Logic and Logical Philosophy
Volume 25 (2016), 521–534
DOI: 10.12775/LLP.2016.015

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INTERPLAYS OF KNOWLEDGE AND NON-CONTINGENCY

Abstract. This paper combines a non-contingency logic with an epistemic logic by means of fusions and products of modal systems. Some consequences of these interplays are pointed out.

Keywords: non-contingency logics, epistemic logics, products of modal logics, Von Wright

1. Introduction

From the perspective of classical propositional logic, a formula φ is contingent if there is a valuation which makes it true, but also a valuation which makes it false. This is a propositional contingency. However, from the viewpoint of modal logic, a formula φ is contingent if φ is possible and $\neg\varphi$, its negation, is possible. This is a modal contingency. These two forms of contingency are closely connected. The present paper is concerned with modal (non-)contingency and its relations with knowledge.

Contingency logics, the mathematics of modal contingency, were defined in [11] and [12]. A survey of them, augmented with further general results about non-contingency, can be found in [8]. These logics are normal modal logics taking the contingency operator as primitive (formalized by ∇ and defined as $\nabla\varphi := \Diamond\varphi \wedge \Diamond\neg\varphi$, where $\Diamond$ means alethically *it is logically possible that*). Given a contingency operator, its dual, non-contingency, formalized by Δ, is obtained by classically

* Work supported by a grant from the National Council for Scientific and Technological Development, CNPq, Brazil (Research project: 486635/2013-9).

Received September 1, 2015. *Revised* June 2, 2016. *Published online* July 13, 2016

denying contingency ($\Delta\varphi := \Box\varphi \vee \Box\neg\varphi$, where $\Box$ has an alethic reading as *it is logically necessary that*). (Non-)contingency logics can be translated into classical normal modal logics and they are sound and complete with respect to some given class of Kripke frames, in the same way normal modal logics are (see [14]).

Since the work developed in [7], a lot of important results have appeared in the domain of epistemic logics. Many authors have studied these systems especially concerning applications in Computer Science or in modeling philosophical concepts (studies in the logics of knowledge and belief can be found in [6, 10]). These epistemic systems formalize concepts of *knowledge* and *belief*, and they are useful in understanding formal properties shared by these notions. Epistemic logics have been used for elucidating paradoxes, definitions and problems involving concepts from formal epistemology. They capture principal attributes of knowledge, and have a regular Kripke semantics, satisfying soundness and completeness with respect to a certain class of Kripke frames.

In this research, the main purpose is to understand how (non-)contingency logics behave in the presence of the knowledge operator. So, it investigates a combination of a non-contingency logic with an epistemic logic by means of fusions of modal logics. This is a method for generating multimodal systems with many non-interdefinable operators. Further, it also investigates the product of these logics, which is a much more complex method, and which contains interactions of operators. These methods have been studied in combination of logics.[1]

The present work in the domain of applied logic uses these procedures for combining logics to investigate two theses proposed by Von Wright in [13] connecting metaphysical and epistemic notions. After which, a product of a non-contingency logic with an epistemic logic is defined and used to study further interplays of knowledge and non-contingency.[2]

2. Fusion of epistemic and contingency logics

Let's construct a logic able to deal with knowledge and (non-)contingency simultaneously. All modal logics considered here are extensions of clas-

[1] For a survey on the nature of fusions and products of modal logics, the reader should check [5].

[2] Some connections between contingency and knowledge were previously studied in [2].

sical propositional logic in a Hilbert-style presentation. To realize this task, let us consider an alphabet composed by a set of propositional variables (let p be one of variables), Greek letters φ, ψ are used for schemas of formulas. Truth-functional classical operators $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication) and $\leftrightarrow$ (biconditional) are adopted. Two non-truth-functional operators are taken as primitives: K (knowledge) and Δ (non-contingency), and contingency operator ∇ is defined with negation and non-contingency. Modal operators are defined straightforwardly. Syntactical logical consequence ($\vdash$), proof, Kripke frame, and all basic logical notions, are standard.

Considering this modal language with non-contingency operator Δ meaning *it is not contingent that* and accepting a Hilbert-style presentation, the non-contingency logic $\mathbf{S5}^{\Delta}$ has the following set of axiom schemas (as defined in [11]):

(A) $\Delta\varphi \leftrightarrow \Delta\neg\varphi$;
(B) $\varphi \rightarrow (\Delta(\varphi \rightarrow \psi) \rightarrow (\Delta\varphi \rightarrow \Delta\psi))$;
(C) $\Delta\Delta\varphi$.

It is usual in (non)-contingency logics to take $\Delta\varphi$ as $\neg\nabla\varphi$, while $\nabla\varphi$ is $\neg\Delta\varphi$. In this context, box is defined as $\Box\varphi := (\varphi \wedge \Delta\varphi)$. In [11], authors consider a modal language with contingency operator ∇ which means *it is contingent that* defined by $\nabla\varphi := \neg\Delta\varphi$, and several (non-)contingency logics are axiomatized and proved to be deductively equivalent to normal modal logics. Thus, diamond is defined as $\Diamond\varphi := (\varphi \vee \nabla\varphi)$ and contingency can be formalized as

(A') $\nabla\varphi \leftrightarrow \nabla\neg\varphi$.

To these axioms, the inference rule

$$\vdash \varphi \;\; \Rightarrow \vdash \Delta\varphi \tag{R1}$$

is added. This rule allows us to put a Δ in any provable formula, i.e., logical truth. Brief comments on these axioms: The first one states that a formula is non-contingent if, and only if, its negation is non-contingent. The same holds for contingency (A'): φ is contingent iff $\neg\varphi$ is contingent. This seems intuitive and corresponding to the very idea of what contingency is. The second one states that non-contingency distributes over implication, if we have the truth of the antecedent of the implication. The third plausibly ensures that "it is not contingent that it is not contingent φ". These axioms are accepted as they seem to reflect essential formal properties of *logical* non-contingency, in the same way the system **S5** is used to model logical possibility and necessity. We use here axiom (C) because we intend to capture the notion of non-contingency based on *logical possibility*.[3]

Now, taking a modal language with knowledge operator K and considering a Hilbert-style presentation, the epistemic logic $\mathbf{S5}^*$ has the following set of axiom schemas (as defined in [6]):

(D) $(\mathsf{K}\,\varphi \wedge \mathsf{K}(\varphi \to \psi)) \to \mathsf{K}\,\psi$;
(E) $\mathsf{K}\,\varphi \to \varphi$;
(F) $\mathsf{K}\,\varphi \to \mathsf{K}\,\mathsf{K}\,\varphi$;
(G) $\neg\,\mathsf{K}\,\varphi \to \mathsf{K}\,\neg\,\mathsf{K}\,\varphi$.

The rule

$$\vdash \varphi \;\; \Rightarrow \vdash \mathsf{K}\,\varphi \tag{R2}$$

is an inference rule of the system. There are many interpretations of these axioms in the literature. As it is known, they reflect logical omniscience (D), axiom of knowledge (E) and forms of introspection — (F) and (G).

So we have two logics and there are two non-interdefinable operators: knowledge (K) and non-contingency (Δ). Thus, in order to talk about both at the same time, we need to put logics together into a single formalism. Whenever there are non-interdefinable concepts in a given situation, combining logics play a crucial role. This is the origin of the fibring problem pointed out in [4]. If we have two concepts to be analyzed in a structure able to interpret only one of them, we need to enhance the expressive power of the structure in order to give the truth-condition for the other concept. Thus we need to combine logics.

For combinations of axiomatic systems, the fusion $\oplus$ of a non-contingency logic with an epistemic logic is a logic containing all axioms and inference rules from both logics. This is a bimodal logic containing two non-interdefinable modalities. Let's call it

$$\mathbf{S5}^{\Delta} \oplus \mathbf{S5}^{*}$$

The logic above contains all the axioms from (A) to (G) and inference rules (R1) and (R2). When required — $\Diamond$, $\Box$ and ∇ — definable operators,

[3] In [11] and [12] the authors define many modal systems with a different variety of modal axioms. For instance, to get **S4** it would be enough to introduce $\Delta\varphi \to \Delta\Delta\varphi$ instead of (C). The reader should also check [8] for a detailed research on non-contingency logics.

and their properties are used, especially because there are immediate conservative translations from **S5** to $\mathbf{S5}^{\Delta}$, and vice-versa (see [14]). A few words on this combined system: technically, it is quite simple to generate this logic, and its semantics is simple. A Kripke frame F^{Δ} for $\mathbf{S5}^{\Delta}$ is defined as $F^{\Delta} = \langle W, R \rangle$, where W is non-empty and R is an accessibility relation, truth-conditions for classical operators remain the same. For non-contingency, truth-condition is:

$$\begin{aligned} w \vDash \Delta\varphi \iff & \text{ for all } w' \text{ such that } wRw',\ w' \vDash \varphi \\ & \text{ or for all } w' \text{ such that } wRw',\ w' \nvDash \varphi. \end{aligned}$$

A Kripke frame for $\mathbf{S5}^{*}$ is $F^{*} = \langle W, P \rangle$, where W is non-empty and P an accessibility relation for knowledge.[4] We take standard truth-condition for K:

$$s \vDash \mathsf{K}\,\varphi \iff \text{ for all } s' \text{ such that } sPs',\ s' \vDash \varphi.$$

A frame for $\mathbf{S5}^{\Delta} \oplus \mathbf{S5}^{*}$ is a combination of F^{Δ} and F^{*} generated by a fusion of frames. This combined structure $F^{\Delta} \oplus F^{*} = \langle W, R, P \rangle$ is such that $F^{\Delta} = \langle W, R \rangle$ is a frame for $\mathbf{S5}^{\Delta}$ and $F^{*} = \langle W, P \rangle$ is a frame for $\mathbf{S5}^{*}$. A result on fusions is that it preserves soundness and completeness of the combined systems. In this sense, if two logics are sound and complete, their fusion is sound and complete, and then these properties are transferred to the combined systems.[5] Given that both $\mathbf{S5}^{\Delta}$ and $\mathbf{S5}^{*}$ are sound and complete with respect to the class of Kripke frames with accessibility relations satisfying reflexivity, symmetry and transitivity, it follows that the combined $\mathbf{S5}^{\Delta} \oplus \mathbf{S5}^{*}$ is sound and complete with respect to the combined class of frames.

An interactive law connecting knowledge and non-contingency arises automatically given axioms (B) of $\mathbf{S5}^{\Delta}$ and (E) of $\mathbf{S5}^{*}$, because implication is classical and, therefore, a transitive relation. This is a new theorem which does not hold in any of the logics used in the fusion but holds in the fusion itself:

$$\mathsf{K}\,\varphi \to (\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)) \tag{KC}$$

A connection like (KC) is accidental because fusions of modal logics in general do not give rise to any interactive law, that is, modal operators

[4] To define a fusion it is mandatory to take domains as equal. When elements of W are connected by P, we refer to them as s to reveal epistemic aspects of worlds.

[5] This result has been proved by [9] and [3]. For more properties preserved by fusions, like finite model property, decidability, see [5].

stay without any communication, or interaction, as we learn from combining logics (see [5]). When two or more logics are fused we normally found no bridge principle. A similar example does occur in [1], where the logic of conjunction is combined with the logic of disjunction giving rise to a new property which does not hold in any of the systems used in the combination: the distributivity of these operators. Another example is verified even in a fusion of reflexive epistemic logic and reflexive alethic modal logic: given $\mathsf{K}\,\varphi \to \varphi$ and $\varphi \to \Diamond\varphi$ (a version for $\Diamond$ of axiom (T)), it follows that

$$\mathsf{K}\,\varphi \to \Diamond\varphi.^{6} \tag{INT}$$

The combined system $\mathbf{S5}^{\Delta} \oplus \mathbf{S5}^{*}$ allows us to reason about interactions of knowledge and non-contingency. Suppose, for instance, that $\vdash \mathsf{K}\,\varphi$. Given that $\vdash \mathsf{K}\,\varphi \to \varphi$, it follows, by *modus ponens* (MP), that $\vdash \varphi$. Thus, by rule (R1), $\vdash \Delta\varphi$. This generates a derived inference rule:

$$\vdash \mathsf{K}\,\varphi \ \Rightarrow \vdash \Delta\varphi.$$

We also get

$$\vdash \mathsf{K}\,\varphi \text{ or } \vdash \mathsf{K}\,\neg\varphi \ \Rightarrow \vdash \Delta\varphi,$$

because from $\vdash \mathsf{K}\,\neg\varphi \to \neg\varphi$ and $\vdash \mathsf{K}\,\neg\varphi$ we have $\vdash \neg\varphi$ and then $\vdash \Delta\neg\varphi$ which, in turn, is equivalent to $\vdash \Delta\varphi$. From $\vdash \mathsf{K}\,\varphi \to \varphi$ and $\vdash \mathsf{K}\,\varphi$ we have $\vdash \varphi$ and then $\vdash \Delta\varphi$.

Now we use the fusion to think about a result from Von Wright on the nature of knowledge with consequences on the metaphysics of epistemology.

2.1. Von Wright on contingent and necessary knowledge

Von Wright in [13] explores connections between metaphysical concepts such as *contingency* and *necessity* with epistemic notions like *knowledge*. He states that:

> [...] if the object of knowledge is contingent, then knowledge of it is contingent too. [13, p. 68]
>
> Knowledge of contingent truths must itself be contingent knowledge. But knowledge of necessary truths may, as far as logic is concerned, be either contingent or itself necessary. [13, p. 69]

Indeed, he proves the following reasoning, assuming an atomic proposition p:

6 (INT) is also valid in the fusion $\mathbf{S5}^{\Delta} \oplus \mathbf{S5}^{*}$.

(CK) Given that p is contingent and known, it follows that knowledge of p is contingent.

If an agent knows contingent truths, then his or her knowledge is contingent. The inference can be developed in a bimodal combined logic with knowledge and logical possibility. He uses a bimodal logic with $\Diamond$ and K, a system equivalent to the fusion $\mathbf{S5}^{\Delta} \oplus \mathbf{S5}^{*}$. Below the result is reproduced.[7]

The inference runs as follows: Given that knowledge implies truth $\mathsf{K}\,p \to p$ is an axiom of the combined logic, then, by necessitation, $\Box(\mathsf{K}\,p \to p)$. Using normality and (MP), we deduce that $\Box\,\mathsf{K}\,p \to \Box p$. By contraposition and box-diamond duality, it follows $\Diamond\neg p \to \Diamond\neg\,\mathsf{K}\,p$. By assumption, p is contingent and, for this reason, $\Diamond\neg p$. Thus, using (MP) again, a consequence is $\Diamond\neg\,\mathsf{K}\,p$. An instance of the dual version of the axiom (T) assures that $\mathsf{K}\,p \to \Diamond\,\mathsf{K}\,p$ and, by hypothesis, using the fact that p is known, it follows $\Diamond\,\mathsf{K}\,p$. Therefore, Von Wright infers contingent knowledge:

$$\Diamond\,\mathsf{K}\,p \wedge \Diamond\neg\,\mathsf{K}\,p$$

from the fact that p is contingent and known. (CK) shows an important fact about the nature of knowledge. If an agent knows an empirical proposition (that is, a contingent one), then knowledge is itself contingent:

$$((\Diamond p \wedge \Diamond\neg p) \wedge \mathsf{K}\,p) \vdash (\Diamond\,\mathsf{K}\,p \wedge \Diamond\neg\,\mathsf{K}\,p)$$

Von Wright realizes the reasoning above using a fusion of a modal logic for possibility with a modal logic for knowledge, although he did not mention anything concerning this logic. The combined system $\mathbf{S5}^{\Delta} \oplus \mathbf{S5}^{*}$ can be applied as a framework to analyze this result on metaphysics of epistemology, but now without using boxes and diamonds, but only resources of the fused language, that is: (non-)contingency and knowledge. So, in the fused language we have to prove (CK):

$$\nabla p, \mathsf{K}\,p \vdash \nabla\,\mathsf{K}\,p$$

Assume that we have ∇p and $\mathsf{K}\,p$. By an instance of axiom (B), we have that $\mathsf{K}\,p \to (\Delta(\mathsf{K}\,p \to p) \to (\Delta\,\mathsf{K}\,p \to \Delta p))$. It follows then, by (MP), $\Delta(\mathsf{K}\,p \to p) \to (\Delta\,\mathsf{K}\,p \to \Delta p)$. Given axiom (E) and (R1), we have

[7] Indeed, Von Wright's argument takes into account temporal dimensions of knowledge, but the result follows even if no temporal aspect is considered. Here we proceed without considering time.

$\Delta(\mathsf{K}\, p \to p)$ and, again by (MP), $\Delta\,\mathsf{K}\, p \to \Delta p$. By contraposition, and the interdefinability of Δ and ∇, it follows $\nabla p \to \nabla\,\mathsf{K}\, p$. Given that p is, by assumption, contingent, we have that $\nabla\,\mathsf{K}\, p$.

This result by G. H. Von Wright on the metaphysical status of knowledge containing connections of knowledge and (non-)contingency, developed in the fusion, is relevant for the metaphysics of epistemology because it elucidates properties of the nature of knowledge. From a metaphysical viewpoint, propositions could be classified as (im)possible, necessary and (non-)contingent. We saw Von Wright's argument connecting knowledge and contingency and producing *contingent knowledge*. In addition, note that given the dual $\varphi \to \Diamond\varphi$ of the famous axiom (T) in alethic shape formulated as $\Box\varphi \to \varphi$ — here called in epistemic version as (E) — it is easy to see that knowledge is a sufficient condition for *possible knowledge*: $(\mathsf{K}\,\varphi \to \Diamond\,\mathsf{K}\,\varphi)$. However, it is not immediate to check what are sufficient conditions for *necessary knowledge* and *non-contingent knowledge*. Von Wright argues that in order to have *necessary knowledge* an agent should be omniscient because we have to assume that this agent "necessarily knows whether any given proposition is true or not [...]" (p. 69). So, let's call (NK) the thesis according to which there is, under certain assumptions, *necessary knowledge*:

(NK) Given that an agent necessarily knows whether p or $\neg p$ and given that p is necessary (not contingent, therefore), then knowledge of p is necessary.

Formally:

$$\Box(\mathsf{K}\, p \vee \mathsf{K}\,\neg p), \Box p \vdash \Box\,\mathsf{K}\, p \tag{NK}$$

Von Wright has a proof for (NK) (p. 69), and it goes here in a slightly different argumentation:[8] Verify first that $\Box(\mathsf{K}\, p \vee \mathsf{K}\,\neg p) \vdash \Box p \to \mathsf{K}\, p$. Assume $\Box p$ and consider that $\Box p \to \neg\Diamond\neg p$. By (INT), we have that $\mathsf{K}\,\neg p \to \Diamond\neg p$ and, by classical reasoning, $\neg\Diamond\neg p \to \neg\,\mathsf{K}\,\neg p$. Given that $\Box p$, it follows $\neg\,\mathsf{K}\,\neg p$. But if $\Box(\mathsf{K}\, p \vee \mathsf{K}\,\neg p)$, then $\mathsf{K}\, p \vee \mathsf{K}\,\neg p$ also holds (by axiom (T)), and by classical reasoning again, $\mathsf{K}\, p$. So, $\Box p \to \mathsf{K}\, p$. Now, we can deduce — using deduction theorem which holds in modal logic[9] — that $\vdash \Box(\mathsf{K}\, p \vee \mathsf{K}\,\neg p) \to (\Box p \to \mathsf{K}\, p)$. Moreover, by the derived rule

[8] Note that the following reasoning is not realized inside our fusion, but in a translation of it with $\Box$ instead of Δ. So, it is realized in the fusion of standard **S5** with its epistemic counterpart.

[9] Assuming a suitable restricted version of it.

$\vdash \varphi \to \psi \Rightarrow \vdash \Box\varphi \to \Box\psi$, it follows $\vdash \Box\Box(\mathsf{K}\,p \vee \mathsf{K}\,\neg p) \to \Box(\Box p \to \mathsf{K}\,p)$. In **S5**, we have reduction of modalities, especially here we use the fact that $\Box\Box\varphi \leftrightarrow \Box\varphi$, and get — using normality $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ — that $\vdash \Box(\mathsf{K}\,p \vee \mathsf{K}\,\neg p) \to (\Box p \to \Box\,\mathsf{K}\,p)$. (NK) follows by two applications of *modus ponens* in the last formula.

So, Von Wright states:

> It used to be one of the disputed things in the philosophy of modal logic whether the necessary entails its own necessity, or not. The answer, in my opinion, depends upon what 'type' of necessity is involved. Some necessity is itself necessary; other necessity is contingent. It is, moreover, feasible to think that *logical* necessity is of the former type, but that *natural* or *physical* necessity is of the latter. Accepting this we could say that God, since he necessarily knows whether any given proposition is true or not, also necessarily knows all logically necessary truths but not all 'natural', i.e., contingent necessities. Knowledge of them is contingent knowledge. [13, pp. 69–70]

According to Von Wright, the above argument for necessary knowledge seems plausible if the agent under consideration is omniscient, but it fails to be the case when the agent in not omniscient, as the case of human beings: "[...] it is hard to see that there are any specific truths which are such that any man necessarily knows *them*." [13, p. 71]

3. Product of epistemic and contingency logics

The product of two modal logics, besides introducing multimodalities, generates multidimensional modal logics in which consequence relation $\vDash$ is defined between n-tuples $\langle x_1, \ldots, x_n\rangle$ of worlds $(n > 1)$ and formulas: $\langle x_1, \ldots, x_n\rangle \vDash \varphi$. We follow [5] (p. 222) in the presentation of products, and we take into account only $n = 2$, that is, two-dimensional products. Consider now that $F^{\Delta} = \langle W, R\rangle$ is a frame for the interpretation of Δ, while $F^{*} = \langle S, P\rangle$ — for K, where S is non-empty, both with behavior like in the fusion, but without restrictions concerning domains (W and S can be or not equal). Thus, a product of them is a frame $F^{\Delta} \times F^{*} = \langle W \times S, R^{\Delta}_{h}, P^{*}_{v}\rangle$ such that:

$$\begin{aligned} \langle w, s\rangle R^{\Delta}_{h} \langle w', s'\rangle \quad &\text{iff} \quad wRw' \text{ and } s = s', \\ \langle w, s\rangle P^{*}_{v} \langle w', s'\rangle \quad &\text{iff} \quad sPs' \text{ and } w = w'. \end{aligned}$$

In this product of frames, worlds are complex pairs $\langle w, s\rangle \in W \times S$ containing metaphysical and epistemic ingredients. Considering a product of Kripke frames $F^{\Delta} \times F^{*}$, we have now to examine the truth-conditions for Δ and K. Truth-condition for Δ is:

$$\langle w, s\rangle \vDash \Delta\varphi \iff \begin{array}{l}\text{for all } w' \text{ such that } wRw', \langle w', s\rangle \vDash \varphi \\ \text{or for all } w' \text{ such that } wRw', \langle w', s\rangle \nvDash \varphi.\end{array}$$ [10]

For K operator, we have that:

$$\langle w, s\rangle \vDash \mathsf{K}\,\varphi \iff \text{for all } s' \text{ such that } sPs', \langle w, s'\rangle \vDash \varphi.$$ [11]

Note that in the truth-condition for Δ, epistemic states are fixed (that is, the condition does not involve a change of s), while the formula φ moves in the 'horizontal' direction. Differently, in the truth-condition for K, possible worlds are fixed (that is, the condition does not contain a change of w) and formula φ moves in the 'vertical' direction. So we have in this particular bidimensional product two new accessibility relations R_h^{Δ} and P_v^{*} generated by the relations R and P respectively. These new relations characterize the two-dimensional aspect of the combined frame.

Concerning products of modal logics, a natural question posed by researchers in the field is: how to axiomatize a product of Kripke frames? Or, as Gabbay puts it in [4] (p. 327), "What kind of modal logics correspond to those frames?". There are some logics which are *product-matching*, this means that they can be axiomatized by a fusion extended by two new axioms: commutativity and Church-Rosser. In the case studied here we are lucky enough to have a bidimensional product based in a fusion which is, indeed, a case of product-matching.[12] For commutativity we would have $(\Diamond\neg\,\mathsf{K}\,\neg\varphi \leftrightarrow \neg\,\mathsf{K}\,\neg\Diamond\varphi)$, and for Church-Rosser $(\Diamond\,\mathsf{K}\,\varphi \rightarrow \mathsf{K}\,\Diamond\varphi)$. The modality $\neg\,\mathsf{K}\,\neg$ is the dual of the operator K. However, in our language we do not have diamonds, so we have to formulate these axioms properly. Δ is the horizontal operator, while K represents the vertical one in standard products:

(H) $(\mathsf{K}\,\varphi \rightarrow (\mathsf{K}\,\Delta\varphi \leftrightarrow \Delta\,\mathsf{K}\,\varphi)$
(I) $(\Delta\,\mathsf{K}\,\varphi \rightarrow \mathsf{K}\,\varphi) \rightarrow \mathsf{K}(\Delta\varphi \rightarrow \varphi)$

Axioms (H) and (I) are obtained with transformations in the commutativity and Church-Rosser using basically classical propositional logic,

[10] In this case, $\langle w, s\rangle R_h^{\Delta} \langle w', s\rangle$.

[11] Similarly, we have here that $\langle w, s\rangle P_v^{*} \langle w, s'\rangle$.

[12] This result is proved in [5], p. 230, Corollary 5.10.

modal axioms of the fused logics and $\Diamond\varphi := (\varphi \vee \nabla\varphi)$. Although previously not mentioned, we make use of two important metatheorems which hold in normal modal logics: replacement of provable equivalents and uniform substitution.

The axiom (H) is obtained from $\Diamond\neg\,\mathsf{K}\,\neg p \leftrightarrow \neg\,\mathsf{K}\,\neg\Diamond p$ (the commutativity law) using the following inference and finally substitution of φ for p:

$$
\begin{array}{rc}
\Diamond\neg\,\mathsf{K}\,\neg p \leftrightarrow \neg\,\mathsf{K}\,\neg\Diamond p & \\
 & \overset{\text{df. of }\Diamond}{\Longleftrightarrow} \\
(\neg\,\mathsf{K}\,\neg p \vee \nabla\neg\,\mathsf{K}\,\neg p) \leftrightarrow (\neg\,\mathsf{K}\,\neg(p \vee \nabla p)) & \\
 & \overset{\text{classical logic}}{\Longleftrightarrow} \\
(\mathsf{K}\,\neg p \to \nabla\neg\,\mathsf{K}\,\neg p) \leftrightarrow (\neg\,\mathsf{K}(\neg p \wedge \Delta p)) & \\
 & \overset{\text{(A') and the replacement}}{\Longleftrightarrow} \\
(\mathsf{K}\,\neg p \to \nabla\,\mathsf{K}\,\neg p) \leftrightarrow (\neg\,\mathsf{K}(\neg p \wedge \Delta p)) & \\
 & \overset{\text{classical logic}}{\Longleftrightarrow} \\
\mathsf{K}(\neg p \wedge \Delta p) \leftrightarrow \neg(\mathsf{K}\,\neg p \to \nabla\,\mathsf{K}\,\neg p) & \\
 & \overset{\text{distributivity of }\mathsf{K}\text{ and classical logic}}{\Longleftrightarrow} \\
(\mathsf{K}\,\neg p \wedge \mathsf{K}\,\Delta p) \leftrightarrow \neg(\neg\,\mathsf{K}\,\neg p \vee \nabla\,\mathsf{K}\,\neg p) & \\
 & \overset{\text{classical logic}}{\Longleftrightarrow} \\
(\mathsf{K}\,\neg p \wedge \mathsf{K}\,\Delta p) \leftrightarrow (\mathsf{K}\,\neg p \wedge \Delta\,\mathsf{K}\,\neg p) & \\
 & \overset{\substack{\text{subst. } p/\neg p\text{, double negation, (A)}\\ \text{and extensionality}}}{\Longleftrightarrow} \\
(\mathsf{K}\,p \wedge \mathsf{K}\,\Delta p) \leftrightarrow (\mathsf{K}\,p \wedge \Delta\,\mathsf{K}\,p) & \\
 & \overset{\text{classical logic}}{\Longleftrightarrow} \\
\mathsf{K}\,p \to (\mathsf{K}\,\Delta p \leftrightarrow \Delta\,\mathsf{K}\,p) &
\end{array}
$$

Axiom (I) is obtained from $(\Diamond\,\mathsf{K}\,\varphi \to \mathsf{K}\,\Diamond\varphi)$ (Church-Rosser) in the following way:

$$
\begin{array}{rc}
(\Diamond\,\mathsf{K}\,\varphi \to \mathsf{K}\,\Diamond\varphi) & \\
 & \overset{\text{df. of }\Diamond}{\Longleftrightarrow} \\
(\mathsf{K}\,\varphi \vee \nabla\,\mathsf{K}\,\varphi) \to \mathsf{K}(\varphi \vee \nabla\varphi) & \\
 & \overset{\text{df. of }\nabla}{\Longleftrightarrow} \\
(\mathsf{K}\,\varphi \vee \neg\Delta\,\mathsf{K}\,\varphi) \to \mathsf{K}(\varphi \vee \neg\Delta\varphi) & \\
 & \overset{\text{classical logic}}{\Longleftrightarrow} \\
(\Delta\,\mathsf{K}\,\varphi \to \mathsf{K}\,\varphi) \to \mathsf{K}(\Delta\varphi \to \varphi) &
\end{array}
$$

Observe that (I) implies $(\Delta\,\mathsf{K}\,\varphi \to \mathsf{K}\,\varphi) \to (\mathsf{K}\,\Delta\varphi \to \mathsf{K}\,\varphi)$, given axiom (D); Thus, the product is the fusion $\mathbf{S5}^{\Delta} \oplus \mathbf{S5}^{*}$ with axioms (H) and (I). Let's denote it $\mathbf{S5}^{\Delta} \oplus \mathbf{S5}^{*} \oplus$ (H) $\oplus$ (I) or, simply, $\mathbf{S5}^{\Delta} \times \mathbf{S5}^{*}$.

From axiom (H), we infer that if φ is any logical theorem or any known proposition, then $\mathsf{K}\,\Delta\varphi$ and $\Delta\,\mathsf{K}\,\varphi$ are equivalents. Moreover, assuming that the fusion is a sublogic of the product, as it is a product-matching, then Von Wright's argument for contingent knowledge also holds in the product, but now we prove it using axiom (H): assume that ∇p and $\mathsf{K}\,p$. Then, by (MP) and the axiom (H), it follows that $\mathsf{K}\,\Delta p \leftrightarrow \Delta\,\mathsf{K}\,p$. As an instance of axiom (E), $\mathsf{K}\,\Delta p \to \Delta p$, and considering transitivity of implication, we have $\Delta \mathsf{K}\, p \to \Delta p$. By contraposition and (MP), we derive $\nabla \mathsf{K}\, p$.

Up to now, we have seen how to get possible, contingent and necessary knowledge. But what about non-contingent knowledge? What are sufficient conditions for it? A natural conjecture is that:

(nCK) Given that p is non-contingent and known, then knowledge of p is non-contingent.

In a language with Δ and K, it is formulated as:

$$\Delta p, \mathsf{K}\, p \vdash \Delta \mathsf{K}\, p \tag{nCK}$$

The conjecture (nCK) is inspired by the two questions above proposed by Von Wright. If $\Box(\mathsf{K}\, p \vee \mathsf{K}\neg p)$ and $\Box p$ are sufficient conditions for necessary knowledge, then they also are for non-contingent knowledge, given that $\Box \mathsf{K}\, p \to (\Box \mathsf{K}\, p \vee \Box \neg \mathsf{K}\, p)$. If the conjecture (nCK) holds, then omniscience can be eliminated even to produce non-contingent knowledge, but we do not have a proof of this fact, and we leave it as an open question.

4. Conclusion

The contributions of this paper are in the domain of philosophical logic. We have proposed logics able to deal with interplays of knowledge and non-contingency. These connections have been obtained by fusions and products of modal logics. Concerning fusions, we saw that the combined logic contains an axiom (i.e. (KC)) which connects in a single formula knowledge and non-contingency. This is another example of the problem mentioned in [1]. Then, as a special case, Von Wright's arguments for contingent and necessary knowledge were presented in distinct approaches inside the fused logic. Indeed, we took his initial argument for contingent and necessary knowledge, and we have formulated similar questions for possible and non-contingent knowledge. With respect to products, we gave truth-conditions for Δ in bidimensional frames and provided an axiomatic system for a two-dimensional modal logic containing interplays of knowledge and non-contingency. In addition, inspired by Von Wright's proofs, we have formulated in the product logic a question of whether non-contingent knowledge can be obtained by non-contingency and knowledge.

The product $\mathbf{S5}^{\Delta} \times \mathbf{S5}^{*}$ is the simplest scenario to take into account connections between knowledge and non-contingency. Logics for knowledge and logics for (non-)contingency have been widely studied in the literature, but it seems that this is the first time both are studied in a single formalism. Axioms like (KC), (H) and (I) show, therefore, what are the basic properties for these relations. Other properties should also be investigated and other philosophical implications of these combinations should also be taken into consideration in the future.

Acknowledgment. Thanks to the anonymous reviewer who really helped to clarify many aspects of this article. Thanks also to Phil Jones for proofreading the English.

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