoms. In particular, we can express that the part-relation is atomistic or, alternatively, atomless:

AT $\quad \forall x \exists y(y \sqsubseteq x \wedge At(y))$,
AF $\quad \forall x \exists y(y \sqsubseteq x \wedge x \not\sqsubseteq y)$.

With these, the induced theories are[8]:

$$\begin{aligned}
&\mathrm{ACI} := \mathrm{CI} + \{\mathrm{AT}\},\\
&\mathrm{FCI} := \mathrm{CI} + \{\mathrm{AF}\},\\
&\mathrm{MCI} := \mathrm{CI} + \{\neg\mathrm{AT}, \neg\mathrm{AF}\},\\
&\mathrm{ACI}_{n+1} := \mathrm{ACI} + \{\exists_{=n+1} At\} \quad (\text{for } n \in \mathbb{N}),\\
&\mathrm{ACI}_{\omega} := \mathrm{ACI} + \{\exists_{\geq n+1} At \mid n \in \mathbb{N}\},\\
&\mathrm{MCI}_{n+1} := \mathrm{MCI} + \{\exists_{=n+1} At\} \quad (\text{for } n \in \mathbb{N}),\\
&\mathrm{MCI}_{\omega} := \mathrm{MCI} + \{\exists_{\geq n+1} At \mid n \in \mathbb{N}\}.
\end{aligned}$$

Moreover, arbitrary instances of FUS may be added to each of these sets as axioms.

2.2. Mereological theorems

For several of the results mentioned later which concern the combination of both mereology and finiteness, theorems of a purely mereological character are useful. Those theorems for which I could not find proofs in the published literature are brought together here.

Lemma 2.1. CI $\vdash \forall xy(x \circ y \rightarrow \exists z \forall u(u \sqsubseteq z \longleftrightarrow u \sqsubseteq x \wedge u \sqsubseteq y))$.

---

[6] In general, if Σ and Σ' are sets of formulas from a first-order language L, I use "$\Sigma + \Sigma'$" for "$\{\psi \mid \psi$ is a formula of L $\wedge\, \Sigma \cup \Sigma' \vdash \psi\}$".

[7] More on the motivation of this stipulation can be found in [Niebergall, 2011b]. Let me point out that other philosophers may understand this predicate in a different way and may also be interested in a wider class of theories when dealing with mereological theories.

[8] Here I employ two abbreviations:
$\exists_{\geq n+1} At :\longleftrightarrow \exists x_1 \ldots x_{n+1}(At(x_1) \wedge \ldots \wedge At(x_{n+1}) \wedge x_1 \neq x_2 \wedge \ldots x_1 \neq x_{n+1} \wedge \ldots \wedge x_n \neq x_{n+1})$,
$\exists_{=n+1} At :\longleftrightarrow \exists_{\geq n+1} At \wedge \neg \exists_{\geq n+2} At$.

The object z whose existence is guaranteed by this lemma (given x and y) is sometimes called "the product of x and y"; I often write "$x \sqcap y$" for it. Similarly, I often write "$x \sqcup y$" for the object z, the so called "sum of x and y", whose existence is guaranteed by SUM (given x and y), and "$-x$" for the object y whose existence is guaranteed by NEG (given x).[9]

Lemma 2.2. (i) *$\mathrm{CI} \vdash \forall x x' y(y \sqsubseteq x \sqcup x' \to y \sqsubseteq x \vee y \circ x')$ and $\mathrm{CI} \vdash \forall x x' y(y \sqsubseteq x \sqcup x' \to y \circ x \vee y \sqsubseteq x')$.*

(ii) *$\mathrm{CI} \vdash \forall x x' y(y \sqsubseteq x \sqcup x' \to y \sqsubseteq x \vee y \sqsubseteq x' \vee (y \circ x \wedge y \circ x'))$.*

(iii) *$\mathrm{CI} \vdash \forall xyz(z \circ x \wedge z \circ y \wedge z \sqsubseteq x \sqcup y \to z \sqsubseteq (z \sqcap x) \sqcup (z \sqcap y))$.*

(iv) *$\mathrm{CI} \vdash \forall xyz(z \sqsubseteq x \sqcup y \to z \sqsubseteq x \vee z \sqsubseteq y \vee z = (z \sqcap x) \sqcup (z \sqcap y))$.*

Proof. (i) We have:

$$\begin{aligned} \mathrm{CI} \vdash y \sqsubseteq x \sqcup x' \wedge \neg y \circ x \wedge z \circ y &\to \exists u(u \sqsubseteq z \wedge u \sqsubseteq y) \wedge \neg y \circ x \\ &\to \exists u(u \circ y \wedge \neg u \circ x \wedge u \sqsubseteq z), \end{aligned}$$

whence:

$$\begin{aligned} \mathrm{CI} \vdash y \sqsubseteq x \sqcup x' \wedge \neg y \circ x \wedge z \circ y &\to \exists u(u \circ y \wedge \neg u \circ x \wedge y \sqsubseteq x \sqcup x' \wedge u \sqsubseteq z) \\ &\to \exists u(u \circ x' \wedge u \sqsubseteq z) \\ &\to z \circ x' . \end{aligned}$$

In sum, $\mathrm{CI} \vdash y \sqsubseteq x \sqcup x' \wedge \neg y \circ x \to y \sqsubseteq x'$, i.e., $\mathrm{CI} \vdash y \sqsubseteq x \sqcup x' \to y \circ x \vee y \sqsubseteq x'$.

Similarly, $\mathrm{CI} \vdash y \sqsubseteq x \sqcup x' \to y \circ x' \vee y \sqsubseteq x$.

(ii) By conjoining the claims from (i).

(iii) First, by SUM in the second, O in the third and Lemma 2.1 in the fourth step:

$$\begin{aligned} \mathrm{CI} \vdash{} & z \circ x \wedge z \circ y \wedge z \sqsubseteq x \sqcup y \wedge \neg(z \sqsubseteq (z \sqcap x) \sqcup (z \sqcap y)) \to \\ & z \circ x \wedge z \circ y \wedge \exists w(w \circ z \wedge \neg w \circ ((z \sqcap x) \sqcup (z \sqcap y))) \\ & \to \exists w(w \circ z \wedge z \circ x \wedge z \circ y \wedge \neg w \circ (z \sqcap x) \wedge \neg w \circ (z \sqcap y)) \\ & \to \exists w(w \circ z \wedge z \circ x \wedge z \circ y \wedge \neg \exists v(v \sqsubseteq w \wedge v \sqsubseteq z \sqcap x) \wedge \neg \exists v(v \sqsubseteq w \wedge v \sqsubseteq z \sqcap y)) \\ & \to \exists w(w \circ z \wedge \forall v(v \sqsubseteq w \wedge v \sqsubseteq z \to \neg v \sqsubseteq x) \wedge \forall v(v \sqsubseteq w \wedge v \sqsubseteq z \to \neg v \sqsubseteq y)) . \end{aligned}$$

Second, by (i) in the first step:

$$\begin{aligned} \mathrm{CI} \vdash{} & u \sqsubseteq w \wedge u \sqsubseteq z \wedge z \sqsubseteq x \sqcup y \wedge \forall v(v \sqsubseteq w \wedge v \sqsubseteq z \to \neg v \sqsubseteq x) \wedge \\ & \forall v(v \sqsubseteq w \wedge v \sqsubseteq z \to \neg v \sqsubseteq y) \end{aligned}$$

[9] Without simplifications such as these, the (official) notation quickly becomes too cumbersome.

$$\rightarrow (u \sqsubseteq x \vee u \circ y) \wedge \neg u \sqsubseteq x \wedge \forall v(v \sqsubseteq w \wedge v \sqsubseteq z \rightarrow \neg v \sqsubseteq y)$$
$$\rightarrow u \circ y \wedge \forall v(v \sqsubseteq w \wedge v \sqsubseteq z \rightarrow \neg v \sqsubseteq y)\,,$$

whence:

$$\begin{aligned}\mathrm{CI} \vdash\ & u \sqsubseteq w \wedge u \sqsubseteq z \wedge z \sqsubseteq x \sqcup y \wedge \forall v(v \sqsubseteq w \wedge v \sqsubseteq z \rightarrow \neg v \sqsubseteq x) \wedge \\ & \quad \forall v(v \sqsubseteq w \wedge v \sqsubseteq z \rightarrow \neg v \sqsubseteq y) \\ & \rightarrow \exists a(a = u \sqcap y \wedge a \sqsubseteq w \wedge a \sqsubseteq z \wedge a \sqsubseteq y) \wedge \\ & \rightarrow \forall v(v \sqsubseteq w \wedge v \sqsubseteq z \rightarrow \neg v \sqsubseteq y)) \\ & \rightarrow \exists a(a = u \sqcap y \wedge a \sqsubseteq y \wedge \neg a \sqsubseteq y) \rightarrow \bot\,,\end{aligned}$$

and therefore $\mathrm{CI} \vdash z \sqsubseteq x \sqcup y \wedge \exists w(w \circ z \wedge \forall v(v \sqsubseteq w \wedge v \sqsubseteq z \rightarrow \neg v \sqsubseteq x) \wedge \forall v(v \sqsubseteq w \wedge v \sqsubseteq z \rightarrow \neg v \sqsubseteq y)) \rightarrow \bot$. Together with the first part, this implies $\mathrm{CI} \vdash z \circ x \wedge z \circ y \wedge z \sqsubseteq x \sqcup y \wedge \neg(z \sqsubseteq (z \sqcap x) \sqcup (z \sqcap y)) \rightarrow \bot$, which yields the claim.

(iv) By (iii), $\mathrm{CI} \vdash z \circ x \wedge z \circ y \wedge z \sqsubseteq x \sqcup y \rightarrow z = (z \sqcap x) \sqcup (z \sqcap y)$. In conjunction with (ii), this implies the claim. □

Lemma 2.3. $\mathrm{ACI}_n \vdash \exists x_1 \ldots x_n(At(x_1) \wedge \cdots \wedge At(x_n) \wedge \forall y(y \sqsubseteq x_1 \sqcup \cdots \sqcup x_n))$.

2.3. The main metatheorems

Let me assemble the metatheorems concerning mereological theories which are relevant for the present study [cf. Niebergall, 2009a, 2011b].

Theorem 2.1. (i) $\mathrm{FUS} \subseteq \mathrm{ACI}$, $\mathrm{FUS} \subseteq \mathrm{FCI}$, $\mathrm{FUS} \subseteq \mathrm{MCI} + \{\mathrm{FUS}_{At}\}$.
(ii) *If for each* $n \in \mathbb{N}$, $\mathrm{ACI}_{n+1} \vdash \psi$, *then* $\mathrm{ACI} \vdash \psi$. *If for each* $n \in \mathbb{N}$, $\mathrm{MCI}_{n+1} \vdash \psi$, *then* $\mathrm{MCI} + \{\mathrm{FUS}_{At}\} \vdash \psi$.
(iii) *The maximal consistent mereological theories are exactly* ACI_{n+1} *and* MCI_{n+1}, *for* $n \in \mathbb{N}$, FCI, ACI_ω, *and* $\mathrm{MCI}_\omega + \{\mathrm{FUS}_{At}\}$. *These theories are decidable.*

3. Mereological axioms of finiteness

3.1. Axioms of finiteness relative to mereological theories

$\mathrm{L}[\circ]$ is now extended by the 1-place predicate "$\mathcal{F}$" to the first-order language $\mathrm{L}[\circ, \mathcal{F}]$. Again, "$\mathcal{F}x$" is read "$x$ is finite".

Sentences from $\mathrm{L}[\circ, \mathcal{F}]$ which, given this reading of $\mathcal{F}$, are plausible axioms of finiteness, are easily forthcoming:

(AxI 1) $\forall x(At(x) \to \mathcal{F}x)$,
(AxI 2) $\forall xy(\mathcal{F}x \wedge \mathcal{F}y \to \mathcal{F}(x \sqcup y))$,[10]
(AxI 3) $\forall xy(\mathcal{F}x \wedge y \sqsubseteq x \to \mathcal{F}y)$,
(AxI 4) $\forall xy(\mathcal{F}x \wedge At(y) \to \mathcal{F}(x \sqcup y))$.

As mereological principles of finiteness, these sentences should be as evident as the ones from the introduction were as set theoretic principles of finiteness. Moreover, they can also be easily obtained from the latter by the procedure of replacing "$\subseteq$" (from $L[\in]$) by its mereological counterpart, "$\sqsubseteq$". In this way, set-theoretic singletons are turned into mereological atoms and set-theoretic unions are transformed into mereological sums. The empty set gets, of course, lost.

What is not claimed is that this list is exhaustive. In particular, principles of induction are missing. But even without them, some useful theorems can be proven.

By induction on n we obtain:

Lemma 3.1. *In* $\mathrm{CI} + \{(\mathrm{AxI1}), (\mathrm{AxI2})\}$ *it is provable*

(i) $\forall xy(\mathcal{F}x \wedge At(y) \to \mathcal{F}(x \sqcup y))$,
(ii) $\forall x_1 ... x_n(At(x_1) \wedge \cdots \wedge At(x_n) \to \mathcal{F}(x_1 \sqcup \cdots \sqcup x_n))$.

Lemma 3.2. *For any* $n \in \mathbb{N}$, $\mathrm{ACI}_{n+1} + \{(\mathrm{AxI1}), (\mathrm{AxI2}), (\mathrm{AxI3})\} \vdash \forall x\, \mathcal{F}x$.

Proof. By Lemma 2.3, for each $n \geq 1$, $\mathrm{ACI}_n \vdash \exists x_1 ... x_n(At(x_1) \wedge \cdots \wedge At(x_n) \wedge \forall y(y \sqsubseteq x_1 \sqcup \cdots \sqcup x_n))$.

By Lemma 3.1(ii), it follows that $\mathrm{ACI}_n + \{(\mathrm{AxI1}), (\mathrm{AxI2})\} \vdash \exists x_1 ... x_n(\mathcal{F}(x_1 \sqcup \cdots \sqcup x_n) \wedge \forall y(y \sqsubseteq x_1 \sqcup \cdots \sqcup x_n))$, which implies with (AxI 3) that $\mathrm{ACI}_n + \{(\mathrm{AxI1}), (\mathrm{AxI2}), (\mathrm{AxI3})\} \vdash \forall y\, \mathcal{F}y$. □

I propose two axiom systems built upon the above list. In both cases, a suitable minimality condition — captured by an induction schema — is added to axioms from that list; that is, a schema whose premisses just have the logical forms of the axioms.

Ax-a(FinI): (AxI 1), (AxI 2), and (IndI-a), that is,

(IndI-a) $\{(\mathrm{IndI-a})_\psi \mid \psi$ is a formula from $L[\circ, \mathcal{F}]\}$,

where $(\mathrm{IndI\text{-}a})_\psi \equiv AxFinI\text{-a}[\psi] \to \forall x(\mathcal{F}x \to \psi(x))$

and for α in $L[\circ, \mathcal{F}]$

$$AxFinI\text{-a}[\alpha] \equiv \forall x(At(x) \to \alpha(x)) \ \wedge \ \forall xy(\alpha(x) \wedge \alpha(y) \to \alpha(x \sqcup y)).$$

[10] That is, in primitive notation, $\forall xy(\mathcal{F}x \wedge \mathcal{F}y \to \exists z(\forall u(u \circ z \leftrightarrow u \circ x \vee u \circ y) \wedge \mathcal{F}z))$.

Ax-b(FinI): (AxI 1), (AxI 2), (AxI 3), and (IndI-b), that is,

(IndI-b) $\{(\text{IndI} - \text{b})_\psi \mid \psi \text{ is a formula from } \mathrm{L}[\circ, \mathcal{F}]\}$,

where $(\text{IndI-b})_\psi \equiv \mathit{AxFinI}\text{-b}[\psi] \to \forall x(\mathcal{F}x \to \psi(x))$

and for α in $\mathrm{L}[\circ, \mathcal{F}]$

$\mathit{AxFinI}\text{-b}[\alpha] \equiv \forall x(At(x) \to \alpha(x)) \wedge \forall xy(\alpha(x) \wedge \alpha(y) \to \alpha(x \sqcup y)) \wedge \forall xy(\alpha(x) \wedge y \sqsubseteq x \to \alpha(y))$.

In the next subsection, the equivalence of these two axiom systems is established (relative to CI).

3.2. Comparison of the axiom systems

Lemma 3.3. $\mathrm{CI} + \text{Ax-b(FinI)} \vdash \text{Ax-a(FinI)}$.

Proof. What has to be shown is, for each formula ψ from $\mathrm{L}[\circ, \mathcal{F}]$:

$\mathrm{CI} + \text{Ax-b(FinI)} \vdash (\text{IndI-a})_\psi$.

For every x, let $\phi(x)$ be the following formula: $\forall y(y \sqsubseteq x \to \psi(y))$.

Claim. $\mathrm{CI} \vdash \mathit{AxFinI}\text{-a}[\psi] \to \mathit{AxFinI}\text{-b}[\varphi]$.

Proof. The claim is established by showing that (i) $\mathrm{CI} \vdash \mathit{AxFinI}\text{-a}[\psi] \to \forall x(At(x) \to \varphi(x))$, (ii) $\mathrm{CI} \vdash \mathit{AxFinI}\text{-a}[\psi] \to \forall xy(\varphi(x) \wedge \varphi(y) \to \varphi(x \sqcup y))$, (iii) $\mathrm{CI} \vdash \mathit{AxFinI}\text{-a}[\psi] \to \forall xy(\varphi(x) \wedge y \sqsubseteq x \to \varphi(y))$.

Ad (i):

$$\begin{aligned}\mathrm{CI} \vdash \mathit{AxFinI}\text{-a}[\psi] \wedge At(x) \wedge y \sqsubseteq x \to At(y)\\ \to \psi(y)\,,\end{aligned}$$

that is:

$$\mathrm{CI} \vdash \mathit{AxFinI}\text{-a}[\psi] \wedge At(x) \to \forall y(y \sqsubseteq x \to \psi(y)) \to \varphi(x)\,.$$

Ad (ii): By the definition of φ, $(*)$ $\mathrm{CI} \vdash \varphi(x) \wedge \varphi(y) \wedge z \sqsubseteq x \to \psi(z)$ and $\mathrm{CI} \vdash \varphi(x) \wedge \varphi(y) \wedge z \sqsubseteq y \to \psi(z)$.

By the same kind of reasoning:

$$\begin{aligned}\mathrm{CI} \vdash \varphi(x) \wedge \varphi(y) \wedge z = (z \sqcap x) \sqcup (z \sqcap y) \to \varphi(x) \wedge \varphi(y) \wedge z \sqcap x \sqsubseteq x \wedge z \sqcap y \sqsubseteq y\\ \to \psi(z \sqcap x) \wedge \psi(z \sqcap y)\,,\end{aligned}$$

whence:

$$\begin{aligned}(**)\ \mathrm{CI} \vdash \mathit{AxFinI}\text{-a}[\psi] \wedge \varphi(x) \wedge \varphi(y) \wedge z = (z \sqcap x) \sqcup (z \sqcap y) \to\\ \mathit{AxFinI}\text{-a}[\psi] \wedge \psi(z \sqcap x) \wedge \psi(z \sqcap y) \wedge z = (z \sqcap x) \sqcup (z \sqcap y)\\ \to \psi(z)\,.\end{aligned}$$

Now $(*)$ and $(**)$ together with Lemma 2.2(iv) yield CI $\vdash AxFinI\text{-a}[\psi] \wedge \varphi(x) \wedge \varphi(y) \wedge z \sqsubseteq x \sqcup y \to \psi(z)$, that is, the desired:

$$\text{CI} \vdash AxFinI\text{-a}[\psi] \wedge \varphi(x) \wedge \varphi(y) \to \forall z(z \sqsubseteq x \sqcup y \to \psi(z)) \\ \to \varphi(x \sqcup y)\,.$$

Ad (iii): CI $\vdash AxFinI\text{-a}[\psi] \wedge \varphi(x) \wedge y \sqsubseteq x \wedge z \sqsubseteq y \to \varphi(x) \wedge z \sqsubseteq x \to \psi(z)$, whence CI $\vdash AxFinI\text{-a}[\psi] \wedge \varphi(x) \wedge y \sqsubseteq x \to \forall z(z \sqsubseteq y \to \psi(z)) \to \varphi(y)$.

Thus the claim is established. And since it implies:

$$\text{CI} + \text{Ax-b(FinI)} \vdash AxFinI\text{-a}[\psi] \to \forall x(\mathcal{F}x \to \varphi(x)),$$

we finally obtain:

$$\text{CI+Ax-b(FinI)} \vdash AxFinI\text{-a}[\psi] \to \forall x(\mathcal{F}x \to \forall y(y \sqsubseteq x \to \psi(y))) \\ \to \forall x(\mathcal{F}x \to \psi(x))\,.$$

□

LEMMA 3.4. CI + Ax-a(FinI) $\vdash$ Ax-b(FinI).

PROOF. What has to be shown is: (i) CI + Ax-a(FinI) $\vdash$ (AxI 3), and (ii) CI + Ax-a(FinI) $\vdash$ (IndI-b).

For every x, let $\psi(x)$ be the following formula: $\forall y(y \sqsubseteq x \to \mathcal{F}y)$. Then the proof of (i) is almost the same as for Lemma 3.3. The proof of (ii) is trivial, since each premiss of an (IndI-b)$_\psi$ is a strengthening of the corresponding premiss of (IndI-a)$_\psi$. □

THEOREM 3.1. *Relative to* CI, Ax-a(FinI), *and* Ax-b(FinI) *are equivalent.*

In the light of the above considerations, it is obvious that from the theories formulated in L$[\circ, \mathcal{F}]$, those are of special interest which contain both a part which is a mereological theory and a part which consists of axioms of finiteness. In what follows, I will use "T^*" for that theory in L$[\circ, \mathcal{F}]$ which results from a mereological theory T by the addition of Ax-a(FinI) (that is, of Ax-b(FinI)). Thus, Lemma 3.2 implies that $\text{ACI}^*_{n+1} \vdash \forall x \mathcal{F}x$. The following lemma is a complementary result.

LEMMA 3.5. FCI* $\vdash \forall x\, \neg\mathcal{F}x$.

PROOF. Set $\psi(x) \equiv x \neq x$. Then with (IndI-a), (i) FCI* $\vdash AxFinI\text{-a}[\psi] \to \forall x(\mathcal{F}x \to \psi(x))$. But also (ii) FCI $\vdash \forall x(At(x) \to \psi(x)) \wedge \forall xy(\psi(x) \wedge \psi(y) \to \psi(x \sqcup y))$. (i)+(ii) imply the claim. □

4. Definitions of "is finite" relative to mereological theories

4.1. A possible definition

(AxI 1) to (AxI 4) are of a special form: they are like clauses in a positive inductive definition. From set theory, it is known how to transform an inductive definition into a real definition, an explicit one. Applying this procedure here leads to:

$$IFin(x) :\longleftrightarrow \forall y(ICl(y) \rightarrow x \in y),$$

with:

$$ICl(y) :\longleftrightarrow \forall z(At(z) \rightarrow z \in y) \wedge \forall zz'(z \in y \wedge z' \in y \rightarrow z \sqcup z' \in y).$$

Of course, we do not have the set-theoretic "$\in$" at our disposal now. Moreover, the result that Q (Robinson Arithmetic; see [Tarski et al., 1953] cannot be relatively interpreted[11] in a consistent mereological theory [see Niebergall, 2011b] suggests that elementhood cannot be mimicked mereologically.

Although it is a crude manoeuvre, let me nonetheless simply replace "$\in$" by "$\sqsubseteq$" in the *definiens* just suggested and thereby put forward as a mereologically stated definition of "x is finite":

Definition 4.1.
$$ICl(y) :\longleftrightarrow \forall z(At(z) \rightarrow z \sqsubseteq y) \wedge \forall zz'(z \sqsubseteq y \wedge z' \sqsubseteq y \rightarrow z \sqcup z' \sqsubseteq y),$$
$$IFin(x) :\longleftrightarrow \forall y(ICl(y) \rightarrow x \sqsubseteq y).$$

Fortunately, this mereological *definiens* of "x is finite" works as it should.

4.2. Consequences of the definition

As has been pointed out in the introduction, there is a test to find out if Definition 4.1 is adequate as an explication of "x is finite": show that Ax-a(FinI) (or Ax-b(FinI)) belongs to CI + $\{FUS_{At}\}$ extended by "$\forall x(\mathcal{F}x \leftrightarrow IFin(x))$". This is done in this subsection (see Theorem 4.1 and Corollary 4.1). Let me first establish some preparatory lemmas, which are also useful later on.

Lemma 4.1. (i) CI $\vdash \forall y(ICl(y) \rightarrow \forall z(At(z) \rightarrow z \sqsubseteq y))$.
(ii) CI $\vdash \forall x(At(x) \rightarrow IFin(x))$.

[11] For relative interpretations, see [Tarski et al., 1953; Feferman, 1960].

(iii) $\mathrm{CI} \vdash \forall x x'(\mathit{IFin}(x) \wedge \mathit{IFin}(x') \rightarrow \mathit{IFin}(x \sqcup x'))$.
(iv) $\mathrm{CI} \vdash \forall xy(\mathit{IFin}(x) \wedge y \sqsubseteq x \rightarrow \mathit{IFin}(y))$.

PROOF. (i) Since "$\forall zz'(z \sqsubseteq y \wedge z' \sqsubseteq y \rightarrow z \sqcup z' \sqsubseteq y)$" is derivable in CI, it may be omitted from the *definiens*. (ii) follows from (i). (iii) $\mathrm{CI} \vdash \mathit{IFin}(x) \wedge \mathit{IFin}(x') \wedge \mathit{ICl}(y) \rightarrow x \sqsubseteq y \wedge x' \sqsubseteq y \rightarrow x \sqcup x' \sqsubseteq y$. □

LEMMA 4.2. (i) $\mathrm{ACI} \vdash \forall y(\mathit{ICl}(y) \leftrightarrow \forall z\, z \sqsubseteq y)$.
(ii) $\mathrm{ACI} \vdash \forall x\, \mathit{IFin}(x)$.
(iii) $\mathrm{FCI} \vdash \forall y\, \mathit{ICl}(y)$.
(iv) $\mathrm{FCI} \vdash \forall x\, \neg \mathit{IFin}(x)$.
(v) $\mathrm{CI} + \{\forall x\, \mathit{IFin}(x)\} \vdash \exists x\, \mathit{At}(x)$.

PROOF. (i) By Lemma 4.1(i) and the fact that $\mathrm{ACI} \vdash \forall z(\mathit{At}(z) \rightarrow z \sqsubseteq y) \rightarrow \forall z\, z \sqsubseteq y$. (ii) From (i). (iii) By Lemma 4.1(i).

(iv) By (iii), $\mathrm{FCI} \vdash \mathit{IFin}(x) \rightarrow \forall y\, x \sqsubseteq y$; hence $\mathrm{FCI} \vdash \neg \mathit{IFin}(x)$.

(v) Since $\mathrm{FCI} \subseteq \mathrm{CI} + \{\forall x\, \mathit{IFin}(x), \neg \exists x\, \mathit{At}(x)\}$, it follows from (iv) that $\mathrm{CI} + \{\forall x\, \mathit{IFin}(x), \neg \exists x\, \mathit{At}(x)\}$ is inconsistent. □

The following lemma is the main step on the way to Corollary 4.1:

LEMMA 4.3. *Let ψ be an arbitrary formula from* $\mathrm{L}[\circ]$, $n > 0$. *Then*:

$$\mathrm{CI} + \{\exists_n \mathit{At}\} \vdash \mathit{AxFinI}\text{-b}[\psi] \rightarrow \forall x(\mathit{IFin}(x) \rightarrow \psi(x)).$$

PROOF. By definition of AxFinI-b$[\psi]$:

$$\mathrm{CI} \vdash \mathit{At}(x_1) \wedge \ldots \wedge \mathit{At}(x_n) \wedge \mathit{AxFinI}\text{-b}[\psi] \rightarrow \psi(x_1) \wedge \ldots \wedge \psi(x_n) \wedge \mathit{AxFinI}\text{-b}[\psi] \rightarrow \psi(x_1 \sqcup \cdots \sqcup x_n),$$

whence $\mathrm{CI} \vdash \mathit{At}(x_1) \wedge \cdots \wedge \mathit{At}(x_n) \wedge \forall z(\mathit{At}(z) \rightarrow z = x_1 \vee \cdots \vee z = x_n) \wedge \mathit{AxFinI}\text{-b}[\psi] \rightarrow \psi(x_1 \sqcup \cdots \sqcup x_n) \wedge \forall z(\mathit{At}(z) \rightarrow z \sqsubseteq x_1 \sqcup \cdots \sqcup x_n)$. Therefore:

$$\begin{aligned}\mathrm{CI} \vdash\ & \mathit{At}(x_1) \wedge \cdots \wedge \mathit{At}(x_n) \wedge \forall z(\mathit{At}(z) \rightarrow z = x_1 \vee \cdots \vee z = x_n) \wedge \\ & \quad \mathit{AxFinI}\text{-b}[\psi] \wedge \mathit{IFin}(x) \\ & \rightarrow \forall y(\forall z(\mathit{At}(z) \rightarrow z \sqsubseteq y) \rightarrow x \sqsubseteq y) \wedge \forall z(\mathit{At}(z) \rightarrow z \sqsubseteq x_1 \sqcup \cdots \sqcup x_n) \wedge \\ & \quad \psi(x_1 \sqcup \cdots \sqcup x_n) \wedge \mathit{AxFinI}\text{-b}[\psi] \\ & \rightarrow x \sqsubseteq x_1 \sqcup \cdots \sqcup x_n \wedge \psi(x_1 \sqcup \cdots \sqcup x_n) \wedge \mathit{AxFinI}\text{-b}[\psi] \\ & \rightarrow \psi(x)\,.\end{aligned}$$

This yields $\mathrm{CI} \vdash \mathit{At}(x_1) \wedge \cdots \wedge \mathit{At}(x_n) \wedge \forall z(\mathit{At}(z) \rightarrow z = x_1 \vee \cdots \vee z = x_n) \wedge \mathit{AxFinI}\text{-b}[\psi] \rightarrow \forall x(\mathit{IFin}(x) \rightarrow \psi(x))$, which implies $\mathrm{CI} + \{\exists_n \mathit{At}\} \vdash \mathit{AxFinI}\text{-b}[\psi] \rightarrow \forall x(\mathit{IFin}(x) \rightarrow \psi(x))$. □

Theorem 4.1. *Let* ψ *be an arbitrary formula from* L[$\circ$]. *Then*:

$$\mathrm{CI} + \{\mathrm{FUS}_{At}\} \vdash AxFinI\text{-b}[\psi] \to \forall x(IFin(x) \to \psi(x)).$$

Proof. Let ψ be an arbitrary formula from L[$\circ$]. It suffices to establish:

Claim.

(a) $\mathrm{ACI} \vdash (\mathrm{IndI\text{-}b})_\psi$,
(b) $\mathrm{MCI} + \{\mathrm{FUS}_{At}\} \vdash (\mathrm{IndI\text{-}b})_\psi$,
(c) $\mathrm{FCI} \vdash (\mathrm{IndI\text{-}b})_\psi$.

Ad (a): By Lemma 4.3, since each ACI_{n+1} is an extension of $\mathrm{CI} + \{\exists_{n+1} x At(x)\}$, $(\mathrm{IndI\text{-}b})_\psi$ is provable in each ACI_{n+1}. (a) follows from Theorem 2.1(ii).

Ad (b): By Lemma 4.3, since each MCI_{n+1} is an extension of $\mathrm{CI} + \{\exists_{n+1} x At(x)\}$, $(\mathrm{IndI\text{-}b})_\psi$ is provable in each MCI_{n+1}. (b) follows from Theorem 2.1(ii).

Ad (c): This follows from Lemma 4.2(iv). □

Let us now define: $\mathcal{F}x :\longleftrightarrow IFin(x)$. Then, by Lemma 4.1 and Theorem 4.1, we obtain:

Corollary 4.1. (i) Ax-b(FinI) *belongs to* $\mathrm{CI} + \{\mathrm{FUS}_{At}\}$ *extended by this definition.*
(ii) Ax-a(FinI) *belongs to* $\mathrm{CI} + \{\mathrm{FUS}_{At}\}$ *extended by this definition.*

Let me close this section by pointing out two equivalence results for "$\forall x\, IFin(x)$", which, though not necessary for the proof of Corollary 4.1, seem to be illuminating and are also useful for later results.

Lemma 4.4. $\mathrm{CI} \vdash \mathrm{AT} \leftrightarrow \forall x\, IFin(x)$.

Proof. "$\to$" is Lemma 4.2(ii). "$\leftarrow$" By Lemma 4.2(v), we have:

$$\mathrm{CI} \vdash \forall x\, IFin(x) \to \forall y(\forall x\, x \sqsubseteq y \to \exists z(At(z) \wedge z \sqsubseteq y)).$$

So it suffices to show:

$$\mathrm{CI} \vdash \forall x\, IFin(x) \to \forall y(\neg\forall x\, x \sqsubseteq y \to \exists z(At(z) \wedge z \sqsubseteq y)).$$

First, by definition of "$IFin(x)$", $\mathrm{CI} \vdash \forall x\, IFin(x) \to \forall yx(\forall z(At(z) \to z \sqsubseteq y) \to x \sqsubseteq y) \to \forall y(\forall z(At(z) \to z \sqsubseteq y) \to \forall x\, x \sqsubseteq y)$, whence

(∗) $\mathrm{CI} \vdash \forall x\, IFin(x) \to \forall y(\neg\forall x\, x \sqsubseteq y \to \exists z(At(z) \wedge \neg z \sqsubseteq y))$.

Second, because of NEG:

(∗∗) $\mathrm{CI} \vdash \forall x\, IFin(x) \wedge \neg\forall x\, x \sqsubseteq y \to \exists u[\neg\forall x\, x \sqsubseteq u \wedge \forall v(\neg v \sqsubseteq u \leftrightarrow v \circ y)]$.

By "specializing" the universally quantified variable "y" in $(*)$ to "u" from $(**)$, we obtain:

$$\begin{aligned}\mathrm{CI} \vdash \forall x\, \mathit{IFin}(x) \wedge \neg\forall x\, x \sqsubseteq y \rightarrow \exists u[\exists z(\mathit{At}(z) \wedge \neg z \sqsubseteq u) \wedge \forall v(\neg v \sqsubseteq u \leftrightarrow v \circ y)]\\ \rightarrow \exists u[\exists z(\mathit{At}(z) \wedge \neg z \sqsubseteq u \wedge (\neg z \sqsubseteq u \leftrightarrow z \circ y))]\\ \rightarrow \exists z(\mathit{At}(z) \wedge z \circ y)\\ \rightarrow \exists z(\mathit{At}(z) \wedge z \sqsubseteq y)\,,\end{aligned}$$

which is the desired result. □

Lemma 4.5. *$\mathrm{CI} + \{\mathrm{FUS}_{At}, \exists x\mathit{At}(x)\} \vdash \mathit{IFin}(x) \leftrightarrow x \sqsubseteq \bigvee \mathit{At}$, where "$z = \bigvee \mathit{At}$" is an abbreviation of "$\forall y(z \circ y \leftrightarrow \exists x(x \circ y \wedge \mathit{At}(x)))$".*

Proof. By Lemma 4.1(i),

$$\mathrm{CI} + \{\mathrm{FUS}_{At}, \exists x\mathit{At}(x)\} \vdash \mathit{ICl}(y) \leftrightarrow \bigvee \mathit{At} \sqsubseteq y.$$

This implies:

$$\begin{aligned}\mathrm{CI} + \{\mathrm{FUS}_{At}, \exists x\mathit{At}(x)\} \vdash \mathit{IFin}(x) &\leftrightarrow \forall y(\mathit{ICl}(y) \rightarrow x \sqsubseteq y)\\ &\leftrightarrow \forall y(\bigvee \mathit{At} \sqsubseteq y \rightarrow x \sqsubseteq y)\\ &\leftrightarrow x \sqsubseteq \bigvee \mathit{At}\,.\end{aligned}$$ □

4.3. Alternatives

"$\forall y(\mathit{ICl}(y) \rightarrow x \sqsubseteq y)$" is not particularly plausible as a mereological *explicans* of "x is finite". In addition, its adoption has consequences which seem to be unwelcome. In particular, by Lemma 4.2(ii), $\mathrm{ACI}_\omega \vdash \forall x\, \mathit{IFin}(x)$. Thus, in each model $\mathcal{M}$ of ACI_ω, its maximal element 1^M satisfies "$\mathit{IFin}(x)$"; yet 1^M has infinitely many parts^M [see also Niebergall, 2014].[12]

Should it not be possible to find formulas $\alpha(x)$ in $\mathrm{L}[\circ]$ which express more adequately than "$\mathit{IFin}(x)$" that x is finite? I will argue that the answer is "no".

To start with, let me first deal with the case where the axiomatization of finiteness, i.e. Ax-a(FinI), is retained unchanged. Here, any attempt to find a formula $\alpha(x)$ in $\mathrm{L}[\circ]$ (for which these axioms are provable) which is superior to "$\mathit{IFin}(x)$" is futile. For it turns out that such an $\alpha(x)$ is provably equivalent to "$\mathit{IFin}(x)$" (in each mereological theory T). This is a consequence of a general lemma, a *characterization lemma*, which immediately follows from Theorem 4.1:

[12] In general, I use "P^M" for the interpretation of the predicate "P" in $\mathcal{M}$.

LEMMA 4.6. *Let T be a mereological theory and let $\alpha(x)$ be a formula from* L[$\circ$] *such that*:

$T \vdash AxFinI\text{-a}[\alpha]$,
$T \vdash AxFinI\text{-a}[\psi] \to \forall x(\alpha(x) \to \psi(x))$,

for each formula ψ from L[$\circ$]. Then $T \vdash \forall x(\alpha(x) \leftrightarrow IFin(x))$.[13]

Now let us consider the case of a changed axiomatization of finiteness (in L[$\in, \mathcal{F}$]). For example, one could envisage "$\exists x \neg\mathcal{F}x$" as being a further axiom of finiteness. Surely, because of Lemma 3.2, this sentence cannot be consistently added to each consistent mereological theory. But in light of the problem mentioned at the beginning of this subsection, it *should* be true in models of ACI_ω. Thus, let's deal only with the theory $\mathrm{ACI}^*_\omega + \{\exists x\, \neg\mathcal{F}x\}$.

Assume then that there exists a formula $\alpha(x)$ in L[$\circ$] such that, when "$\mathcal{F}x$" is replaced by $\alpha(x)$, the theorems of $\mathrm{ACI}^*_\omega + \{\exists x\, \neg\mathcal{F}x\}$ are turned into L[$\circ$]-sentences which are derivable in ACI_ω. That is:

(i) $\mathrm{ACI}_\omega \vdash AxFinI\text{-a}[\alpha]$,
(ii) $\mathrm{ACI}_\omega \vdash AxFinI\text{-a}[\psi] \to \forall x(\alpha(x) \to \psi(x))$, for each formula ψ from L[$\circ$]), and
(iii) $\mathrm{ACI}_\omega \vdash \exists x\, \neg\alpha(x)$.

(i) and (ii) and Lemma 4.6 yield $\mathrm{ACI}_\omega \vdash \forall x(\alpha(x) \leftrightarrow IFin(x))$. Together with Lemma 4.2(ii), it follows that $\mathrm{ACI}_\omega \vdash \forall x\, \alpha(x)$. Yet this together with (iii) implies that ACI_ω is inconsistent (which is not the case). In sum, there is no formula α as the one which was assumed to exist.

The question whether there may exist better axiomatizations FIN of finiteness (in L[$\in, \mathcal{F}$]), then Ax-a(FinI) is certainly not uninteresting (though I doubt that the example just dealt with is one[14]). It should be kept in mind, however, that our task is not to find such an axiomatization, but to figure out whether there exists a formula $\alpha(x)$ in L[$\circ$] which

[13] A variant of Lemma 4.6 (which is not used in this paper), which is a characterization lemma for the extended language, can also be shown: LEMMA. *Let T be a mereological theory and $\alpha(x)$ be a formula from* L[$\in, \mathcal{F}$] *such that $T^* \vdash AxFinI\text{-a}[\alpha]$ and $T^* \vdash AxFinI\text{-a}[\psi] \to \forall x(\alpha(x) \to \psi(x))$, for each formula ψ from* L[$\in, \mathcal{F}$]. *Then $T \vdash \forall x(\alpha(x) \leftrightarrow \mathcal{F}x)$.*

[14] For example, since we have "$\exists x\, \neg\mathcal{F}x$" as a new axiom of finiteness, it seems to be more natural to adopt an induction schema which is different from (IndI-a); i.e., $\{(\text{IndI-})_\psi \mid \psi$ is a formula from L[$\circ, \mathcal{F}$]$\}$, where $(\text{IndI-})_\psi \equiv AxFinI\text{-}[\psi] \to \forall x(\mathcal{F}x \to \psi(x))$ and (for α in L[$\circ, \mathcal{F}$]) $AxFinI\text{-}[\alpha] \equiv \forall x(At(x) \to \alpha(x)) \wedge \forall xy(\alpha(x) \wedge \alpha(y) \to \alpha(x \sqcup y)) \wedge \exists x\, \neg\alpha(x)$.

more adequately expresses that x is finite than "$IFin(x)$". For this, the axiom system just dealt with is simply of no help.

Coming back to the general case, let T be a consistent mereological theory and let FIN be some set of axioms of finiteness in $\mathrm{L}[\circ, \mathcal{F}]$. For FIN I will only assume — and this I take to be evident — that T + FIN must have (AxI 1) and (AxI 2) as theorems. In addition, let $\alpha(x)$ be a formula in $\mathrm{L}[\circ]$ which is taken into account as an *explicans* of "x is finite" — but which now corresponds to FIN. Then it will still be the case that $T \vdash AxFinI\text{-a}[\alpha]$. Whence, by Corollary 4.1, it follows that:

$$T \vdash \forall x(IFin(x) \to \alpha(x)).$$

Yet this means in particular that in each model $\mathcal{M}$ of ACI_ω, 1^M also satisfies $\alpha(x)$. Thus, even with a changed axiomatization of finiteness, we could not free ourselves from the problem which was used as a motivation for this very change.

In sum, then, whether the axiomatization of finiteness is changed or not, a formula $\alpha(x)$ in $\mathrm{L}[\circ]$ which more adequately expresses that x is finite than "$IFin(x)$" is hardly forthcoming.

5. Assumptions of infinity

5.1. "T makes an assumption of infinity": the framework

In [Niebergall, 2011a, 2014], I considered several possible explications of "T makes an assumption of infinity". Most of them were rejected as inadequate. What had remained in these papers is:[15]

(DIi) T makes an assumption of infinity $:\Longleftrightarrow$ $\forall \mathcal{M}(\mathcal{M} \models T \Longrightarrow M$ is infinite).

(DIiii) T makes an assumption of infinity $:\Longleftrightarrow T \models \exists x\, x$ is infinite.

In the *definiens* of (DIiii), the sentence "$\exists x\, x$ is infinite" has to belong to $\mathrm{L}[T]$. Yet, $\mathrm{L}[T]$ need not contain the predicate "is infinite"; in fact, it most probably fails to do so. Thus, as it stands, (DIiii) is rather questionable. It may be defended by pointing out that "x is infinite" functions as some kind of placeholder in it: for a given theory T, it should be replaced by a specific formula $\alpha(x)$ (with sole free variable x) from $\mathrm{L}[T]$; by a formula $\alpha(x)$, that is, which expresses that x is infinite.

[15] The numbering follows that of [Niebergall, 2014].

Thus, a more complete rendering of (DIiii) is rather an equivalence like this one:

(DIiii+) T makes an assumption of infinity $:\Longleftrightarrow \exists\alpha(\alpha(x)$ expresses that x is infinite and $T \models \exists x\, \alpha(x))$.

The conjunct "$\alpha(x)$ expresses that x is infinite" in the *definiens* of (DIiii+) seems to be indispensable. For if it were not included and if, for example, $\alpha(x)$ expressed that x is a planet, a theory T proving "$\exists x\, \alpha(x)$" would assume planets rather than the infinite.

Whether (DIiii+) is reasonable can, of course, eventually only be decided if an explication can be provided for "$\alpha(x)$ expresses that x is infinite". Moreover, I think that a further modification of this phrase, i.e., its relativization to a theory T (in which $\alpha(x)$ may express that x is finite) is advisable.

Thus, following the procedure elaborated in more detail in [Niebergall, 2011a, 2014], I will first present *explicantia* of

"$\alpha(x)$ expresses that x is finite relative to T"

(where T has to be a theory and $\alpha(x)$ has to be a formula (with x as its sole free variable) in L[T]). These, in turn, are then employed to formulate *explicantia* of "T makes an assumption of infinity" in the style envisaged in (DIiii+).

5.2. "$\alpha(x)$ expresses that x is finite" and "T makes an assumption of infinity": definitions

In this subsection, I recall the precise renderings of (DIiii+) presented in [Niebergall, 2014] which have not shown to be inadequate in this paper.

Let α be a formula from L[$\circ$] and T be a mereological theory. As already hinted at in the introduction, the *explicantia* for "T makes an assumption of infinity" put forward in this section rest on axioms of finiteness. Ax-a(FinI), in particular, leads to these definitions:

$\alpha(x)$ strongly expresses that x is finite relative to T $:\Longleftrightarrow$
$T \vdash AxFinI\text{-a}[\alpha]$ and for each formula ψ in L[$\circ$],
$T \vdash AxFinI\text{-a}[\psi] \rightarrow \forall x(\alpha(x) \rightarrow \psi(x))$;

$\alpha(x)$ very strongly expresses that x is finite relative to T $:\Longleftrightarrow$
$\alpha(x)$ strongly expresses that x is finite relative to T and $T \nvdash \forall x \alpha(x)$.

For a different type of definitions, a further axiom system of finiteness is used:

$$(\mathrm{AxFinI}^-) \qquad \{\ulcorner \forall x(\neg \mathcal{F}x \to \exists^{\geqslant n} y\, y \sqsubseteq x)\urcorner \mid n \in \mathbb{N}\}.$$ [16]

(AxFinI^-) is an obvious necessary, but hardly a sufficient choice for a set of axioms of finiteness: on its own, it seems to be too weak. All the same, it can be usefully employed in *explicantia* of "T makes an assumption of infinity". Thus, let's define:

> $\alpha(x)$ very weakly expresses that x is finite relative to T $:\Longleftrightarrow$ $\forall n(n \in \mathbb{N} \Longrightarrow T \vdash \forall x(\neg\alpha(x) \to \exists^{\geqslant n} y\, y \sqsubseteq x))$;
>
> $\alpha(x)$ weakly expresses that x is finite relative to T $:\Longleftrightarrow$ $\alpha(x)$ very weakly expresses that x is finite relative to T and $T \nvdash \forall x \alpha(x)$.[17]

In these definitions, we have free variables for theories; that is, we have a dependency on theories. In order to obtain *explicantia* for "α expresses that x is finite", it is a natural step to bind these variables. However, it turns out that existential quantifiers are not suitable for this task (see Footnote 18).

$\alpha(x)$ universally very strongly expresses that x is finite $:\Longleftrightarrow$ $\forall T(T$ is a mereological theory $\Longrightarrow$ $\alpha(x)$ very strongly expresses that x is finite relative to $T)$;

$\alpha(x)$ universally strongly expresses that x is finite $:\Longleftrightarrow$ $\forall T(T$ is a mereological theory $\Longrightarrow$ $\alpha(x)$ strongly expresses that x is finite relative to $T)$;

$\alpha(x)$ universally weakly expresses that x is finite $:\Longleftrightarrow$ $\forall T(T$ is a mereological theory $\Longrightarrow$ $\alpha(x)$ weakly expresses that x is finite relative to $T)$;

$\alpha(x)$ universally very weakly expresses that x is finite $:\Longleftrightarrow$ $\forall T(T$ is a mereological theory$\Longrightarrow$ $\alpha(x)$ very weakly expresses that x is finite relative to $T)$.

Finally, these predicates are used in the following *definientia* of possible explications of "T makes an assumption of infinity" (with T in $\mathrm{L}[\circ]$):

T makes a universal very strong assumption of infinity $:\Longleftrightarrow$ $\exists\alpha(\alpha \in \mathrm{L}[\circ]$ $\wedge T \vdash \exists x \neg\alpha(x) \wedge \alpha(x)$ universally very strongly expresses that x is finite);

[16] $\exists^{\geqslant n} y\, y \sqsubseteq x :\longleftrightarrow \exists y_1 \ldots y_n(\bigwedge_{i=1}^{n} y_i \sqsubseteq x \wedge \bigwedge_{\substack{i,j=1\\ i<j}}^{n} y_i \neq y_j))$.

[17] Note that "$x = x$" very weakly expresses that x is finite relative to each mereological theory T. To circumvent this trivialization, "$T \nvdash \forall x\, \alpha(x)$" is added.

T makes a universal strong assumption of infinity $:\Longleftrightarrow \exists\alpha(\alpha \in \mathrm{L}[\circ] \wedge T \vdash \exists x \neg\alpha(x) \wedge \alpha(x)$ universally strongly expresses that x is finite);

T makes a universal weak assumption of infinity $:\Longleftrightarrow \exists\alpha(\alpha \in \mathrm{L}[\circ] \wedge T \vdash \exists x \neg\alpha(x) \wedge \alpha(x)$ universally weakly expresses that x is finite);

T makes a universal very weak assumption of infinity $:\Longleftrightarrow \exists\alpha(\alpha \in \mathrm{L}[\circ] \wedge T \vdash \exists x \neg\alpha(x) \wedge \alpha(x)$ universally very weakly expresses that x is finite);

T makes a strong assumption of infinity $:\Longleftrightarrow \exists\alpha(\alpha \in \mathrm{L}[\circ] \wedge T \vdash \exists x \neg\alpha(x) \wedge \alpha(x)$ strongly expresses that x is finite relative to T);

T makes a weak assumption of infinity $:\Longleftrightarrow \exists\alpha(\alpha \in \mathrm{L}[\circ] \wedge T \vdash \exists x \neg\alpha(x) \wedge \alpha(x)$ weakly expresses that x is finite relative to T).[18]

5.3. Evaluating the definitions

In [Niebergall, 2011a, 2014], I compared the merits of *definientia* such as the above as *explicantia* of "T makes an assumption of infinity" mainly for set theories. Mereological theories were also considered, and it seemed that the picture for them was somewhat simpler than that for set theories. I can now show that it is *much* simpler: due to Corollary 5.2 in particular (the main new result of this part), only two options remain for the mereological setting investigated here: "T makes an assumption of infinity" can be explicated either via (DIi) or by "T makes a strong assumption of infinity".

[18] The variants of these definitions with existential quantifiers in place of universal quantifiers are

T makes an existential very strong assumption of infinity $:\Longleftrightarrow \exists\alpha(\alpha \in \mathrm{L}[\circ] \wedge T \vdash \exists x \neg\alpha(x) \wedge \exists S(S$ is a mereological theory $\wedge\ \alpha(x)$ very strongly expresses that x is finite relative to S)).

T makes an existential strong assumption of infinity $:\Longleftrightarrow \exists\alpha(\alpha \in \mathrm{L}[\circ] \wedge T \vdash \exists x \neg\alpha(x) \wedge \exists S(S$ is a mereological theory $\wedge\ \alpha(x)$ strongly expresses that x is finite relative to S)).

T makes an existential weak assumption of infinity $:\Longleftrightarrow \exists\alpha(\alpha \in \mathrm{L}[\circ] \wedge T \vdash \exists x \neg\alpha(x) \wedge \exists S(S$ is a mereological theory $\wedge\ \alpha(x)$ weakly expresses that x is finite relative to S)).

T makes an existential very weak assumption of infinity $:\Longleftrightarrow \exists\alpha(\alpha \in \mathrm{L}[\circ] \wedge T \vdash \exists x \neg\alpha(x) \wedge \exists S(S$ is a mereological theory $\wedge\ \alpha(x)$ very weakly expresses that x is finite relative to S)).

Now in [Niebergall, 2014, lemmas 6 and 11] it has been shown that each mereological theory makes an assumption of infinity in each of the four ways considered here. In particular, if any of them were accepted as an *explicans* of "T makes an assumption of infinity", even a theory like ACI_1, which only has models with one element, would make an assumption of infinity.

But let me first quote the relevant results from [Niebergall, 2014, lemmas A to D].

LEMMA A. *Let T be a mereological theory. Then:*

(i) *If T makes a universal strong assumption of infinity, then T makes a strong assumption of infinity and T makes a universal very weak assumption of infinity.*

(ii) *If T makes a universal very weak assumption of infinity, then T makes a weak assumption of infinity.*

LEMMA B. (i) *"$IFin(x)$" universally strongly expresses that x is finite.*

(ii) *Equivalent are for mereological theories T: $T \vdash \exists x\, \neg IFin(x)$, T makes a universal strong assumption of infinity, T makes a strong assumption of infinity.*

By Lemma 4.4 and Lemma B we obtain:

COROLLARY 5.1. *Equivalent are for mereological theories T: $T \vdash \neg\mathrm{AT}$, $T \vdash \exists x\, \neg IFin(x)$, T makes a universal strong assumption of infinity, T makes a strong assumption of infinity.*

LEMMA C. (i) *No mereological theory makes a universal very strong assumption of infinity.*

(ii) *No mereological theory makes a universal weak assumption of infinity.*

LEMMA D. *Let T be a mereological theory. Then: T makes a weak assumption of infinity $\Longleftrightarrow$ all models of T are infinite.*

To move forward, examples are helpful:

Example ACI_n. For each $n \geqslant 1$, ACI_n does not make a weak assumption of infinity (by Lemma D).

Example FCI. FCI makes a strong assumption of infinity (by Corollary 5.1).

Example MCI_n. For each $n \geqslant 1$, MCI_n makes a strong assumption of infinity (by Corollary 5.1).

Example $\mathrm{MCI}_\omega + \{\mathrm{FUS}_{At}\}$. $\mathrm{MCI}_\omega + \{\mathrm{FUS}_{At}\}$ makes a strong assumption of infinity (by Corollary 5.1).

All of this is as it should be. It is only ACI_ω which is out of line.

Example ACI_ω. (i) ACI_ω does not make a strong assumption of infinity (by Corollary 5.1).

(ii) ACI_ω makes a weak assumption of infinity (by Lemma D).
(iii) ACI_ω does not make a universal very weak assumption of infinity.

Indeed, assume that ACI_ω makes a universal very weak assumption of infinity, i.e., for some α from $\mathrm{L}[\circ]$, $\mathrm{ACI}_\omega \vdash \exists x \neg\alpha(x)$ and $\forall S(S$ is a mereological theory $\Longrightarrow \forall n(n \in \mathbb{N} \to S \vdash \forall x(\neg\alpha(x) \to \exists^{\geqslant n} y\, y \sqsubseteq x)))$. Then for some $k \in \mathbb{N}$:

$(*)$ $\mathrm{ACI}_k \vdash \exists x \neg\alpha(x)$,

and, specializing S to ACI_k, $\forall n(n \in \mathbb{N} \Longrightarrow \mathrm{ACI}_k \vdash \forall x(\neg\alpha(x) \to \exists^{\geqslant n} y\, y \sqsubseteq x))$, whence:

$(**)$ $\mathrm{ACI}_k \vdash \forall x(\neg\alpha(x) \to \exists^{\geq 2^{k+1}} y\, y \sqsubseteq x)$.

(*) and (**) imply:

$(***)$ $\mathrm{ACI}_k \vdash \exists x \exists^{\geqslant 2^{k+1}} y\, y \sqsubseteq x$.

Now let $\mathcal{M}$ be a model for ACI_k. Then $|M| = 2^k - 1$. But by $(***)$, $|M| \geqslant 2^{k+1}$. Contradiction. □

Making a strong assumption of infinity and making a universal very weak assumption of infinity seem to be conceptually quite distinct. The fact that they are equivalent for maximal consistent mereological theories may seem to be a special feature of the latter. It can, however, been shown that this equivalence holds for all mereological theories.

LEMMA 5.1. *Let T be a mereological theory which makes a weak assumption of infinity. Then: T does not make a strong assumption of infinity $\Longleftrightarrow T \subseteq \mathrm{ACI}_\omega$.*

PROOF. "$\Leftarrow$" If $T \subseteq \mathrm{ACI}_\omega$ and T makes a strong assumption of infinity, then by Corollary 5.1, $T \vdash \neg\mathrm{AT}$; whence $\mathrm{ACI}_\omega \vdash \neg\mathrm{AT}$. But since ACI_ω is consistent, this cannot be the case.

"$\Rightarrow$" Assume that T makes a weak assumption of infinity but fails to make a strong assumption of infinity. Then by Corollary 5.1, $T \nvdash \neg\mathrm{AT}$. Let's consider the (therefore consistent) theory: $T' := T + \{\mathrm{AT}\}$.

Claim. $\mathrm{ACI}_\omega \subseteq T'$.

Proof. Since T is a mereological theory, $\mathrm{ACI} \subseteq T'$. Moreover:

$(*)$ $\forall \mathcal{M}(\mathcal{M} \models T' \Longrightarrow M$ is infinite$)$.

This is the case because, by assumption, T makes a weak assumption of infinity and has therefore only infinite models (by Lemma D). By $(*)$

and since ACI $\subseteq T'$, each model of T' contains infinitely many atoms^M, and the claim follows.

Now ACI_ω is maximal consistent in L[$\circ$]. Therefore, $\mathrm{ACI}_\omega = T'$, whence $T \subseteq \mathrm{ACI}_\omega$. $\square$

COROLLARY 5.2. *Let T be a mereological theory. Then: T makes a strong assumption of infinity $\Longleftrightarrow$ T makes a universal very weak assumption of infinity.*

PROOF. "$\Rightarrow$" It follows from Lemma A and Lemma B(ii). "$\Leftarrow$" Assume that T makes a universal very weak assumption of infinity but fails to make a strong assumption of infinity. Then by Lemma A(ii), T makes a weak assumption of infinity. So, by Lemma 5.1, $T \subseteq \mathrm{ACI}_\omega$. But then, by the assumption that T makes a universal very weak assumption of infinity, ACI_ω has to make a universal very weak assumption of infinity, too. This, however, contradicts (iii) from Example ACI_ω. $\square$

Let me sum up. When for mereological theories T, "T makes an assumption of infinity" is explained by (DIiii+), the potentially acceptable *definientia* are "T makes a universal strong assumption of infinity", "T makes a strong assumption of infinity", "T makes a universal very weak assumption of infinity" and "T makes a weak assumption of infinity". Now, it turned out that "T makes a universal strong assumption of infinity", "T makes a strong assumption of infinity" and "T makes a universal very weak assumption of infinity" are equivalent with each other, each of them being stronger than "T makes a weak assumption of infinity". Moreover, "T makes a weak assumption of infinity" is equivalent to the *definiens* of (DIi). Thus, the above mentioned options eventually lead to only two possible *explicantia* of "T makes an assumption of infinity":

(DIi) T makes an assumption of infinity $:\Longleftrightarrow$
$\forall \mathcal{M}(\mathcal{M} \models T \Longrightarrow M$ is infinite).

(DIiii) T makes an assumption of infinity $:\Longleftrightarrow T \models \neg$AT.

Under (DIi), ACI_ω makes an assumption of infinity; yet under (DIiii), ACI_ω fails to make an assumption of infinity. In [Niebergall, 2014], I have opted for the first alternative: ACI_ω *makes* an assumption of infinity. The findings of the present paper simplify the picture up to the point that only these two *explicantia* have to be taken into account; but I do not think that they provide any reasons against the assessment in [Niebergall, 2014].

6. Theories of finiteness extending mereological theories

Let $\mathcal{C}$ be the class of theories in $\mathrm{L}[\circ, \mathcal{F}]$ which extend a theory of the type T^*, T being a mereological theory. In this (and also in the following) section, theories in $\mathcal{C}$ are studied for their own sake. Of course, the T^*'s deserve to be investigated closer, and they belong to $\mathcal{C}$. There are, however, certain further theories over and above the theories T^* which are both helpful for the understanding of the T^*'s and play a distinguished role on their own. These are the T^+'s, where:

$T^+ :=$ the deductive closure (in $\mathrm{L}[\circ, \mathcal{F}]$) of $T \cup \{\forall x(\mathcal{F}x \leftrightarrow \mathit{IFin}(x))\}$

(if T is a mereological theory).

In what follows, for the theories in $\mathcal{C}$, metatheoretical topics such as maximal consistency, proof theoretic strength and their relation to mereological theories are addressed. For the rest of Section 6, let T be a consistent mereological theory.

6.1. More on T^* and T^+

LEMMA 6.1. (i) *For each formula ψ from $\mathrm{L}[\circ, \mathcal{F}]$ there is a formula $\psi^{-\mathcal{F}}$ from $\mathrm{L}[\circ]$ such that $T^+ \vdash \psi \leftrightarrow \psi^{-\mathcal{F}}$. If ψ is a sentence, then $\psi^{-\mathcal{F}}$ is a sentence.*

(ii) *Let T be maximal consistent in $\mathrm{L}[\circ]$. Then T^+ is maximal consistent in $\mathrm{L}[\circ, \mathcal{F}]$ and decidable.*

(iii) $T^* \subseteq T^+$.

PROOF. (i) If ψ is a formula from $\mathrm{L}[\circ, \mathcal{F}]$, let $\psi^{-\mathcal{F}}$ result from ψ by replacing each occurrence of "$\mathcal{F}x$" in ψ by "$\mathit{IFin}(x)$" (for each variable x). Then $\psi^{-\mathcal{F}}$ is a formula from $\mathrm{L}[\circ]$, ψ and $\psi^{-\mathcal{F}}$ have the same free variables, and it can be shown by induction on the complexity of ψ that $T^+ \vdash \psi \leftrightarrow \psi^{-\mathcal{F}}$.

(ii) As to the maximal consistency, let φ be a sentence from $\mathrm{L}[\circ, \mathcal{F}]$, and assume $T^+ \nvdash \varphi$. Then by (i), $T^+ \nvdash \varphi^{-\mathcal{F}}$, whence $(*)$ $T \nvdash \varphi^{-\mathcal{F}}$.

Note that by (i), $\varphi^{-\mathcal{F}}$ is a sentence from $\mathrm{L}[\circ]$. And since T is assumed to be maximal consistent (for $\mathrm{L}[\circ]$), we obtain from (*), $T \vdash \neg\varphi^{-\mathcal{F}}$. Therefore, $T^+ \vdash \neg\varphi^{-\mathcal{F}}$, and finally by (i) again, $T^+ \vdash \neg\varphi$.

Moreover, since each maximal consistent mereological theory is decidable (by Theorem 2.1(iii)), each T^+, resulting from T through the addition of one axiom, is recursively enumerable. Thus, by its maximal consistency, it is decidable.

(iii) By Lemma 4.1, $T^+ \vdash$ (AxI 1), (AxI 2). In order to show (IndI-a), let ψ be a formula from L$[\circ, \mathcal{F}]$. Then by (i), $\psi^{-\mathcal{F}}$ is from L$[\circ]$. Now because of Corollary 4.1, $T \vdash AxFinI\text{-a}[\psi^{-\mathcal{F}}] \rightarrow \forall x(IFin(x) \rightarrow \psi^{-\mathcal{F}}(x))$. But then by (i), $T^+ \vdash AxFinI\text{-a}[\psi] \rightarrow \forall x(IFin(x) \rightarrow \psi(x))$. □

6.2. Models for L$[\circ, \mathcal{F}]$

Models for L$[\circ, \mathcal{F}]$ are of the form $\langle M, \circ^M, \mathcal{F}^M \rangle$, with $\mathcal{F}^M \subseteq M$. Two special choices of $\mathcal{F}^M$ deserve attention (for models $\langle M, \circ^M \rangle$ of CI):

$\mathcal{E}^M := \{a \mid a \in M \wedge a$ is a finite sum^M of atoms$^M\}$,
$\mathcal{A}^M := \{a \mid a \in M \wedge \langle M, \circ^M \rangle, \beta(x : a) \models IFin(x)\}$.

LEMMA 6.2. (i) *If* $\langle M, \circ^M \rangle \models T$, *then* $\langle M, \circ^M, \mathcal{A}^M \rangle \models T^+$.
(ii) T^+ *is a conservative extension of* T.

PROOF. (i) Follows immediately from the definition of $\mathcal{A}^M$.
(ii) As usual, from (i). □

LEMMA 6.3. (i) *If* $\langle M, \circ^M \rangle \models T$, *then* $\langle M, \circ^M, \mathcal{E}^M \rangle \models T^*$.
(ii) T^* *is a conservative extension of* T.

PROOF. (i) Each atom^M is a finite sum^M of atoms^M; and binary sums^M of finite sums^M of atoms^M are finite sums^M of atoms^M. Moreover, assume the hypotheses of (IndI-a) for ψ and let a satisfy "$\mathcal{F}x$" in $\langle M, \circ^M, \mathcal{E}^M \rangle$. Then each atom^M satisfies ψ and, therefore, each finite sum^M of atoms^M satisfies ψ. But a, being an element of $\mathcal{E}^M$, is such a finite sum^M.
(ii) As usual, from (i). □

LEMMA 6.4. *If* $\langle M, \circ^M \rangle \models CI + \{\mathrm{FUS}_{At}, \exists x\, At(x)\}$, *then*:
(i) $\mathcal{A}^M = \{a \mid a \in M \wedge a$ is a part^M of the sum^M of all atoms$^M\}$,
(ii) $\mathcal{E}^M \subseteq \mathcal{A}^M$.

PROOF. (i) follows from Lemma 4.5. □

LEMMA 6.5. *If* $\langle M, \circ^M, \mathcal{F}^M \rangle \models T^*$, *then*: (i) $\mathcal{E}^M \subseteq \mathcal{F}^M$; (ii) $\mathcal{F}^M \subseteq \mathcal{A}^M$.

PROOF. (i) By Lemma 3.1(ii).
(ii) By Lemma 4.1(ii) and (iii), $T^* \vdash \forall x(\mathcal{F}x \rightarrow IFin(x))$. □

LEMMA 6.6. *Assume* $\forall M \forall \circ^M\, (\langle M, \circ^M \rangle \models T \Longrightarrow \mathcal{E}^M = \mathcal{A}^M)$. *Then*:
(i) $T^+ \subseteq T^*$,
(ii) $T^+ = T^*$.

Proof. (i) Let $\mathcal{B} \models T^*$, with $\mathcal{B} = \langle B, \circ^B, \mathcal{F}^B \rangle$. Then $\langle B, \circ^B \rangle \models T$, and by assumption we have $\mathcal{E}^B = \mathcal{A}^B$. But then, by Lemma 6.5, $\mathcal{F}^B = \mathcal{A}^B$.

Therefore $\mathcal{B} = \langle B, \circ^B, \mathcal{A}^B \rangle$. Moreover, since $\langle B, \circ^B, \mathcal{A}^B \rangle \models T^+$, by Lemma 6.2(i), it follows that $\mathcal{B} \models T^+$.

(ii) By (i) and Lemma 6.1(iii). □

6.3. Extensions of maximal consistent mereological theories: examples

Examples, step 1. (ACI_{n+1}) If $\langle M, \circ^M \rangle \models \mathrm{ACI}_{n+1}$, then $\mathcal{E}^M = \mathcal{A}^M = M$ (for each $n \in \mathbb{N}$).

(FCI) If $\langle M, \circ^M \rangle \models \mathrm{FCI}$, then $\mathcal{E}^M = \mathcal{A}^M = \emptyset$.

(MCI_{n+1}) If $\langle M, \circ^M \rangle \models \mathrm{MCI}_{n+1}$, then $\mathcal{E}^M = \mathcal{A}^M = \{a \mid a \in M \wedge a \text{ is a part}^M \text{ of the sum}^M \text{ of all atoms}^M\} \subset M$ (for each $n \in \mathbb{N}$).

The maximal element of M is not an element of $\mathcal{E}^M$.

(ACI_ω) If $\langle M, \circ^M \rangle \models \mathrm{ACI}_\omega$, then $\mathcal{E}^M \subset \mathcal{A}^M = M$.

The maximal element of M is not an element of $\mathcal{E}^M$.

($\mathrm{MCI}_\omega + \{\mathrm{FUS}_{At}\}$) If $\langle M, \circ^M \rangle \models \mathrm{MCI}_\omega + \{\mathrm{FUS}_{At}\}$, then $\mathcal{E}^M \subset \mathcal{A}^M = \{a \mid a \in M \wedge a \text{ is a part}^M \text{ of the sum}^M \text{ of all atoms}^M\} \subset M$.

The sum^M of all atoms^M is an element of $\mathcal{A}^M$, but not of $\mathcal{E}^M$. The maximal element of M is not an element of $\mathcal{A}^M$.

Examples, step 2. By Lemma 6.6 and the examples, step 1, we obtain:

(ACI_{n+1}) $\mathrm{ACI}^+_{n+1} = \mathrm{ACI}^*_{n+1}$ (for each $n \in \mathbb{N}$).
(FCI) $\mathrm{FCI}^+ = \mathrm{FCI}^*$.
(MCI_{n+1}) $\mathrm{MCI}^+_{n+1} = \mathrm{MCI}^*_{n+1}$ (for each $n \in \mathbb{N}$).

Therefore, by Lemma 6.1, each of theories ACI^*_{n+1}, FCI^*, and MCI^*_{n+1} is maximal consistent (in $\mathrm{L}[\circ, \mathcal{F}]$), finitely axiomatizable and decidable.

Examples, step 3. (ACI_{n+1}) $\mathrm{ACI}^+_{n+1} = \mathrm{ACI}_{n+1} + \{\forall x\, \mathcal{F}x\}$ (for each $n \in \mathbb{N}$).

Proof. Choose a model $\langle M, \circ^M \rangle$ of ACI_{n+1}. Then, by Theorem 2.1(iii) and Lemmas 6.1(ii) and 6.2(i), plus Examples, step 1:

$$\mathrm{ACI}^+_{n+1} = \mathrm{Th}(\langle M, \circ^M, \mathcal{A}^M \rangle) = \mathrm{Th}(\langle M, \circ^M, M \rangle).$$[19]

Since $\langle M, \circ^M, M \rangle \models \forall x\, \mathcal{F}x$, it follows that $\mathrm{ACI}^+_{n+1} \vdash \forall x\, \mathcal{F}x$.

Moreover, $\mathrm{ACI} + \{\forall x \mathcal{F}x\} \vdash \forall x \mathcal{F}x$, and $\mathrm{ACI} + \{\forall x \mathcal{F}x\} \vdash \forall x\, \mathit{IFin}(x)$, by Lemma 4.2(ii). Taken together:

[19] If $\mathcal{B}$ is a structure for language L, $\mathrm{Th}(\mathcal{B}) := \{\psi \mid \psi \text{ is a sentence from L} \wedge \mathcal{B} \models \psi\}$.

$\mathrm{ACI}_{n+1} + \{\forall x\, \mathcal{F}x\} \vdash \forall x(\mathcal{F}x \leftrightarrow \mathit{IFin}(x))$. □

By the same kind of reasoning, it can be shown that:

(FCI) $\mathrm{FCI}^+ = \mathrm{FCI} + \{\forall x\, \neg\mathcal{F}x\}$,
(ACI_ω) $\mathrm{ACI}_\omega^+ = \mathrm{ACI}_\omega + \{\forall x\, \mathcal{F}x\}$.

6.4. Extensions of ACI_ω^*

The results of the previous subsections imply for each consistent mereological theory T: if U is a maximal consistent extension of T^* (in $\mathrm{L}[\circ, \mathcal{F}]$), then $U = \mathrm{FCI}^*$, or $U = \mathrm{ACI}_n^*$, or $U = \mathrm{MCI}_n^*$ (for some $n \geqslant 1$); or U is a maximal consistent extension of ACI_ω^* or $\mathrm{MCI}_\omega^* + \{\mathrm{FUS}_{At}\}$. But which theories are the maximal consistent extensions (in $\mathrm{L}[\circ, \mathcal{F}]$) of these two? ACI_ω^+ and $\mathrm{MCI}_\omega^+ + \{\mathrm{FUS}_{At}\}$ must be among them; but it turns out that there are more.

Let me say a few words about the maximal consistent extensions of ACI_ω^*.

First, as has just been pointed out, $\mathrm{ACI}_\omega^* + \{\forall x\, \mathcal{F}x\}$ is a maximal consistent extension of ACI_ω^* (in $\mathrm{L}[\circ, \mathcal{F}]$).

Second, $\mathrm{ACI}_\omega^* + \{\neg\forall x\, \mathcal{F}x\}$ is consistent, and therefore a proper consistent extension of ACI_ω^*. For consider $\langle \wp(\mathbb{N}) \setminus \{\emptyset\}, \circ^{\wp(\mathbb{N})\setminus\{\emptyset\}} \rangle$, with:

$$A \circ^{\wp(\mathbb{N})\setminus\{\emptyset\}} B :\Longleftrightarrow A \cap B \neq \emptyset \text{ (for } A, B \subseteq \mathbb{N},\ A, B \neq \emptyset).$$

This is a model of ACI_ω, whence $\mathcal{P} := \langle \wp(\mathbb{N})\setminus\{\emptyset\}, \circ^{\wp(\mathbb{N})\setminus\{\emptyset\}}, \mathcal{E}^{\wp(\mathbb{N})\setminus\{\emptyset\}} \rangle \models \mathrm{ACI}_\omega^*$ (by Lemma 6.3(i)). But also $\mathcal{P} \models \neg\forall x\, \mathcal{F}x$ (see Examples, step 1).

Third: $\mathrm{ACI}_\omega^* + \{\neg\forall x\, \mathcal{F}x\}$ is not maximal consistent (in $\mathrm{L}[\circ, \mathcal{F}]$).

For $\mathrm{ACI}_\omega^* + \{\neg\forall x\, \mathcal{F}x\}$ does not decide the sentence:

(DIS) $\forall x(\neg\forall y\, y \sqsubseteq x \to \mathcal{F}x \vee \mathcal{F}(-x))$.

On the one side, (DIS) is false in $\mathcal{P}$ (specialize "x" to the set of even numbers). On the other side, it holds in this structure: $\mathcal{FC} := \langle \mathrm{FC}, \circ^{\mathrm{FC}}, \mathcal{E}^{\mathrm{FC}} \rangle$, with $\mathrm{FC} := \{A \subseteq \mathbb{N} \mid A \text{ is finite} \vee \mathbb{N} \setminus A \text{ is finite}\} \setminus \{\emptyset\}$ and $A \circ^{\mathrm{FC}} B :\Longleftrightarrow A \cap B \neq \emptyset$ (for $A, B \in FC$).

$\langle \mathrm{FC}, \circ^{\mathrm{FC}} \rangle$ is induced by a Boolean algebra, is atomistic and has infinitely many atoms (i.e., the singletons $\{k\}$, with $k \in \mathbb{N}$). Therefore, it is a model of ACI_ω, and $\mathcal{FC} \models \mathrm{ACI}_\omega^*$. Moreover, if $A \in \mathrm{FC}$ satisfies "$\neg\forall y\, y \sqsubseteq x$" (in $\mathcal{FC}$), it is either finite, or it has a finite complement $\mathbb{N} \setminus A$, which is also an element of FC. Now, X ($\in FC$) is finite iff X is a finite union of singletons of natural numbers, which means that X is a finite sum$^{\mathcal{FC}}$ of atoms$^{\mathcal{FC}}$, i.e., that X is an element of $\mathcal{E}^{\mathrm{FC}}$. Therefore, $A \in \mathcal{E}^{\mathrm{FC}}$ or $\mathbb{N} \setminus A \in \mathcal{E}^{\mathrm{FC}}$; which shows that (DIS) holds in $\mathcal{FC}$.

In sum, ACI^*_ω is a proper subtheory of ACI^+_ω; and there are at least three maximal consistent extensions of ACI^*_ω (in $\mathrm{L}[\circ, \mathcal{F}]$): ACI^+_ω, $\mathrm{Th}(\mathcal{P})$ and $\mathrm{Th}(\mathcal{FC})$.

6.5. Relative interpretability

When a language L is extended to a language L^+ by adding new vocabulary, this is often done with the aim of enriching the expressional resources of L: in L^+, it should be possible to express more, to make finer distinctions. Yet, what is a language? Let's deal with $\mathrm{L}[\circ]$ and its extension $\mathrm{L}[\circ, \mathcal{F}]$. If $\mathrm{L}[\circ]$ and $\mathrm{L}[\circ, \mathcal{F}]$ are individuated by their vocabularies (which has been taken for granted in the present paper), it should be plausible that the second language is richer (at least not poorer) than the first one. But now consider two theories: ZF°, which results from ZF by replacing "$\in$" by "$\circ$" and is therefore formulated in $\mathrm{L}[\circ]$; and the set PL_1 of first-order logical truths stated in $\mathrm{L}[\circ, \mathcal{F}]$. Certainly, ZF° is stronger than PL_1 (in whatever way "stronger" may be reasonably understood here). And, using ZF°, it should be possible to express more than by employing merely logical truths. So what happened to the supposed gain in expressional richness when moving from $\mathrm{L}[\circ]$ to $\mathrm{L}[\circ, \mathcal{F}]$?

Inspired by this, admittedly sketchily presented, line of thought let me not deal with $\mathrm{L}[\circ]$ and $\mathrm{L}[\circ, \mathcal{F}]$, but rather with the framework which is given by both $\mathrm{L}[\circ]$ and the mereological theories T, and in addition with the framework supplied by $\mathrm{L}[\circ, \mathcal{F}]$ together with the extensions of mereological theories formulated therein. I suggest to formulate the claim that the second framework constitutes an enrichment of the expressional resources of the first one as follows:

> Among the consistent extensions of mereological theories in $\mathrm{L}[\circ, \mathcal{F}]$, there is at least one T such that for all consistent mereological theories S, T is not reducible to S.

If this cannot be attained, everything that can be done in such an extension of a mereological theory can be simulated in a mereological theory itself. The extension to the new framework which rests on $\mathrm{L}[\circ, \mathcal{F}]$ becomes superfluous.

In my opinion, the best precise *explicans* for "(a theory) S is reducible to (a theory) T" is "S is relatively interpretable to T" (or something close to it; reasons are given Bonevac [1982] and Niebergall [2000]). In what follows, let me therefore investigate this:

> *Conjecture.* Among the consistent extensions of mereological theories in $\mathrm{L}[\circ, \mathcal{F}]$, there is at least one T such that for all consistent mereological theories S, T is not relatively interpretable in S.

Given what has been shown in the present paper, it is clear that for each consistent mereological theory T, T^+ is a subtheory of a definitional extension of T (employing the definition "$\mathcal{F}x :\longleftrightarrow IFin(x)$"). Thus, by Lemma 6.1(iii) and well known results on relative interpretability, T, T^* and T^+ are relatively interpretable in each other. Here, we find no enrichment or strengthening of the original framework; the theories T^* and T^+ deliver no witnesses for the conjecture.

The conjecture is nonetheless true. An example is provided by $\mathrm{ACI}^*_\omega + \{\neg\forall x\, \mathcal{F}x\}$.

First, by [Niebergall, 2009a, Theorem 5], $\mathrm{ACI}_\omega \not\preceq \mathrm{ACI}_k$, $\mathrm{ACI}_\omega \not\preceq \mathrm{MCI}_k$ (for $k \geqslant 1$), $\mathrm{ACI}_\omega \not\preceq \mathrm{FCI}$; whence by what has just been remarked $\mathrm{ACI}^*_\omega + \{\neg\forall x\, \mathcal{F}x\} \not\preceq \mathrm{ACI}_k$, $\mathrm{ACI}^*_\omega + \{\neg\forall x\, \mathcal{F}x\} \not\preceq \mathrm{MCI}_k$ (for $k \geqslant 1$), $\mathrm{ACI}^*_\omega + \{\neg\forall x\, \mathcal{F}x\} \not\preceq \mathrm{FCI}$.

Second, only the relation between $\mathrm{ACI}^*_\omega + \{\neg\forall x\, \mathcal{F}x\}$ and ACI_ω or $\mathrm{MCI}_\omega + \{\mathrm{FUS}_{At}\}$ needs to be discussed. Here, a lemma about $\mathrm{ACI}_\omega + \{\neg\forall x\, \mathcal{F}x, \text{(AxI 1)}, \text{(AxI 2)}, \text{(AxI 3)}\}$ is helpful.

LEMMA 6.7. (i) *For each* $n \in \mathbb{N}$,
$\mathrm{ACI} + \{\neg\forall x\, \mathcal{F}x, \text{(AxI 1)}, \text{(AxI 2)}, \text{(AxI 3)}\} \vdash \exists^{>n} At$.
(ii) $\mathrm{ACI}_\omega + \{\neg\forall x\, \mathcal{F}x, \text{(AxI 1)}, \text{(AxI 2)}, \text{(AxI 3)}\} = \mathrm{ACI} + \{\neg\forall x\, \mathcal{F}x, \text{(AxI 1)}, \text{(AxI 2)}, \text{(AxI 3)}\}$.
(iii) $\mathrm{ACI}_\omega + \{\neg\forall x\, \mathcal{F}x, \text{(AxI 1)}, \text{(AxI 2)}, \text{(AxI 3)}\}$ *is finitely axiomatizable.*[20]

PROOF. (i) follows from Lemma 3.2. (ii) by (i). (iii) by (ii). □

COROLLARY 6.1. (i) $\mathrm{ACI}^*_\omega + \{\neg\forall x\, \mathcal{F}x\} \not\preceq \mathrm{ACI}_\omega$.
(ii) $\mathrm{ACI}^*_\omega + \{\neg\forall x\, \mathcal{F}x\} \not\preceq \mathrm{MCI}_\omega + \mathrm{FUS}_{At}$.

[20] ACI^*_ω and ACI^+_ω, and also $\mathrm{MCI}^*_\omega + \mathrm{FUS}_{At}$ and $\mathrm{MCI}^+_\omega + \mathrm{FUS}_{At}$, are not finitely axiomatizable.

If, for example, ACI^*_ω or ACI^+_ω were finitely axiomatizable, each of them would be a subtheory of some ACI^+_{n+1}. Since this theory is relatively interpretable in ACI_{n+1}, ACI^*_ω, and ACI^+_ω would eventually be relatively interpretable in ACI_{n+1} — which is not the case.

Proof. (ii)[21] Assume $\mathrm{ACI}^*_\omega + \{\neg\forall x \mathcal{F}x\} \preceq \mathrm{MCI}_\omega + \mathrm{FUS}_{At}$. Then $\mathrm{ACI}_\omega +$ $\{\neg\forall x \mathcal{F}x, (\mathrm{AxI}\ 1), (\mathrm{AxI}\ 2), (\mathrm{AxI}\ 3)\} \preceq \mathrm{MCI}_\omega + \mathrm{FUS}_{At}$, and there is a $k \in \mathbb{N}$ such that $\mathrm{ACI}_\omega + \{\neg\forall x \mathcal{F}x\} \preceq \mathrm{MCI}_k + \mathrm{FUS}_{At}$, since by Lemma 6.7(iii), $\mathrm{ACI}_\omega + \{\neg\forall x\, \mathcal{F}x, (\mathrm{AxI}\ 1), (\mathrm{AxI}\ 2), (\mathrm{AxI}\ 3)\}$ is finitely axiomatizable.

By Theorem 2.1(i), $\mathrm{MCI}_k + \mathrm{FUS}_{At} = \mathrm{MCI}_k$, and therefore $\mathrm{ACI}_\omega +$ $\{\neg\forall x\, \mathcal{F}x\} \preceq \mathrm{MCI}_k$. □

Yet this contradicts *first* from above.

Corollary 6.2. *There is no consistent mereological theory in which* $\mathrm{ACI}^*_\omega + \{\neg\forall x\, \mathcal{F}x\}$ *is relatively interpretable.*

Proof. If T were a consistent mereological theory such that $\mathrm{ACI}^*_\omega +$ $\{\neg\forall x\, \mathcal{F}x\} \preceq T$, then $\mathrm{ACI}^*_\omega + \{\neg\forall x\, \mathcal{F}x\}$ would be relatively interpretable in a maximal consistent extension of T (in $\mathrm{L}[\circ]$). Yet this is excluded by *first* and *second* (i.e., Corollary 6.1) in conjunction with Theorem 2.1(iii). □

7. A second-order treatment of "x is finite"

Since the set-theoretic *definiens* of "x is finite" mentioned in Section 4 uses only two levels of the von Neumann hierarchy, it should be possible to formulate it in a second-order language. In this section, I will give such a definition in $\mathrm{L}^2[\circ]$, the monadic second-order extension of $\mathrm{L}[\circ]$, and address some of its consequences.[22]

7.1. The definition of "x is finite" in $\mathrm{L}^2[\circ]$

$\mathrm{L}^2[\circ]$ is the second-order language which results from $\mathrm{L}[\circ]$ through the addition of *one-place* second-order variables ("X", ...); in particular, $\mathrm{L}^2[\circ]$ and $\mathrm{L}[\circ]$ have the same vocabulary. In $\mathrm{L}^2[\circ]$, we have classical second-order logic: roughly put, the axioms and rules known from common axiomatizations of first-order logic (with *modus ponens* and generalization as rules of inference) are transferred from first-order to second-order variables. In particular, with these, "$\Sigma \vdash_2 \psi$" is defined for formulas ψ and sets of formulas Σ from $\mathrm{L}^2[\circ]$ along the common lines.[23]

[21] The proof of (i) is similar to the proof of (ii), but easier.

[22] At several places, this section builts on [Niebergall, 2008b].

[23] For explicitnes, one may consider Shapiro [1991], for example.

Sometimes, instances of the *comprehension schema*, that is, $L^2[\circ]$-formulas of the form:

$$\exists X \forall z(Xz \longleftrightarrow \psi(z))$$

(or their universal closures), where ψ is from $L^2[\circ]$ and X does not occur in ψ, are also regarded as logical truths of second-order logic. They will be included in most of the theories (in $L^2[\circ]$) considered in this section.

Relative to extensions of CI,[24] "x is finite" can now be defined as follows in $L^2[\circ]$:

DEFINITION 7.1. $\mathit{Fin}^2(x) :\longleftrightarrow \forall Y(\forall z(\mathit{At}(z) \to Yz) \wedge \forall zz'(Yz \wedge Yz' \to Y(z \sqcup z')) \to Yx)$.

7.2. Metatheorems and adequacy results

In my opinion, Definition 7.1 is from an intuitive perspective *the* natural definition in $L^2[\circ]$ of "x is finite". The same can certainly not be claimed for the definition that can be found in [Lewis, 1991], which amounts to:

$$\mathit{Fin}^L(x) :\longleftrightarrow \neg\exists X(\exists y\, Xy \wedge x = \bigvee X \wedge \forall y(Xy \to \exists z(Xz \wedge y \sqsubseteq z \wedge \neg z \sqsubseteq y)))\,.$$[25]

In addition, Definition 7.1 has some salient consequences of a partly formal nature, which also suggest its adequacy.

First, as was to be expected, it straightforwardly yields the axioms of finiteness from Section 3. Actually, given Definition 7.1, they follow from quite weak axioms stated in $L^2[\circ]$. More explicitly, let ψ be a formula in $L^2[\circ]$ which does not contain the variable X; set:

$\mathrm{Comp}_\psi :=$ (the universal closure of) $\exists X \forall z(Xz \leftrightarrow \psi(z))$,

$\mathrm{Comp} := \{\mathrm{Comp}_\psi \mid \psi$ is an X-free $L^2[\circ]$-formula$\}$.

Then we have:

LEMMA 7.1. *(i)* $\mathrm{CI} \vdash_2 \forall x(\mathit{At}(x) \to \mathit{Fin}^2(x))$;

(ii) $\mathrm{CI} \vdash_2 \forall xy(\mathit{Fin}^2(x) \wedge \mathit{Fin}^2(y) \to \mathit{Fin}^2(x \sqcup y))$;

(iii) $\mathrm{CI} \cup \mathrm{Comp} \vdash_2 \forall z(\mathit{At}(z) \to \psi(z)) \wedge \forall zz'(\psi(z) \wedge \psi(z') \to \psi(z \sqcup z')) \to \forall x(\mathit{Fin}^2(x) \to \psi(x))$ *(for each formula ψ from* $L^2[\circ]$*)*.

PROOF. (i) Purely logically, $\mathrm{CI} \vdash_2 \mathit{At}(x) \to (\forall z(\mathit{At}(z) \to Yz) \wedge \forall zz'(Yz \wedge Yz' \to Y(z \sqcup z')) \to Yx)$.

[24] The notation used in Definition 7.1 presupposes SUM.

[25] With $x = \bigvee X :\longleftrightarrow \forall y(y \circ x \leftrightarrow \exists z(Xz \wedge y \circ z))$.

Universal generalization with respect to "Y" and distribution yield $\mathrm{CI} \vdash_2 At(x) \to \forall Y(\forall z(At(z) \to Yz) \wedge \forall zz'(Yz \wedge Yz' \to Y(z \sqcup z')) \to Yx)$.

(ii) $\mathrm{CI} \vdash_2 \forall Y(\forall z(At(z) \to Yz) \wedge \forall zz'(Yz \wedge Yz' \to Y(z \sqcup z')) \to Yx) \wedge \forall Y(\forall z(At(z) \to Yz) \wedge \forall zz'(Yz \wedge Yz' \to Y(z \sqcup z')) \to Yy) \wedge \forall z(At(z) \to Yz) \wedge \forall zz'(Yz \wedge Yz' \to Y(z \sqcup z')) \to Yx \wedge Yy \wedge \forall zz'(Yz \wedge Yz' \to Y(z \sqcup z')) \to Y(x \sqcup y)$. Whence $\mathrm{CI} \vdash_2 Fin^2(x) \wedge Fin^2(y) \to (\forall z(At(z) \to Yz) \wedge \forall zz'(Yz \wedge Yz' \to Y(z \sqcup z')) \to Y(x \sqcup y))$. Therefore $\mathrm{CI} \vdash_2 Fin^2(x) \wedge Fin^2(y) \to Fin^2(x \sqcup y)$.

(iii) Let $\psi(u)$ be a formula from $\mathrm{L}^2[\circ]$. By comprehension with respect to ψ we have: $\mathrm{CI} \cup \mathrm{Comp} \vdash_2 Fin^2(x) \to \forall Y(\forall z(At(z) \to Yz) \wedge \forall zz'(Yz \wedge Yz' \to Y(z \sqcup z')) \to Yx) \wedge \exists Y \forall u(Yu \leftrightarrow \psi(u)) \to \exists Y\, [\forall u(Yu \leftrightarrow \psi(u)) \wedge (\forall z(At(z) \to Yz) \wedge \forall zz'(Yz \wedge Yz' \to Y(z \sqcup z')) \to Yx)]$. Whence $\mathrm{CI} \cup \mathrm{Comp} \vdash_2 Fin^2(x) \to (\forall z(At(z) \to \psi(z)) \wedge \forall zz'(\psi(z) \wedge \psi(z') \to \psi(z \sqcup z')) \to \psi(x))$. Therefore $\mathrm{CI} \cup \mathrm{Comp} \vdash_2 \forall z(At(z) \to \psi(z)) \wedge \forall zz'(\psi(z) \wedge \psi(z') \to \psi(z \sqcup z')) \to \forall x(Fin^2(x) \to \psi(x))$. $\square$

Second, there is a model theoretic adequacy result for Definition 7.1. Since it is dependent on the distinction between standard second-order models and generalized second-order models, let me quickly recapitulate that distinction. Thus, let ψ be a formula from $\mathrm{L}^2[\circ]$: and let's discuss how to define "variable assignment β satisfies ψ in structure $\mathcal{M}$". Similarly to the first-order case, such a structure has to provide domains of the variables of $\mathrm{L}^2[\circ]$ and an interpretation function $\mathcal{I}$. More explicitly, there has to be a nonempty set M for the first-order variables, a nonempty set Ω for the second-order variables, and we must have $\mathcal{I}(\circ) \subseteq M^2$. In addition, since there are only monadic second-order variables in $\mathrm{L}^2[\circ]$, Ω should be a subset of $\wp(M)$.

Accordingly, one can define *generalized second-order structures* (in short: g2-structures) as follows:

$$x \text{ is a g2-structure } :\Longleftrightarrow$$
$$\exists M \Omega \mathcal{I}(x = \langle M, \Omega, \mathcal{I}\rangle \ \wedge\ M \neq \emptyset \ \wedge\ \emptyset \neq \Omega \subseteq \wp(M) \ \wedge\ \mathcal{I}(\circ) \subseteq M^2).$$

As regards variable assignments β, I assume that they have both first- and second-order variables in their domains, mapping the second-order variables to elements of Ω. Given that, satisfaction in a g2-structure is defined as in the first-order case, with the following clauses added:

$$\langle M, \Omega, \mathcal{I}\rangle, \beta \models Xy \iff \beta(y) \in \beta(X)\,,$$
$$\langle M, \Omega, \mathcal{I}\rangle, \beta \models \forall X\psi \iff \forall C(C \in \Omega \Rightarrow \langle M, \Omega, \mathcal{I}\rangle, \beta(X:C) \models \psi)\,.$$

A special case, which is sometimes regarded as particularly natural and important, obtains when $\Omega = \wp(M)$. Here, we have the so-called "standard second-order structures" (in short: s2-structures):

$$x \text{ is a s2-structure} :\Longleftrightarrow \exists M\mathcal{I}(x = \langle M, \wp(M), \mathcal{I}\rangle \wedge M \neq \emptyset \wedge \mathcal{I}(\circ) \subseteq M^2)\,.$$

Since each s2-structure is a g2-structure, satisfaction in a s2-structure is already defined. Let me merely reformulate the quantifier-case:

$$\langle M, \wp(M), \mathcal{I}\rangle, \beta \models \forall X\psi \iff \forall C(C \subseteq M \Longrightarrow \langle M, \wp(M), \mathcal{I}\rangle, \beta(X : C) \models \psi)\,.$$

Consequence with respect to g2-structures and with respect to s2-structures is then defined as follows (for $\mathrm{L}^2[\circ]$-sentences ψ and sets Σ of $\mathrm{L}^2[\circ]$-sentences):

$$\Sigma \models^{\mathrm{g2}} \psi :\Longleftrightarrow \forall\mathcal{M}(\mathcal{M} \text{ is a } g2\text{-structure} \Longrightarrow (\mathcal{M} \models \Sigma \Longrightarrow \mathcal{M} \models \psi))\,,$$
$$\Sigma \models^{\mathrm{s2}} \psi :\Longleftrightarrow \forall\mathcal{M}(\mathcal{M} \text{ is a } s2\text{-structure} \Longrightarrow (\mathcal{M} \models \Sigma \Longrightarrow \mathcal{M} \models \psi))\,.$$ [26]

What can be shown now is that for each s2-structure $\mathcal{M}$ which is a model of CI, a is in the extension of "Fin^2" in $\mathcal{M}$ iff a has finitely many parts^M. That is

LEMMA 7.2. *Let $\mathcal{M}$ $(= \langle \mathcal{M}, \Omega, \circ^{\mathcal{M}}\rangle)$ be a g2-structure for $\mathrm{L}^2[\circ]$ which satisfies* $\mathrm{Ax(CI)}$*. Then (if β is an assignment over M):*

$$\beta(x) \text{ is the sum}^M \text{ of finitely many atoms}^M \Longrightarrow \mathcal{M}, \beta \models Fin^2(x).$$

PROOF. By assumption, there are $a_1, \ldots, a_k \in M$ such that:

$(*)$ $a_1, \ldots, a_k$ are atoms^M $\wedge\, \beta(x) = \bigvee^M\{a_1, \ldots, a_k\}$.

Now let $C \in \Omega$ be arbitrary, and assume:

$(**)$ $\mathcal{M}, \beta(Y : C) \models \forall z(At(z) \to Yz)$,
$(***)$ $\mathcal{M}, \beta(Y : C) \models \forall zz'(Yz \wedge Yz' \to Y(z \sqcup z'))$.

By $(*)$ and $(**)$: $\mathcal{M}, \beta(Y : C)(z_1 : a_1) \models Yz_1, \ldots, \mathcal{M}, \beta(Y : C)(z_k : a_k) \models Yz_k$. And these imply by repeated use of $(***)$: $\mathcal{M}, \beta(Y : C)(z_1 : a_1)\ldots(z_k : a_k) \models Y(z_1 \sqcup \ldots \sqcup z_k)$. Whence $\bigvee^M\{a_1, \ldots, a_k\} \in C$ and, by $(*)$, $\mathcal{M}, \beta(Y : C) \models Yx$. □

[26] We have the completeness theorem: $\Sigma \vdash_2 \psi \iff \Sigma \models^{\mathrm{g2}} \psi$.

LEMMA 7.3. *Let* $\mathcal{M}$ $(= \langle \mathcal{M}, \wp(\mathcal{M}), \circ^{\mathcal{M}} \rangle)$ *be a s2-structure for* $\mathrm{L}^2[\circ]$ *which satisfies Ax(CI). Then (if* β *is an assignment over* M*):*[27]

$$\mathcal{M}, \beta \models Fin^2(x) \implies \beta(x) \text{ is the sum}^M \text{ of finitely many atoms}^M.$$

PROOF. If $\mathcal{M}, \beta \models Fin^2(x)$, then $\forall C(C \subseteq M \implies \mathcal{M}, \beta(Y : C) \models \forall z(At(z) \to Yz) \wedge \forall zz'(Yz \wedge Yz' \to Y(z \sqcup z')) \to Yx)$.

Now specialize "C" to B:

$$B := \{b \in M \mid \exists a_1 ... a_k(a_1, ..., a_k \text{ are atoms}^M \wedge b = \bigvee^M \{a_1, ..., a_k\})\}.$$

Then we obtain $(*)$ $\mathcal{M}, \beta(Y : B) \models \forall z(At(z) \to Yz) \wedge \forall zz'(Yz \wedge Yz' \to Y(z \sqcup z')) \to Yx$. Moreover, since obviously $(**)$ $\mathcal{M}, \beta(Y : B) \models \forall z(At(z) \to Yz)$ and $\mathcal{M}, \beta(Y : B) \models \forall zz'(Yz \wedge Yz' \to Y(z \sqcup z'))$, it follows with $(*)$ and $(**)$ that $\mathcal{M}, \beta(Y : B) \models Yx$. That is, $\beta(x)$ is the sum^M of finitely many atoms^M. □

COROLLARY 7.1. (i) *Let* $\mathcal{M}$ $(= \langle \mathcal{M}, \Omega, \circ^{\mathcal{M}} \rangle)$ *be a g2-structure for* $\mathrm{L}^2[\circ]$ *which satisfies* Ax(CI), *and let* β *be an assignment over* M*. Then:* $\{a \in M \mid a \sqsubseteq^M \beta(x)\}$ *is finite* $\implies \mathcal{M}, \beta \models Fin^2(x)$.

(ii) *Let* $\mathcal{M}$ $(= \langle \mathcal{M}, \wp(\mathcal{M}), \circ^{\mathcal{M}} \rangle)$ *be a s2-structure for* $\mathrm{L}^2[\circ]$ *which satisfies* Ax(CI), *and let* β *be an assignment over* M*. Then:* $\mathcal{M}, \beta \models Fin^2(x) \implies \{a \in M \mid a \sqsubseteq^M \beta(x)\}$ *is finite.*

PROOF. It can be shown that: $\{a \in M \mid a \sqsubseteq^M \beta(x)\}$ is finite iff $\beta(x)$ is the sum^M of finitely many atoms^M. □

Let's compare now Definition 7.1 with the one given in Lewis 1991 with respect to Lemmas 7.2 and 7.3, and Corollary 7.1.

First, analogues of these lemmas, can be shown for "$Fin^L(x)$" [see Niebergall, 2008b, lemmas 23 and 24]. Yet, Lemma 24, in particular, is: *Let* $\mathcal{M}$ $(= \langle \mathcal{M}, \wp(\mathcal{M}), \circ^{\mathcal{M}} \rangle)$ *be a s2-structure for* $\mathrm{L}^2[\circ]$ *which satisfies* Ax(ACI) *and* FUS-Ax, *and let* β *be an assignment over* M*. Then:* $\mathcal{M}, \beta \models Fin^L(x) \implies \{a \in M \mid a \sqsubseteq^M \beta(x)\}$ *is finite.*

Thus, we have additional assumptions here when compared with Lemma 7.3 or Corollary 7.1: there are "fewer" s2-structures in which "$Fin^L(x)$" "expresses" finiteness.

Now, I see a decent chance to prove a version of Lemma 24 where "Ax(ACI)" is replaced by "Ax(CI)". But I have no idea how one could get rid of the assumption that FUS-Ax has to hold in $\mathcal{M}$.

---

[27] In this case, we also have: $\mathcal{M}, \beta \models Fin^2(x) \implies \mathcal{M}, \beta \models \exists x\, At(x)$.

FUS-Ax is the *second-order fusion axiom*:

$$\forall X(\exists x\, Xx \to \exists z \forall y(z \circ y \leftrightarrow \exists x(x \circ y \wedge Xx))).$$

As a mereological principle, I take it to be as plausible as the first-order schema FUS. Therefore, for s2-structures, "$Fin^L(x)$" may turn out to be almost as good as "$Fin^2(x)$". But what is the relation between "$Fin^L(x)$" and "$Fin^2(x)$" in general, i.e., with respect to arbitrary g2-structures?

It is of importance here that an analogue of Lemma 7.1, with "$Fin^L(x)$" replacing "$Fin^2(x)$", can be established:

LEMMA 7.4. (i) $\mathrm{ACI} \cup \mathrm{Comp} \cup \{\text{FUS-Ax}\} \vdash_2 \forall x(At(x) \to Fin^L(x))$;
(ii) $\mathrm{ACI} \cup \mathrm{Comp} \cup \{\text{FUS-Ax}\} \vdash_2 \forall xy(Fin^L(x) \wedge Fin^L(y) \to Fin^L(x \sqcup y))$;
(iii) $\mathrm{ACI} \cup \mathrm{Comp} \cup \{\text{FUS-Ax}\} \vdash_2 \forall z(At(z) \to \psi(z)) \wedge \forall zz'(\psi(z) \wedge \psi(z') \to \psi(z \sqcup z')) \to \forall x(Fin^L(x) \to \psi(x))$ *(for each formula* ψ *from* $\mathrm{L}^2[\circ]$*)*.

This lemma (which is stated here without proof) has been obtained by Werner (unplublished). Let me remark that Werner's proof is much more complicated than the one for Lemma 7.1. Moreover, it is not easy to see whether the use of FUS-Ax could be eliminated from it. In this sense, Definition 7.1 is again superior to the definition from [Lewis, 1991], at least from a practical point of view. Yet on a more abstract level, there is not much to choose between Definition 7.1 for "x is finite" and the one found by Lewis. For as a direct consequence of lemmas 7.1 and 7.4 (cf. Lemma 4.6, the characterization lemma), one obtains

COROLLARY 7.2. $\mathrm{ACI} \cup \mathrm{Comp} \cup \{\text{FUS-Ax}\} \vdash_2 \forall x(Fin^L(x) \leftrightarrow Fin^2(x))$.

7.3. Connection with extensions of $\mathbf{ACI}^*_\omega$

In this subsection, results about extensions of mereological theories in $\mathrm{L}^2[\circ]$ are used as a means to shed some light on the maximal consistent extensions of ACI_ω in $\mathrm{L}[\circ, \mathcal{F}]$.

Given earlier remarks in this section, I take it that analogues of mereological theories formulated in $\mathrm{L}^2[\circ]$ should contain mereological theories, Comp and FUS-Ax. Moreover, they should be "closed"; that is, closed under g2-consequence or closed under s2-consequence. Only s2-closure is relevant for the present paper; it is defined as follows (if Σ is a set of $\mathrm{L}^2[\circ]$-sentences; "Comp" need not be added, because $\mathrm{Comp} \subseteq \Sigma^{s2}$ anyway):

$$\Sigma^{s2} := \{\psi \mid \psi \text{ is a sentence from } \mathrm{L}^2[\circ] \wedge \Sigma \cup \mathrm{CI} \cup \{\text{FUS-Ax}\} \models^{s2} \psi\}.$$

In addition, I assume that Definition 7.1 is in force.

Thus, let us deal with supersets of $((\mathrm{ACI}_\omega)^{s2})^D)$, where $((\mathrm{ACI}_\omega)^{s2})^D)$ being $(\mathrm{ACI}_\omega)^{s2}$, extended by "$\forall x(\mathcal{F}x \leftrightarrow \mathit{Fin}^2(x))$". By Lemma 7.1, $\mathrm{ACI}^*_\omega \subseteq ((\mathrm{ACI}_\omega)^{s2})^D)$. But also, "$\neg\forall x\, \mathcal{F}x$" belongs to $((\mathrm{ACI}_\omega)^{s2})^D)$. From this point of view, ACI^+_ω would be excluded as a natural maximal consistent extension of ACI^*_ω. Given Section 6.4, let's therefore deal only with $\mathrm{Th}(\mathcal{P})$ and $\mathrm{Th}(\mathcal{FC})$.

I start with some general lemmas.

LEMMA 7.5. *Let* $\mathcal{M}$ $(= \langle \mathcal{M}, \wp(\mathcal{M}), \circ^{\mathcal{M}}\rangle)$ *be a s2-structure for* $\mathrm{L}^2[\circ]$ *which satisfies* Ax(CI). *Then*:

(i) $\mathcal{M}, \beta(x : a) \models \mathit{Fin}^2(x) \iff a \in \mathcal{E}^M$.
(ii) $\langle M, \wp(M), \circ^M, \mathcal{E}^M\rangle \models \forall x(\mathcal{F}x \leftrightarrow \mathit{Fin}^2(x))$.[28]

PROOF. (i) By lemmas 7.2 and 7.3. (ii) For every $a \in M$, by (i):

$$\begin{aligned}\langle M, \wp(M), \circ^M, \mathcal{E}^M\rangle, \beta(x : a) \models \mathcal{F}x \iff\ & a \in \mathcal{E}^M \\ \iff\ & \langle M, \wp(M), \circ^M\rangle, \beta(x : a) \models \mathit{Fin}^2(x) \\ \iff\ & \langle M, \wp(M), \circ^M, \mathcal{E}^M\rangle, \beta(x : a) \models \mathit{Fin}^2(x). \square\end{aligned}$$

If ψ is a formula from $\mathrm{L}[\circ, \mathcal{F}]$, then let $\psi^{-2\mathcal{F}}$ result from ψ by replacing each occurrence of "$\mathcal{F}x$" in ψ by "$\mathit{Fin}^2(x)$" (for each variable x).

LEMMA 7.6. (i) $\psi^{-2\mathcal{F}}$ *is a formula from* $\mathrm{L}^2[\circ]$. *If* ψ *is a sentence, then* $\psi^{-2\mathcal{F}}$ *is a sentence.*
(ii) *If* ψ *is a formula from* $\mathrm{L}[\circ, \mathcal{F}]$, $\langle M, \wp(M), \circ^M, \mathcal{E}^M\rangle, \beta \models \psi \leftrightarrow \psi^{-2\mathcal{F}}$.
(iii) *If* ψ *is a sentence from* $\mathrm{L}[\circ, \mathcal{F}]$, *then* $\langle M, \circ^M, \mathcal{E}^M\rangle \models \psi \iff \langle M, \wp(M), \circ^M\rangle \models \psi^{-2\mathcal{F}}$.

PROOF. (ii) By Lemma 7.5(ii) and induction on the built-up of ψ.
(iii) By (i) and (ii). $\square$

Let $\mathrm{Th}^2(\langle M, \Omega, \circ^M\rangle)$ be the set of sentences from $\mathrm{L}^2[\circ]$ which hold in $\langle M, \Omega, \circ^M\rangle$. Then we have as a consequence of Lemma 7.6(iii):

COROLLARY 7.3. $\mathrm{Th}^2(\langle \wp(\mathbb{N}) \setminus \{\emptyset\}, \wp(\wp(\mathbb{N}) \setminus \{\emptyset\}), \circ^{\wp(\mathbb{N})\setminus\{\emptyset\}}\rangle)$ *extended by the definition "*$\mathcal{F}x :\longleftrightarrow \mathit{Fin}^2(x)$*" contains the same sentences from* $\mathrm{L}[\circ, \mathcal{F}]$ *as* $\mathrm{Th}(\mathcal{P})$.

[28] The semantical vocabulary introduced for $\mathrm{L}^2[\circ]$ is supposed to be explained similarly for $\mathrm{L}^2[\circ, \mathcal{F}]$.

Consider the $L^2[\circ]$-sentence:

(CountAt) $\quad \forall X x(x = \bigvee X \wedge \forall y(Xy \to At(y)) \wedge \neg Fin^2(x) \to large(x)),$

with:

$$\begin{aligned} large(x) :\longleftrightarrow \exists X[&\exists y\, Xy \wedge \forall yz(Xy \wedge Xz \wedge y \circ z \to y = z) \wedge \\ &\exists y(y = \bigvee X \wedge \forall z\, z \sqsubseteq y) \wedge \\ &\forall y(Xy \to \exists_1 z(At(z) \wedge z \sqsubseteq y \wedge z \sqsubseteq x) \wedge \exists^{\leqslant 2} z(At(z) \wedge z \sqsubseteq y))]. \end{aligned}$$

In [Niebergall, 2008b, Theorem 26] it is shown that:

$$\mathrm{Th}^2(\langle \wp(\mathbb{N}) \setminus \{\emptyset\}, \wp(\wp(\mathbb{N}) \setminus \{\emptyset\}), \circ^{\wp(\mathbb{N})\setminus\{\emptyset\}}\rangle) = (\mathrm{ACI}_\omega \cup \{(\mathrm{CountAt})\})^{s2}.$$

With Corollary 7.3, this yields:

COROLLARY 7.4. $(\mathrm{ACI}_\omega \cup \{(\mathrm{CountAt})\})^{s2}$ *extended by the definition "$\mathcal{F}x :\longleftrightarrow Fin^2(x)$" contains the same sentences from* $L[\circ, \mathcal{F}]$ *as* $\mathrm{Th}(\mathcal{P})$.

Similarly, one can deal with $\mathcal{FC}$ instead of $\mathcal{P}$ and obtain from Lemma 7.6(iii):

COROLLARY 7.5. $\mathrm{Th}^2(\langle \mathrm{FC}, \wp(\mathrm{FC}), \circ^{\mathrm{FC}}\rangle)$ *extended by the definition "$\mathcal{F}x :\longleftrightarrow Fin^2(x)$" contains the same sentences from* $L[\circ, \mathcal{F}]$ *as* $\mathrm{Th}(\mathcal{FC})$.

Yet, what could be the analogue of Corollary 7.4 in this case? $\mathrm{Th}^2(\langle \wp(\mathbb{N}) \setminus \{\emptyset\}, \wp(\wp(\mathbb{N}) \setminus \{\emptyset\}), \circ^{\wp(\mathbb{N})\setminus\{\emptyset\}}\rangle)$ was presented there in a quasi-axiomatic way as $(\mathrm{ACI}_\omega \cup \{(\mathrm{CountAt})\})^{s2}$. How could such a presentation for $\mathrm{Th}^2(\langle \mathrm{FC}, \wp(\mathrm{FC}), \circ^{\mathrm{FC}}\rangle)$ look like? — The answer is: there is no such presentation (in $L^2[\circ]$).

LEMMA 7.7. (i) $\langle \mathrm{FC}, \wp(\mathrm{FC}), \circ^{\mathrm{FC}}\rangle \not\models$ FUS-Ax.
(ii) *There is no set of sentences Σ in* $L^2[\circ]$ *such that* $\mathrm{Th}^2(\langle \mathrm{FC}, \wp(\mathrm{FC}), \circ^{\mathrm{FC}}\rangle) = \Sigma^{s2}$.

PROOF. (i) If $\langle \mathrm{FC}, \wp(\mathrm{FC}), \circ^{\mathrm{FC}}\rangle \models$ FUS-Ax, it is a s2-structure which satisfies ACI plus FUS-Ax. But then, by Lemma 16 in [Niebergall, 2008b], FC plus an additional object is the domain of a complete Boolean algebra which is atomistic. In light of Lemma 21 in [Niebergall, 2008b], there exists then a set A — which must be infinite, since $\langle \mathrm{FC}, \wp(\mathrm{FC}), \circ^{\mathrm{FC}}\rangle \models \mathrm{ACI}_\omega$ — such that FC plus that additional object has the same cardinality as $\wp(A)$. Therefore, FC itself must be uncountable; but is is countable.

(ii) Assume there is such a set of sentences Σ. Since $\mathrm{Th}^2(\langle \mathrm{FC}, \wp(\mathrm{FC}), \circ^{\mathrm{FC}}\rangle)$ holds in $\langle \mathrm{FC}, \wp(\mathrm{FC}), \circ^{\mathrm{FC}}\rangle$, each element of Σ^{s2} must be true in $\langle \mathrm{FC}, \wp(\mathrm{FC}), \circ^{\mathrm{FC}}\rangle$ as well. Yet, FUS-Ax is such an element. This contradicts (i). $\square$

In sum, given the considerations of this subsection, it should be $\mathrm{Th}(\mathcal{P})$ which is distinguished among the maximal consistent extensions of ACI^*_ω known from Section 6.

8. Conclusion

The present paper dealt with mereological theories, as I have called them, and with theories belonging to a class $\mathcal{C}$ of extensions of them (see Section 6). Yet why, it may be asked, are these theories philosophically interesting and worthy of study? I see three answers to this question:[29]

First, these theories are simply taken to be that: philosophically interesting and worthy of study; no further reason is given (and need be given).

Second, these theories are philosophically interesting and worthy of study because they are *nominalistic* theories.[30]

Third, these theories are philosophically interesting and worthy of study because they are non-mathematical theories which may be able to replace mathematical theories as parts of empirical theories.

Those who prefer the first answer may appreciate the development and study of extensions of mereological theories as a widening, refinement and strengthening of the *mereological paradigm*. However, since I lean towards the second and third answer, let me close this paper with some comments on them.

The second answer presupposes that the elements of $\mathcal{C}$ deserve to be regarded as nominalistic theories. Now, it has to be granted that no accepted precise general *explicans* of "T is a nominalistic theory" is available. This notwithstanding, theories are known to us which are commonly accepted as nominalistic theories: among them are at least

[29] In particular, I agree that mereological theories need not be interpreted nominalistically and their study need not be motivated by nominalistic concerns.

[30] As regards the advantages of a nominalistic position over a platonistic one, I have no really new ingredients to add to what has been discussed by Goodman, Quine and their critics and followers (see [Goodman and Quine, 1947; Goodman, 1951] and, with an emphasis on nominalistic *theories*, [Niebergall, 2005]).

some of the mereological theories, such as Goodman's *calulus of individuals* [see Goodman, 1951], but also *token concatention theories* [see Goodman and Quine, 1947; Niebergall, 2005]. In fact, the first ones are *the* examples of nominalistic theories.

I follow those philosophers who take the nominalistic position to consist in the avoidance of the assumption of abstract objects. Then, the criterion for T being a nominalistic theory is roughly this: the preferred reading of the vocabulary of L[T] deals with concrete objects; and T is true, given that reading. From this perspective, too, at least some mereological theories, but also some theories in $\mathcal{C}$, should be viewed as nominalistic theories. This leads to the third answer, which, though closely related to the second one, should be clearly distinguished from it. I have mainly two reasons for this assessment.

The first one is simply that one may be a platonist and assume the existence of abstract objects, and still agree that the empirical objects known from our everyday experience are only concrete objects. Accordingly, speaking about theories instead of ontology, even a platonist may prefer *empirical* theories to be free from mathematics.

Actually, it seems that more than a few empirical theories — in particular physical theories — have, as they are commonly stated, what may be called a "mathematical core": if such a theory T is explicitly presented, mathematical expressions, such as "+" or "$\in$", belong to its vocabulary; and T contains mathematical theories as subtheories. Admittedly, a general distinction between mathematical and non-mathematical theories is not that easy to state. Yet, for the discussion of this section, it should suffice to assume (a) that the elements of $\mathcal{C}$ are non-mathematical theories (which should be particularly plausible if they are regarded as nominalistic theories)[31] and (b) that theories such as PA and Q, ZF and ZF minus the axiom of infinity and perhaps theories of real numbers (such as RCF, the theory of real closed fields [see Schwabhäuser, 1983]) plus, for example, second-order extensions of each of them, are mathematical theories. *De facto*, mathematics permeates the empirical sciences. But sets and numbers are foreign bodies when it comes to our world of concrete objects. It therefore seems to be a natural reaction to aim at the development and study of empirical theories which are free

[31] Probably each nominalistic theory should be taken to be a non-mathematical theory. A theory of properties, however, may be a non-mathematical theory which is no nominalistic theory.

from mathematical vocabulary and mathematical principles. One need not be a nominalist to concede that much.

Let me come to the second reason for distinguishing answer two from answer three. To start with, recall that a reductive programme has often been a component of the nominalistic programme. In particular, the nominalist (when standing in a Quinean tradition) has to face the challenge of construing mathematics nominalistically. In the approach promoted here, with its emphasis on theories, this task is more precisely rendered as that of reducing, say, ZF to a nominalistic theory. Now, it is not at all clear that mathematical theories of the (excessive) strength of ZF are actually used as mathematical cores of empirical theories. Moreover, it has been claimed repeatedly that even for physical theories, theories which are of about the same strength as PA — for example, theories which are conservative extensions of PA — *suffice* as their mathematical cores. Thus, it seems that the mathematical theories which should be replaced by non-mathematical ones, as mentioned in answer three, are allowed to be much weaker than those which have to be construed nominalistically, as addressed in answer two.

Actually, I think that there is a further, in some sense more fundamental, difference between the programmes underlying the second and the third answer: the replacement programme expressed in the third answer need not be understood as implying a reductive programme. I agree that when the mathematical core of an empirical theory T is replaced by a non-mathematical theory, it is plausible to assume that the theory T' resulting from T should be able to *play the same role* as T. But this way of putting it is — deliberately — vague and open. In particular, it does not follow from it that T has to be relatively interpretable in T', or that the mathematical core of T has to be relatively interpretable in the non-mathematical theory which replaces it. It could also mean that T and T' have to be, for example, *empirically equivalent* with each other (whatever that means exactly [cf. Quine, 1975].

In sum, it might be said that answers two and three are connected with two different programmes: a reduction programme and a replacement programme. Now, what are the prospects for a realization of these two programmes? And, in particular, which roles may theories belonging to $\mathcal{C}$ play for them?

Two results delimit the nominalistic reduction programme. First, even the rather weak theory Q (not to mention ZF) is not relatively interpretable in any consistent mereological theory [see Niebergall, 2011b].

Second, token concatenation theories can be developed which *are* strong enough to relatively interpret ZF [see Niebergall, 2005]; but the theories of this type that are known to me seem to be rather unattractive when compared to the mereological theories.

In this situation, it should be highly interesting for the nominalist to develop and investigate further theories; theories, that is, which are both rightfully regarded as nominalistic and are strong enough for the realization of the nominalistic reduction programme with respect to mathematical theories. Theories belonging to $\mathcal{C}$ may be welcome examples of such theories. But are they? I do not know; but I doubt it. Ultimately, it may well be that the theories in $\mathcal{C}$ are no gain over the mereological theories when it comes to the nominalistic reduction programme for mathematical theories.

Again, for the replacement programme involved in the third answer, it is certainly desirable to have more non-mathematical theories T at one's disposal than only the mereological theories. It should here by much easier to find suitable examples. In particular, even if the theories in $\mathcal{C}$ should turn out to be worthless for the nominalistic reduction programme, they may be examples of theories mentioned in the third answer.

At this point, some philosophers may be impressed by indispensability arguments and argue that, plausible as the *aim* of going along without mathematics in empirical theories might be, it just cannot be realized. As understood here, such arguments are supposed to establish the thesis that the adoption of mathematical theories is indispensable for the scientific enterprise.[32] Let me simply answer that I am not aware of a cogent argument for this thesis. As an example, consider a theory T of real numbers or of, e.g., 4-tuples of real numbers as the mathematical core of an empirical theory. RCF could be a formalized version of it. When being a part of an empirical theory, T typically plays the role of a theory of space-time: space-time points have been replaced or simulated by the 4-tuples of real numbers. Now withdraw this simulation, come back to

[32] This is a weak version of an indispensabilty thesis. A stronger one would be

> The adoption of mathematical theories and the assumption of mathematical objects are indispensable for the scientific enterprise.

I think that this is the more common version. I deal with the weak version because it has a greater chance of being true.

space-time points we started with (being individuals of their own kind) and replace RCF by a suitable axiomatic theory of geometry.[33]

This should be an example of the execution of the replacement programme as addressed in the third answer. It is granted that the axiomatic theories of geometry considered here are no elements of $\mathcal{C}$. But the move from mereological theories to $\mathcal{C}$ is only one example for the extension of the mereological paradigm in the pure form, anyway.

Whether this replacement programme and also the nominalistic reduction programme are ultimately feasible can hardly be decided *a priori* (e.g., with an indispensability argument). The answers depend on which theories are conceived of as non-mathematical theories and as nominalistic theories; and in order to attain a reasonable assessment of this, potential theories of these kinds have to be invented, developed and investigated.

Acknowledgements. I would like to thank Maike Albertzart, Sebastian Paasch and the editors of this special issue of *Logic and Logical Philosophy* for their support and their comments.

References

Bonevac, D, A., 1982, *Reduction in the Abstract Sciences*, Ridgeview Publishing Company.

Feferman, S., 1960, "Arithmetization of metamathematics in a general setting", *Fundamenta Mathematicae* 49: 35–92.

Fraenkel1, A., 1927, *Zehn Vorlesungen über die Grundlegung der Mengenlehre*; reproduced by Wissenschaftliche Buchgesellschaft, Darmstadt, 1972.

Goodman, N., 1951, *Structure of Appearance*, Reidel, Dordrecht.

Goodman, N., and W. V. O. Quine, 1947, "Steps toward a constructive nominalism", *The Journal of Symbolic Logic*, 12: 105–122. DOI: 10.2307/2266485

Klaua, D., 1973, *Grundbegriffe der axiomatischen Mengenlehre, Teil 2*, Braunschweig.

Levy, A., 1958, "The independence of various definitions of finiteness", *Fundamenta Mathematicae*, 46: 1–13.

Lewis, D., 1991, *Parts of Classes*, Blackwell, Oxford.

Niebergall, K.-G., 2000, "On the logic of reducibility: Axioms and examples", *Erkenntnis* 53: 27–61.

[33] Of the kind developed by Tarski and his followers [see Schwabhäuser, 1983].

Niebergall, K.-G., 2005, "Zur nominalistischen Behandlung der Mathematik", pages 235–260 in J. Steinbrenner, O. Scholz und G. Ernst (eds.), *Symbole, Systeme, Welten: Studien zur Philosophie Nelson Goodmans*, Heidelberg, Synchron Wissenschaftsverlag der Autoren.

Niebergall, K.-G., 2009a, "Calculi of individuals and some extensions: An overview", pages 335–354 in A. Hieke and H. Leitgeb (eds.): *Reduction – Abstraction – Analysis*, Proceedings of the 31th International Ludwig Wittgenstein-Symposium in Kirchberg 2008, Ontos Verlag, Frankfurt–Paris–Lancaster–New Brunswick.

Niebergall, K.-G., 2009b, "On 2nd order calculi of individuals", *Theoria* 24: 169–202.

Niebergall, K.-G., 2011a, "Unendlichkeit ausdrücken und Unendlichkeitsannahmen machen", pages 957–976 in C. F. Gethmann (ed.), *Lebenswelt und Wissenschaft. Deutsches Jahrbuch Philosophie*, 2.

Niebergall, K.-G., 2011b, "Mereology", pages 271–298 in R. Pettigrew and L. Horsten (eds.), *The Continuum Companion to Philosophical Logic*, Continuum, London.

Niebergall, K.-G., 2014, "Assumptions of Infinity", pages 229–274 in G. Link (ed.), *Formalism and Beyond. On the nature of mathematical discourse*, de Gruyter, Boston–Berlin. DOI: 10.1515/9781614518471.229

Parsons, Ch., 1987, "Developing arithmetic in set theory without infinity: Some historical remarks", *History and Philosophy of Logic*, 8: 201–213. DOI: 10.1080/01445348708837116

Quine, W. V. O, 1975, "On empirically equivalent systems of the world", *Erkenntnis*, 9: 313–328. DOI: 10.1007/BF00178004

Schmidt, J., 1966, *Mengenlehre* I, Bibliographisches Institut Mannheim.

Schwabhäuser, W., with W. Szmielew and A. Tarski, 1983, *Metamathematische Methoden in der Geometrie*, Springer-Verlag.

Shapiro, S., 1991, *Foundations without Foundationalism*, Clarendon Press, Oxford. DOI: 10.1093/0198250290.001.0001

Takeuti, G., and W. Zaring, 1982: *Introduction to Axiomatic Set Theory*, 2nd edition, Springer, New York–Heidelberg–Berlin. DOI: 10.1007/978-1-4613-8168-6

Tarski, A., A. Mostowski, and R. M. Robinson, 1953, *Undecidable Theories*, North-Holland, Amsterdam.

KARL-GEORG NIEBERGALL
Department of Philosophy
Humboldt-Universität zu Berlin
Germany
niebergk@cms.hu-berlin.de