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(DIiii+)  $T$  makes an assumption of infinity : $\iff \exists\alpha(\alpha(x)$  expresses that  $x$  is infinite and  $T \models \exists x \alpha(x)$ ).

The conjunct “ $\alpha(x)$  expresses that  $x$  is infinite” in the *definiens* of (DIiii+) seems to be indispensable. For if it were not included and if, for example,  $\alpha(x)$  expressed that  $x$  is a planet, a theory  $T$  proving “ $\exists x \alpha(x)$ ” would assume planets rather than the infinite.

Whether (DIiii+) is reasonable can, of course, eventually only be decided if an explication can be provided for “ $\alpha(x)$  expresses that  $x$  is infinite”. Moreover, I think that a further modification of this phrase, i.e., its relativization to a theory  $T$  (in which  $\alpha(x)$  may express that  $x$  is finite) is advisable.

Thus, following the procedure elaborated in more detail in [Niebergall, 2011a, 2014], I will first present *explicantia* of

“ $\alpha(x)$  expresses that  $x$  is finite relative to  $T$ ”

(where  $T$  has to be a theory and  $\alpha(x)$  has to be a formula (with  $x$  as its sole free variable) in  $L[T]$ ). These, in turn, are then employed to formulate *explicantia* of “ $T$  makes an assumption of infinity” in the style envisaged in (DIiii+).

### 5.2. “ $\alpha(x)$ expresses that $x$ is finite” and “ $T$ makes an assumption of infinity”: definitions

In this subsection, I recall the precise renderings of (DIiii+) presented in [Niebergall, 2014] which have not shown to be inadequate in this paper.

Let  $\alpha$  be a formula from  $L[\circ]$  and  $T$  be a mereological theory. As already hinted at in the introduction, the *explicantia* for “ $T$  makes an assumption of infinity” put forward in this section rest on axioms of finiteness. Ax-a(FinI), in particular, leads to these definitions:

$\alpha(x)$  strongly expresses that  $x$  is finite relative to  $T$  : $\iff$   
 $T \vdash AxFinI\text{-a}[\alpha]$  and for each formula  $\psi$  in  $L[\circ]$ ,  
 $T \vdash AxFinI\text{-a}[\psi] \rightarrow \forall x(\alpha(x) \rightarrow \psi(x))$ ;

$\alpha(x)$  very strongly expresses that  $x$  is finite relative to  $T$  : $\iff$

$\alpha(x)$  strongly expresses that  $x$  is finite relative to  $T$  and  $T \not\models \forall x\alpha(x)$ .

For a different type of definitions, a further axiom system of finiteness is used:

$$(\text{AxFinI}^-) \quad \{\ulcorner \forall x(\neg \mathcal{F}x \rightarrow \exists^{\geq n} y y \sqsubseteq x) \urcorner \mid n \in \mathbb{N}\}.^{16}$$

$(\text{AxFinI}^-)$  is an obvious necessary, but hardly a sufficient choice for a set of axioms of finiteness: on its own, it seems to be too weak. All the same, it can be usefully employed in *explicantia* of “ $T$  makes an assumption of infinity”. Thus, let’s define:

$$\alpha(x) \text{ very weakly expresses that } x \text{ is finite relative to } T : \iff \\ \forall n(n \in \mathbb{N} \implies T \vdash \forall x(\neg \alpha(x) \rightarrow \exists^{\geq n} y y \sqsubseteq x));$$

$$\alpha(x) \text{ weakly expresses that } x \text{ is finite relative to } T : \iff \\ \alpha(x) \text{ very weakly expresses that } x \text{ is finite relative to } T \text{ and} \\ T \not\vdash \forall x \alpha(x).^{17}$$

In these definitions, we have free variables for theories; that is, we have a dependency on theories. In order to obtain *explicantia* for “ $\alpha$  expresses that  $x$  is finite”, it is a natural step to bind these variables. However, it turns out that existential quantifiers are not suitable for this task (see Footnote 18).

$\alpha(x)$  universally very strongly expresses that  $x$  is finite :  $\iff \forall T(T \text{ is a mereological theory} \implies \alpha(x) \text{ very strongly expresses that } x \text{ is finite relative to } T)$ ;

$\alpha(x)$  universally strongly expresses that  $x$  is finite :  $\iff \forall T(T \text{ is a mereological theory} \implies \alpha(x) \text{ strongly expresses that } x \text{ is finite relative to } T)$ ;

$\alpha(x)$  universally weakly expresses that  $x$  is finite :  $\iff \forall T(T \text{ is a mereological theory} \implies \alpha(x) \text{ weakly expresses that } x \text{ is finite relative to } T)$ ;

$\alpha(x)$  universally very weakly expresses that  $x$  is finite :  $\iff \forall T(T \text{ is a mereological theory} \implies \alpha(x) \text{ very weakly expresses that } x \text{ is finite relative to } T)$ .

Finally, these predicates are used in the following *defnientia* of possible explications of “ $T$  makes an assumption of infinity” (with  $T$  in  $L[\circ]$ ):

$T$  makes a universal very strong assumption of infinity :  $\iff \exists \alpha(\alpha \in L[\circ] \wedge T \vdash \exists x \neg \alpha(x) \wedge \alpha(x) \text{ universally very strongly expresses that } x \text{ is finite})$ ;

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<sup>16</sup>  $\exists^{\geq n} y y \sqsubseteq x : \iff \exists y_1 \dots y_n (\bigwedge_{i=1}^n y_i \sqsubseteq x \wedge \bigwedge_{\substack{i,j=1 \\ i < j}}^n y_i \neq y_j)$ .

<sup>17</sup> Note that “ $x = x$ ” very weakly expresses that  $x$  is finite relative to each mereological theory  $T$ . To circumvent this trivialization, “ $T \not\vdash \forall x \alpha(x)$ ” is added.

$T$  makes a universal strong assumption of infinity  $:\iff \exists\alpha(\alpha \in L[o] \wedge T \vdash \exists x \neg\alpha(x) \wedge \alpha(x)$  universally strongly expresses that  $x$  is finite);

$T$  makes a universal weak assumption of infinity  $:\iff \exists\alpha(\alpha \in L[o] \wedge T \vdash \exists x \neg\alpha(x) \wedge \alpha(x)$  universally weakly expresses that  $x$  is finite);

$T$  makes a universal very weak assumption of infinity  $:\iff \exists\alpha(\alpha \in L[o] \wedge T \vdash \exists x \neg\alpha(x) \wedge \alpha(x)$  universally very weakly expresses that  $x$  is finite);

$T$  makes a strong assumption of infinity  $:\iff \exists\alpha(\alpha \in L[o] \wedge T \vdash \exists x \neg\alpha(x) \wedge \alpha(x)$  strongly expresses that  $x$  is finite relative to  $T$ );

$T$  makes a weak assumption of infinity  $:\iff \exists\alpha(\alpha \in L[o] \wedge T \vdash \exists x \neg\alpha(x) \wedge \alpha(x)$  weakly expresses that  $x$  is finite relative to  $T$ ).<sup>18</sup>

### 5.3. Evaluating the definitions

In [Niebergall, 2011a, 2014], I compared the merits of *definiencia* such as the above as *explicantia* of “ $T$  makes an assumption of infinity” mainly for set theories. Mereological theories were also considered, and it seemed that the picture for them was somewhat simpler than that for set theories. I can now show that it is *much* simpler: due to Corollary 5.2 in particular (the main new result of this part), only two options remain for the mereological setting investigated here: “ $T$  makes an assumption of infinity” can be explicated either via (DIi) or by “ $T$  makes a strong assumption of infinity”.

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<sup>18</sup> The variants of these definitions with existential quantifiers in place of universal quantifiers are

$T$  makes an existential very strong assumption of infinity  $:\iff \exists\alpha(\alpha \in L[o] \wedge T \vdash \exists x \neg\alpha(x) \wedge \exists S(S \text{ is a mereological theory} \wedge \alpha(x) \text{ very strongly expresses that } x \text{ is finite relative to } S))$ .

$T$  makes an existential strong assumption of infinity  $:\iff \exists\alpha(\alpha \in L[o] \wedge T \vdash \exists x \neg\alpha(x) \wedge \exists S(S \text{ is a mereological theory} \wedge \alpha(x) \text{ strongly expresses that } x \text{ is finite relative to } S))$ .

$T$  makes an existential weak assumption of infinity  $:\iff \exists\alpha(\alpha \in L[o] \wedge T \vdash \exists x \neg\alpha(x) \wedge \exists S(S \text{ is a mereological theory} \wedge \alpha(x) \text{ weakly expresses that } x \text{ is finite relative to } S))$ .

$T$  makes an existential very weak assumption of infinity  $:\iff \exists\alpha(\alpha \in L[o] \wedge T \vdash \exists x \neg\alpha(x) \wedge \exists S(S \text{ is a mereological theory} \wedge \alpha(x) \text{ very weakly expresses that } x \text{ is finite relative to } S))$ .

Now in [Niebergall, 2014, lemmas 6 and 11] it has been shown that each mereological theory makes an assumption of infinity in each of the four ways considered here. In particular, if any of them were accepted as an *explicans* of “ $T$  makes an assumption of infinity”, even a theory like  $ACI_1$ , which only has models with one element, would make an assumption of infinity.

But let me first quote the relevant results from [Niebergall, 2014, lemmas A to D].

LEMMA A. *Let  $T$  be a mereological theory. Then:*

- (i) *If  $T$  makes a universal strong assumption of infinity, then  $T$  makes a strong assumption of infinity and  $T$  makes a universal very weak assumption of infinity.*
- (ii) *If  $T$  makes a universal very weak assumption of infinity, then  $T$  makes a weak assumption of infinity.*

LEMMA B. (i) *“ $IFin(x)$ ” universally strongly expresses that  $x$  is finite.*

- (ii) *Equivalent are for mereological theories  $T$ :  $T \vdash \exists x \neg IFin(x)$ ,  $T$  makes a universal strong assumption of infinity,  $T$  makes a strong assumption of infinity.*

By Lemma 4.4 and Lemma B we obtain:

COROLLARY 5.1. *Equivalent are for mereological theories  $T$ :  $T \vdash \neg AT$ ,  $T \vdash \exists x \neg IFin(x)$ ,  $T$  makes a universal strong assumption of infinity,  $T$  makes a strong assumption of infinity.*

LEMMA C. (i) *No mereological theory makes a universal very strong assumption of infinity.*

- (ii) *No mereological theory makes a universal weak assumption of infinity.*

LEMMA D. *Let  $T$  be a mereological theory. Then:  $T$  makes a weak assumption of infinity  $\iff$  all models of  $T$  are infinite.*

To move forward, examples are helpful:

*Example  $ACI_n$ .* For each  $n \geq 1$ ,  $ACI_n$  does not make a weak assumption of infinity (by Lemma D).

*Example FCI.* FCI makes a strong assumption of infinity (by Corollary 5.1).

*Example  $MCI_n$ .* For each  $n \geq 1$ ,  $MCI_n$  makes a strong assumption of infinity (by Corollary 5.1).

*Example  $MCI_\omega + \{FUS_{At}\}$ .*  $MCI_\omega + \{FUS_{At}\}$  makes a strong assumption of infinity (by Corollary 5.1).

All of this is as it should be. It is only  $ACI_\omega$  which is out of line.

*Example  $ACI_\omega$ .* (i)  $ACI_\omega$  does not make a strong assumption of infinity (by Corollary 5.1).

- (ii)  $ACI_\omega$  makes a weak assumption of infinity (by Lemma D).
- (iii)  $ACI_\omega$  does not make a universal very weak assumption of infinity.

Indeed, assume that  $ACI_\omega$  makes a universal very weak assumption of infinity, i.e., for some  $\alpha$  from  $L[o]$ ,  $ACI_\omega \vdash \exists x \neg\alpha(x)$  and  $\forall S(S \text{ is a mereological theory} \implies \forall n(n \in \mathbb{N} \rightarrow S \vdash \forall x(\neg\alpha(x) \rightarrow \exists^{\geq n} y y \sqsubseteq x)))$ . Then for some  $k \in \mathbb{N}$ :

$$(*) \quad ACI_k \vdash \exists x \neg\alpha(x),$$

and, specializing  $S$  to  $ACI_k$ ,  $\forall n(n \in \mathbb{N} \implies ACI_k \vdash \forall x(\neg\alpha(x) \rightarrow \exists^{\geq n} y y \sqsubseteq x))$ , whence:

$$(**) \quad ACI_k \vdash \forall x(\neg\alpha(x) \rightarrow \exists^{\geq 2^{k+1}} y y \sqsubseteq x).$$

(\*) and (\*\*) imply:

$$(***) \quad ACI_k \vdash \exists x \exists^{\geq 2^{k+1}} y y \sqsubseteq x.$$

Now let  $\mathcal{M}$  be a model for  $ACI_k$ . Then  $|M| = 2^k - 1$ . But by (\*\*\*),  $|M| \geq 2^{k+1}$ . Contradiction.  $\square$

Making a strong assumption of infinity and making a universal very weak assumption of infinity seem to be conceptually quite distinct. The fact that they are equivalent for maximal consistent mereological theories may seem to be a special feature of the latter. It can, however, be shown that this equivalence holds for all mereological theories.

LEMMA 5.1. *Let  $T$  be a mereological theory which makes a weak assumption of infinity. Then:  $T$  does not make a strong assumption of infinity  $\iff T \subseteq ACI_\omega$ .*

PROOF. “ $\Leftarrow$ ” If  $T \subseteq ACI_\omega$  and  $T$  makes a strong assumption of infinity, then by Corollary 5.1,  $T \vdash \neg AT$ ; whence  $ACI_\omega \vdash \neg AT$ . But since  $ACI_\omega$  is consistent, this cannot be the case.

“ $\Rightarrow$ ” Assume that  $T$  makes a weak assumption of infinity but fails to make a strong assumption of infinity. Then by Corollary 5.1,  $T \not\vdash \neg AT$ . Let’s consider the (therefore consistent) theory:  $T' := T + \{AT\}$ .

*Claim.*  $ACI_\omega \subseteq T'$ .

*Proof.* Since  $T$  is a mereological theory,  $ACI \subseteq T'$ . Moreover:

$$(*) \quad \forall \mathcal{M}(\mathcal{M} \models T' \implies M \text{ is infinite}).$$

This is the case because, by assumption,  $T$  makes a weak assumption of infinity and has therefore only infinite models (by Lemma D). By (\*)

and since  $ACI \subseteq T'$ , each model of  $T'$  contains infinitely many atoms<sup>*M*</sup>, and the claim follows.

Now  $ACI_\omega$  is maximal consistent in  $L[o]$ . Therefore,  $ACI_\omega = T'$ , whence  $T \subseteq ACI_\omega$ .  $\square$

**COROLLARY 5.2.** *Let  $T$  be a mereological theory. Then:  $T$  makes a strong assumption of infinity  $\iff T$  makes a universal very weak assumption of infinity.*

**PROOF.** “ $\implies$ ” It follows from Lemma A and Lemma B(ii). “ $\impliedby$ ” Assume that  $T$  makes a universal very weak assumption of infinity but fails to make a strong assumption of infinity. Then by Lemma A(ii),  $T$  makes a weak assumption of infinity. So, by Lemma 5.1,  $T \subseteq ACI_\omega$ . But then, by the assumption that  $T$  makes a universal very weak assumption of infinity,  $ACI_\omega$  has to make a universal very weak assumption of infinity, too. This, however, contradicts (iii) from Example  $ACI_\omega$ .  $\square$

Let me sum up. When for mereological theories  $T$ , “ $T$  makes an assumption of infinity” is explained by (DIiii+), the potentially acceptable *definiencia* are “ $T$  makes a universal strong assumption of infinity”, “ $T$  makes a strong assumption of infinity”, “ $T$  makes a universal very weak assumption of infinity” and “ $T$  makes a weak assumption of infinity”. Now, it turned out that “ $T$  makes a universal strong assumption of infinity”, “ $T$  makes a strong assumption of infinity” and “ $T$  makes a universal very weak assumption of infinity” are equivalent with each other, each of them being stronger than “ $T$  makes a weak assumption of infinity”. Moreover, “ $T$  makes a weak assumption of infinity” is equivalent to the *definiens* of (DIi). Thus, the above mentioned options eventually lead to only two possible *explicantia* of “ $T$  makes an assumption of infinity”:

(DIi)  $T$  makes an assumption of infinity : $\iff$

$\forall \mathcal{M}(\mathcal{M} \models T \implies M \text{ is infinite}).$

(DIiii)  $T$  makes an assumption of infinity : $\iff T \models \neg AT.$

Under (DIi),  $ACI_\omega$  makes an assumption of infinity; yet under (DIiii),  $ACI_\omega$  fails to make an assumption of infinity. In [Niebergall, 2014], I have opted for the first alternative:  $ACI_\omega$  makes an assumption of infinity. The findings of the present paper simplify the picture up to the point that only these two *explicantia* have to be taken into account; but I do not think that they provide any reasons against the assessment in [Niebergall, 2014].

## 6. Theories of finiteness extending mereological theories

Let  $\mathcal{C}$  be the class of theories in  $L[\circ, \mathcal{F}]$  which extend a theory of the type  $T^*$ ,  $T$  being a mereological theory. In this (and also in the following) section, theories in  $\mathcal{C}$  are studied for their own sake. Of course, the  $T^*$ 's deserve to be investigated closer, and they belong to  $\mathcal{C}$ . There are, however, certain further theories over and above the theories  $T^*$  which are both helpful for the understanding of the  $T^*$ 's and play a distinguished role on their own. These are the  $T^+$ 's, where:

$T^+ :=$  the deductive closure (in  $L[\circ, \mathcal{F}]$ ) of  $T \cup \{\forall x(\mathcal{F}x \leftrightarrow IFin(x))\}$   
(if  $T$  is a mereological theory).

In what follows, for the theories in  $\mathcal{C}$ , metatheoretical topics such as maximal consistency, proof theoretic strength and their relation to mereological theories are addressed. For the rest of Section 6, let  $T$  be a consistent mereological theory.

### 6.1. More on $T^*$ and $T^+$

- LEMMA 6.1. (i) For each formula  $\psi$  from  $L[\circ, \mathcal{F}]$  there is a formula  $\psi^{-\mathcal{F}}$  from  $L[\circ]$  such that  $T^+ \vdash \psi \leftrightarrow \psi^{-\mathcal{F}}$ . If  $\psi$  is a sentence, then  $\psi^{-\mathcal{F}}$  is a sentence.
- (ii) Let  $T$  be maximal consistent in  $L[\circ]$ . Then  $T^+$  is maximal consistent in  $L[\circ, \mathcal{F}]$  and decidable.
- (iii)  $T^* \subseteq T^+$ .

PROOF. (i) If  $\psi$  is a formula from  $L[\circ, \mathcal{F}]$ , let  $\psi^{-\mathcal{F}}$  result from  $\psi$  by replacing each occurrence of " $\mathcal{F}x$ " in  $\psi$  by " $IFin(x)$ " (for each variable  $x$ ). Then  $\psi^{-\mathcal{F}}$  is a formula from  $L[\circ]$ ,  $\psi$  and  $\psi^{-\mathcal{F}}$  have the same free variables, and it can be shown by induction on the complexity of  $\psi$  that  $T^+ \vdash \psi \leftrightarrow \psi^{-\mathcal{F}}$ .

(ii) As to the maximal consistency, let  $\varphi$  be a sentence from  $L[\circ, \mathcal{F}]$ , and assume  $T^+ \not\vdash \varphi$ . Then by (i),  $T^+ \not\vdash \varphi^{-\mathcal{F}}$ , whence (\*)  $T \not\vdash \varphi^{-\mathcal{F}}$ .

Note that by (i),  $\varphi^{-\mathcal{F}}$  is a sentence from  $L[\circ]$ . And since  $T$  is assumed to be maximal consistent (for  $L[\circ]$ ), we obtain from (\*),  $T \vdash \neg\varphi^{-\mathcal{F}}$ . Therefore,  $T^+ \vdash \neg\varphi^{-\mathcal{F}}$ , and finally by (i) again,  $T^+ \vdash \neg\varphi$ .

Moreover, since each maximal consistent mereological theory is decidable (by Theorem 2.1(iii)), each  $T^+$ , resulting from  $T$  through the addition of one axiom, is recursively enumerable. Thus, by its maximal consistency, it is decidable.

(iii) By Lemma 4.1,  $T^+ \vdash (\text{AxI } 1), (\text{AxI } 2)$ . In order to show (IndI-a), let  $\psi$  be a formula from  $L[\circ, \mathcal{F}]$ . Then by (i),  $\psi^{-\mathcal{F}}$  is from  $L[\circ]$ . Now because of Corollary 4.1,  $T \vdash \text{AxFinI-a}[\psi^{-\mathcal{F}}] \rightarrow \forall x(\text{IFin}(x) \rightarrow \psi^{-\mathcal{F}}(x))$ . But then by (i),  $T^+ \vdash \text{AxFinI-a}[\psi] \rightarrow \forall x(\text{IFin}(x) \rightarrow \psi(x))$ .  $\square$

## 6.2. Models for $L[\circ, \mathcal{F}]$

Models for  $L[\circ, \mathcal{F}]$  are of the form  $\langle M, \circ^M, \mathcal{F}^M \rangle$ , with  $\mathcal{F}^M \subseteq M$ . Two special choices of  $\mathcal{F}^M$  deserve attention (for models  $\langle M, \circ^M \rangle$  of CI):

$$\begin{aligned} \mathcal{E}^M &:= \{a \mid a \in M \wedge a \text{ is a finite sum}^M \text{ of atoms}^M\}, \\ \mathcal{A}^M &:= \{a \mid a \in M \wedge \langle M, \circ^M \rangle, \beta(x : a) \models \text{IFin}(x)\}. \end{aligned}$$

LEMMA 6.2. (i) If  $\langle M, \circ^M \rangle \models T$ , then  $\langle M, \circ^M, \mathcal{A}^M \rangle \models T^+$ .  
(ii)  $T^+$  is a conservative extension of  $T$ .

PROOF. (i) Follows immediately from the definition of  $\mathcal{A}^M$ .  
(ii) As usual, from (i).  $\square$

LEMMA 6.3. (i) If  $\langle M, \circ^M \rangle \models T$ , then  $\langle M, \circ^M, \mathcal{E}^M \rangle \models T^*$ .  
(ii)  $T^*$  is a conservative extension of  $T$ .

PROOF. (i) Each atom<sup>M</sup> is a finite sum<sup>M</sup> of atoms<sup>M</sup>; and binary sums<sup>M</sup> of finite sums<sup>M</sup> of atoms<sup>M</sup> are finite sums<sup>M</sup> of atoms<sup>M</sup>. Moreover, assume the hypotheses of (IndI-a) for  $\psi$  and let  $a$  satisfy “ $\mathcal{F}x$ ” in  $\langle M, \circ^M, \mathcal{E}^M \rangle$ . Then each atom<sup>M</sup> satisfies  $\psi$  and, therefore, each finite sum<sup>M</sup> of atoms<sup>M</sup> satisfies  $\psi$ . But  $a$ , being an element of  $\mathcal{E}^M$ , is such a finite sum<sup>M</sup>.

(ii) As usual, from (i).  $\square$

LEMMA 6.4. If  $\langle M, \circ^M \rangle \models \text{CI} + \{\text{FUS}_{At}, \exists x \text{At}(x)\}$ , then:

- (i)  $\mathcal{A}^M = \{a \mid a \in M \wedge a \text{ is a part}^M \text{ of the sum}^M \text{ of all atoms}^M\}$ ,
- (ii)  $\mathcal{E}^M \subseteq \mathcal{A}^M$ .

PROOF. (i) follows from Lemma 4.5.  $\square$

LEMMA 6.5. If  $\langle M, \circ^M, \mathcal{F}^M \rangle \models T^*$ , then: (i)  $\mathcal{E}^M \subseteq \mathcal{F}^M$ ; (ii)  $\mathcal{F}^M \subseteq \mathcal{A}^M$ .

PROOF. (i) By Lemma 3.1(ii).

(ii) By Lemma 4.1(ii) and (iii),  $T^* \vdash \forall x(\mathcal{F}x \rightarrow \text{IFin}(x))$ .  $\square$

LEMMA 6.6. Assume  $\forall M \forall \circ^M (\langle M, \circ^M \rangle \models T \implies \mathcal{E}^M = \mathcal{A}^M)$ . Then:

- (i)  $T^+ \subseteq T^*$ ,
- (ii)  $T^+ = T^*$ .



PROOF. (i) Let  $\mathcal{B} \models T^*$ , with  $\mathcal{B} = \langle B, \circ^B, \mathcal{F}^B \rangle$ . Then  $\langle B, \circ^B \rangle \models T$ , and by assumption we have  $\mathcal{E}^B = \mathcal{A}^B$ . But then, by Lemma 6.5,  $\mathcal{F}^B = \mathcal{A}^B$ .

Therefore  $\mathcal{B} = \langle B, \circ^B, \mathcal{A}^B \rangle$ . Moreover, since  $\langle B, \circ^B, \mathcal{A}^B \rangle \models T^+$ , by Lemma 6.2(i), it follows that  $\mathcal{B} \models T^+$ .

(ii) By (i) and Lemma 6.1(iii).  $\square$

### 6.3. Extensions of maximal consistent mereological theories: examples

*Examples, step 1.* (ACI $_{n+1}$ ) If  $\langle M, \circ^M \rangle \models \text{ACI}_{n+1}$ , then  $\mathcal{E}^M = \mathcal{A}^M = M$  (for each  $n \in \mathbb{N}$ ).

(FCI) If  $\langle M, \circ^M \rangle \models \text{FCI}$ , then  $\mathcal{E}^M = \mathcal{A}^M = \emptyset$ .

(MCI $_{n+1}$ ) If  $\langle M, \circ^M \rangle \models \text{MCI}_{n+1}$ , then  $\mathcal{E}^M = \mathcal{A}^M = \{a \mid a \in M \wedge a \text{ is a part}^M \text{ of the sum}^M \text{ of all atoms}^M\} \subset M$  (for each  $n \in \mathbb{N}$ ).

The maximal element of  $M$  is not an element of  $\mathcal{E}^M$ .

(ACI $_{\omega}$ ) If  $\langle M, \circ^M \rangle \models \text{ACI}_{\omega}$ , then  $\mathcal{E}^M \subset \mathcal{A}^M = M$ .

The maximal element of  $M$  is not an element of  $\mathcal{E}^M$ .

(MCI $_{\omega}$  + {FUS $_{At}$ }) If  $\langle M, \circ^M \rangle \models \text{MCI}_{\omega} + \{\text{FUS}_{At}\}$ , then  $\mathcal{E}^M \subset \mathcal{A}^M = \{a \mid a \in M \wedge a \text{ is a part}^M \text{ of the sum}^M \text{ of all atoms}^M\} \subset M$ .

The sum $^M$  of all atoms $^M$  is an element of  $\mathcal{A}^M$ , but not of  $\mathcal{E}^M$ . The maximal element of  $M$  is not an element of  $\mathcal{A}^M$ .

*Examples, step 2.* By Lemma 6.6 and the examples, step 1, we obtain:

(ACI $_{n+1}$ ) ACI $_{n+1}^+ = \text{ACI}_{n+1}^*$  (for each  $n \in \mathbb{N}$ ).

(FCI) FCI $^+ = \text{FCI}^*$ .

(MCI $_{n+1}$ ) MCI $_{n+1}^+ = \text{MCI}_{n+1}^*$  (for each  $n \in \mathbb{N}$ ).

Therefore, by Lemma 6.1, each of theories ACI $_{n+1}^*$ , FCI $^*$ , and MCI $_{n+1}^*$  is maximal consistent (in  $L[\circ, \mathcal{F}]$ ), finitely axiomatizable and decidable.

*Examples, step 3.* (ACI $_{n+1}$ ) ACI $_{n+1}^+ = \text{ACI}_{n+1} + \{\forall x \mathcal{F}x\}$  (for each  $n \in \mathbb{N}$ ).

PROOF. Choose a model  $\langle M, \circ^M \rangle$  of ACI $_{n+1}$ . Then, by Theorem 2.1(iii) and Lemmas 6.1(ii) and 6.2(i), plus Examples, step 1:

$$\text{ACI}_{n+1}^+ = \text{Th}(\langle M, \circ^M, \mathcal{A}^M \rangle) = \text{Th}(\langle M, \circ^M, M \rangle).^{19}$$

Since  $\langle M, \circ^M, M \rangle \models \forall x \mathcal{F}x$ , it follows that ACI $_{n+1}^+ \vdash \forall x \mathcal{F}x$ .

Moreover, ACI +  $\{\forall x \mathcal{F}x\} \vdash \forall x \mathcal{F}x$ , and ACI +  $\{\forall x \mathcal{F}x\} \vdash \forall x \text{IFin}(x)$ , by Lemma 4.2(ii). Taken together:

<sup>19</sup> If  $\mathcal{B}$  is a structure for language  $L$ ,  $\text{Th}(\mathcal{B}) := \{\psi \mid \psi \text{ is a sentence from } L \wedge \mathcal{B} \models \psi\}$ .

$ACI_{n+1} + \{\forall x \mathcal{F}x\} \vdash \forall x(\mathcal{F}x \leftrightarrow IFin(x))$ . □

By the same kind of reasoning, it can be shown that:

- (FCI)  $FCI^+ = FCI + \{\forall x \neg \mathcal{F}x\}$ ,
- ( $ACI_\omega$ )  $ACI_\omega^+ = ACI_\omega + \{\forall x \mathcal{F}x\}$ .

### 6.4. Extensions of $ACI_\omega^*$

The results of the previous subsections imply for each consistent mereological theory  $T$ : if  $U$  is a maximal consistent extension of  $T^*$  (in  $L[\circ, \mathcal{F}]$ ), then  $U = FCI^*$ , or  $U = ACI_n^*$ , or  $U = MCI_n^*$  (for some  $n \geq 1$ ); or  $U$  is a maximal consistent extension of  $ACI_\omega^*$  or  $MCI_\omega^* + \{FUS_{At}\}$ . But which theories are the maximal consistent extensions (in  $L[\circ, \mathcal{F}]$ ) of these two?  $ACI_\omega^+$  and  $MCI_\omega^+ + \{FUS_{At}\}$  must be among them; but it turns out that there are more.

Let me say a few words about the maximal consistent extensions of  $ACI_\omega^*$ .

*First*, as has just been pointed out,  $ACI_\omega^* + \{\forall x \mathcal{F}x\}$  is a maximal consistent extension of  $ACI_\omega^*$  (in  $L[\circ, \mathcal{F}]$ ).

*Second*,  $ACI_\omega^* + \{\neg \forall x \mathcal{F}x\}$  is consistent, and therefore a proper consistent extension of  $ACI_\omega^*$ . For consider  $\langle \wp(\mathbb{N}) \setminus \{\emptyset\}, \circ^{\wp(\mathbb{N}) \setminus \{\emptyset\}} \rangle$ , with:

$$A \circ^{\wp(\mathbb{N}) \setminus \{\emptyset\}} B := \iff A \cap B \neq \emptyset \text{ (for } A, B \subseteq \mathbb{N}, A, B \neq \emptyset \text{)}.$$

This is a model of  $ACI_\omega$ , whence  $\mathcal{P} := \langle \wp(\mathbb{N}) \setminus \{\emptyset\}, \circ^{\wp(\mathbb{N}) \setminus \{\emptyset\}}, \mathcal{E}^{\wp(\mathbb{N}) \setminus \{\emptyset\}} \rangle \models ACI_\omega^*$  (by Lemma 6.3(i)). But also  $\mathcal{P} \models \neg \forall x \mathcal{F}x$  (see Examples, step 1).

*Third*:  $ACI_\omega^* + \{\neg \forall x \mathcal{F}x\}$  is not maximal consistent (in  $L[\circ, \mathcal{F}]$ ).

For  $ACI_\omega^* + \{\neg \forall x \mathcal{F}x\}$  does not decide the sentence:

$$(DIS) \quad \forall x(\neg \forall y y \sqsubseteq x \rightarrow \mathcal{F}x \vee \mathcal{F}(-x)).$$

On the one side, (DIS) is false in  $\mathcal{P}$  (specialize “ $x$ ” to the set of even numbers). On the other side, it holds in this structure:  $\mathcal{FC} := \langle FC, \circ^{FC}, \mathcal{E}^{FC} \rangle$ , with  $FC := \{A \subseteq \mathbb{N} \mid A \text{ is finite} \vee \mathbb{N} \setminus A \text{ is finite}\} \setminus \{\emptyset\}$  and  $A \circ^{FC} B := \iff A \cap B \neq \emptyset$  (for  $A, B \in FC$ ).

$\langle FC, \circ^{FC} \rangle$  is induced by a Boolean algebra, is atomistic and has infinitely many atoms (i.e., the singletons  $\{k\}$ , with  $k \in \mathbb{N}$ ). Therefore, it is a model of  $ACI_\omega$ , and  $\mathcal{FC} \models ACI_\omega^*$ . Moreover, if  $A \in FC$  satisfies “ $\neg \forall y y \sqsubseteq x$ ” (in  $\mathcal{FC}$ ), it is either finite, or it has a finite complement  $\mathbb{N} \setminus A$ , which is also an element of  $FC$ . Now,  $X (\in FC)$  is finite iff  $X$  is a finite union of singletons of natural numbers, which means that  $X$  is a

finite sum <sup>$\mathcal{FC}$</sup>  of atoms <sup>$\mathcal{FC}$</sup> , i.e., that  $X$  is an element of  $\mathcal{E}^{\mathcal{FC}}$ . Therefore,  $A \in \mathcal{E}^{\mathcal{FC}}$  or  $\mathbb{N} \setminus A \in \mathcal{E}^{\mathcal{FC}}$ ; which shows that (DIS) holds in  $\mathcal{FC}$ .

In sum,  $\text{ACI}_\omega^*$  is a proper subtheory of  $\text{ACI}_\omega^+$ ; and there are at least three maximal consistent extensions of  $\text{ACI}_\omega^*$  (in  $L[\circ, \mathcal{F}]$ ):  $\text{ACI}_\omega^+$ ,  $\text{Th}(\mathcal{P})$  and  $\text{Th}(\mathcal{FC})$ .

### 6.5. Relative interpretability

When a language  $L$  is extended to a language  $L^+$  by adding new vocabulary, this is often done with the aim of enriching the expressional resources of  $L$ : in  $L^+$ , it should be possible to express more, to make finer distinctions. Yet, what is a language? Let's deal with  $L[\circ]$  and its extension  $L[\circ, \mathcal{F}]$ . If  $L[\circ]$  and  $L[\circ, \mathcal{F}]$  are individuated by their vocabularies (which has been taken for granted in the present paper), it should be plausible that the second language is richer (at least not poorer) than the first one. But now consider two theories:  $\text{ZF}^\circ$ , which results from  $\text{ZF}$  by replacing “ $\in$ ” by “ $\circ$ ” and is therefore formulated in  $L[\circ]$ ; and the set  $\text{PL}_1$  of first-order logical truths stated in  $L[\circ, \mathcal{F}]$ . Certainly,  $\text{ZF}^\circ$  is stronger than  $\text{PL}_1$  (in whatever way “stronger” may be reasonably understood here). And, using  $\text{ZF}^\circ$ , it should be possible to express more than by employing merely logical truths. So what happened to the supposed gain in expressional richness when moving from  $L[\circ]$  to  $L[\circ, \mathcal{F}]$ ?

Inspired by this, admittedly sketchily presented, line of thought let me not deal with  $L[\circ]$  and  $L[\circ, \mathcal{F}]$ , but rather with the framework which is given by both  $L[\circ]$  and the mereological theories  $T$ , and in addition with the framework supplied by  $L[\circ, \mathcal{F}]$  together with the extensions of mereological theories formulated therein. I suggest to formulate the claim that the second framework constitutes an enrichment of the expressional resources of the first one as follows:

Among the consistent extensions of mereological theories in  $L[\circ, \mathcal{F}]$ , there is at least one  $T$  such that for all consistent mereological theories  $S$ ,  $T$  is not reducible to  $S$ .

If this cannot be attained, everything that can be done in such an extension of a mereological theory can be simulated in a mereological theory itself. The extension to the new framework which rests on  $L[\circ, \mathcal{F}]$  becomes superfluous.

In my opinion, the best precise *explicans* for “(a theory)  $S$  is reducible to (a theory)  $T$ ” is “ $S$  is relatively interpretable to  $T$ ” (or something close

to it; reasons are given Bonevac [1982] and Niebergall [2000]). In what follows, let me therefore investigate this:

*Conjecture.* Among the consistent extensions of mereological theories in  $L[\circ, \mathcal{F}]$ , there is at least one  $T$  such that for all consistent mereological theories  $S$ ,  $T$  is not relatively interpretable in  $S$ .

Given what has been shown in the present paper, it is clear that for each consistent mereological theory  $T$ ,  $T^+$  is a subtheory of a definitional extension of  $T$  (employing the definition “ $\mathcal{F}x : \leftarrow \text{IFin}(x)$ ”). Thus, by Lemma 6.1(iii) and well known results on relative interpretability,  $T$ ,  $T^*$  and  $T^+$  are relatively interpretable in each other. Here, we find no enrichment or strengthening of the original framework; the theories  $T^*$  and  $T^+$  deliver no witnesses for the conjecture.

The conjecture is nonetheless true. An example is provided by  $\text{ACI}_\omega^* + \{\neg\forall x \mathcal{F}x\}$ .

*First*, by [Niebergall, 2009a, Theorem 5],  $\text{ACI}_\omega \not\leq \text{ACI}_k$ ,  $\text{ACI}_\omega \not\leq \text{MCI}_k$  (for  $k \geq 1$ ),  $\text{ACI}_\omega \not\leq \text{FCI}$ ; whence by what has just been remarked  $\text{ACI}_\omega^* + \{\neg\forall x \mathcal{F}x\} \not\leq \text{ACI}_k$ ,  $\text{ACI}_\omega^* + \{\neg\forall x \mathcal{F}x\} \not\leq \text{MCI}_k$  (for  $k \geq 1$ ),  $\text{ACI}_\omega^* + \{\neg\forall x \mathcal{F}x\} \not\leq \text{FCI}$ .

*Second*, only the relation between  $\text{ACI}_\omega^* + \{\neg\forall x \mathcal{F}x\}$  and  $\text{ACI}_\omega$  or  $\text{MCI}_\omega + \{\text{FUS}_{At}\}$  needs to be discussed. Here, a lemma about  $\text{ACI}_\omega + \{\neg\forall x \mathcal{F}x, (\text{AxI } 1), (\text{AxI } 2), (\text{AxI } 3)\}$  is helpful.

LEMMA 6.7. (i) For each  $n \in \mathbb{N}$ ,

$$\text{ACI} + \{\neg\forall x \mathcal{F}x, (\text{AxI } 1), (\text{AxI } 2), (\text{AxI } 3)\} \vdash \exists^n \text{At}.$$

$$(ii) \text{ACI}_\omega + \{\neg\forall x \mathcal{F}x, (\text{AxI } 1), (\text{AxI } 2), (\text{AxI } 3)\} = \text{ACI} + \{\neg\forall x \mathcal{F}x, (\text{AxI } 1), (\text{AxI } 2), (\text{AxI } 3)\}.$$

(iii)  $\text{ACI}_\omega + \{\neg\forall x \mathcal{F}x, (\text{AxI } 1), (\text{AxI } 2), (\text{AxI } 3)\}$  is finitely axiomatizable.<sup>20</sup>

PROOF. (i) follows from Lemma 3.2. (ii) by (i). (iii) by (ii).  $\square$

COROLLARY 6.1. (i)  $\text{ACI}_\omega^* + \{\neg\forall x \mathcal{F}x\} \not\leq \text{ACI}_\omega$ .

(ii)  $\text{ACI}_\omega^* + \{\neg\forall x \mathcal{F}x\} \not\leq \text{MCI}_\omega + \text{FUS}_{At}$ .

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<sup>20</sup>  $\text{ACI}_\omega^*$  and  $\text{ACI}_\omega^+$ , and also  $\text{MCI}_\omega^* + \text{FUS}_{At}$  and  $\text{MCI}_\omega^+ + \text{FUS}_{At}$ , are not finitely axiomatizable.

If, for example,  $\text{ACI}_\omega^*$  or  $\text{ACI}_\omega^+$  were finitely axiomatizable, each of them would be a subtheory of some  $\text{ACI}_{n+1}^+$ . Since this theory is relatively interpretable in  $\text{ACI}_{n+1}$ ,  $\text{ACI}_\omega^*$ , and  $\text{ACI}_\omega^+$  would eventually be relatively interpretable in  $\text{ACI}_{n+1}$  — which is not the case.

PROOF. (ii)<sup>21</sup> Assume  $\text{ACI}_\omega^* + \{\neg\forall x \mathcal{F}x\} \preceq \text{MCI}_\omega + \text{FUS}_{At}$ . Then  $\text{ACI}_\omega + \{\neg\forall x \mathcal{F}x, (\text{AxI } 1), (\text{AxI } 2), (\text{AxI } 3)\} \preceq \text{MCI}_\omega + \text{FUS}_{At}$ , and there is a  $k \in \mathbb{N}$  such that  $\text{ACI}_\omega + \{\neg\forall x \mathcal{F}x\} \preceq \text{MCI}_k + \text{FUS}_{At}$ , since by Lemma 6.7(iii),  $\text{ACI}_\omega + \{\neg\forall x \mathcal{F}x, (\text{AxI } 1), (\text{AxI } 2), (\text{AxI } 3)\}$  is finitely axiomatizable.

By Theorem 2.1(i),  $\text{MCI}_k + \text{FUS}_{At} = \text{MCI}_k$ , and therefore  $\text{ACI}_\omega + \{\neg\forall x \mathcal{F}x\} \preceq \text{MCI}_k$ .  $\square$

Yet this contradicts *first* from above.

COROLLARY 6.2. *There is no consistent mereological theory in which  $\text{ACI}_\omega^* + \{\neg\forall x \mathcal{F}x\}$  is relatively interpretable.*

PROOF. If  $T$  were a consistent mereological theory such that  $\text{ACI}_\omega^* + \{\neg\forall x \mathcal{F}x\} \preceq T$ , then  $\text{ACI}_\omega^* + \{\neg\forall x \mathcal{F}x\}$  would be relatively interpretable in a maximal consistent extension of  $T$  (in  $L[\circ]$ ). Yet this is excluded by *first* and *second* (i.e., Corollary 6.1) in conjunction with Theorem 2.1(iii).  $\square$

## 7. A second-order treatment of “ $x$ is finite”

Since the set-theoretic *definiens* of “ $x$  is finite” mentioned in Section 4 uses only two levels of the von Neumann hierarchy, it should be possible to formulate it in a second-order language. In this section, I will give such a definition in  $L^2[\circ]$ , the monadic second-order extension of  $L[\circ]$ , and address some of its consequences.<sup>22</sup>

### 7.1. The definition of “ $x$ is finite” in $L^2[\circ]$

$L^2[\circ]$  is the second-order language which results from  $L[\circ]$  through the addition of *one-place* second-order variables (“ $X$ ”, ...); in particular,  $L^2[\circ]$  and  $L[\circ]$  have the same vocabulary. In  $L^2[\circ]$ , we have classical second-order logic: roughly put, the axioms and rules known from common axiomatizations of first-order logic (with *modus ponens* and generalization as rules of inference) are transferred from first-order to second-order variables. In particular, with these, “ $\Sigma \vdash_2 \psi$ ” is defined for formulas  $\psi$  and sets of formulas  $\Sigma$  from  $L^2[\circ]$  along the common lines.<sup>23</sup>

<sup>21</sup> The proof of (i) is similar to the proof of (ii), but easier.

<sup>22</sup> At several places, this section builds on [Niebergall, 2008b].

<sup>23</sup> For explicitnes, one may consider Shapiro [1991], for example.

Sometimes, instances of the *comprehension schema*, that is,  $L^2[o]$ -formulas of the form:

$$\exists X \forall z (Xz \longleftrightarrow \psi(z))$$

(or their universal closures), where  $\psi$  is from  $L^2[o]$  and  $X$  does not occur in  $\psi$ , are also regarded as logical truths of second-order logic. They will be included in most of the theories (in  $L^2[o]$ ) considered in this section.

Relative to extensions of CI,<sup>24</sup> “ $x$  is finite” can now be defined as follows in  $L^2[o]$ :

DEFINITION 7.1.  $Fin^2(x) := \longleftrightarrow \forall Y (\forall z (At(z) \rightarrow Yz) \wedge \forall z z' (Yz \wedge Yz' \rightarrow Y(z \sqcup z'))) \rightarrow Yx$ .

## 7.2. Metatheorems and adequacy results

In my opinion, Definition 7.1 is from an intuitive perspective *the* natural definition in  $L^2[o]$  of “ $x$  is finite”. The same can certainly not be claimed for the definition that can be found in [Lewis, 1991], which amounts to:

$$Fin^L(x) := \longleftrightarrow \neg \exists X (\exists y Xy \wedge x = \bigvee X \wedge \forall y (Xy \rightarrow \exists z (Xz \wedge y \sqsubseteq z \wedge \neg z \sqsubseteq y))).$$
<sup>25</sup>

In addition, Definition 7.1 has some salient consequences of a partly formal nature, which also suggest its adequacy.

First, as was to be expected, it straightforwardly yields the axioms of finiteness from Section 3. Actually, given Definition 7.1, they follow from quite weak axioms stated in  $L^2[o]$ . More explicitly, let  $\psi$  be a formula in  $L^2[o]$  which does not contain the variable  $X$ ; set:

$$\text{Comp}_\psi := (\text{the universal closure of } \exists X \forall z (Xz \leftrightarrow \psi(z)),$$

$$\text{Comp} := \{\text{Comp}_\psi \mid \psi \text{ is an } X\text{-free } L^2[o]\text{-formula}\}.$$

Then we have:

- LEMMA 7.1. (i)  $CI \vdash_2 \forall x (At(x) \rightarrow Fin^2(x))$ ;  
(ii)  $CI \vdash_2 \forall xy (Fin^2(x) \wedge Fin^2(y) \rightarrow Fin^2(x \sqcup y))$ ;  
(iii)  $CI \cup \text{Comp} \vdash_2 \forall z (At(z) \rightarrow \psi(z)) \wedge \forall z z' (\psi(z) \wedge \psi(z') \rightarrow \psi(z \sqcup z')) \rightarrow \forall x (Fin^2(x) \rightarrow \psi(x))$  (for each formula  $\psi$  from  $L^2[o]$ ).

PROOF. (i) Purely logically,  $CI \vdash_2 At(x) \rightarrow (\forall z (At(z) \rightarrow Yz) \wedge \forall z z' (Yz \wedge Yz' \rightarrow Y(z \sqcup z'))) \rightarrow Yx$ .

<sup>24</sup> The notation used in Definition 7.1 presupposes SUM.

<sup>25</sup> With  $x = \bigvee X := \longleftrightarrow \forall y (y \circ x \leftrightarrow \exists z (Xz \wedge y \circ z))$ .

Universal generalization with respect to “ $Y$ ” and distribution yield  $\text{CI} \vdash_2 At(x) \rightarrow \forall Y(\forall z(At(z) \rightarrow Yz) \wedge \forall zz'(Yz \wedge Yz' \rightarrow Y(z \sqcup z'))) \rightarrow Yx$ .

(ii)  $\text{CI} \vdash_2 \forall Y(\forall z(At(z) \rightarrow Yz) \wedge \forall zz'(Yz \wedge Yz' \rightarrow Y(z \sqcup z'))) \rightarrow Yx) \wedge \forall Y(\forall z(At(z) \rightarrow Yz) \wedge \forall zz'(Yz \wedge Yz' \rightarrow Y(z \sqcup z'))) \rightarrow Yy) \wedge \forall z(At(z) \rightarrow Yz) \wedge \forall zz'(Yz \wedge Yz' \rightarrow Y(z \sqcup z')) \rightarrow Yx \wedge Yy \wedge \forall zz'(Yz \wedge Yz' \rightarrow Y(z \sqcup z')) \rightarrow Y(x \sqcup y)$ . Whence  $\text{CI} \vdash_2 Fin^2(x) \wedge Fin^2(y) \rightarrow (\forall z(At(z) \rightarrow Yz) \wedge \forall zz'(Yz \wedge Yz' \rightarrow Y(z \sqcup z'))) \rightarrow Y(x \sqcup y)$ . Therefore  $\text{CI} \vdash_2 Fin^2(x) \wedge Fin^2(y) \rightarrow Fin^2(x \sqcup y)$ .

(iii) Let  $\psi(u)$  be a formula from  $L^2[o]$ . By comprehension with respect to  $\psi$  we have:  $\text{CI} \cup \text{Comp} \vdash_2 Fin^2(x) \rightarrow \forall Y(\forall z(At(z) \rightarrow Yz) \wedge \forall zz'(Yz \wedge Yz' \rightarrow Y(z \sqcup z'))) \rightarrow Yx) \wedge \exists Y \forall u(Yu \leftrightarrow \psi(u)) \rightarrow \exists Y [\forall u(Yu \leftrightarrow \psi(u)) \wedge (\forall z(At(z) \rightarrow Yz) \wedge \forall zz'(Yz \wedge Yz' \rightarrow Y(z \sqcup z'))) \rightarrow Yx]$ . Whence  $\text{CI} \cup \text{Comp} \vdash_2 Fin^2(x) \rightarrow (\forall z(At(z) \rightarrow \psi(z)) \wedge \forall zz'(\psi(z) \wedge \psi(z') \rightarrow \psi(z \sqcup z'))) \rightarrow \psi(x)$ . Therefore  $\text{CI} \cup \text{Comp} \vdash_2 \forall z(At(z) \rightarrow \psi(z)) \wedge \forall zz'(\psi(z) \wedge \psi(z') \rightarrow \psi(z \sqcup z')) \rightarrow \forall x(Fin^2(x) \rightarrow \psi(x))$ .  $\square$

Second, there is a model theoretic adequacy result for Definition 7.1. Since it is dependent on the distinction between standard second-order models and generalized second-order models, let me quickly recapitulate that distinction. Thus, let  $\psi$  be a formula from  $L^2[o]$ : and let’s discuss how to define “variable assignment  $\beta$  satisfies  $\psi$  in structure  $\mathcal{M}$ ”. Similarly to the first-order case, such a structure has to provide domains of the variables of  $L^2[o]$  and an interpretation function  $\mathcal{I}$ . More explicitly, there has to be a nonempty set  $M$  for the first-order variables, a nonempty set  $\Omega$  for the second-order variables, and we must have  $\mathcal{I}(o) \subseteq M^2$ . In addition, since there are only monadic second-order variables in  $L^2[o]$ ,  $\Omega$  should be a subset of  $\wp(M)$ .

Accordingly, one can define *generalized second-order structures* (in short: *g2-structures*) as follows:

$$x \text{ is a g2-structure} \iff \exists M \Omega \mathcal{I}(x = \langle M, \Omega, \mathcal{I} \rangle \wedge M \neq \emptyset \wedge \emptyset \neq \Omega \subseteq \wp(M) \wedge \mathcal{I}(o) \subseteq M^2).$$

As regards variable assignments  $\beta$ , I assume that they have both first- and second-order variables in their domains, mapping the second-order variables to elements of  $\Omega$ . Given that, satisfaction in a g2-structure is defined as in the first-order case, with the following clauses added:

$$\begin{aligned} \langle M, \Omega, \mathcal{I} \rangle, \beta \models Xy &\iff \beta(y) \in \beta(X), \\ \langle M, \Omega, \mathcal{I} \rangle, \beta \models \forall X \psi &\iff \forall C(C \in \Omega \Rightarrow \langle M, \Omega, \mathcal{I} \rangle, \beta(X : C) \models \psi). \end{aligned}$$

A special case, which is sometimes regarded as particularly natural and important, obtains when  $\Omega = \wp(M)$ . Here, we have the so-called “standard second-order structures” (in short: s2-structures):

$x$  is a s2-structure  $:\Leftrightarrow$

$$\exists M\mathcal{I}(x = \langle M, \wp(M), \mathcal{I} \rangle \wedge M \neq \emptyset \wedge \mathcal{I}(\circ) \subseteq M^2).$$

Since each s2-structure is a g2-structure, satisfaction in a s2-structure is already defined. Let me merely reformulate the quantifier-case:

$$\begin{aligned} \langle M, \wp(M), \mathcal{I} \rangle, \beta \models \forall X\psi &\iff \\ \forall C(C \subseteq M \implies \langle M, \wp(M), \mathcal{I} \rangle, \beta(X : C) \models \psi). \end{aligned}$$

Consequence with respect to g2-structures and with respect to s2-structures is then defined as follows (for  $L^2[o]$ -sentences  $\psi$  and sets  $\Sigma$  of  $L^2[o]$ -sentences):

$$\begin{aligned} \Sigma \models^{g^2} \psi &:\iff \forall \mathcal{M}(\mathcal{M} \text{ is a } g^2\text{-structure} \implies (\mathcal{M} \models \Sigma \implies \mathcal{M} \models \psi)), \\ \Sigma \models^{s^2} \psi &:\iff \forall \mathcal{M}(\mathcal{M} \text{ is a } s^2\text{-structure} \implies (\mathcal{M} \models \Sigma \implies \mathcal{M} \models \psi)). \end{aligned} \quad ^{26}$$

What can be shown now is that for each s2-structure  $\mathcal{M}$  which is a model of CI,  $a$  is in the extension of “ $Fin^2$ ” in  $\mathcal{M}$  iff  $a$  has finitely many parts<sup>M</sup>. That is

LEMMA 7.2. *Let  $\mathcal{M}$  ( $= \langle \mathcal{M}, \Omega, \circ^{\mathcal{M}} \rangle$ ) be a g2-structure for  $L^2[o]$  which satisfies Ax(CI). Then (if  $\beta$  is an assignment over  $M$ ):*

$$\beta(x) \text{ is the sum}^M \text{ of finitely many atoms}^M \implies \mathcal{M}, \beta \models Fin^2(x).$$

PROOF. By assumption, there are  $a_1, \dots, a_k \in M$  such that:

$$(*) \ a_1, \dots, a_k \text{ are atoms}^M \wedge \beta(x) = \bigvee^M \{a_1, \dots, a_k\}.$$

Now let  $C \in \Omega$  be arbitrary, and assume:

$$(**) \ \mathcal{M}, \beta(Y : C) \models \forall z(At(z) \rightarrow Yz),$$

$$(***) \ \mathcal{M}, \beta(Y : C) \models \forall zz'(Yz \wedge Yz' \rightarrow Y(z \sqcup z')).$$

By (\*) and (\*\*):  $\mathcal{M}, \beta(Y : C)(z_1 : a_1) \models Yz_1, \dots, \mathcal{M}, \beta(Y : C)(z_k : a_k) \models Yz_k$ . And these imply by repeated use of (\*\*\*):  $\mathcal{M}, \beta(Y : C)(z_1 : a_1) \dots (z_k : a_k) \models Y(z_1 \sqcup \dots \sqcup z_k)$ . Whence  $\bigvee^M \{a_1, \dots, a_k\} \in C$  and, by (\*),  $\mathcal{M}, \beta(Y : C) \models Yx$ .  $\square$

<sup>26</sup> We have the completeness theorem:  $\Sigma \vdash_2 \psi \iff \Sigma \models^{g^2} \psi$ .



LEMMA 7.3. Let  $\mathcal{M}$  ( $= \langle \mathcal{M}, \wp(\mathcal{M}), \circ^{\mathcal{M}} \rangle$ ) be a s2-structure for  $L^2[\circ]$  which satisfies Ax(CI). Then (if  $\beta$  is an assignment over  $M$ ):<sup>27</sup>

$$\mathcal{M}, \beta \models Fin^2(x) \implies \beta(x) \text{ is the sum}^M \text{ of finitely many atoms}^M.$$

PROOF. If  $\mathcal{M}, \beta \models Fin^2(x)$ , then  $\forall C(C \subseteq M \implies \mathcal{M}, \beta(Y : C) \models \forall z(At(z) \rightarrow Yz) \wedge \forall zz'(Yz \wedge Yz' \rightarrow Y(z \sqcup z')) \rightarrow Yx)$ .

Now specialize “ $C$ ” to  $B$ :

$$B := \{b \in M \mid \exists a_1 \dots a_k (a_1, \dots, a_k \text{ are atoms}^M \wedge b = \bigvee^M \{a_1, \dots, a_k\})\}.$$

Then we obtain (\*)  $\mathcal{M}, \beta(Y : B) \models \forall z(At(z) \rightarrow Yz) \wedge \forall zz'(Yz \wedge Yz' \rightarrow Y(z \sqcup z')) \rightarrow Yx$ . Moreover, since obviously (\*\*)  $\mathcal{M}, \beta(Y : B) \models \forall z(At(z) \rightarrow Yz)$  and  $\mathcal{M}, \beta(Y : B) \models \forall zz'(Yz \wedge Yz' \rightarrow Y(z \sqcup z'))$ , it follows with (\*) and (\*\*) that  $\mathcal{M}, \beta(Y : B) \models Yx$ . That is,  $\beta(x)$  is the sum <sup>$M$</sup>  of finitely many atoms <sup>$M$</sup> . □

COROLLARY 7.1. (i) Let  $\mathcal{M}$  ( $= \langle \mathcal{M}, \Omega, \circ^{\mathcal{M}} \rangle$ ) be a g2-structure for  $L^2[\circ]$  which satisfies Ax(CI), and let  $\beta$  be an assignment over  $M$ .

Then:  $\{a \in M \mid a \sqsubseteq^M \beta(x)\}$  is finite  $\implies \mathcal{M}, \beta \models Fin^2(x)$ .

(ii) Let  $\mathcal{M}$  ( $= \langle \mathcal{M}, \wp(\mathcal{M}), \circ^{\mathcal{M}} \rangle$ ) be a s2-structure for  $L^2[\circ]$  which satisfies Ax(CI), and let  $\beta$  be an assignment over  $M$ . Then:  $\mathcal{M}, \beta \models Fin^2(x) \implies \{a \in M \mid a \sqsubseteq^M \beta(x)\}$  is finite.

PROOF. It can be shown that:  $\{a \in M \mid a \sqsubseteq^M \beta(x)\}$  is finite iff  $\beta(x)$  is the sum <sup>$M$</sup>  of finitely many atoms <sup>$M$</sup> . □

Let’s compare now Definition 7.1 with the one given in Lewis 1991 with respect to Lemmas 7.2 and 7.3, and Corollary 7.1.

First, analogues of these lemmas, can be shown for “ $Fin^L(x)$ ” [see Niebergall, 2008b, lemmas 23 and 24]. Yet, Lemma 24, in particular, is: Let  $\mathcal{M}$  ( $= \langle \mathcal{M}, \wp(\mathcal{M}), \circ^{\mathcal{M}} \rangle$ ) be a s2-structure for  $L^2[\circ]$  which satisfies Ax(ACI) and FUS-Ax, and let  $\beta$  be an assignment over  $M$ . Then:  $\mathcal{M}, \beta \models Fin^L(x) \implies \{a \in M \mid a \sqsubseteq^M \beta(x)\}$  is finite.

Thus, we have additional assumptions here when compared with Lemma 7.3 or Corollary 7.1: there are “fewer” s2-structures in which “ $Fin^L(x)$ ” “expresses” finiteness.

Now, I see a decent chance to prove a version of Lemma 24 where “Ax(ACI)” is replaced by “Ax(CI)”. But I have no idea how one could get rid of the assumption that FUS-Ax has to hold in  $\mathcal{M}$ .

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<sup>27</sup> In this case, we also have:  $\mathcal{M}, \beta \models Fin^2(x) \implies \mathcal{M}, \beta \models \exists x At(x)$ .

FUS-Ax is the *second-order fusion axiom*:

$$\forall X(\exists x Xx \rightarrow \exists z\forall y(z \circ y \leftrightarrow \exists x(x \circ y \wedge Xx))).$$

As a mereological principle, I take it to be as plausible as the first-order schema FUS. Therefore, for s2-structures, “ $Fin^L(x)$ ” may turn out to be almost as good as “ $Fin^2(x)$ ”. But what is the relation between “ $Fin^L(x)$ ” and “ $Fin^2(x)$ ” in general, i.e., with respect to arbitrary g2-structures?

It is of importance here that an analogue of Lemma 7.1, with “ $Fin^L(x)$ ” replacing “ $Fin^2(x)$ ”, can be established:

LEMMA 7.4. (i)  $ACI \cup \text{Comp} \cup \{\text{FUS-Ax}\} \vdash_2 \forall x(At(x) \rightarrow Fin^L(x))$ ;  
 (ii)  $ACI \cup \text{Comp} \cup \{\text{FUS-Ax}\} \vdash_2 \forall xy(Fin^L(x) \wedge Fin^L(y) \rightarrow Fin^L(x \sqcup y))$ ;  
 (iii)  $ACI \cup \text{Comp} \cup \{\text{FUS-Ax}\} \vdash_2 \forall z(At(z) \rightarrow \psi(z)) \wedge \forall z z'(\psi(z) \wedge \psi(z') \rightarrow \psi(z \sqcup z')) \rightarrow \forall x(Fin^L(x) \rightarrow \psi(x))$  (for each formula  $\psi$  from  $L^2[o]$ ).

This lemma (which is stated here without proof) has been obtained by Werner (unpublished). Let me remark that Werner’s proof is much more complicated than the one for Lemma 7.1. Moreover, it is not easy to see whether the use of FUS-Ax could be eliminated from it. In this sense, Definition 7.1 is again superior to the definition from [Lewis, 1991], at least from a practical point of view. Yet on a more abstract level, there is not much to choose between Definition 7.1 for “ $x$  is finite” and the one found by Lewis. For as a direct consequence of lemmas 7.1 and 7.4 (cf. Lemma 4.6, the characterization lemma), one obtains

COROLLARY 7.2.  $ACI \cup \text{Comp} \cup \{\text{FUS-Ax}\} \vdash_2 \forall x(Fin^L(x) \leftrightarrow Fin^2(x))$ .

### 7.3. Connection with extensions of $ACI_\omega^*$

In this subsection, results about extensions of mereological theories in  $L^2[o]$  are used as a means to shed some light on the maximal consistent extensions of  $ACI_\omega$  in  $L[o, \mathcal{F}]$ .

Given earlier remarks in this section, I take it that analogues of mereological theories formulated in  $L^2[o]$  should contain mereological theories, Comp and FUS-Ax. Moreover, they should be “closed”; that is, closed under g2-consequence or closed under s2-consequence. Only s2-closure is relevant for the present paper; it is defined as follows (if  $\Sigma$  is a set of  $L^2[o]$ -sentences; “Comp” need not be added, because  $\text{Comp} \subseteq \Sigma^{s2}$  anyway):

$$\Sigma^{s2} := \{\psi \mid \psi \text{ is a sentence from } L^2[o] \wedge \Sigma \cup \text{CI} \cup \{\text{FUS-Ax}\} \models^{s2} \psi\}.$$

In addition, I assume that Definition 7.1 is in force.

Thus, let us deal with supersets of  $((\text{ACI}_\omega)^{\text{s}2})^D$ , where  $((\text{ACI}_\omega)^{\text{s}2})^D$  being  $(\text{ACI}_\omega)^{\text{s}2}$ , extended by “ $\forall x(\mathcal{F}x \leftrightarrow \text{Fin}^2(x))$ ”. By Lemma 7.1,  $\text{ACI}_\omega^* \subseteq ((\text{ACI}_\omega)^{\text{s}2})^D$ . But also, “ $\neg \forall x \mathcal{F}x$ ” belongs to  $((\text{ACI}_\omega)^{\text{s}2})^D$ . From this point of view,  $\text{ACI}_\omega^+$  would be excluded as a natural maximal consistent extension of  $\text{ACI}_\omega^*$ . Given Section 6.4, let's therefore deal only with  $\text{Th}(\mathcal{P})$  and  $\text{Th}(\mathcal{FC})$ .

I start with some general lemmas.

LEMMA 7.5. *Let  $\mathcal{M}$  ( $= \langle \mathcal{M}, \wp(\mathcal{M}), \circ^{\mathcal{M}} \rangle$ ) be a s2-structure for  $L^2[\circ]$  which satisfies Ax(CI). Then:*

- (i)  $\mathcal{M}, \beta(x : a) \models \text{Fin}^2(x) \iff a \in \mathcal{E}^{\mathcal{M}}$ .
- (ii)  $\langle \mathcal{M}, \wp(\mathcal{M}), \circ^{\mathcal{M}}, \mathcal{E}^{\mathcal{M}} \rangle \models \forall x(\mathcal{F}x \leftrightarrow \text{Fin}^2(x))$ .<sup>28</sup>

PROOF. (i) By lemmas 7.2 and 7.3. (ii) For every  $a \in M$ , by (i):

$$\begin{aligned} \langle \mathcal{M}, \wp(\mathcal{M}), \circ^{\mathcal{M}}, \mathcal{E}^{\mathcal{M}} \rangle, \beta(x : a) \models \mathcal{F}x &\iff a \in \mathcal{E}^{\mathcal{M}} \\ &\iff \langle \mathcal{M}, \wp(\mathcal{M}), \circ^{\mathcal{M}} \rangle, \beta(x : a) \models \text{Fin}^2(x) \\ &\iff \langle \mathcal{M}, \wp(\mathcal{M}), \circ^{\mathcal{M}}, \mathcal{E}^{\mathcal{M}} \rangle, \beta(x : a) \models \text{Fin}^2(x). \square \end{aligned}$$

If  $\psi$  is a formula from  $L[\circ, \mathcal{F}]$ , then let  $\psi^{-2\mathcal{F}}$  result from  $\psi$  by replacing each occurrence of “ $\mathcal{F}x$ ” in  $\psi$  by “ $\text{Fin}^2(x)$ ” (for each variable  $x$ ).

LEMMA 7.6. (i)  $\psi^{-2\mathcal{F}}$  is a formula from  $L^2[\circ]$ . If  $\psi$  is a sentence, then  $\psi^{-2\mathcal{F}}$  is a sentence.

- (ii) If  $\psi$  is a formula from  $L[\circ, \mathcal{F}]$ ,  $\langle \mathcal{M}, \wp(\mathcal{M}), \circ^{\mathcal{M}}, \mathcal{E}^{\mathcal{M}} \rangle, \beta \models \psi \leftrightarrow \psi^{-2\mathcal{F}}$ .
- (iii) If  $\psi$  is a sentence from  $L[\circ, \mathcal{F}]$ , then  $\langle \mathcal{M}, \circ^{\mathcal{M}}, \mathcal{E}^{\mathcal{M}} \rangle \models \psi \iff \langle \mathcal{M}, \wp(\mathcal{M}), \circ^{\mathcal{M}} \rangle \models \psi^{-2\mathcal{F}}$ .

PROOF. (ii) By Lemma 7.5(ii) and induction on the built-up of  $\psi$ .

(iii) By (i) and (ii). □

Let  $\text{Th}^2(\langle \mathcal{M}, \Omega, \circ^{\mathcal{M}} \rangle)$  be the set of sentences from  $L^2[\circ]$  which hold in  $\langle \mathcal{M}, \Omega, \circ^{\mathcal{M}} \rangle$ . Then we have as a consequence of Lemma 7.6(iii):

COROLLARY 7.3.  $\text{Th}^2(\langle \wp(\mathbb{N}) \setminus \{\emptyset\}, \wp(\wp(\mathbb{N}) \setminus \{\emptyset\}), \circ^{\wp(\mathbb{N}) \setminus \{\emptyset\}} \rangle)$  extended by the definition “ $\mathcal{F}x := \leftrightarrow \text{Fin}^2(x)$ ” contains the same sentences from  $L[\circ, \mathcal{F}]$  as  $\text{Th}(\mathcal{P})$ .

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<sup>28</sup> The semantical vocabulary introduced for  $L^2[\circ]$  is supposed to be explained similarly for  $L^2[\circ, \mathcal{F}]$ .

Consider the  $L^2[o]$ -sentence:

$$(\text{CountAt}) \quad \forall Xx(x = \bigvee X \wedge \forall y(Xy \rightarrow \text{At}(y)) \wedge \neg \text{Fin}^2(x) \rightarrow \text{large}(x)),$$

with:

$$\begin{aligned} \text{large}(x) : \longleftrightarrow & \exists X[\exists y Xy \wedge \forall yz(Xy \wedge Xz \wedge y \circ z \rightarrow y = z) \wedge \\ & \exists y(y = \bigvee X \wedge \forall z z \sqsubseteq y) \wedge \\ & \forall y(Xy \rightarrow \exists_1 z(\text{At}(z) \wedge z \sqsubseteq y \wedge z \sqsubseteq x) \wedge \exists^{\leq 2} z(\text{At}(z) \wedge z \sqsubseteq y))]. \end{aligned}$$

In [Niebergall, 2008b, Theorem 26] it is shown that:

$$\text{Th}^2(\langle \wp(\mathbb{N}) \setminus \{\emptyset\}, \wp(\wp(\mathbb{N}) \setminus \{\emptyset\}), \circ^{\wp(\mathbb{N}) \setminus \{\emptyset\}} \rangle) = (\text{ACI}_\omega \cup \{(\text{CountAt})\})^{\text{s}2}.$$

With Corollary 7.3, this yields:

**COROLLARY 7.4.**  $(\text{ACI}_\omega \cup \{(\text{CountAt})\})^{\text{s}2}$  extended by the definition “ $\mathcal{F}x : \longleftrightarrow \text{Fin}^2(x)$ ” contains the same sentences from  $L[o, \mathcal{F}]$  as  $\text{Th}(\mathcal{P})$ .

Similarly, one can deal with  $\mathcal{FC}$  instead of  $\mathcal{P}$  and obtain from Lemma 7.6(iii):

**COROLLARY 7.5.**  $\text{Th}^2(\langle \text{FC}, \wp(\text{FC}), \circ^{\text{FC}} \rangle)$  extended by the definition “ $\mathcal{F}x : \longleftrightarrow \text{Fin}^2(x)$ ” contains the same sentences from  $L[o, \mathcal{F}]$  as  $\text{Th}(\mathcal{FC})$ .

Yet, what could be the analogue of Corollary 7.4 in this case?  $\text{Th}^2(\langle \wp(\mathbb{N}) \setminus \{\emptyset\}, \wp(\wp(\mathbb{N}) \setminus \{\emptyset\}), \circ^{\wp(\mathbb{N}) \setminus \{\emptyset\}} \rangle)$  was presented there in a quasi-axiomatic way as  $(\text{ACI}_\omega \cup \{(\text{CountAt})\})^{\text{s}2}$ . How could such a presentation for  $\text{Th}^2(\langle \text{FC}, \wp(\text{FC}), \circ^{\text{FC}} \rangle)$  look like? — The answer is: there is no such presentation (in  $L^2[o]$ ).

**LEMMA 7.7.** (i)  $\langle \text{FC}, \wp(\text{FC}), \circ^{\text{FC}} \rangle \not\models \text{FUS-Ax}$ .

(ii) There is no set of sentences  $\Sigma$  in  $L^2[o]$  such that  $\text{Th}^2(\langle \text{FC}, \wp(\text{FC}), \circ^{\text{FC}} \rangle) = \Sigma^{\text{s}2}$ .

**PROOF.** (i) If  $\langle \text{FC}, \wp(\text{FC}), \circ^{\text{FC}} \rangle \models \text{FUS-Ax}$ , it is a  $\text{s}2$ -structure which satisfies  $\text{ACI}$  plus  $\text{FUS-Ax}$ . But then, by Lemma 16 in [Niebergall, 2008b],  $\text{FC}$  plus an additional object is the domain of a complete Boolean algebra which is atomistic. In light of Lemma 21 in [Niebergall, 2008b], there exists then a set  $A$  — which must be infinite, since  $\langle \text{FC}, \wp(\text{FC}), \circ^{\text{FC}} \rangle \models \text{ACI}_\omega$  — such that  $\text{FC}$  plus that additional object has the same cardinality as  $\wp(A)$ . Therefore,  $\text{FC}$  itself must be uncountable; but is countable.

(ii) Assume there is such a set of sentences  $\Sigma$ . Since  $\text{Th}^2(\langle \text{FC}, \wp(\text{FC}), \circ^{\text{FC}} \rangle)$  holds in  $\langle \text{FC}, \wp(\text{FC}), \circ^{\text{FC}} \rangle$ , each element of  $\Sigma^{\text{s}2}$  must be true in  $\langle \text{FC}, \wp(\text{FC}), \circ^{\text{FC}} \rangle$  as well. Yet, FUS-Ax is such an element. This contradicts (i).  $\square$

In sum, given the considerations of this subsection, it should be  $\text{Th}(\mathcal{P})$  which is distinguished among the maximal consistent extensions of  $\text{ACI}_\omega^*$  known from Section 6.

## 8. Conclusion

The present paper dealt with mereological theories, as I have called them, and with theories belonging to a class  $\mathcal{C}$  of extensions of them (see Section 6). Yet why, it may be asked, are these theories philosophically interesting and worthy of study? I see three answers to this question:<sup>29</sup>

*First*, these theories are simply taken to be that: philosophically interesting and worthy of study; no further reason is given (and need be given).

*Second*, these theories are philosophically interesting and worthy of study because they are *nominalistic* theories.<sup>30</sup>

*Third*, these theories are philosophically interesting and worthy of study because they are non-mathematical theories which may be able to replace mathematical theories as parts of empirical theories.

Those who prefer the first answer may appreciate the development and study of extensions of mereological theories as a widening, refinement and strengthening of the *mereological paradigm*. However, since I lean towards the second and third answer, let me close this paper with some comments on them.

The second answer presupposes that the elements of  $\mathcal{C}$  deserve to be regarded as nominalistic theories. Now, it has to be granted that no accepted precise general *explicans* of “ $T$  is a nominalistic theory” is available. This notwithstanding, theories are known to us which are commonly accepted as nominalistic theories: among them are at least

<sup>29</sup> In particular, I agree that mereological theories need not be interpreted nominalistically and their study need not be motivated by nominalistic concerns.

<sup>30</sup> As regards the advantages of a nominalistic position over a platonistic one, I have no really new ingredients to add to what has been discussed by Goodman, Quine and their critics and followers (see [Goodman and Quine, 1947; Goodman, 1951] and, with an emphasis on nominalistic *theories*, [Niebergall, 2005]).

some of the mereological theories, such as Goodman's *calculus of individuals* [see [Goodman, 1951](#)], but also *token concatenation theories* [see [Goodman and Quine, 1947](#); [Niebergall, 2005](#)]. In fact, the first ones are the examples of nominalistic theories.

I follow those philosophers who take the nominalistic position to consist in the avoidance of the assumption of abstract objects. Then, the criterion for  $T$  being a nominalistic theory is roughly this: the preferred reading of the vocabulary of  $L[T]$  deals with concrete objects; and  $T$  is true, given that reading. From this perspective, too, at least some mereological theories, but also some theories in  $\mathcal{C}$ , should be viewed as nominalistic theories. This leads to the third answer, which, though closely related to the second one, should be clearly distinguished from it. I have mainly two reasons for this assessment.

The first one is simply that one may be a platonist and assume the existence of abstract objects, and still agree that the empirical objects known from our everyday experience are only concrete objects. Accordingly, speaking about theories instead of ontology, even a platonist may prefer *empirical* theories to be free from mathematics.

Actually, it seems that more than a few empirical theories — in particular physical theories — have, as they are commonly stated, what may be called a “mathematical core”: if such a theory  $T$  is explicitly presented, mathematical expressions, such as “+” or “ $\in$ ”, belong to its vocabulary; and  $T$  contains mathematical theories as subtheories. Admittedly, a general distinction between mathematical and non-mathematical theories is not that easy to state. Yet, for the discussion of this section, it should suffice to assume (a) that the elements of  $\mathcal{C}$  are non-mathematical theories (which should be particularly plausible if they are regarded as nominalistic theories)<sup>31</sup> and (b) that theories such as PA and Q, ZF and ZF minus the axiom of infinity and perhaps theories of real numbers (such as RCF, the theory of real closed fields [see [Schwabhäuser, 1983](#)]) plus, for example, second-order extensions of each of them, are mathematical theories. *De facto*, mathematics permeates the empirical sciences. But sets and numbers are foreign bodies when it comes to our world of concrete objects. It therefore seems to be a natural reaction to aim at the development and study of empirical theories which are free

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<sup>31</sup> Probably each nominalistic theory should be taken to be a non-mathematical theory. A theory of properties, however, may be a non-mathematical theory which is no nominalistic theory.

from mathematical vocabulary and mathematical principles. One need not be a nominalist to concede that much.

Let me come to the second reason for distinguishing answer two from answer three. To start with, recall that a reductive programme has often been a component of the nominalistic programme. In particular, the nominalist (when standing in a Quinean tradition) has to face the challenge of construing mathematics nominalistically. In the approach promoted here, with its emphasis on theories, this task is more precisely rendered as that of reducing, say, ZF to a nominalistic theory. Now, it is not at all clear that mathematical theories of the (excessive) strength of ZF are actually used as mathematical cores of empirical theories. Moreover, it has been claimed repeatedly that even for physical theories, theories which are of about the same strength as PA — for example, theories which are conservative extensions of PA — *suffice* as their mathematical cores. Thus, it seems that the mathematical theories which should be replaced by non-mathematical ones, as mentioned in answer three, are allowed to be much weaker than those which have to be construed nominalistically, as addressed in answer two.

Actually, I think that there is a further, in some sense more fundamental, difference between the programmes underlying the second and the third answer: the replacement programme expressed in the third answer need not be understood as implying a reductive programme. I agree that when the mathematical core of an empirical theory  $T$  is replaced by a non-mathematical theory, it is plausible to assume that the theory  $T'$  resulting from  $T$  should be able to *play the same role* as  $T$ . But this way of putting it is — deliberately — vague and open. In particular, it does not follow from it that  $T$  has to be relatively interpretable in  $T'$ , or that the mathematical core of  $T$  has to be relatively interpretable in the non-mathematical theory which replaces it. It could also mean that  $T$  and  $T'$  have to be, for example, *empirically equivalent* with each other (whatever that means exactly [cf. Quine, 1975]).

In sum, it might be said that answers two and three are connected with two different programmes: a reduction programme and a replacement programme. Now, what are the prospects for a realization of these two programmes? And, in particular, which roles may theories belonging to  $\mathcal{C}$  play for them?

Two results delimit the nominalistic reduction programme. First, even the rather weak theory Q (not to mention ZF) is not relatively interpretable in any consistent mereological theory [see Niebergall, 2011b].

Second, token concatenation theories can be developed which *are* strong enough to relatively interpret ZF [see Niebergall, 2005]; but the theories of this type that are known to me seem to be rather unattractive when compared to the mereological theories.

In this situation, it should be highly interesting for the nominalist to develop and investigate further theories; theories, that is, which are both rightfully regarded as nominalistic and are strong enough for the realization of the nominalistic reduction programme with respect to mathematical theories. Theories belonging to  $\mathcal{C}$  may be welcome examples of such theories. But are they? I do not know; but I doubt it. Ultimately, it may well be that the theories in  $\mathcal{C}$  are no gain over the mereological theories when it comes to the nominalistic reduction programme for mathematical theories.

Again, for the replacement programme involved in the third answer, it is certainly desirable to have more non-mathematical theories  $T$  at one's disposal than only the mereological theories. It should here be much easier to find suitable examples. In particular, even if the theories in  $\mathcal{C}$  should turn out to be worthless for the nominalistic reduction programme, they may be examples of theories mentioned in the third answer.

At this point, some philosophers may be impressed by indispensability arguments and argue that, plausible as the *aim* of going along without mathematics in empirical theories might be, it just cannot be realized. As understood here, such arguments are supposed to establish the thesis that the adoption of mathematical theories is indispensable for the scientific enterprise.<sup>32</sup> Let me simply answer that I am not aware of a cogent argument for this thesis. As an example, consider a theory  $T$  of real numbers or of, e.g., 4-tuples of real numbers as the mathematical core of an empirical theory. RCF could be a formalized version of it. When being a part of an empirical theory,  $T$  typically plays the role of a theory of space-time: space-time points have been replaced or simulated by the 4-tuples of real numbers. Now withdraw this simulation, come back to

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<sup>32</sup> This is a weak version of an indispensability thesis. A stronger one would be

The adoption of mathematical theories and the assumption of mathematical objects are indispensable for the scientific enterprise.

I think that this is the more common version. I deal with the weak version because it has a greater chance of being true.



space-time points we started with (being individuals of their own kind) and replace RCF by a suitable axiomatic theory of geometry.<sup>33</sup>

This should be an example of the execution of the replacement programme as addressed in the third answer. It is granted that the axiomatic theories of geometry considered here are no elements of  $\mathcal{C}$ . But the move from mereological theories to  $\mathcal{C}$  is only one example for the extension of the mereological paradigm in the pure form, anyway.

Whether this replacement programme and also the nominalistic reduction programme are ultimately feasible can hardly be decided *a priori* (e.g., with an indispensability argument). The answers depend on which theories are conceived of as non-mathematical theories and as nominalistic theories; and in order to attain a reasonable assessment of this, potential theories of these kinds have to be invented, developed and investigated.

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<sup>33</sup> Of the kind developed by Tarski and his followers [see [Schwabhäuser, 1983](#)].

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