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"THE WHOLE IS GREATER THAN THE PART." MEREOLOGY IN EUCLID'S *ELEMENTS*

Abstract. The present article provides a mereological analysis of Euclid's planar geometry as presented in the first two books of his *Elements*. As a standard of comparison, a brief survey of the basic concepts of planar geometry formulated in a set-theoretic framework is given in Section 2. Section 3.2, then, develops the theories of incidence and order (of points on a line) using a blend of mereology and convex geometry. Section 3.3 explains Euclid's "megethology", i.e., his theory of magnitudes. In Euclid's system of geometry, megethology takes over the role played by the theory of congruence in modern accounts of geometry. Mereology and megethology are connected by Euclid's Axiom 5: "The whole is greater than the part." Section 4 compares Euclid's theory of polygonal area, based on his "Whole-Greater-Than-Part" principle, to the account provided by Hilbert in his *Grundlagen der Geometrie*. An hypothesis is set forth why modern treatments of geometry abandon Euclid's Axiom 5. Finally, in Section 5, the adequacy of atomistic mereology as a framework for a formal reconstruction of Euclid's system of geometry is discussed.

Keywords: atomistic mereology; convex geometry; Euclidean plane; polygons; points; continuum; measure theory

1. Introduction

When Walter Prenowitz in 1961 presented his "contemporary approach to classical geometry" [18], he introduced his monographic article by an interesting thought experiment: What would Euclid, could he return to earth (in 1961), say about the (then) present state of geometry after having examined "the famous work of Hilbert on the foundations" [11]

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of this discipline. Prenowitz surmised that Euclid would not find modern geometry very different from his own and he concludes that Euclid "would have more trouble with the German than with the geometry"; [18, p. 1]. By this he suggests that the difference of the languages is marginal whereas Euclid's and Hilbert's agreement on the method of synthetic geometry is essential and sets their axiomatic systems apart from that inspired by algebra which he presents in his little monograph.

What differences between Euclid and Hilbert are essential rather than marginal is of course as delicate a question as that how the system of geometry developed by the historical Euclid "really" looked like.[1] In the following I shall ignore all issues of textual criticism and shall consider Euclid's geometry as presented in his *Elements* — in Heath' well-known translation [8] — as a contemporary mathematical theory. Though Prenowitz sees a basic difference "in spirit" between his own system on the one hand, and Euclid's and Hilbert's on the other, there is one trait which his system shares with Hilbert's and which is missing from Euclid's. Both Hilbert and Prenowitz make implicit use of set theory. Euclid, of course, does not employ the notion of a set. But already in the very first sentence of the *Elements* he makes use of the notion of a part: "A point is that which has no part"; [8, p. 153].[2] The notion of a part then re-occurs in Euclid's list of axioms (his "common notions"); the fifth axiom is "The whole is bigger than the part"; [8, p. 155].

When Prenowitz draws the distinction between his "algebra-inspired" approach to geometry and Euclid's and Hilbert's axiomatic procedure, he obviously does not recognize the use of mereological rather than set-theoretic notions as essential. For him, this difference seems to be as unimportant as that between Euclid's ancient Greek and Hilbert's German. In contrast to this, I think that Euclid's use of the mereological notion of a part marks an important difference between his system of geometry and that of Hilbert. In the following I shall provide a reconstruction of Euclid's planar geometry within a system of atomistic mereology. Whereas most mereological approaches to geometry eschew points, Euclid, as his definition of a point shows, has no problem with them. Thus an atomistic mereology seems to be a plausible foundational

[1] For comparisons of Euclid's and Hilbert's works cf., e.g., Mueller [13, pp. 1–16] and Hartshorne [7, chs. 1, 2, 5].

[2] If no volume is indicated in a reference to Heath' three-volume edition of the *Elements*, the first volume is always meant.

framework for his geometry. I shall return to this question in the final section of the present article. First, however, I shall start in the following section with a brief overview of Hilbert-type plane geometry in order to provide a standard of comparison. Section 3 is concerned with the relationships of incidence and betweeness first (Section 3.2) and then with Euclid's theory of magnitudes which is the counterpart in his system to the modern theory of congruence (Section 3.3). That section prepares a comparison of Euclid's treatment of polygonal area with that of Hilbert in Section 4. The final section 5 takes up the question of the adequacy of atomistic mereology for the sake of a reconstruction of Euclid's geometry and links up his conception of area to Caratheodory's [4] account to measure theory.

2. Line Plane and H-Plane

Since the modern theory of the plane is mainly due to three researchers whose names start by the letter "H" — namely Hilbert, Hessenberg, and Hjelmslev, it is called the "H-plane" by Diller and Hessenberg [10, p. 96]. We introduce the notion of an H-plane in two steps: first we define line planes and then we add axioms for the relationship(s) of congruence. Following Hartshorne [7, ch. 2], we use a set-theoretic framework though the theory of the plane could also be developed within first-order logic; cf. Tarski [23].

Definition 1. A *line plane* is a triple $\mathcal{P} = \langle P, L, B\rangle$ where P is some set whose members are called the *points* of $\mathcal{P}$, $L \subseteq \mathfrak{P}(L)$ is the set of *lines* of $\mathcal{P}$, and $B \subseteq P \times P \times P$ is the *betweenness-relation* of $\mathcal{P}$. We denote points by lower case italics from the second half of the alphabet and lines by such letters from its first half. $\mathcal{P}$ fulfills the following axioms.

I1 $p \neq q \to \exists^1 a.\ p, q \in a$
I2 $\exists p, q.[p \neq q \wedge p, q \in a]$
I3 $\exists p, q, r.[p \neq q \wedge p \neq r \wedge q \neq r \wedge \neg\exists a.p, q, r \in a]$
B1 $\langle p, q, r\rangle \in B \to p \neq q \wedge p \neq r \wedge q \neq r \wedge \exists a.p, q, r \in a$
B2 $p \neq q \to \exists r.\langle p, r, q\rangle \in B$
B3 $\langle p, q, r\rangle \in B \to \langle r, q, p\rangle \in B \wedge \langle q, p, r\rangle \notin B$
B4 $\neg\exists b.p, q, r \in b \wedge p, q, r \notin a \wedge \exists s \in a.\langle p, s, q\rangle \in B \to \exists t \in a.[\langle p, t, r\rangle \in B \vee \langle r, t, q\rangle \in B]$ (Pasch-Axiom)[3]

[3] Harsthorne [7, p. 74] adds that the line a cannot intersect both edges pr and qr of the triangle $\triangle pqr$. This, however, is derivable; cf. Hessenberg [10, p. 40].

Requirements **I1**–**I3** of Def. 1 correspond to the first three items in Hilbert's "Group I: Axioms of connection"; **B1**–**B4** are counterparts of the four axioms of his "Group II: Axioms of order"; cf. [11, ch. 1, §§ 3,4]. Structures very similar to those defined in Def. 1 are known as "line spaces"; cf. van der Vel [24, p. 156]. The (according to clause 1 of Def. 1) unique line determined by two points p and q will be denoted by "$\overline{pq}$". The description of the congruence relation(s) in Def. 3 below makes use of the notions of a line segment, a ray, an angle, and a triangle. We thus have to define these concepts in advance.

Definitions 2. Let $\mathcal{P} = \langle P, L, B \rangle$ be a line plane.

1. The line *segment* pq ($p, q \in P,\ p \neq q$) is the set $\{r \in P \mid \langle p, r, q \rangle \in B \vee r = p \vee r = q\}$. $\mathbf{Seg}(\mathcal{P})$ is the class of all line segments of $\mathcal{P}$. We use lower case bold face letters such as "$\boldsymbol{s}$", "$\boldsymbol{t}$", etc. as variables for segments.
2. The *ray* $\overrightarrow{pq}$ ($p, q \in P, p \neq q$) is the set $\{r \in P \mid r \in pq \vee q \in pr\}$. We denote the class of all rays of $\mathcal{P}$ by "$\mathbf{Ray}(\mathcal{P})$" and use lower case Gothic letters ("$\mathfrak{r}$", "$\mathfrak{s}''$, ...) as variables for rays.
3. If $p,\ q,\ r \in P$ are distinct and not collinear, then the *angle* $\angle pqr$ is the set $\overrightarrow{qp} \cup \overrightarrow{qr}$. $\mathbf{Ang}(\mathcal{P})$ is the class of all angles of $\mathcal{P}$. Small Greek letters ("α", "β", ...) are used (though first in following sections) for angles.
4. For distinct and non-collinear $p, q, r \in P$ the *triangle* $\triangle pqr$ is the set $pq \cup qr \cup rp$. The class of all triangles of $\mathcal{P}$ is denoted by "$\mathbf{Tri}(\mathcal{P})$".
5. On the basis of **B4** of Def. 1 it can be proved that each line $a \in L$ dissects the points not lying on it — thus the set $P \setminus a$ — into two disjoint subsets S_1 and S_2 such that two points $p, q \in P \setminus a$ belong to the same of these subsets (either both to S_1 or both to S_2) iff pq does not intersect a. We call these two subsets the *sides* of a.

An H-plane is the extension of an underlying line plane by two relations: congruence of segments and congruence of angles. Using the relation of congruence between segments, we define circles in Def. 4 below.

Definition 3. An *H-plane* is a quintuple $\mathcal{H} = \langle P, L, B, \cong_{\mathrm{s}}, \cong_{\mathrm{a}} \rangle$ where the contraction $\mathcal{P}' = \langle P, L, B \rangle$ of $\mathcal{H}$ is a line plane, $\cong_{\mathrm{s}} \subseteq \mathbf{Seg}(\mathcal{P}) \times \mathbf{Seg}(\mathcal{P})$, and $\cong_{\mathrm{a}} \subseteq \mathbf{Ang}(\mathcal{P}) \times \mathbf{Ang}(\mathcal{P})$. Furthermore $\mathcal{P}$ fulfills the following axioms.

C1 Both $\cong_{\mathrm{s}}$ and $\cong_{\mathrm{a}}$ are equivalence relations.
C2 $pq \in \mathbf{Seg}(\mathcal{P}) \wedge \overrightarrow{rs} \in \mathbf{Ray}(\mathcal{P}) \to \exists^1 t \in \overrightarrow{rs}.pq \cong_{\mathrm{s}} rt$
C3 $\langle p, q, r \rangle, \langle s, t, v \rangle \in B \wedge pq \cong_{\mathrm{s}} st \wedge qr \cong_{\mathrm{s}} tv \to pr \cong sv$

C4 Let $\angle qpr \in \mathbf{Ang}(\mathcal{P})$, $\overrightarrow{st} \in \mathbf{Ray}(\mathcal{P})$, and $S \subseteq P$ be a side of $\overline{st}$; then $\exists^1 \mathfrak{r} \in \mathbf{Ray}(\mathcal{P}).[\mathfrak{r} \setminus \{s\} \subseteq S \wedge \angle qpr \cong_{\mathrm{a}} \overrightarrow{st} \cup \mathfrak{r}]$

C5 $\triangle pqr, \triangle stv \in \mathbf{Tri}(\mathcal{P}) \wedge pq \cong_{\mathrm{s}} st \wedge pr \cong sv \wedge \angle qpr \cong_{\mathrm{a}} \angle tsv \rightarrow qr \cong_{\mathrm{s}} tv \wedge \angle pqr \cong_{\mathrm{a}} \angle stv \wedge \angle prq \cong_{\mathrm{a}} \angle svt$

Definition 4. Let $\mathcal{H} = \langle P, L, B, \cong_{\mathrm{s}}, \cong_{\mathrm{a}} \rangle$ be an H-plane. A subset $\Gamma \subseteq P$ is a *circle* of $\mathcal{H}$ iff there is a $p \in P$ and an $\boldsymbol{s} \in \mathbf{Seg}(\mathcal{H})$ such that $\Gamma = \{q \in H \mid pq \cong \boldsymbol{s}\}$. The point p is uniquely determined; it is the *center* of the circle Γ. The segment $\boldsymbol{s}$ specifies the *radius* of the circle. The circle with center p and radius $\boldsymbol{s}$ is denoted by "$\bigcirc p\boldsymbol{s}$". $\mathbf{Circ}(\mathcal{H})$ is the class of circles of $\mathcal{H}$. Capital Greek letters ("Γ", "Δ", ...) are variables ranging over circles.

Following Schreiber [21, pp. 99, 105f] we finally define two subtypes of H-planes.

Definition 5 (Euclidean and Platonic planes).

1. An H-plane $\mathcal{H} = \langle P, L, B, \cong_{\mathrm{s}}, \cong_{\mathrm{a}} \rangle$ is an *Euclidean plane* iff it fulfills the Playfair axiom: $p \notin a \rightarrow \exists^1 b.[p \in b \wedge a \cap b = \emptyset]$.
2. An Euclidean plane is a *Platonic* one iff a line cutting through the interior of a circle intersects the periphery exactly twice: $\exists p, r.[\Gamma = \bigcirc p(pr) \wedge [p \in a \vee \exists q.(q \in a \wedge \langle p, q, r \rangle \in B)]] \rightarrow \exists^2 t.t \in a \cap \Gamma$.

3. Reconstructing Euclid's System of Plane Geometry

3.1. Basics: Logic and Mereology

In order to make explicit the mereology underlying Euclid's system of geometry, we employ Hellman and Shapiro's modification [9] of Tarski's [22] axiom system for extensional mereology. Hellman and Shapiro deviate from Tarski by using second-order logic instead of the simple theory of types. The first-order fragment of their second-order system is one-sorted. Here we adopt a many-sorted system instead since it allows for a more succinct representation of geometric facts. We retain the conventions concerning the use of different sorts of variables which have been introduced in the previous section and will extend them whenever necessary. We do not require sorts to be entirely disjoint from each other but allow for flexibility by admitting subsorts in the same way as, for instance, Cohn [6]. We refrain from describing the systems of

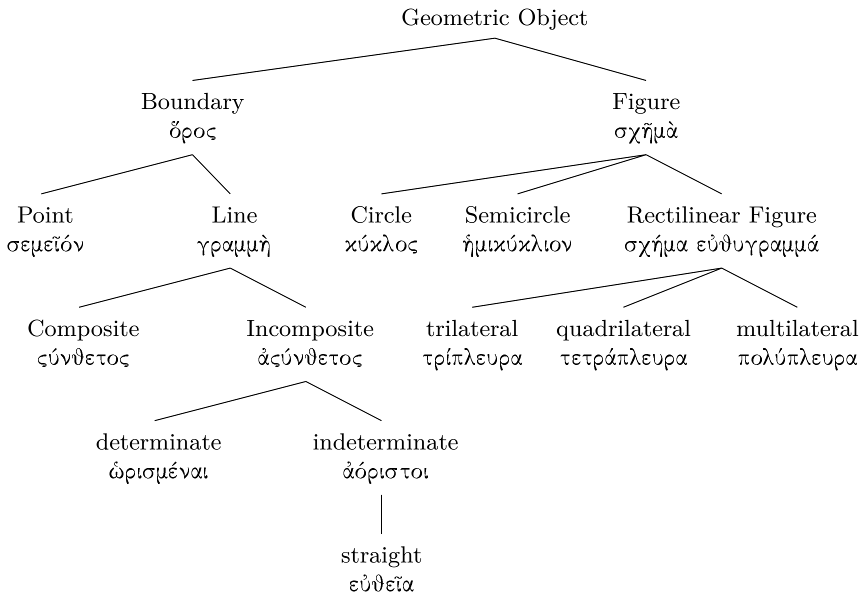

Figure 1. Euclid's ontology of geometric objects. Terms in parentheses do not occur in the *Elements* but are common in other ancient presentations of geometry, cf. Heath [8, p. 160f].

sorts in all detail; but we assume that it accords to that fragment of Euclid's geometric ontology which is relevant for the first two books of the *Elements* and which is in part displayed in Fig. 1.

Geometric objects (cf. the root of the tree in Fig. 1) are a subsort of regions; and a region, in turn, is conceived here as a mereological whole of points. We reserve the letters "x", "y", and "z" (if necessary with subscripts) as variables for regions. The only undefined concept of Hellman and Shapiro's version of mereology is the binary part-of-relation $\sqsubseteq$. Given that relation, points may be defined as $\sqsubseteq$-minima; cf. Def. 6-1 below which echos Euclid's Definition I.1 in the *Elements*. The mereological axioms are formulated by means of the binary relation $\circ$ of overlapping; cf. Def. 6-2.

DEFINITIONS 6 (Point and overlapping).

1. $\mathrm{Pnt}(x) \overset{\mathrm{df}}{\iff} \forall y.[y \sqsubseteq x \rightarrow y = x]$
2. $x \circ y \overset{\mathrm{df}}{\iff} \exists z.[z \sqsubseteq x \wedge z \sqsubseteq y]$

Axioms 1 (Axioms of atomistic mereology).

1. $\sqsubseteq$ *is a partial order.*
2. $\forall z.[z \circ x \to z \circ y] \to x \sqsubseteq y$
3. $\exists x.P(x) \to \exists x.\forall y.[y \circ x \leftrightarrow \exists z.[P(z) \land y \circ z]]$
4. $\exists p.p \sqsubseteq x$

(Remember from the previous section that "p" ranges over points.)

Def. 7 introduces some mereological notions used in the following.

Definitions 7 (Mereological concepts).

1. $x \sqsubset y \overset{\text{df}}{\iff} x \sqsubseteq y \land x \neq y$
2. $x|y \overset{\text{df}}{\iff} \neg\, x \circ y$
3. $\Sigma x.\varphi \overset{\text{df}}{=} \imath y.\forall z.[z \circ y \leftrightarrow \exists x.[\varphi \land z \circ x]]$
4. $x + y \overset{\text{df}}{=} \Sigma z.[z = x \lor z = y]$
5. $\Pi x.\varphi \overset{\text{df}}{=} \Sigma y.\forall z.[\varphi(z) \to y \sqsubseteq z]$
6. $x \cdot y \overset{\text{df}}{=} \Pi z.[z = x \lor z = y]$
7. $x - y \overset{\text{df}}{=} \Sigma z.[z \sqsubseteq x \land z|y]$
8. $\top \overset{\text{df}}{=} \Sigma x.x = x$

3.2. Incidence and Betweenness

Euclid's First Postulate, cf. [8, pp. 154, 195f] requires that for each pair of distinct points p and q there is a segment connecting the two. Suppose there were another such segment; then the two segments connecting p and q would enclose an area. This, however, is excluded by the (probably interpolated) Axiom I.9 of the *Elements*: "Two straight lines do not enclose (*or* contain) a space"; [8, p. 232].[4] We may thus assume that there is a binary operation which assign to pairs of points the unique segments joining these points. In the following, the value of this operation for the argument points p and q is denoted by "pq"; for $p = q$ we let $pq = pp = p$. Segments are not directed; hence we postulate that the operation of joining points by segments is commutative; cf. Ax. 2-1.[5]

[4] Heath [8, p. 232] considers this axiom redundant "since the fact which it states is included in Postulate 1". However, as it stands — and as Heath himself translates it — the First Postulate only requires the existence of the line segment, not its uniqueness. That the uniqueness of the segment is guaranteed by Postulate 9 has already been observed by the Italian mathematician Saccheri; cf. [8, p. 198].

[5] We do not explicitly postulate the idempotency of the joining operation since this property will follow from Decomposability, i.e., Ax. 4, below.

Euclid himself would probably have accepted the commutative law since he does not mind to interchange the order of labels of points in his denotations of segments; cf., for example, his proof of Proposition I.23 where he refers to the same segment by both "CD" and "DC"; cf. [8, p. 295]. The part-of-relation takes over the role of the incidence relations; thus pq comprises, in particular, its boundary points p and q as parts; cf. Ax. 2-2. Ax. 2 says that the points under the operation of joining make up (the mereological counterpart of) an interval space; cf. van der Vel [24, pp. 71–90]. "Proper" segments containing more than one point are defined in Def. 8. Lower case bold face letter exclusively range over proper segments; a complex term pq, however, does not necessarily stand for such a segment.

AXIOMS 2 (Interval space).

1. $pq = qp$
2. $p, q \sqsubseteq pq$

DEFINITION 8 ((Proper) segments).

$$\mathrm{Seg}(x) \overset{\mathrm{df}}{\Longleftrightarrow} \exists p, q.[p \neq q \wedge x = pq]$$

Segments and infinite straight lines (which we shall define in Def. 11-3 below) are both examples of the sort of indeterminate lines in the ontology displayed in Fig. 1. They are indeterminate since they do not delimit a figure as, for example, the periphery of a circle does. They both belong to the sort of straight lines, too. But what exactly renders a segment pq straight? Euclid tries to explain this in his Definition I.4: "A straight line is a line which lies evenly with the points on itself"; [8, pp. 153, 165–169]. The proper interpretation of this explanation is an issue much discussed among Euclid's commentators. Instead of joining the debate, we suggest below two principles spelling out what it means for a line to be straight. The first principle states that segments sharing two points cannot bend away from each other; hence their sum is straight, too.

AXIOMS 3 (Straightness).

$$\exists p, q.[p \neq q \wedge p, q \sqsubseteq \boldsymbol{r}, \boldsymbol{s}] \rightarrow \exists \boldsymbol{t}.\boldsymbol{t} = \boldsymbol{r} + \boldsymbol{s}$$

This property has been called "straightness" in the literature on convex geometry; cf. van de Vel [24, p. 143]. Since by Ax. 3 the sum segment $\boldsymbol{r} + \boldsymbol{s}$ extends both of its summands $\boldsymbol{s}$ and $\boldsymbol{t}$, part of the content of Ax. 3 seems to be covered by Euclid's Postulate I.2 according to which one can "produce a finite straight line continuously in a straight line"; [8, pp. 154, 196–199]. This postulate is commonly interpreted as requiring the possibility to extend a segment beyond its boundary points to an infinite line; cf., e.g., [8, p. 196], [14, p. 329]. In the constellation considered in Ax. 3, however, we have only to do with "finite" entities. In the non-trivial case where $\boldsymbol{r} \neq \boldsymbol{s}$ and none of the segments is a part of the other, their sum $\boldsymbol{r} + \boldsymbol{s}$ extends both and is straight. The first fact is plain mereology, the second is geometric in nature and seems to be envisaged by Euclid's Postulate I.2, too, something straight is extended into something straight again.

The second principle contributing to the explanation of the term "straight" ascribes to segments the property called "decomposability"; cf. van de Vel [24, p. 193]: each point of a segment divides this segment into two parts sharing only the dividing point as their common boundary.

AXIOM 4 (Decomposability of segments).
$r \sqsubseteq pq \rightarrow pq = pr + rq \wedge pr \cdot rq = r$

Ax. 4 excludes the occurrence of loops in segments. Suppose that there were a single loop in pq starting from the point r and returning to it. Then pq would comprise three parts pr, rq, and the loop at r. In that case $pr + rq \neq pq$ which contradicts Ax. 4. From Ax. 4 we have Th. 1 as immediate consequences.

THEOREM 1 (Geometricity of straight interval spaces).

1. *Idempotency*: $pp = p$
2. *Monotony*: $r \sqsubseteq pq \rightarrow pr \sqsubseteq pq$
3. *Inversion*: $r, s \sqsubseteq pq \wedge r \sqsubseteq ps \rightarrow s \sqsubseteq rq$
4. *Convexity of segments*: $r, s \sqsubseteq pq \rightarrow rs \sqsubseteq pq$

An interval space possessing the first three properties mentioned in Th. 1 (or rather their set-theoretic counterparts) is called "geometric"; cf. van de Vel [24, p. 74]. A geometric object is "convex" iff all segments joining points of that object are parts of it; cf. Def. 9, where also some related notions are defined which will be used in the following. The convex hull $[x]$ of a geometric entity x is the smallest convex entity containing x as a part. Triangles are the convex hulls of triples of non-collinear points.

DEFINITION 9 (Convexity and related notions).

1. $\mathrm{Cvx}(x) \overset{\mathrm{df}}{\Longleftrightarrow} \forall p, q \sqsubseteq x . pq \sqsubseteq x$

2. $[x] \stackrel{\text{df}}{=} \Pi y.[\mathrm{Cvx}(y) \wedge x \sqsubseteq y]$
3. $[x_1, x_2, \ldots, x_m] \stackrel{\text{df}}{=} [x_1 + x_2 + \cdots + x_m]$
4. $\triangle pqr \stackrel{\text{df}}{=} [p, q, r]$
5. $\mathrm{Tri}(x) \stackrel{\text{df}}{=} \exists p, q, r.[x = \triangle pqr \wedge p \neq q \wedge p \neq r \wedge q \neq r \wedge \neg\exists \boldsymbol{s}.p, q, r \sqsubseteq \boldsymbol{s}]$

It has often been pointed out that Euclid's system of the *Elements* lacks the concept of order (of points on a line) and, consequently, does not comprise any principles for that important concept; cf., e.g., Hartshorne [7, pp. 65, 73] or Neuenschwander [14, p. 326]. Of course, Euclid does not deal explicitly with the order of the points of a segment. However, any system of geometry starting with segments rather than "complete" lines (and embracing the notion of incidence) comprises the notion of betweenness: r lies between p and q iff r belongs to pq; cf. Def. 10. The order axiom **B3** of Def. 1 from sec. 2 above then has Th. 2 as its direct mereological mirror image.

Definitions 10 (Betweenness).

1. $\mathrm{Betw}(p, q, r) \stackrel{\text{df}}{\Longleftrightarrow} q \sqsubseteq pr$
2. $\mathrm{Betw}^+(p, q, r) \stackrel{\text{df}}{\Longleftrightarrow} p \neq q \wedge p \neq r \wedge p \neq r \wedge \mathrm{Betw}(p, q, r)$

Theorem 2 (The mereological counterpart of **B3**).
$\mathrm{Betw}^+(p, q, r) \to \mathrm{Betw}^+(r, q, p) \wedge \neg\mathrm{Betw}^+(q, p, r)$

Proof. Assume $\mathrm{Betw}^+(p, q, r)$, then p, q, and r are distinct from each other and $q \sqsubseteq pr$. By Ax. 2-1 it follows from this that $\mathrm{Betw}^+(r, q, p)$. Furthermore, $pr = pq + qr$ and $pq \cdot qr = q$ by Ax. 4. But then $p \not\sqsubseteq qr$. Hence $\neg\mathrm{Betw}^+(q, p, r)$. □

Euclid's Proposition I.10 explains how "[t]o bisect a given finite straight line"; [8, p. 267], i.e., how to find for a segment pq a point $r \sqsubseteq pq$ such that pr and rq are of equal size. There are thus points in a segment which differ from its boundaries. We postulate this by Ax. 5.

Axiom 5 (Denseness).
$p \neq q \to pq \neq p + q$

It is immediate then that two points are always separated by a third one lying between them; cf. Th. 3. Th. 3 is the mereological version of **B2** from Def. 1.

Theorem 3 (The mereological counterpart of **B2**).
$p \neq q \to \exists r.\mathrm{Betw}^+(p, r, q)$

All the axioms of Def. 1 different from **B2** and **B3** make use of the notion of a straight line (rather than that of a straight line segment). Thus in order to provide mereological mirror images of these principles, we have first to define the notion of a line within our framework.

If one wants to stick to an Euclidean conceptual framework, this is not as unproblematic as it may seem since Euclid quite generally eschews "infinite" entities like straight lines; cf. Mueller [13, p. 56, Fn. 43]: "[...] Euclid uses 'straight line' to designate line segments of determinate and known extent."[6] Actually, Postulate I.2 is among the three exceptions to this listed by Mueller; and Neuenschwander [14, p. 329] notes that Postulate I.2 is never directly cited in the first four books of the *Elements*. However, it is already used in the proof of Euclid's Proposition I.2. In that proof two segments pq and pr are extended to more comprehensive segments pu and pv — such that, respectively, $\mathrm{Betw}^+(p, q, u)$ and $\mathrm{Betw}^+(p, r, v)$ — in order to make sure that the thus prolonged segments intersect with a circle with center p and a certain radius ps.[7] Of course, Euclid could ensure the existence of the desired points of intersection by using complete lines. Assuming the standard interpretation of his Axiom I.2, he would be entitled to do this since this axiom claims that a segment can be infinitely extended to both sides supplementing it thus to an infinite line. Since our definition of the notion of an angle in the next section will make use of the notion of a ray, we shall admit extensions of segments into lines and rays. This presupposes that there always exists a point r outside a given segment pq which can be reached from pq by a straight prolongation. The following Ax. 6 is thus our interpretation of Euclid's Postulate I.2.

AXIOM 6 (Extendability).
$\exists r.\mathrm{Betw}^+(p, q, r)$

Ax. 6 ensures that rays and lines prolonging segments are proper extensions of them and contain them as their proper parts.

DEFINITIONS 11 (Ray and straight line).

1. $\overrightarrow{pq} \stackrel{\mathrm{df}}{=} pq + \Sigma r.\mathrm{Betw}^+(p, q, r)$

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[6] The same skeptic attitude towards "infinite" geometric objects can be found in Pasch' work [16].

[7] Here and in the following I tacitly rename the points occurring in Euclid's constructions in order to make the labelling of points comply with the notational conventions adopted in the present article.

2. $\mathrm{Ray}(x) \overset{\text{df}}{\Longleftrightarrow} \exists pq.[p \neq q \wedge x = \overrightarrow{pq}]$
3. $\overline{pq} \overset{\text{df}}{=} \overrightarrow{qp} + \overrightarrow{pq}$
4. $\mathrm{Line}(x) \overset{\text{df}}{\Longleftrightarrow} \exists pq.[p \neq q \wedge x = \overline{pq}]$

Given these definition, the following counterparts to Axs. **I1**, **I2**, and **B1** are immediate.

THEOREM 4 (Mereological counterparts of **I1**, **I2**, and**B1**).

1. $p \neq q \rightarrow \exists^1 a.p, q \sqsubseteq a$
2. $\exists p, q.[p \neq q \wedge p, q \sqsubseteq a]$
3. $\mathrm{Betw}^+(p, q, r) \rightarrow p \neq q \wedge p \neq r \wedge q \neq r \wedge \exists a.p, q, r \sqsubseteq a$

Two axioms from Def. 1 are still lacking mereological counterparts: **I3** and the Pasch-axiom **B4**. As Mueller [13, p. 14f] explains, we should not expect to find something resembling **I3** in the *Elements*. For Euclid, the geometric objects come into existence by construction. Hence an absolute existence axiom like **I3** does not make sense in the framework of the *Elements*: "In the geometry of the *Elements* the existence of one object is always inferred from the existence of another by means of a construction"; [13, p. 15]. Given this attitude, Euclid probably would have considered **I3** too trivial to be worth stating. Given some planar piece of the plane, one can always single out three non-collinear points in it by marking them.[8] We, however, have to be more explicit on this and thus add the following axiom.

AXIOM 7 (Existence of three non-collinear points).
$\exists p, q, r.[p \neq q \wedge p \neq r \wedge q \neq r \wedge \neg\exists a.p, q, r \sqsubseteq a]$

What remains then is the Pasch-axiom **B4** of Def. 1. As formulated by **B4**, the axiom fails in 3-space; hence one of its tasks in Def. 1 is to fix the dimension of the geometrical structure defined to 2. This presupposed, it describes certain constellations in the plane. As regards dimension, Euclid explains in the definitions of the first book of the *Elements* that lines have length but no breadth (Definition I.2) and that surfaces have length and breadth only (Definition I.5); cf. [8, p. 153]. In Book XI, it is added that solids have depth in addition to length and breadth. These sparse remarks are hardly sufficient for the development of a theory of

[8] Euclid's Greek term for "point" is σημεῖον meaning (besides 'point') also 'sign', and 'mark' (and still other things).

dimension which would allow us to formulate a condition which excludes all three-dimensional entities and thus renders the universe $\top$ a plane. Within a mereological framework, however, a rather natural condition for achieving this is the requirement that lines dissect $\top$ into two disjoint parts; cf. Def. 2-5. This is an analogue to Th. 5 below. Furthermore, it is known that, given the incidence axioms **I1** and **I2** of Def. 3, Th. 5 is equivalent to the axiom of Pasch; cf. [10, p. 41].

AXIOM 8 (Bipartitions of the plane by its lines). *For each line a there are regions x and y such that:*

1. $x|y$;
2. $x + y = \top - a$;
3. $\forall p, q \not\sqsubseteq a.[pq|a \leftrightarrow p, q \sqsubseteq x \vee p, q \sqsubseteq y]$

THEOREM 5 (Equivalence relation induced by a line).

1. *For each line a and pair of regions x and y as described in* Ax. 8: $\forall p, q \not\sqsubseteq a.[pq \circ a \leftrightarrow [p \sqsubseteq x \wedge q \sqsubseteq y] \vee [p \sqsubseteq y \wedge q \sqsubseteq x]]$.
2. *The relation $\sim_a$ defined by $p \sim_a q \overset{\text{df}}{\Longleftrightarrow} pq|a$ is an equivalence relation on the class of points external to a.*
3. *The two regions x and y belonging to the line a according to* Ax. 8 *are the mereological sums of the equivalence classes modulo $\sim_a$ and hence uniquely determined by a.*

PROOF. 1. According to Def. 7-2 $pq \circ a$ iff $\neg pq|a$. By Ax. 8-3 this is the case iff both $p \not\sqsubseteq x \vee q \not\sqsubseteq x$ and $p \not\sqsubseteq y \vee q \not\sqsubseteq y$. Combinatorially, this leaves four possibilities: (a) $p \not\sqsubseteq x \wedge p \not\sqsubseteq y$, or (b) $p \not\sqsubseteq x \wedge q \not\sqsubseteq y$, or (c) $q \not\sqsubseteq x \wedge p \not\sqsubseteq y$, or, finally, (d) $q \not\sqsubseteq x \wedge q \not\sqsubseteq y$. But (a) and (d) are excluded by Ax. 8-2. By the same axiom and by Ax. 8-1, the disjunction of (b) and (c) is paramount to $[p \sqsubseteq y \wedge q \sqsubseteq x] \vee [q \sqsubseteq y \wedge p \sqsubseteq x]$.

2. The reflexivity and symmetry of the relation of belonging to the same side is obvious. Assume that $p \sim_a q$ and $q \sim_a r$. By Ax. 8-3 we have $p, q \sqsubseteq x \vee p, q \sqsubseteq y$ and $q, r \sqsubseteq x \vee q, r \sqsubseteq y$ where x and y are regions as described in Ax. 8. Assume for the case $p, q \sqsubseteq x$ that we had $q, r \not\sqsubseteq x$ (the case of $p, q \sqsubseteq y$ is analogous). Then $q, r \sqsubseteq y$. Thus both $q \sqsubseteq x$ and $q \sqsubseteq y$ which contradicts Ax. 8-1.

3. We have to show that the class of punctual parts of x and the class of punctual parts of y are the only equivalence classes of $\sim_a$. We first show that they are in fact equivalence classes. Let $p, q \sqsubseteq x$ or $p, q \sqsubseteq y$ where again x and y are as described in Ax. 8. In both cases we have $p \sim_a q$ from Ax. 8-3. Let now, conversely, $p \sim_a q$ and assume $p \sqsubseteq x$.

(The case for $p \sqsubseteq y$ is analogous.) Then, according to Ax. 8-3 q is part of the same of the two regions x and y as p. But since p as a part of x cannot by Ax. 8-1 belong to y, both are parts of x. Since, according to Ax. 8-2, x and y contain all points external to a, there cannot be equivalence classes of $\sim_a$ other than the two classes of points of these regions. □

DEFINITION 12. We call the two regions which are the sums of the two equivalence classes of the relation $\sim_a$ determined by the line a the *sides* of a. If, furthermore, $p \not\sqsubseteq a$, then $(a; p)$ is that side of a to which p belongs as a part.

The mereological counterpart of the Pasch-Principle follows now readily from Th. 5.

THEOREM 6 (The Pasch-principle).
$\mathrm{Tri}(\triangle pqr) \wedge a \circ pq \wedge p, q, r \not\sqsubseteq a \rightarrow a \circ pr \vee a \circ qr$

PROOF. Since $a \circ pq$, the points p and q are according to Th. 5-1 on different sides of the line a, i.e., $p \not\sim_a q$. Hence by Th. 5-3 r must be either of the same side as p or as q (but cannot be both). □

In the following section, we shall have to consider angles. Angles may be defined as the intersections of two halfplanes — i.e., sides of lines — bounded by two rays. The mereological sum of these rays is an example of a composite (ςύνθετος) line; cf. Fig. 1.

DEFINITIONS 13 (Angles and their components). [9]

1. $\angle pqr \overset{\mathrm{df}}{=} \overrightarrow{qp} + \overrightarrow{qr} + (\overline{qp}; r) \cdot (\overline{qr}; p)$
2. $\mathrm{Angle}(x) \overset{\mathrm{df}}{\Longleftrightarrow} \exists p, q, r.[\mathrm{Tri}(\triangle pqr) \wedge x = \angle pqr]$
3. We call the point q of an angle $\angle pqr$ the *vertex* of the angle, the rays $\overrightarrow{qp}$ and $\overrightarrow{qr}$ its *legs* and the region $\angle^{\mathrm{i}}pqr \overset{\mathrm{df}}{=} (\overline{qp}; r) \cdot (\overline{qr}; p)$ its *interior*.

We note the following theorem about angles. It is known as the "crossbar theorem" and can be proven by means theorems 5 and 6; cf. Hartshorne [7, p. 77f].

THEOREM 7 (The crossbar theorem (cf. Fig. 2-(a))).
$s \sqsubseteq \angle^{\mathrm{i}}pqr \rightarrow pr \circ \overrightarrow{qs}$

[9] Euclid excludes straight angles; cf. his Definition I.8, [8, p. 154] — as Hilbert does, too; cf. [11, p. 13].

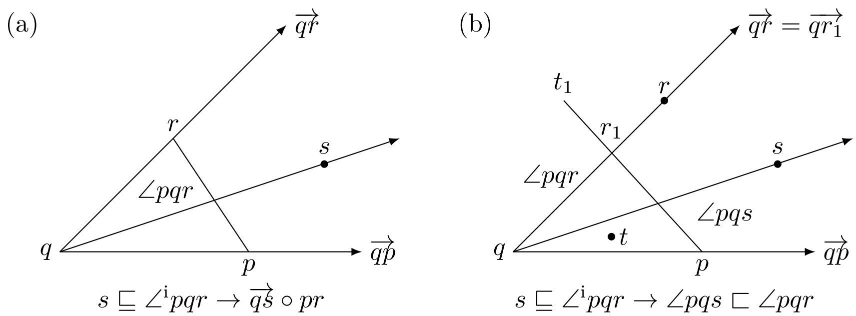

Figure 2. The crossbar theorem and parts of an angle

The following theorem which will be used in the next section is proved with the help of the crossbar theorem.

THEOREM 8 (Proper parts of an angle (cf. Fig. 2-(b))).
$s \sqsubseteq \angle^{\mathrm{i}} pqr \to \angle pqs \sqsubset \angle pqr$

PROOF. Let $s \sqsubseteq \angle^{\mathrm{i}} pqr$ and $t \sqsubseteq \angle pqs$. We show that $t \sqsubseteq \angle pqr$, too. This is trivial for the case that t is a point of the common leg $\overrightarrow{qp}$ of the two angles. Furthermore, each point on the leg $\overrightarrow{qs}$ (other than the common vertex q of both angles) is part of both $(\overline{qp}; r)$ and of $(\overline{qr}; p)$; hence $\overrightarrow{qs} - q \sqsubseteq \angle^{\mathrm{i}} pqr$. So the assertion remains to be shown for the case that $t \sqsubseteq \angle^{\mathrm{i}} pqs$, i.e., under the assumption that $t \sqsubseteq (\overline{qp}; s) \cdot (\overline{qs}; p)$ $(*)$. Since s and r are on the same side of $\overline{qp}$ (namely the r-side), it follows from $t \sqsubseteq (\overline{qp}; s)$ that $t \sqsubseteq (\overline{qp}; r)$ (a). The only thing which remains to be shown is thus $t \sqsubseteq (\overline{qr}; p)$ (b). Suppose therefore that $t \not\sqsubseteq (\overline{qr}; p)$ (cf. point $t1$ in Fig. 2-(b)); then $pt \circ \overline{qr}$ according to Th. 5-1. Let r_1 be the point of intersection. Then $\angle pqr = \angle pqr_1$ (since r_1 lies on the leg $\overrightarrow{qr}$ of $\angle pqr$). So s is in the interior of $\angle pqr_1$ and $\overrightarrow{qs} \circ pr_1$ according to the Crossbar Theorem (Th. 7). Hence $\overline{qs} \circ pr_1$ and, furthermore, $\overline{qs} \circ pt$ since pt is the elongation of pr_1 by $r_1 t$. Thus $t \not\sqsubseteq (\overline{qr}; p)$ implies that p and t belong to different sides of the line $\overline{qs}$ in contradiction to our assumption $(*)$. Since, furthermore, the points on the leg $\overrightarrow{pr}$ are external to $\angle pqs$, the latter angle is a proper part of $\angle pqr$. □

3.3. Euclid's Theory of Magnitudes

In the previous section, it has been shown how Euclid's treatment of line segments can be systematized within a mereological framework in such a way that the resulting system matches the modern exposition provided by Def. 1. Mereology was, however, merely used for modeling geometric objects as collections of points. In effect, talk about sets of points merely has been replaced by talk about collective wholes of points. When we now turn to that part of Euclid's systems that corresponds to the modern theory of congruence, we shall meet a more essential use of mereology. In the first book of the *Elements* Euclid proves three propositions which are commonly interpreted as dealing with the relation of congruence: propositions I.4, I.8, and I.26. His proofs rely on a set of principles which Euclid distinguishes as "Common Notions" (κοιναὶ ἔννοιαι) from his Postulates (αἰτήματα). Here we follow Euclid's antique commentator Proclus, and call them "axioms". The list given below states them in Heath's [8, p. 155] translation.

1. *Things which are equal to the same thing are also equal to one another.*
2. *If equals are added to equals, the wholes are equal.*
3. *If equals are subtracted from equals, the remainders are equal.*
4. *Things which coincide with one another are equal to one another.*
5. *The whole is greater than the part.*

In the three "congruence theorems" I.4, I.8, and I.26, Euclid states conditions under which certain geometric entities (lines, angles, and triangle) are "equal". The Greek adjective which he uses is ἴσος. Since the same word is used by him in the principles listed above, it is suggestive to interpret these principles as his congruence axioms. I shall, however, not follow this interpretation. Euclid's axioms deal with four relationships: equality (1–4), coincidence (4), the greater-than relation and the part-of relationship (5). Entities which are related by the last two relationships are called "magnitudes" (gr. singular μεγέθος) by Euclid. In Definition V.1 he explains: "A magnitude is part of a magnitude, the less of the greater, when it measures the greater"; cf. [8, II, pp. 113, 115].[10] Examples of magnitudes occurring in the first two books of the

[10] According to Heath [8, p. 115] the sense of the word *part* in this definition differs "from the more general sense in which it is used in the Common Notion (5) which says 'the whole is greater than the part' ". He refers to Aristotle's *Metaphysics* (1023 b 12, [1, II, p. 1616] where he detects a similar distinction. Aristotle explains that the concept of a "measuring" part to which Heath alludes is a subconcept of the general notion according to which a part is "that into which a quantity can be in any way divided". Thus, if one wants to follow Heath, one could call that which above has been labelled "magnitudes" "magnitudes in a general sense".

Elements, are segments, angles, and polygons.[11] A denotation for the relation of congruence occurs only in Book 6 in which Euclid develops his theory of (geometric) similarity. In the proof of Proposition VI.28, it is said of two parallelograms that they are "equal and similar" (gr. ἴσον ἄρα καὶ ὅμοιον): they are equal in both size and shape; cf. [8, II, p. 261] The technical term *equal and similar* is then introduced for the special case of solids in Definition XI.9; cf. [8, II, pp. 261, 267f].

Using Euclid's Greek denotation (μεγέθος) for a magnitude as a basis, one could call the theory of magnitude "megethology" — as one uses "mereology", derived from the Greek word μέρος for "part" — as a name for the theory of the part-of-relationship.[12] As is evident from Euclid's Axiom I.5, his megethology is an extension of mereology. Given the "part-less-than-whole" principle expressed by this axiom, there is an obvious method for checking whether two (planar) magnitudes are equal in size or not: one has to "place" a "true copy" of one of them upon the other and has just to check whether the coverage is total or only partial. Euclid's Axiom I.4 asserts just the legitimacy of this "method of superposition". Nevertheless, its status as a proof procedure has been debated and many commentators detect a reluctance in Euclid's work to apply it; cf. Heath' comments on Proposition I.4 in [8, p. 249f] and Neuenschwander's [14, p. 361] discussion of this issue. Mueller [13, p. 23], on the other hand, concludes his discussion of that method by the résumé that "there is very little evidence that Euclid found such movements problematic". It is well known that superposition can be described in as precise and exact a manner as one could wish by employing the concept of a function which preserves the relationship of order (of points on a line) and those of equality in size of segments and angles; cf., for instance, Hartshorne [7, pp. 148–155]. The gain in deductive economy, however, in the framework

[11] As Mueller [13, p. 121] states, "there has been some disagreement among scholars concerning the exact nature of magnitudes". According to him, magnitudes are (or, at least, "involve") "abstractions from geometric objects" rather than such objects themselves. However, Euclid actually calls segments, angles, and triangles equal to, or greater, or lesser than other items of the same kind. But since equality (in size) is an equivalence relation (and the greater-than-relation is invariant with respect to it), one may concede to Mueller that Euclid uses the objects as representatives of the equivalence classes to which they belong.

[12] The German word "Megethologie" has been used as a name for mathematics by the philosopher Franz Brentano. Lewis [12] calls so the extension of mereology by Boolos' plural quantification and defends the thesis that "mathematics is megethology".

of the present re-construction of Euclid's system would be rather low. Propositions I.4 and I.8 are actually the only ones in the first two books whose proofs involve the method of superposition[13] and it can even be avoided, as Mueller [13, p. 22f] explains, in the case of Proposition I.8. Hence we simply drop Euclid's Axiom I.4 justifying superposition and accept his Proposition I.4 as a new axiom instead; cf. below Ax. 10.

The greater-than-relation mentioned in Axiom I.5 is obviously a strict one. Formally it is more comfortable to start with the relation "at most as great than" which we shall denote by $\preceq$. Equality and strict order can be defined then as in Def. 14 below. Magnitudes are objects bearing this relation to each other. We reserve the letters "l", "m" and "n" (subscripted when necessary) as variables for magnitudes. Formulas containing variables for magnitudes should always be understood in such a way that these variable all refer to individuals of the same subsort thus, for instance, all to segments or all to angles.

Definitions 14 (Magnitudes and their ordering).

1. $\mathrm{Mag}(x) \overset{\mathrm{df}}{\Longleftrightarrow} \exists y.[x \preceq y \vee y \preceq x]$
2. $m \prec n \overset{\mathrm{df}}{\Longleftrightarrow} m \preceq n \wedge n \not\preceq m$
3. $m \simeq n \overset{\mathrm{df}}{\Longleftrightarrow} m \preceq n \wedge n \preceq m$

Since we conceive of megethology as an extension of mereology, it is suggestive to understand the operations of addition and subtraction mentioned in Axioms I.2 and I.3 as the mereological operations defined above in Def. 7. Then, however, Euclid's formulations need an amendment. Even under the condition that, for instance, both $\boldsymbol{s}_1 \simeq \boldsymbol{s}_2$ and $\boldsymbol{t}_1 \simeq \boldsymbol{t}_2$, we only expect the sums $\boldsymbol{s}_1 + \boldsymbol{t}_1$ and $\boldsymbol{s}_2 + \boldsymbol{t}_2$ to be equal in size when the respective products ("overlaps") of the summands are so, too, which is especially the case when the two summands are apart from each other, i.e., do not overlap. Non-overlapping obtains almost always when Euclid adds magnitudes.[14] Also in the case of mereological subtractions, $\boldsymbol{s}_1 \simeq \boldsymbol{s}_2$ and $\boldsymbol{t}_1 \simeq \boldsymbol{t}_2$ will imply $\boldsymbol{s}_1 - \boldsymbol{t}_1 \simeq \boldsymbol{s}_2 - \boldsymbol{t}_2$ again only if the products $\boldsymbol{s}_1 \cdot \boldsymbol{t}_1$ and $\boldsymbol{s}_2 \cdot \boldsymbol{t}_2$ are of equal size. This will be especially the case when the subtrahends are parts of the respective minuends since then

[13] There is another application of this method in Proposition III.24.

[14] A notable exception is Proposition II.7 where two overlapping rectangles of equal size are "added" and it is said that the sum is twice as big as one of the summands; cf. [8, p. 388f]. Obviously, in this case the "areas" are added rather than the rectangular regions and the overlap of the added regions is counted twice.

the products coincide with the subtrahends which are equal in size by hypothesis.

In the discussion above, apartness should not be taken to be the relation $|$ defined in Def. 7-2. Euclid does not mind to glue segments sharing a common boundary point or triangles with a common edge. Apartness in the sense relevant in the present context means that magnitudes, when overlapping at all, only do so in boundary points. As Fig. 1 shows, Euclid distinguishes between figures on the one hand (like, e.g., triangles and circles) and boundaries on the other. According to Definition I.13, "[a] boundary is that which is an extremity of anything" and Definition I.14 continues that "[a] figure is that which is contained by any boundary or boundaries"; [8, p. 154]. A circle's boundary is its periphery[15], that of a polygon a composite line (cf. Fig. 1) consisting of several segments. Definition I.3 explains that "[t]he extremities of a line are points"; [8, p. 154]. Since Euclid normally means "segments" when he is talking about "lines", his Definition I.13 suggests that the boundaries of a segments are its endpoints and that its other points are interior. When considered as parts of a line, segments have interior points in the sense of modern topology, too. However, our segments are part of the Euclidean plane; and considered as subsets of the plane, they lack interior points in the standard topology of the plane.

Euclid's explanations about "extremities" and "boundaries" are obviously not sufficient for setting up a mereotopological theory of this topic as it has been developed in our times by, e.g., Casati and Varzi [5, ch. 5]. However, it is completely clear for the magnitudes considered in the first two books of the *Elements* what their boundaries are. Hence we postpone the problem how to determine the boundary points in order to define first notions of overlap and apartness apt for the formal rendering of Euclid's common notions. In Def. 15, we use "BPnt" for the relation between boundary points and the magnitudes they delimit. Magnitudes sharing at most boundary points are "separated". By means of this newly defined concept, we provide formal renderings of Euclid's common notions in Ax. 9.

Definition 15 (Separation).

$$m \wr n \overset{\text{df}}{\iff} \forall p.[p \sqsubseteq m, n \to \mathrm{BPnt}(p, m) \land \mathrm{BPnt}(p, n)]$$

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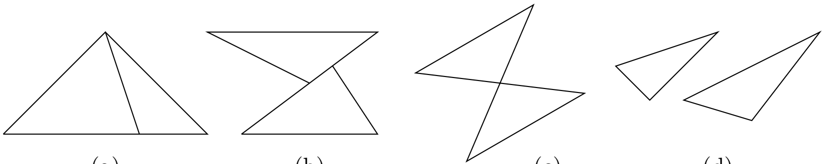
[15] The boundary of a circle is described in its Definition I.15. The term *periphery* (gr. περιφέρεια), however, has probably be added by a later editor of the *Elements*; cf. [8, p. 184].



(a) (b) (c) (d)

Figure 3. Simple rectilinear figures

AXIOMS 9 (Axioms for magnitudes).

1. $\preceq$ *is a linear order.*
2. $m_1 \wr m_2 \wedge n_1 \wr n_2 \to [m_1 \preceq n_1 \wedge m_2 \preceq n_2 \to m_1 + m_2 \preceq n_1 + n_2]$
3. $m_2 \sqsubseteq m_1 \wedge n_2 \sqsubseteq n_1 \to [m_1 \preceq n_1 \wedge n_2 \preceq m_2 \to m_1 - m_2 \preceq n_1 - n_2]$
4. $m \sqsubset n \to m \prec n$.

We still have to determine the magnitudes at issue in the first two books of the *Elements*. These are — besides segments and angles — polygons and circles. Polygons and circles are subsorts of figures; cf. the right branch of the tree graph of Fig. 1. In Definition I.19 Euclid defines rectilinear figures as those whose boundaries are line segments. The following Def. 16-1 of polygons generalizes Euclid's definition by admitting some "degenerated" cases such as (c) and (d) in Fig. 3. We use bold face capitals for polygons, e.g., "$\boldsymbol{P}$", "$\boldsymbol{Q}$", "$\boldsymbol{R}$". The formula "Finite $y.P(y)$" in Def. 16-1 should be read as "there are (at least one but only) finitely many Ps". As is well-known, the notion of finiteness can be defined in 2nd-order logic.

DEFINITIONS 16 (Polygons and circles).

1. $\mathrm{PGon}(m) \overset{\mathrm{df}}{\Longleftrightarrow} \exists P.[\mathrm{Finite}\, y.P(y) \wedge$
$$\forall y.[P(y) \to \mathrm{Tri}(y)] \wedge m = \Sigma y.P(y)]$$
2. $\bigcirc p\boldsymbol{r} \overset{\mathrm{df}}{=} p + \Sigma q.(pq \preceq \boldsymbol{r})$
3. $\mathrm{Circ}(m) = \exists p, \boldsymbol{r}.m = \bigcirc p\boldsymbol{r}$

A circle[16] is defined by Euclid as "a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another"; cf. [8, p. 153]. Thus an Euclidean circle is really that what in modern mathematical parlance

[16] According to our convention concerning the use the relation sign $\preceq$, this sign may only be flanked of terms of equal sort. Hence pq in Def. 16-2 has to be segment which implies that $p \neq q$.

is called a (circular) "disk". Def. 16-3 follows Euclid by conceiving of each point between the center and the periphery as a part of the circle. The next definitions explains for each of the four sorts of magnitudes what are the boundary points of items of that sort; cf. Def. 17. Non-boundary points are interior (cf. Def. 18-1) and the whole consisting of the boundary points of a circle Γ is its periphery $\mathring{\Gamma}$ (Def. 18-2).

DEFINITION 17 (Boundary points).
BPnt(p, m) iff either

1. m is a segment qr and $p = q \vee p = r$; or
2. m is an angle $\angle qrs$ and $p \sqsubseteq \overrightarrow{rq} + \overrightarrow{rs}$; or
3. m is a polygon $\Sigma n.P(n)$, $\exists n.[P(n) \wedge p \sqsubseteq n]$ but $\neg\exists q, r, s.[\mathrm{Tri}(\triangle qrs) \wedge p \sqsubseteq \triangle qrs - (qr + qs + rs) \wedge \triangle qrs \sqsubseteq m]$; or
4. m is a circle $\bigcirc q\boldsymbol{s}$ and $qp \simeq \boldsymbol{s}$.

DEFINITION 18 (Interior points and periphery).

1. $\mathrm{IPnt}(p, m) \stackrel{\mathrm{df}}{\Longleftrightarrow} p \sqsubseteq m \wedge \neg\mathrm{BPnt}(p, m)$
2. $\mathring{\Gamma} = \Sigma p.\mathrm{BPnt}(p, \Gamma)$

Our definitions and axioms immediately yield the following theorem.

THEOREM 9 (Megethological counterparts of **C1** and **C3**).

1. *The relations $\simeq$ of equality is an equivalence relation.*
2. $\mathrm{Betw}^+(p, q, r) \wedge \mathrm{Betw}^+(s, t, v) \wedge pq \simeq st \wedge qr \cong tv \to pr \simeq sv$

PROOF. Th. 9-1 readily follows from Ax. 9-1 and Def. 14-3. — In the case of Th. 9-2 we have from $\mathrm{Betw}^+(p, q, r)$ and Ax. 4 that $pr = pq + qr$ and $pq \wr qr$ and similarly from $\mathrm{Betw}(s, t, v)$ that $sv = st + tv$ and $st \wr tv$. Applying Ax. 9-2 yields $pr \simeq tv$. □

Analogues of the remaining congruence axioms from Def. 3 — thus **C2**, **C4**, and **C5** — are proved by Euclid. As already explained above, we accept his Proposition I.4, which corresponds to **C5**, as an axiom.

AXIOM 10 (The SAS (side-angle-side) equality criterion).

$$\begin{aligned}&\mathrm{Tri}(\triangle p_1q_1r_1) \wedge \mathrm{Tri}(\triangle p_2q_2r_2) \wedge p_1q_1 \simeq p_2q_2 \wedge p_1r_1 \simeq p_2r_2 \wedge \\ &\angle r_1p_1q_1 \simeq \angle r_2p_2q_2 \to q_1r_1 \cong q_2r_2 \wedge \angle p_1r_1q_1 \simeq \angle p_2r_2q_2 \\ &\wedge \angle p_1q_1r_1 \simeq \angle p_2q_2r_2 \wedge \triangle p_1q_1r_1 \simeq \triangle p_2q_2r_2\end{aligned}$$

Counterparts of **C2** are proved by Euclid in propositions I.2, I.3 by means of constructions by ruler and compasses. In his Postulate I.3, Euclid declares it always to be possible "[t]o describe a circle with any centre and distance"; [8, pp. 154, 169f]. In our mereological reconstruction of Euclid's framework, the existence of the circle with center p and radius $\boldsymbol{r}$ is guaranteed by Def. 16-2 since $\bigcirc p\boldsymbol{r}$ contains at least its center p. That $\bigcirc p\boldsymbol{r}$ also contains further points and especially those of the periphery will be ensured by axioms which we are going to introduce now. It is well known that Euclid's Postulate I.3 is not sufficient to prove the existence of points of intersection between circles close enough to each other and between a circle and the segments and lines crossing through it. In his proofs of propositions I.2 and I.3, Euclid simply assumes the points of intersection to exist. To ensure that the periphery of a circle intersects a line crossing through its interior in exactly two points, we add the axiom Ax. 11-1, which corresponds to the axiom characteristic of a platonic plane; cf. Def. 5-2. Given the axioms for an H-plane, Ax. 11-1 implies that the peripheries of two circles, none of which is a part of the other and having a common interior point, share exactly two points; cf. the discussion by Schreiber [21, pp. 103–107]. In our framework, we have to adopt Ax. 11-2 as an additional axiom because we lack a counterpart of **C4** which Euclid, making use of his silent presuppositions concerning the intersection points of circles, proves by means of a construction by ruler and compasses, cf. Th. 11 below. Beyond the existence of a unique pair of common peripheral points, our Ax. 11-2 postulates that these points lie on opposite sides of the line connecting the centers of the circles. The latter will be used in the proof of Th. 12-2 (= Euclid's Proposition I.7) below.

AXIOM 11 (Circle axioms).

1. $\exists p.[p \sqsubseteq a, \mathit{\Gamma} \wedge \mathrm{IPnt}(p, \mathit{\Gamma})] \rightarrow \exists^2 q.q \sqsubseteq a, \mathring{\mathit{\Gamma}}$
2. *Let* $\mathit{\Gamma} = \bigcirc p\boldsymbol{r}$ *and* $\mathit{\Delta} = \bigcirc q\boldsymbol{s}$ *be such that* $\exists r.[\mathrm{IPnt}(r, \mathit{\Delta}) \wedge \mathrm{IPnt}(r, \mathit{\Gamma})]$ *but neither* $\mathit{\Gamma} \sqsubseteq \mathit{\Delta}$ *nor* $\mathit{\Delta} \sqsubseteq \mathit{\Gamma}$. *Then:* $\exists uv.[u \neq v \wedge u, v \sqsubseteq \mathring{\mathit{\Gamma}} \cdot \mathring{\mathit{\Delta}} \wedge \forall w \sqsubseteq \mathring{\mathit{\Gamma}} \cdot \mathring{\mathit{\Delta}}.[w = u \vee w = v] \wedge u \not\sqsubseteq (\overline{pq}; v)]$

From Ax. 11-1 we derive Th. 10 as an analogue to clause **C2** of Def. 3. The proof makes use of the "whole-greater-than-part" principle Ax. 9-4.

THEOREM 10 (The counterpart of the congruence axiom **C2**).
$p \neq q \rightarrow \exists^1 r \sqsubseteq \overrightarrow{pq}.pr \simeq \boldsymbol{s}$

PROOF. The center p of the circle $\mathit{\Gamma} = \bigcirc p\boldsymbol{s}$ is not a boundary point of it (since pp is a point rather than a segment; cf. Def. 17-4). Hence, according to Ax. 11 the line $\overline{pq}$ intersects $\mathit{\Gamma}$ in two points, say r_1 and r_2. Suppose that none of them lies on $\overrightarrow{pq}$, so both of them are on that side on

$\overline{pq}$ which is, relative to p, opposite to q. Then it follows from the ordering principles of sec. 2 that either $\mathrm{Betw}^+(p, r_1, r_2)$ or $\mathrm{Betw}^+(p, r_2, r_1)$. Assume the first alternative (the case for the second is analogous). It follows then, that both $pr_1 \sqsubset pr_2$ and $pr_1 \simeq \boldsymbol{s} \simeq pr_2$ which contradicts Ax. 9-4. Thus one of the two points r_1 and r_2 lies on the ray $\overrightarrow{pq}$ (and the other on the opposite ray). □

The only axiom from Section 2 still awaiting its counterpart in our re-construction of Euclid's system is **C4**. So what remains to be proved is the following proposition (remember that lower case Greek letters range over angles).

Theorem 11 (The counterpart of **C4**).
$\mathrm{Tri}(\triangle pqr) \to \exists^1 \mathfrak{r}.\exists s \sqsubseteq (\overline{pq}; r).[\mathfrak{r} = \overrightarrow{ps} \wedge \angle spq \simeq \alpha]$

An (almost) identical assertion is proven by Euclid in his Proposition I.23. There he shows how to construct "[o]n a given straight line and at a point on it [...] a rectilinear angle equal to a given rectilinear angle"; [8, p. 294].[17] Let a be the line in question and p_1 the point where an angle is to be placed which is congruent to the angle α with vertex p. Euclid chooses two points, q and r, on the legs of α and then constructs a triangle $\triangle p_1q_1r_1$ with $p_1q_1 \sqsubseteq a$, $p_1p_2 \cong pq$, $p_1r_1 \cong pr$, and $q_1r_1 \cong qr$. That one always can construct such a triangle (at every place) with three given sides (fulfilling the triangle inequality) has been proven by Euclid before in Proposition I.22.

It is not possible here to follow the deductive route back from Proposition I.23 to Euclid's postulates and axioms and filling all the lacunae left by him. It deserves to be noted, however, that Euclid does not use his Postulate 5 — i.e., his formulation of the axiom of parallels — on his way to Proposition I.23. The first use of that axiom is made in the proof of Proposition I.29. We may add that axiom in the following form thus providing our universe $\top$ with the structure of a Platonic plane; cf. clause 2 of Def. 5.

Axiom 12 (The axiom of parallels).
$p \not\sqsubseteq a \to \exists^1 b.[\neg\, a \circ b \wedge p \sqsubseteq b]$

[17] Euclid allows also for angles with arcs rather than lines as legs. A rectilinear angle is an angle with linear legs.

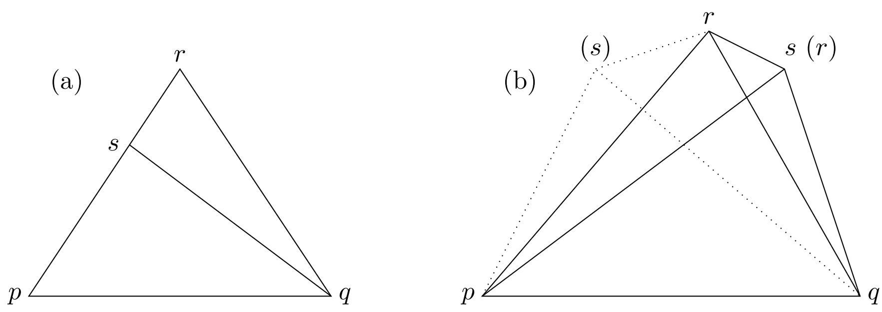


Figure 4. Euclid's propositions I.6 and I.7

On the way to Proposition I.23 the "whole-greater-than-part" principle Ax. 9-4 is used several times.[18] Since we cannot follow here Euclid's argumentation step by step, we illustrate his use of Ax. 9-4 by two typical examples instead, namely Proposition I.6 and I.7; cf. Fig. 4. These are the first propositions in the *Elements* proved by applications of the "whole-greater-than-part" principle. Whereas Proposition I.6 is only used again in later books, Proposition I.7 is a link in the deductive chain leading to Proposition I.23. It is used in order to prove the succeeding Proposition I.8 corresponding to the SSS criterion for the congruence of triangles. Furthermore, it plays also an important role in the theory of polygonal areas, as we shall see later; cf. p. 402. Euclid, when applying the "whole-greater-than-part" principle, never proves that one of the geometric objects at issue is the proper part of the other but relies, presumably, on the figures accompanying the propositions. Filling such lacunae requires in most cases considerable efforts as the following examples will show.

THEOREM 12 (Propositions depending on the "whole-greater-than-part" principle).

1. $\mathrm{Tri}(pqr) \wedge \angle qpr \simeq \angle pqr \rightarrow pr \simeq qr$
2. $\mathrm{Tri}(pqr) \wedge \mathrm{Tri}(pqs) \wedge s \sqsubseteq (\overline{pq}; r) \wedge pr \simeq ps \wedge qr \simeq qs \rightarrow r = s$

PROOF. 1. Assume $pr \not\simeq qr$; then $pr \prec qr \vee qr \prec pr$ by Ax. 9-1. We consider the case that $qr \prec pr$; the other case is analogous. By Th. 10 there is a unique point $s \sqsubseteq \overrightarrow{pr}$ with $ps \simeq qr$; cf. Fig. 4-(a). Using

[18] According to Neuenschwander's tables of the deductive dependencies in the *Elements* [14], Ax. 9-4 is used in the proof of three propositions needed for the proof of Proposition I.23.

the Pasch-Principle (Th. 6) one shows that each point of a triangle lies on a segment connecting a corner of that triangle to a point of the opposite edge. So each point u of $\triangle pqs$ is part of a segment sv with $v \sqsubseteq pq$. Since s and v lie on edges of $\triangle pqr$ and triangles are convex by Def. 9-5 we have $sv \sqsubseteq \triangle pqr$ and thus $u \sqsubseteq \triangle pqr$. Because each point of $\triangle pqs$ belongs, as we just have seen, to $\triangle pqr$, too, $\triangle pqs \sqsubseteq \triangle pqr$. From Ax. 9-4 it follows that $\mathrm{Betw}^+(p, s, r)$ and thus by Th. 2 that $r \not\sqsubseteq ps$ and therefore $r \not\sqsubseteq \triangle pqs$. Hence $\triangle pqs \sqsubset \triangle pqr$. By Ax. 10, however, we have $\triangle pqr \simeq \triangle pqs$. The part would equal the whole; which is excluded by Ax. 9-4.

2. Euclid's proof is indirect once more and considers the constellation displayed in Fig. 4 by the solid lines. The case represented (partly) by the dotted lines can be handled analogously. Because of the hypothesis $pr \simeq ps$ of the theorem, the triangle $\triangle prs$ is isosceles. Using his Proposition I.5 stating that isosceles triangles have equal base angles, Euclid concludes that $\angle prs \simeq \angle psr$. From this he infers that $\angle srq \prec \angle psr$ justifying this inference by his "whole-greater-than-part"-principle. Presumably, he does so since he takes it for obvious that $\angle srq \sqsubset \angle prs$ and hence $\angle srq \prec \angle prs$ by Ax. 9-4. He continues by stating that $\angle rsq$ "is much greater" [8, p. 259] than $\angle psr$, presumably again because $\angle psr \sqsubset \angle rsq$ and thus $\angle psr \prec \angle rsq$. From $\angle srq \prec \angle psr$ and $\angle psr \prec \angle rsq$, he infers $\angle srq \prec \angle rsq$. But an application of Proposition I.5 to $\triangle qrs$ yields $\angle rsq \simeq \angle srq$ contradicting the just inferred $\angle srq \prec \angle rsq$. Euclid does not give any justification for $\angle srq \sqsubset \angle prs$ and $\angle psr \sqsubset \angle rsq$. We postpone this question for considering first the possibility that $\triangle prq \sqsubseteq \triangle psq$ or, conversely, $\triangle psq \sqsubseteq \triangle prq$.

Perhaps Euclid ignores these possibilities because they are at odds with his "Whole-Greater-Than-Part" principle. We have $\triangle prq \simeq \triangle psq$ according to the SSS criterion of congruence. Thus both $\triangle prq \sqsubset \triangle psq$ and $\triangle psq \sqsubset \triangle prq$ are excluded by Ax. 9-4. However, the SSS criterion is only proved in Proposition I.8 by means of the theorem at issue now. Instead of using the SSS criterion, we argue as follows. If one of the triangles were a part of the other, p and q, would be on opposite sides of $\overline{rs}$; cf. Fig. 5-(a) where the case $\triangle psq \sqsubset \triangle prq$ is presented; the case for $\triangle prq \sqsubset \triangle psq$ is analogous. Because of the hypotheses $pr \simeq ps$ and $qr \simeq qs$ of our theorem, the points r and s would lie then on the peripheries of two circles Δ with center p and radius qr and Γ with center q and radius qs; cf. Fig. 5-(a). Γ and Δ would thus intersect in two points both lying on the same side of the line $\overline{pq}$ through their centers. This, however, is excluded by Ax. 11-2. Hence we may assume that p, and q are on the same side of $\overline{rs}$.

Let us now fill the gaps left by Euclid in his proof. First we have to justify that $\angle srq \sqsubset \angle prs$. We do so by showing that $q \sqsubseteq \angle^{\mathrm{i}} prs$ and applying Th. 8 to this result. In order to prove $q \sqsubseteq \angle^{\mathrm{i}} prs$ we have to show (1) $q \sqsubseteq (\overline{rp}; s)$ and (2) $q \sqsubseteq (\overline{rs}; p)$. The last has just been proven in the previous paragraph. Formula (1), on the other hand, describes merely the fact that we are concerned with the constellation dealt with by Euclid, namely that s is to the right of r. The "dotted" constellation in Fig. 4-(b) is analogously characterized by (3) $p \sqsubseteq (\overline{rq}; s)$. So what we really have to prove is that (1) and (3) make up an exhaustive disjunction. They cannot both be true since then $s \sqsubseteq \angle^{\mathrm{i}} prq$ and p and q would be on different sides of $\overline{sr}$ which we have already seen to be impossible. But (1) and (3) cannot both be false either. If both $q \not\sqsubseteq (\overline{rp}; s)$ and $q \not\sqsubseteq (\overline{rs}; p)$, then s would be an interior point of the angle vertical to $\angle prq$. Again, $\overline{rs}$ would separate p and q in this case.

We have still to justify Euclid's assumption that $\angle psr \sqsubset \angle rsq$. We do this again by referring to Th. 8 and by proving p to be an interior point of $\angle rsq$. Thus it has to be shown that both $p \sqsubseteq (\overline{sq}; r)$ and $p \sqsubseteq (\overline{sr}; q)$. The latter, however, has already been proved. For the first formula, assume that p and r were on different sides of $\overline{sq}$. Since s and r are according to the hypothesis of the theorem on the same side of $\overline{pq}$, the segment joining p and r had to meet $\overline{sq}$ either in the segment sq — cf. the segment sr_1 in Fig. 5-(b) — or in the ray starting at s and directed away from q — cf. the segment sr_2 in Fig. 5-(b). In the first case, however, q and s would be on opposite sides of $\overline{pr}$; but we are considering the case that "s is to right of r", i.e., $q \sqsubseteq (\overline{pr}; s)$. In the second case $\overline{rs}$ enters $\triangle pqs$ at its vertex s. By means of the Pasch Principle one shows that this line must leave $\triangle pqs$ through its edge pq. But then p and q lie on different sides of $\overline{rs}$ which already has been shown to be impossible. Hence we have shown that $p \sqsubseteq \angle^{\mathrm{i}} rsq$ and therefore both $\angle psq \sqsubset \angle rsq$ and $\angle psq \prec \angle rsq$. This, finally, completes the proof. $\square$

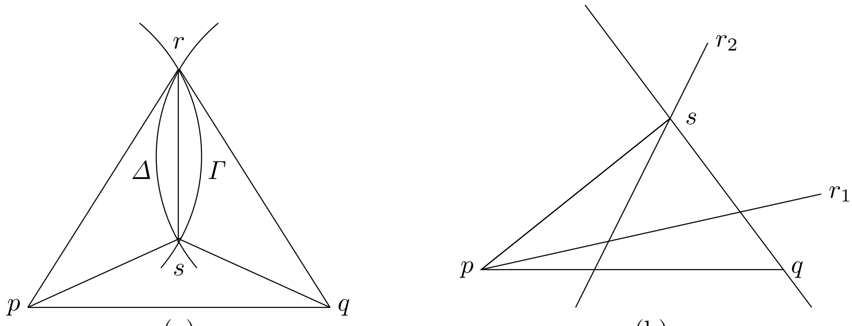


Figure 5. Constellations considered in the proof of Proposition I.7

4. The Mereology and Megethology of Polygons

One of Euclid's central concerns in the first two books of the *Elements* is the determination of the areas of polygons.[19] The same topic is treated by Hilbert in Chapter 4 of his *Grundlagen*. Unlike Hilbert, who measures the area of polygons by means of line segments conceived as the elements of "an algebra of segments, based on Pascal's theorem" [11, pp. 53–59/30–33], Euclid determines the area of such a figure by proving it to be equal in area to a square. His ultimate result is Proposition II.14, the last one of Book II, saying that it is always possible "[t]o construct a square equal to a given rectilinear figure"; [8, p. 409]. The square $\boldsymbol{S}$ equalling in size a given polygon $\boldsymbol{Q}$ is reached in two steps: first a parallelogram $\boldsymbol{P}$ with $\boldsymbol{P} \simeq \boldsymbol{Q}$ is constructed and then transformed into the square $\boldsymbol{S} \simeq \boldsymbol{P} \simeq \boldsymbol{Q}$. For constructing the intermediate parallelogram $\boldsymbol{P}$ in Proposition I.45, Euclid dissects the polygon $\boldsymbol{Q}$ into triangles and constructs for each of the triangular parts a corresponding parallelogram following a procedure specified in the proof of his Proposition I.44. The consecutive parallelograms then are attached side by side to each other thus constituting a final big parallelogram corresponding to the original polygon $\boldsymbol{Q}$. By this procedure the first triangle receives a privileged status since it fixes one edge and the base angle of the final big parallelogram $\boldsymbol{P}$; cf. Fig. 6.

Quite obviously this procedure is based on a mereological analysis of the task to be solved. Consider the special case displayed in

[19] The mereotopology of polygons have been investigated by Pratt and Lemon in [17].

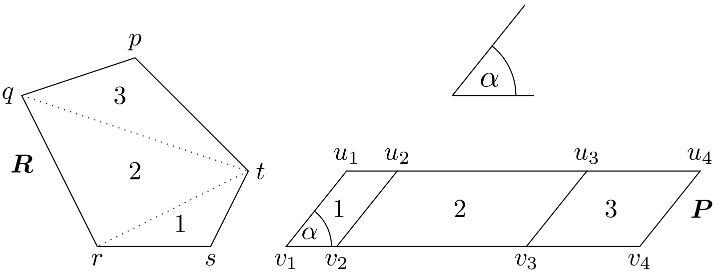


Figure 6. Euclid's Proposition I.45

Fig. 6. The pentagon $\pentagon pqrst$ is conceived as the mereological sum of the three non-overlapping triangles $\triangle rst$, $\triangle rtq$, and $\triangle qtp$: $\pentagon pqrst = \triangle rst + \triangle rtq + \triangle qtp$. In Proposition I.42 it is explained how "[t]o construct, in a given rectilinear angle, a parallelogram equal to a given triangle"; cf. [8, p. 339f]. This procedure is used for constructing the first small parallelogram $\varparallelogram u_1u_2v_1v_2$ with some given base angle α and $\varparallelogram u_1u_2v_1v_2 \simeq \triangle rst$. This fixes, as said above, the edge u_1v_1 of the final parallelogram yet to be constructed. The next task is to construct a parallelogram — again with base angle α and attached to the line segment u_2v_2 — equal in area to $\triangle rtq$. This is exactly the task which is solved by Proposition I.45. In Fig. 6, the parallelogram added is $\varparallelogram u_2u_3v_2v_3$. By Ax. 9-2 we know that the sum $\triangle rst + \triangle rtq \simeq \varparallelogram u_2u_3v_2v_3$. In the concluding third step of the construction the parallelogram $\varparallelogram u_3u_4v_3v_4$ equal in area to $\triangle qtp$ is added. This step yields $\varparallelogram u_1u_4v_1v_4$ which by a further application of Ax. 9-2 is equal in size to the original $\pentagon pqrst$.

Once again, there are several gaps in Euclid's proofs. He tacitly assumes, e.g., that it is always possible to dissect "rectilinear figures" into triangles. This difficulty is circumvented here by replacing Euclid's notion of a rectilinear figure by that of a polygon which is so dissectable by definition; cf. Def. 16-1. Furthermore, Euclid does not consider the general case of an "n-gon" but only treats the special case of a rectangle leaving it to the reader to reduce the general case by an inductive argument to the case $n = 4$. Logically, this issue is not a minor point. As Max Dehn remarks in his appendix to Pasch' well-known book on geometry, the theory of areas of polygons cannot be developed without induction and thus "is not really elementary"; cf. [16, p. 267].[20]

[20] An inductive proof that each polygon is dissectable into a number of triangles is provided by Paul Bernays in a supplement to Hilbert's *Grundlagen*; cf. [11, p. 249f].

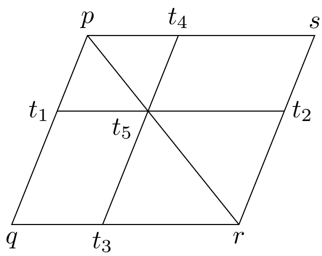


Figure 7. Euclid's Proposition I.43

We shall not try to fill the caveats left by Euclid, but want to conclude our discussion of his procedure by considering his proof of Proposition I.43 which is essential for the proof of Proposition I.45 used to "sum up" all the small parallelograms equalling the component triangles of a polygon; cf. Fig. 6. The proof of Proposition I.43 contains an application of Ax. 9-3 which is the only one of Euclid's axioms (i.e., common notions) which has not been used yet. Let t_5 be a point on the diagonal pr of the parallelogram $\varparallelogram pqrs$; cf. Fig. 7. Furthermore, let t_1t_2 and t_3t_4 be line segments parallel to the sides ps and pq of $\varparallelogram pqrs$, respectively. Euclid calls the parallelograms $\varparallelogram pt_1t_5t_4$ and $\varparallelogram t_5t_3rt_2$ whose diagonals are parts of the diagonal pr of the original big parallelogram the "parallelograms about the diameter" and the two parallelograms $\varparallelogram t_4t_5t_2s$ and $\varparallelogram t_1qt_3t_5$ their "complements". Proposition 43 says: "In any parallelogram the complements of the parallelograms about the diameter are equal to one another"; cf. [8, p. 340]. So, using the labels introduced above, what Euclid has to prove is $\varparallelogram t_4t_5t_2s \simeq \varparallelogram t_1qt_3t_5$.

In Proposition I.34 Euclid had already shown that the diagonal of a parallelogram dissects this figure into two triangles of equal size, hence $\triangle prs \simeq \triangle pqr$. Since, furthermore, $\triangle prs = \triangle t_4pt_5 + \varparallelogram t_4t_5t_2s + \triangle t_2t_5r$ and $\triangle pqr = \triangle pt_1t_5 + \varparallelogram t_1qt_3t_5 + \triangle t_5t_3r$, we have $\triangle t_4pt_5 + \varparallelogram t_4t_5t_2s + \triangle t_2t_5r \simeq \triangle pt_1t_5 + \varparallelogram t_1qt_3t_5 + \triangle t_5t_3r$ $(*)$. Again by Proposition I.34, it follows that both $\triangle pt_1t_5 \simeq \triangle t_4pt_5$ and $\triangle t_2t_5r \simeq \triangle t_5t_3r$. Applying then Axiom 9-2, we have $\triangle t_4pt_5 + \triangle t_2t_5r \simeq \triangle pt_1t_5 + \triangle t_5t_3r$ $(**)$. But $(*)$ and $(**)$, by Axiom 9-3 yield that $\varparallelogram t_4t_5t_2s \simeq \varparallelogram t_1qt_5t_3$. Euclid's clever strategy here is to supplement the two figures $\varparallelogram t_4t_5t_2s$ and $\varparallelogram t_1qt_5t_3$ whose equality is at issue by figures whose equality in area is already

This supplement has been added to a later edition of Hilbert's book so that it is not included in Townsend's translation available on the internet.

known in such a way that the equality of the thus resulting figures is known, too. The subtraction of the added parts then yields the equality of the supplemented areas.

The proofs of the two propositions I.43 and I.45 exemplify Euclid's strategies for proving the equality of polygons $\boldsymbol{P}$ and $\boldsymbol{Q}$. Either both are dissected into an equal numbers of polygonal (non-overlapping) parts which are pairwise equal to each other; respective summation of those parts then yield $\boldsymbol{P} \simeq \boldsymbol{Q}$ by Ax. 9-2. Or $\boldsymbol{P}$ and $\boldsymbol{Q}$ are extended to polygons $\boldsymbol{P} + \boldsymbol{P_1}$ and $\boldsymbol{Q} + \boldsymbol{Q_1}$ and it is shown then that both the thus extended polygons and the extenders $\boldsymbol{P_1}$ and $\boldsymbol{Q_1}$ are equal to each other. $\boldsymbol{P} \simeq \boldsymbol{Q}$ follows then by Ax. 9-3. Both strategies are also applied in Hilbert's treatment of area in Chapter IV of his *Foundations of Geometry*, cf. [11, pp. 69–82/38–47]. Hilbert [11, p. 70/38] calls polygons dissectable in pairwise congruent[21] triangles, *equidecomposable* (German: "zerlegungsgleich").[22] Polygons which can be supplemented by pairwise congruent polygons in such a way that the supplemented figures are equidecomposable are "of equal content" (German: "ergänzungsgleich", i.e., "supplementation equivalent").[23]

As Euclid, Hilbert is not very explicit as regards the mereological details concerning the dissection of polygons and their combination into more comprehensive ones. He leaves the operation of dissection ("Zerlegung") and composition ("Zusammensetzung") undefined relying on a sympathetic reader's intuition with respect to these matters. A more explicit exposition of Hilbert's theory of polygonal area, however, has been added to a later edition of Hilbert's book by his disciple Paul Bernays in two appendices: Supplement III and V. For precise definitions of the operations mentioned, Bernays found it necessary to generalize Hilbert's notion of a polygon[24] to that of a "polygonal". He first explains what it means for a set of triangles to satisfy the "triangulation condition";

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[21] We have now left Euclid's system working with the relation $\simeq$ of equality of size; the role of this relation is now taken over by the relation $\cong$ of congruence.

[22] Townsend, in his translation of Hilbert's book, renders *zerlegungsgleich* by *(being) of equal area*. The term *equidecomposable* — precisely matching the German term — is used by Hartshorne [7, p. 197].

[23] The two notions are equivalent in a euclidean plane for which the Archimedian Axiom (for the addition of segments) is valid; cf. Hartshorne [7, p. 216].

[24] Hilbert defines a polygon as a closed track of line segments (ger. "Streckenzug", eng. "broken line"). He allows for such polygons as that shown in Fig. 3-(c) but disallows non-connected polygons like that shown in Fig. 3-(d)

namely: two members of this set are either (1) completely disjoint, or (2) have only one vertex in common, or (3) share an edge but no further point. The decomposition of a Hilbertian polygon into triangles will satisfy the triangulation condition; conversely, however, a set of triangles fulfilling this condition need not be the decomposition of a Hilbertian polygon. A "polygonal", then, is obtained from a set of triangle satisfying the triangulation condition by the following two operation: (1) removing common edges of the triangles and (2) joining remaining edges lying on the same line. Bernays provides (rather complicated) definitions for the union $p+q$ and the intersection $p \cdot q$ of two triangular complexes p and q.

Bernays' results

1. The restriction of the relation of being supplementation equivalent to the class of Hilbertian polygons is an equivalence relation.
2. That relation is also additive: if $\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$, $\boldsymbol{S}$ are two pairs of disjoint supplementation equivalent polygonals, then $\boldsymbol{P}+\boldsymbol{Q}$ and $\boldsymbol{R}+\boldsymbol{S}$ are supplementation equivalent, too.
3. For each pair of polygonals $\boldsymbol{P}$ and $\boldsymbol{Q}$ there are always polygonals $\boldsymbol{P_1}$ and $\boldsymbol{Q_1}$ such that $\boldsymbol{P_1}$, $\boldsymbol{Q_1}$, and $\boldsymbol{D} = \boldsymbol{P} \cdot \boldsymbol{Q}$ do not overlap and
$$\boldsymbol{P} = \boldsymbol{P_1} + \boldsymbol{D} \qquad \boldsymbol{Q} = \boldsymbol{Q_1} + \boldsymbol{D} \qquad \boldsymbol{P} + \boldsymbol{Q} = \boldsymbol{P_1} + \boldsymbol{Q_1} + \boldsymbol{D}.$$

On the basis of these results, a measure function for polygons can be determined, i.e., a function mapping polygons in such a way to Hilbert's "algebra of segments" (which is an Archimedian ordered field) that polygons are assigned the same item from that algebra iff they are supplementation equivalent.[25] As he himself notes, such a measure would be useless if all polygons had the same content (would all be supplementation equivalent). Relying on simple school geometry, two rectangles $\square p_1q_1r_1s_1$ and $\square p_2q_2r_2s_2$ such that $p_1q_1 \cong p_2q_2$ but $q_1r_1 \ncong q_2r_2$ would constitute a simple counterexample showing that this is not the case. Assume to the contrary that the two rectangles are of equal content. Then the rectangular triangles $\triangle p_1q_1r_1$ and $\triangle p_2q_2r_2$ would be so, too. They have congruent bases but differ with respect to their "heights" (catheti) q_1r_1 and q_2r_2. Hence we could reject the hypothesis that all polygons are of equal area when we could prove that triangles of equal content

[25] Hilbert calls this measure function "Inhaltsmaß", i.e., "measure of content"; Townsend renders this by "measure of area"; cf. [11, p. 77/42].

with congruent bases have also congruent heights. That this is actually the case is proven by both Hilbert and, as Hilbert [11, p. 74/41] himself points out, by Euclid, too.

THEOREM 13 (Bases and heights of supplementation equivalent triangles).

1. *Hilbert's version*: *If two supplementation equivalent triangles have equal bases, they have equal heights, too.*
2. *Euclid's version* (*Proposition I.39*): *Equal triangles which are on the same base and on the same side are also in the same parallels.*

Euclid's proof of Proposition I.39 makes use of the "whole-greater-than-part" principle and depends furthermore upon Proposition I.7, i.e., our Th. 12-2, which was proven by that principle, too. Hilbert — citing Euclid's Axiom I.5 in the original Greek — explains: "In the demonstration of this theorem, however, Euclid appeals to the general proposition relating to magnitudes: 'Καὶ το ὅλον τοῦ μέρους μεῖζόν [έστιν]' — a method of procedure which amounts to the same thing as introducing a new geometrical axiom concerning areas"; [11, p. 74/41]. Hilbert actually copes without it. The reason why he wants to avoid the Euclidean principle, however, does not become quite clear. As he puts the matter, his reason is just the strive for axiomatic economy. Hessenberg [10, p. 57][26] is more explicit on this issue. He defines an ordering relation between polygons in a way reminiscent of Ax. 9-4: "We call a polygon $\boldsymbol{P}$ larger than a polygon $\boldsymbol{Q}$ (and $\boldsymbol{Q}$ smaller than $\boldsymbol{P}$) if $\boldsymbol{P}$ includes a polygon which is supplementation equivalent to $\boldsymbol{Q}$"; [10, p. 57]. Then he formulates a question which is equivalent to Hilbert's concern about the (non-) triviality of his measure function: "Does the validity of one of the three relationships $\boldsymbol{P} < \boldsymbol{Q}$, $\boldsymbol{P} = \boldsymbol{Q}$, $\boldsymbol{P} > \boldsymbol{Q}$ always imply the invalidity of the two others?" Citing Euclid, as Hilbert does, Hessenberg adds the following comment to his question: "Euclid believed that it could be safely answered by yes. The whole is greater than the part"; [10, p. 57]. He goes on to explain, however, that Euclid's attitude would be an intolerable naivety after the discovery of uncountable sets which have proper subsets of a cardinality equal to their own. For Hessenberg, thus, the findings of set theory gave rise to an uncertainty concerning

[26] The first edition of that book is from 1930. According to Diller's explanation [10, p. 3] in the preface to its second edition, the text passage cited in the main text above belongs to a section which has been taken over from the first edition.

the part-of-relation which also renders the whole Euclidean theory of magnitudes problematic.

5. Points, Boundaries, and Measure Theory

We started this article by Prenowitz' thought experiment whether Euclid would recognize a major difference between his approach to geometry and Hilbert's modern one. It is thus appropriate to finish with another question of the same kind: would Euclid accept our amendments of his system as in line with his general conception of geometry? Though I think that Euclid would appreciate some of them, I expect him nevertheless to be skeptic on one major point. As we have seen at the end of the previous section, the discovery of sets equal in cardinality to some of their proper subsets may have induced a skeptic attitude towards Euclid's "whole-greater-than-part" principle and to his "mereologico-megethological" approach to geometry in general. Set theory, on the other hand, has become the standard framework for mathematics (with category theory as perhaps the only serious competitor) and hence for geometry, too. As we shall explain in a moment, there is one problem which any formulation of Euclidean geometry using atomistic mereology will share with formulations using set theory. Since this problem is absent in Euclid's conception of geometry, we may conclude that he would be dissatisfied with both reformulations of his geometry: the set-theoretic one and that using atomistic mereology.

The problem at issue has been pointed out by Carathéodory in his book on measure theory and integration [4].[27] In Proposition I.34, Euclid proves that each diagonal of a parallelogram bisects that figure into two equal triangles; cf. [8, p. 323]. Commenting on Euclid's proof procedure, Caratheodory [4, 10] explains that

> [...] it is impossible to conceive of geometric figures cut into pieces in this way or composed out of separate pieces as sets of points. A decision had to be made in that case whether the points of the edges of a triangle have to be reckoned among that triangle's points or whether one prefers to conceive of the triangle as the set of its inner points, and each possible

[27] It did not remain unrecognized in the mereological literature though; cf., e.g., Randell, Cui, and Cohn's well-known article on their Region Connection Calculus, [19, p. 167], as well as the extensive discussions of boundaries by Casati and Varzi [5, ch. 5].

decision leads up to a contradiction: depending on the option chosen, one will, in the case of the composition of a parallelogram out of two triangles, either count the diagonal twice or it completely drops out of the figure.

Obviously the same dilemma which Carathéodory describes for the set-theoretic conceptions of regions arises also for the mereological one identifying regions with collections of points. Given a parallelogram $\square pqrs$, one has to decide to which of the triangles $\triangle pqs$ or $\triangle qrs$ the diagonal qs belongs. In our re-construction of the Euclidean system, it belongs to both. Distinguishing between the relations of apartness $|$ (cf. Def. 7-2) and that of separation $\wr$ (cf. Def. 15), however, we declared the overlap between $\triangle pqs$ and $\triangle qrs$ for irrelevant for the question of the equality (in size) of the two triangles. This is, perhaps, more hitting through the dilemma than solving it.

Obviously, Euclid does not see any problem in the procedure applied in his proof of Proposition I.34 (nor does Hilbert when dissecting and composing figures in Chapter 4 of the *Grundlagen*). Presumably the reason for this is that he adopts an Aristotelian theory of continua. In Book V of the *Physics* Aristotle distinguishes three basic types of modes how entities can follow each other: succession, contact, and continuity. An object succeeds another one of the same kind if there is nothing of the same kind between them. Objects are in contact with each other if their "extremities" are together, and they are continuous if their extremities are one.[28] Using these distinction then, Aristotle [1, 231 a 31, I, p. 390f] goes on to argue that, for instance, "a line cannot be composed of points, the line being continuous and the point indivisible".[29] The two "extremity" points of a segment pq are its boundaries, cf. the leftmost branch in the tree of Fig. 1, demarcating it from the plane. An interior point r of pq is the boundary separating the two subsegments pr and rq being at the same time their common border. According to

[28] The English word *extremity* is also used in translations of Euclid's explanations about the boundaries of a segment (Definition I.3) and the general notion of a boundary (Definition I.13), cf. p. 389 above. Euclid's Greek term (πέρας), however, differs from Aristotle's (ἔσχατος).

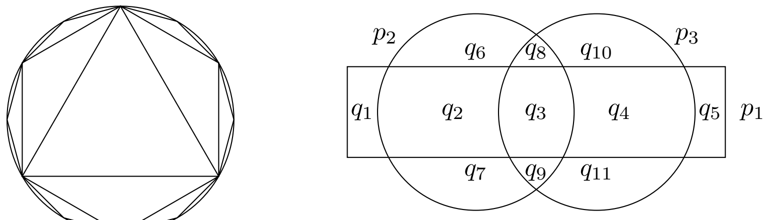
[29] Brentano [3] explains the differences between the modern conception of the continuum deriving from the works of Dedekind and Cantor on the one hand and the Aristotelian on the other, defending the latter conception. A re-construction of "the classical Dedekind-Cantor continuum" is provided by Hellman and Shapiro [9] within the framework of a point-free mereology.

this conception, however, points are definitely not parts of the line: how could something exclusively consists just of boundaries? Similarly, in the case of Proposition I.34 the diagonal is the common border of the two triangles but is not a part of either.[30] Euclid does not see any problems in such boundary items as points (delimiting segments), lines (delimiting surfaces), and surfaces (delimiting solids). Furthermore, it is surely no accident that his definitions (at the beginning of Book I for the first three items and of Book XI for surfaces) proceed exactly in that order. He would probably have rejected definitions of points, lines, and surfaces in terms of solids as an instance of the failure of "the exhibition of the prior through the posterior", as Aristotle [1, I, 142 a 17, p. 240] calls it in the *Topica*. Nevertheless Euclidean mereology seems to be point-free rather than atomistic.

Interestingly enough, Carathéodory builds his approach to measurement and integration upon a point-free (quasi-) mereology.[31] His aim is to lay the foundations for a "general theory of content which comprises both the Euclidean doctrine as a special case and the most general theories of measure which have been developed in this century"; cf. [4, p. 10]. Actually he develops two versions of a point-free mereology in the first chapter of his book and proves them equivalent in section 17 of that chapter. The basic objects of his theory, which he started to develop in the late 1930s, are called "somata" (from the Greek word σῶμα meaning "body") by him. As is explained by Bélanger and Marquis [2, p. 12f], he seems to have influenced Nöbeling's [15] point-free topology both by this terminology and by the idea to "eliminate" points. In his first formulation of his point-free mereology, Carathéodory [4, pp. 11-17] motivates his axioms by considering the special application of it to the analysis of planar figures. This generalizes the case for polygons dealt with in section 4 of the present article. His sole undefined relation

[30] Cf. Hellman and Shapiro [9, p. 513f] for a reconstruction of (part of) Aristotle's theory of continua and boundaries. Roeper [20] is another attempt to model Aristotle's intuition within a mereological framework. Roeper's approach is interesting by making use of ideas also used in the formal semantics of mass terms. Aristotle's conception of continua may be inspired by the existence of "homogeneous masses" as is evidenced by the introductory sentence of his *Historia Animaliae* where he mentions flesh as a part of animals "that divide[s] into parts uniform with themselves"; cf. [1, I, p. 774].

[31] I add the qualification "quasi" since he postulates the existence of a null soma, something strongly disliked by most mereologists.



(a) (b)

Figure 8. Sums and dissections

between somata is that of being apart (ger. "fremd") to each other. He symbolizes this relation by "$\circ$" which we have already used in Def. 6-2 for overlapping. Hence we shall symbolize his relation by the sign $\wr$ which, however, is now to be understood as a basic sign rather than as introduced by Def. 15. Non-overlapping somata may be combined by a sum operation Σ. Infinite sums are admitted since one wants, e.g., exhaust a circle by an infinite numbers of triangles; cf. Fig. 8-(a). For reasons coming from measure theory, however, infinite sums may have an at most denumerable number of summands; cf. [4, pp. 11f, 24].

The sum operation Σ is thus defined for (possibly doubly) indexed systems of somata which are pairwise apart from each other. Using lower case italics for somata and small Greek letters for systems of them, we may state the first six of Carathéodory's axioms as below.

Caratheodory's first six (quasi-) mereological axioms

1. $x \wr y = y \wr x$
2. *If φ is a permutation of ψ, then* $\sum_i \varphi(i) = \sum_i \psi(i)$.
3. *If both* $x = \sum_{ij} \varphi(i,j)$ *and, for each* i, $y_i = \sum_j \varphi(i,j)$, *then* $x = \sum_i y_i$.
4. *There exists a soma* $\perp$ *such that* $\perp \wr x$ *and* $\perp + x = x$.
5. $(\forall i. x \wr \varphi(i)) \rightarrow x \wr \sum_i \varphi(i)$
6. $(\forall i. \varphi(i) \neq \perp) \rightarrow \neg(\varphi(i) \wr \sum_j \varphi(j))$.

The theory based on these axioms is made applicable also to cases involving overlapping somata by adding an axiom postulating that in such a case always suitable non-overlapping somata can be found. Thus, for instance, the soma displayed in Fig. 8-(b) is the sum of the rectangular soma p_1 and the two circular somata p_2 and p_3 which all overlap which each other. However, it can also be analyzed as the sum of the pairwise disjunct somata q_1, q_2, ..., q_{11}.

Carathéodory's dissectability axiom

7. *If* x_1, x_2, ..., x_m *are arbitrary somata, then there are somata* y_1, y_2, ..., y_n *which are pairwise apart from each other such that for each* x_j $(1 \leqslant j \leqslant m)$ *which does not equal an* y_k $(1 \leqslant k \leqslant n)$ *there are* y_{j_1}, y_{j_2}, ..., y_{j_k} $(1 \leqslant j_1, j_2, \ldots, j_k \leqslant n)$ *such that* $x_j = y_{j_1} + y_{j_2} + \cdots + y_{j_k}$.

Carathéodory [4, pp. 28–32] proves that structures fulfilling the seven axioms just stated are Boolean rings in which the (lattice-theoretic counterpart of the) symmetric difference of somata takes the role of addition and in which there always exist the (ordinary) sum of denumerably many somata. In his approach to measure theory and integration, these structures take over the role played by σ-algebras of subsets of some basic set X in standard, set-theory based accounts to these disciplines. Cartheodory's main innovation consists in his demonstration how the work of the point functions defined over the base set of a σ-algebra can be done by special "local" functions (ger. "Ortsfunktionen") defined on somata.

A numerical function measuring the areas of polygons could obviously be developed in Carathéodory's framework. Unlike the reconstruction of Euclid's approach which has been proposed in the present article, this would not equate figures with collections of points. This, surely, is closer to Euclid's original conception. Conversely, however it suffers from the lack of any account of the boundaries of somata. How, one could imagine Euclid having asked, could a soma exist without being delimited from its environment?

References

[1] Aristotle, *The Complete Works of Aristotle. The Revised Oxford Translation*, J. Barnes (ed.) 2 vols., Princeton University Press, Princeton NJ, 1984.

[2] Bélanger, Matthieu, and Jean-Pierre Marquis, "Menger and Nöbeling on pointless topology" *Logic and Logical Philosophy*, 22 (2013): 145–165. DOI: 10.12775/LLP.2013.009

[3] Brentano, Franz, *Philosophische Untersuchungen zu Raum, Zeit und Kontinuum*, Meiner, Hamburg, 1976.

[4] Carathéodory, Constantin, *Mass und Integral und ihre Algebraisierung*, Birkhäuser, Basel und Stuttgart, 1956.

[5] Casati, Roberto, and Achille C. Varzi, *Parts and Places*, MIT Press, Boston MA, 1999.

[6] Cohn, Anthony, "A more expressive formulation of many sorted logic", *Journal of Automated Reasoning*, 3 (1987): 113–200. DOI: 10.1007/BF00243207

[7] Hartshorne, Robin, *Geometry: Euclid and Beyond*, Springer, New York, 2000.

[8] Heath, Thomas L., *The Thirteen Books of the Elements*, 3 vols, Dover Publications, Newburyport, 2012–2013. Reprint of the 2nd edition 1956. First edition: Cambridge University Press, Cambridge GB, 1908.

[9] Hellman, Geoffrey, and Stewart Shapiro, "Regions-based two dimensional continua: the Euclidean case", *Logic and Logical Philosophy*, 24 (2015): 499–534. DOI: 10.1305/ndjfl/1153858647

[10] Hessenberg, Gerhard, and Justus Diller, *Grundlagen der Geometrie*, de Gruyter, Berlin and New York, 1967.

[11] Hilbert, David, *Grundlagen der Geometrie*, Teubner, Leipzig, 1899, 14th edition, M. Toepell (ed.), Teubner 1999, Stuttgart. English translation by J. E. Townsend: *Foundations of Geometry*, Open Court 1971, LaSalle IL. 10th printing 1999. https://archive.org/details/thefoundationsof17384gut

[12] Lewis, David, "Mathematics is megethology", *Philosophia Mathematica*, 1, 1 (1993): 3–23. DOI: 10.1093/philmat/1.1.3

[13] Mueller, Ian, *Philosophy of Mathematics and Deductive Structure in Euclid's* Elements, MIT Press, Boston MA, 1981. Paperback edition: Dover, Newburyport, 2006.

[14] Neuenschwander, Erwin, "Die ersten vier Bücher der *Elemente* Euklids", *Archive for the History of Exact Sciences*, 9 (1973): 325–380. http://www.jstor.org/stable/41133382

[15] Nöbeling, Georg, *Grundlagen der analytischen Topologie*, Springer, Berlin, 1954.

[16] Pasch, Moritz, *Vorlesungen über neuere Geoemtrie*, Teubner, Leipzig, 1882. 2nd edition with an appendix by Max Dehn, Berlin: Springer 1926. Reprinted 1976.

[17] Pratt, Ian, and Oliver Lemon, "Ontologies for plane, polygonal mereotopology", *Notre Dame Journal of Formal Logic*, 38, 2 (1997): 225–145. DOI: 10.1305/ndjfl/1039724888

[18] Prenowitz, Walter, "A contemporary approach to classical geometry", *The American Mathematical Monthly*, 68 (1961): 1–67. http://www.jstor.org/stable/2311902

[19] Randell, David A., Zhan Cui, and Anthony G. Cohn, "A spatial logic based on regions and connections", pages 165–176 in B. Nebel, Ch. Rich, and W. Swartout (eds.), *Proceedings of the 3rd International Conference on Knowledge Representation and Reasoning*, Los Altos CA, 1992. Morgan Kaufmann. http://wenxion.net/ac/randell92spatial.pdf

[20] Roeper, Peter, "The Aristotelian continuum. A formal characterization", *Notre Dame Journal of Formal Logic*, 47, 2 (2006): 211–232. DOI: 10.1305/ndjfl/1153858647

[21] Schreiber, Peter, *Theorie der geometrischen Konstruktion*, Deutscher Verlag der Wissenschaften, Berlin, 1975.

[22] Tarski, Alfred, "Appendix E", pages 161–172 in J. Woodger, *The Axiomatic Method in Biology*, Cambridge UK: Cambridge University Press 1937.

[23] Tarski, Alfred, "What is elementary geometry?", pages 16–29 in L. Henkin, P. Suppes, and A. Tarski (eds.), *The Axiomatic Method. With Special Reference to Geometry and Physics*, North-Holland Publishing Company, Amsterdam, 1959. http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.477.3340&rep=rep1&type=pdf

[24] van der Vel, M. L. J., *Theory of Convex Structures*, North-Holland Publishing Company, Amsterdam, 1993.

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