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RELATIONAL SEMANTICS FOR THE 4-VALUED RELEVANT LOGICS BN4 AND E4

Abstract. The logic BN4 was defined by R. T. Brady in 1982. It can be considered as the 4-valued logic of the relevant conditional. E4 is a variant of BN4 that can be considered as the 4-valued logic of (relevant) entailment. The aim of this paper is to define reduced general Routley-Meyer semantics for BN4 and E4. It is proved that BN4 and E4 are strongly sound and complete w.r.t. their respective semantics.

Keywords: relevant logics; many-valued logics; 4-valued logics; Routley-Meyer semantics

1. Introduction

The logic BN4 was defined by Brady in [6]. The matrix MBN4 (cf. Definition 2.3 below) upon which BN4 is built is a modification of Smiley’s matrix MSm4, characteristic of Anderson and Belnap’s First Degree Entailment logic FDE (cf. [1, pp. 161–162]; cf. the proof of Proposition 2.12 below where FDE is defined). According to Dunn [9, p. 8], the matrix MSm4 is in its turn a simplification of Anderson and Belnap’s 8-element matrix M_0 (cf. [1, 3]), which has played an important role in the development of relevant logics (cf. [17, pp. 176, ff.]).

The logic E4 is built upon a modification of Brady’s matrix MBN4; in particular, upon a modification of the function f_{\rightarrow} defining the conditional (cf. Definition 2.4 below).

According to Meyer *et al.*, “BN4 is the correct logic for the 4-valued situation where the extra values are interpreted in the both and neither

senses” [11, p. 25]. On his part, Slaney considers this logic as having the truth-functional implication most naturally associated with the logic FDE referred to above (cf. [18, p. 284]).

In [16], it is claimed that E4 is related to BN4 in a similar way to which Anderson and Belnap’s logic of entailment E is related to the relevant logic R (cf. [1] about these logics): while BN4 can be considered as the “4-valued logic of the relevant conditional”, E4 can be viewed as the “4-valued logic of (relevant) entailment”. BN4 can intuitively be described as a 4-valued extension of contractionless relevant logic R (RW); and E4, on its part, as a 4-valued extension of reductioless logic Er. RW is intuitively the result of dropping the *axiom contraction* (i.e., $[A \rightarrow (A \rightarrow B)] \rightarrow (A \rightarrow B)$) from R, while Er results from dropping the *axiom reductio* (i.e., $(A \rightarrow \neg A) \rightarrow \neg A$) from E.

The aim of this paper is to endow BN4 and E4 with a general reduced Routley-Meyer semantics. In this way BN4 and E4 can be related to the wealth of logics (relevant and non-relevant — i.e., lacking the variable-sharing property) that have been interpreted with this type of semantics. In particular, our results in the present paper are connected to some applications of the Routley-Meyer semantics for non-relevant logics in general (cf. [15]) and to 3-valued logics in particular (cf. [13, 14]).

Both BN4 and E4 have been interpreted with a Belnap-Dunn type semantics (cf. [6, 16]). Furthermore, BN4 has been endowed with a 2-set-up Routley-Meyer semantics (cf. [6]) while in [16] it is summarily indicated how to provide a semantics of this type for E4. But the 2-set-up semantics in [6] is briefly described in scarcely two pages and a half (pp. 29–31) with some essential proofs referring to the fundamental Chapter 4 of [17]. On the other hand, the treatment of the 2-set-up semantics in [16] is limited to a still briefer description in a few lines, as pointed out above. However, the semantics in the present paper are developed in detail, the key theorems being fully proved. Moreover, (although we do not have space here to do it) it will not be difficult to show that the 2-set-up models are special cases of the general models to be investigated in the sequel. Then, soundness w.r.t. the 2-set-up semantics is immediate from soundness w.r.t. the general models, whereas completeness w.r.t. the former is not hard to prove from completeness w.r.t. the latter. Now, whereas the 2-set-up models are only appropriate to certain 3-valued and 4-valued logics, the general ones are, as remarked above, adequate for an ample class of logics, both relevant and non-relevant. In particular, the general models investigated in the present paper are

suitable for modeling any logic including the system b4 (“basic 4-valued logic included in BN4 and E4”). The system b4 is an extension of the logic Cm. The logic Cm is, in its turn, the result of dropping the *Modus Ponens Axiom* ($[A \wedge (A \rightarrow B)] \rightarrow B$) from the logic C, the minimal logic (without weak rules) that can be endowed with a reduced Routley-Meyer semantics.

As remarked above, the aim of this paper is to endow BN4 and E4 with a general Routley-Meyer semantics. In particular, with a reduced general Routley-Meyer semantics. As it is known, reduced and unreduced general Routley-Meyer semantics are distinguished as follows: a set of designated points is used to decide validity in the latter although this set is restricted to a single element in the former. In [17] or in [8], it is argued at length why reduced models are preferable when it is possible to defined them. Concerning the logics investigated in this paper, contrary to what is the case with E4, BN4 presents some problems when defining reduced models. But these problems can be solved according to the method suggested in [17, 7, 8].

Taken from a general point of view, we think that the results in the present paper are a contribution to enhancing what is shown in [17, Chapter 4]: Routley-Meyer semantics is a malleable and powerful instrument for interpreting non-classical logics.

The structure of the paper is as follows. In Section 2, the logics BN4 and E4 are defined as Hilbert-style axiomatic systems. Actually, they are defined from two sublogics of both BN4 and E4, the logics Cm and b4. The latter logics are used to simplify the soundness and completeness proofs of BN4 and E4. In Section 3, reduced models for Cm are introduced. Then, reduced models for BN4 and E4 are defined from Cm-models and soundness of BN4 and E4 w.r.t. their respective models is proved from the soundness of Cm. In sections 4 and 5, the logic b4 is used as a basis for developing the completeness proofs. In Section 4, in particular, following [17, Chapter 4], we prove extensions and primeness lemmas for any extension of b4, Eb4, where theories are defined as sets closed under Adjunction and Eb4-entailment. In Section 5, on the other hand, a series of preliminary lemmas to the completeness proofs are proved. These lemmas work for extensions of b4 where theories are defined as in the previous section. In Section 6, we prove the strong completeness of E4 w.r.t. the reduced models defined in Section 3. The proof is an easy consequence of the results in sections 4 and 5. Finally, in Section 7, a strong completeness theorem for BN4 is proved. Unlike

it was the case with E4, this theorem cannot entirely be based upon the work in sections 4 and 5. In particular, we need a more strict notion of a theory and new extension and primeness lemmas that are defined following the method described in [17, 7, 8].

2. The logics BN4 and E4

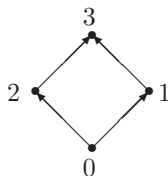
In this section, we define the logics BN4 and E4. Firstly, we define the logical language and the notion of logic used in the paper.

DEFINITION 2.1 (Languages). The propositional language consists of a denumerable set of propositional variables $p_0, p_1, \dots, p_n, \dots$ and some or all of the following connectives \rightarrow (conditional), \wedge (conjunction), \vee (disjunction), \neg (negation). The biconditional (\leftrightarrow) and the set of wffs are defined in the customary way. A, B (possibly with subscripts $0, 1, \dots, n$), etc. are metalinguistic variables.

DEFINITION 2.2 (Logics). A logic \mathcal{S} is a structure $\langle L, \vdash_{\mathcal{S}} \rangle$ where L is a propositional language and $\vdash_{\mathcal{S}}$ is a (proof-theoretical) consequence relation defined on L by a set of axioms and a set of rules of derivation. The notions of *proof* and *theorem* are understood as it is customary in Hilbert-style axiomatic systems ($\Gamma \vdash_{\mathcal{S}} A$ means that A is derivable from the set of wffs Γ in \mathcal{S} ; and $\vdash_{\mathcal{S}} A$ means that A is a theorem of \mathcal{S}).

Next, we introduce the matrices upon which BN4 and E4 are defined. Consider now the following matrices (cf. [6, 16]).

DEFINITION 2.3 (The matrix MBN4). The propositional language L consists of the connectives $\rightarrow, \wedge, \vee,$ and \neg . The matrix MBN4 is the structure $\langle \mathcal{V}, D, F \rangle$, where (i) \mathcal{V} is $\{0, 1, 2, 3\}$ and it is partially ordered as shown in the following diagram:



(ii) $D = \{3, 2\}$; (iii) $F = \{f_{\rightarrow}, f_{\wedge}, f_{\vee}, f_{\neg}\}$, where $f_{\rightarrow}, f_{\wedge}, f_{\vee}, f_{\neg}$ are defined according to the truth-tables 1.

\rightarrow	0	1	2	3	\wedge	0	1	2	3	\vee	0	1	2	3	\neg	
0	3	3	3	3	0	0	0	0	0	0	0	1	2	3	0	3
1	1	3	1	3	1	0	1	0	1	1	1	1	3	3	1	1
2	0	1	2	3	2	0	0	2	2	2	2	3	2	3	2	2
3	0	1	0	3	3	0	1	2	3	3	3	3	3	3	3	0

 Table 1. Truth-tables for: \rightarrow , \wedge , \vee , and \neg

DEFINITION 2.4 (The matrix E4). The propositional language is the same as in MBN4. The matrix E4 is the structure $\langle \mathcal{V}, D, \mathbf{F} \rangle$, where \mathcal{V} , D , and \mathbf{F} are defined exactly as in MBN4 except for f_{\rightarrow} which is defined according to the following truth-table:

\rightarrow	0	1	2	3
0	3	3	3	3
1	0	2	0	3
2	0	0	2	3
3	0	0	0	3

The notions of *interpretation*, *consequence*, and *validity* are defined as it is customary in any matrix M , viz.

DEFINITION 2.5 (M -interpretations, M -consequence, M -validity). Let M be a matrix for (a propositional language) L . An M -interpretation I is a function from \mathcal{F} to \mathcal{V} according to the functions in \mathbf{F} . Then, for any set of wffs Γ and wff A , $\Gamma \models_M A$ (A is a consequence of Γ according to M) iff $I(A) \in D$ whenever $I(\Gamma) \in D$ for all M -interpretations I ($I(\Gamma) \in D$ iff $I(B) \in D$ for each $B \in \Gamma$). In particular, $\models_M A$ (A is M -valid; A is valid in the matrix M) iff $I(A) \in D$ for all M -interpretations I .

Concerning the intuitive meaning of the truth-values in MBN4 and E4, we note the following remark:

Remark 2.6 (On the intuitive meaning of the truth values). The truth values 0, 1, 2, and 3 can intuitively be interpreted in MBN4 and ME4 as follows. Let T and F represent truth and falsity. Then, $0 = F$, $1 = N(\text{either})$, $2 = B(\text{oth})$ and $3 = T$ (cf. [4, 5]) Or, in terms of subsets of $\{T, F\}$, we have: $0 = \{F\}$, $1 = \emptyset$, $2 = \{T, F\}$ and $3 = \{T\}$ (cf. [9] and references therein). It is in this sense that we speak of “bivalent semantics”, when referring to the Belnap-Dunn semantics: there are only two truth values and the possibility of assigning both or neither to propositions. (We use the symbols 0, 1, 2, and 3 because they are convenient for using the tester in [10] in case the reader needs one.)

BN4 and E4 are the logics *determined* by the matrices MBN4 and ME4, respectively (cf. [6] and [16]), where this notion of determination is defined as follows:

DEFINITION 2.7 (Logics determined by matrices). Let L be a propositional language, M a matrix for L and $\vdash_{\mathcal{S}}$ a (proof theoretical) consequence relation defined on L . Then, the logic \mathcal{S} (cf. Definition 2.2) is determined by M iff for every set of wffs Γ and wff A , $\Gamma \vdash_{\mathcal{S}} A$ iff $\Gamma \vDash_M A$. In particular, the logic \mathcal{S} (considered as the set of its theorems) is determined by M iff for every wff A , $\vdash_{\mathcal{S}} A$ iff $\vDash_M A$ (cf. Definition 2.5).

Next, BN4 and E4 are defined. In order to do this, it will be useful to previously define two logics contained in both BN4 and E4. The first of them is the logic Cm, a restriction of the logic C, which is important because it is the minimal logic that can be endowed with reduced RM-models (cf. [17, Chapter 4]). The logic Cm is defined when dropping the axiom *Modus Ponens* (i.e., $[A \wedge (A \rightarrow B)] \rightarrow B$) from C. The second one, b4, is, as far as we know, a new logic serving a mere instrumental role in the present paper.

DEFINITION 2.8 (The logic Cm). The logic Cm can be axiomatized with the following axioms and rules of inference.

Axioms:

$$A \rightarrow A \tag{a1}$$

$$(A \rightarrow B) \rightarrow [(B \rightarrow C) \rightarrow (A \rightarrow C)] \tag{a2}$$

$$(A \wedge B) \rightarrow A / (A \wedge B) \rightarrow B \tag{a3}$$

$$[(A \rightarrow B) \wedge (A \rightarrow C)] \rightarrow [A \rightarrow (B \wedge C)] \tag{a4}$$

$$A \rightarrow (A \vee B) / B \rightarrow (A \vee B) \tag{a5}$$

$$[(A \rightarrow C) \wedge (B \rightarrow C)] \rightarrow [(A \vee B) \rightarrow C] \tag{a6}$$

$$[A \wedge (B \vee C)] \rightarrow [(A \wedge B) \vee (A \wedge C)] \tag{a7}$$

$$(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A) \tag{a8}$$

$$(\neg A \rightarrow B) \rightarrow (\neg B \rightarrow A) \tag{a9}$$

Rules of derivation:

$$\text{Adjunction: } A \ \& \ B \Rightarrow A \wedge B \tag{Adj}$$

$$\text{Modus Ponens: } A \ \& \ A \rightarrow B \Rightarrow B \tag{MP}$$

Notice that Cm is the result of adding (a2) and (a8) — or equivalently, (a9) — to Routley and Meyer's basic logic B (cf. [17, Chapter 4]).

DEFINITION 2.9 (The logic b4). The logic b4 is axiomatized by adding to Cm the following axioms:

$$\neg A \rightarrow [A \vee (A \rightarrow B)] \quad (\text{a10})$$

$$A \rightarrow [B \rightarrow [[(A \vee B) \vee \neg(A \vee B)] \vee (A \rightarrow B)]] \quad (\text{a11})$$

The label ‘b4’ is intended to abbreviate “basic logic contained in BN4 and E4” (the label ‘B4’ has been used to refer to Belnap and Dunn’s well-known 4-valued logic – cf., e.g., [12, p. 282]). The axiom a10 is a conspicuous thesis in some 3-valued and 4-valued logics; the axiom a11 serves an instrumental purpose: it is useful to prove that theories in the canonical models are non-empty (cf. Lemma 5.5).

By an Eb4-logic, or simply by Eb4, we refer to any extension of b4, that is, to a strengthening of b4 in the language of this logic.

BN4 and E4 are defined upon b4 as follows.

DEFINITION 2.10 (The logic BN4). The logic BN4 is the result of adding the axioms (A11)–(A14) listed below to b4. That is, BN4 is axiomatized as follows:

Axioms

$$A \rightarrow A \quad (\text{A1})$$

$$(A \wedge B) \rightarrow A / (A \wedge B) \rightarrow B \quad (\text{A2})$$

$$[(A \rightarrow B) \wedge (A \rightarrow C)] \rightarrow [A \rightarrow (B \wedge C)] \quad (\text{A3})$$

$$A \rightarrow (A \vee B) / B \rightarrow (A \vee B) \quad (\text{A4})$$

$$[(A \rightarrow C) \wedge (B \rightarrow C)] \rightarrow [(A \vee B) \rightarrow C] \quad (\text{A5})$$

$$[A \wedge (B \vee C)] \rightarrow [(A \wedge B) \vee (A \wedge C)] \quad (\text{A6})$$

$$(A \rightarrow B) \rightarrow [(B \rightarrow C) \rightarrow (A \rightarrow C)] \quad (\text{A7})$$

$$(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A) \quad (\text{A8})$$

$$(\neg A \rightarrow B) \rightarrow (\neg B \rightarrow A) \quad (\text{A9})$$

$$\neg A \rightarrow [A \vee (A \rightarrow B)] \quad (\text{A10})$$

$$A \rightarrow [(A \rightarrow B) \rightarrow B] \quad (\text{A11})$$

$$(\neg A \wedge B) \rightarrow (A \rightarrow B) \quad (\text{A12})$$

$$(A \vee \neg B) \vee (A \rightarrow B) \quad (\text{A13})$$

$$A \vee [\neg A \rightarrow (A \rightarrow B)] \quad (\text{A14})$$

Rules of inference: (Adj), (MP), and:

Disjunctive Modus Ponens: $C \vee (A \rightarrow B) \ \& \ C \vee A \Rightarrow C \vee B$ (dMP)

DEFINITION 2.11 (The logic E4). The logic E4 is axiomatized with (A1)–(A10) of BN4 and, in addition:

$$[A \rightarrow (A \rightarrow B)] \rightarrow (A \rightarrow B) \quad (\text{A15})$$

$$[[[(A \rightarrow A) \wedge (B \rightarrow B)] \rightarrow C]] \rightarrow C \quad (\text{A16})$$

$$[\neg(A \rightarrow B) \wedge (\neg A \wedge B)] \rightarrow (A \rightarrow B) \quad (\text{A17})$$

$$[(A \rightarrow B) \wedge (A \wedge \neg B)] \rightarrow \neg(A \rightarrow B) \quad (\text{A18})$$

$$(A \rightarrow B) \vee \neg(A \rightarrow B) \quad (\text{A19})$$

The rules of inference are Adj and MP.

We remark some facts about BN4 and E4.

1. The axiom (a11) is derivable in BN4 and in E4 since it is valid in MBN4 and ME4 (in case a tester is needed the reader can use that in [10]). So, Cm and b4 are sublogics of both BN4 and E4.

2. The axiomatization of BN4 presented in Definition 2.10 is slightly different from Brady's original one (it is proved that both formulations are equivalent in [16, Appendix 2]).

3. Notice that *contractionless relevant logic*, RW, can be axiomatized with (A1)–(A9), (A11), (Adj), and (MP). Thus, BN4 can be considered as a 4-valued extension of RW. On the other hand, *reductioless entailment logic*, Er, can be axiomatized with (A1)–(A9), (A15), (A16), (Adj), and (MP). Thus, E4 can in its turn be considered as a 4-valued extension of Er.

The section is ended with a proposition including some theorems and rules of b4 that will be useful in the completeness proofs.

PROPOSITION 2.12 (Some theorems and rules of B4). *The following theorems and rules are provable in b4.*

$$A \rightarrow (B \vee C) \Rightarrow (A \wedge D) \rightarrow [(B \vee E) \vee C] \quad (\text{r1})$$

$$(A \wedge B) \rightarrow C \Rightarrow [(A \wedge D) \wedge B] \rightarrow (C \vee E) \quad (\text{r2})$$

$$A \rightarrow (B \vee C) \ \& \ (A \wedge C) \rightarrow B \Rightarrow A \rightarrow B \quad (\text{r3})$$

$$A \rightarrow B \Rightarrow (C \vee A) \rightarrow (C \vee B) \quad (\text{r4})$$

$$A \leftrightarrow (A \vee A) \quad (\text{T1})$$

$$[A \vee (B \vee C)] \leftrightarrow [(A \vee B) \vee C] \quad (\text{T2})$$

$$[A \vee (B \wedge C)] \leftrightarrow [(A \vee B) \wedge (A \vee C)] \quad (\text{T3})$$

$$(A \rightarrow B) \rightarrow [(A \wedge C) \rightarrow (B \vee D)] \quad (\text{T4})$$

$$\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B) \quad (\text{T5})$$

$$\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B) \quad (\text{T6})$$

$$A \rightarrow \neg\neg A \quad (\text{T7})$$

$$\neg\neg A \rightarrow A \quad (\text{T8})$$

$$(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A) \quad (\text{T9})$$

$$A \rightarrow [\neg A \vee (\neg A \rightarrow B)] \quad (\text{T10})$$

$$\neg A \rightarrow [B \vee ((A \wedge B) \rightarrow C)] \quad (\text{T11})$$

PROOF. Theorems (r1)–(r4), (T1)–(T3), and (T5)–(T8) are in fact rules and theorems of Anderson and Belnap’s First Degree Entailment logic, FDE (cf. [1]), a weak logic that can be axiomatized with (A1), (A2), (A4), (A6), (T7), (T8), and the rules:

Transitivity ($A \rightarrow B \ \& \ B \rightarrow C \Rightarrow A \rightarrow C$),

Conditioned introduction of conjunction

$$(A \rightarrow B \ \& \ A \rightarrow C \Rightarrow A \rightarrow (B \wedge C)),$$

Elimination of disjunction ($A \rightarrow C \ \& \ B \rightarrow C \Rightarrow (A \vee B) \rightarrow C$)

Contraposition ($A \rightarrow B \Rightarrow \neg B \rightarrow \neg A$).

Then, (T4) is provable in Routley and Meyer’s basic positive logic B_+ , that, as note above, is included in Cm. The theorem (T10) is immediate by (T7), (a10), and (a2). Finally, (T11) follows from (a3), (T9), (a2), (r4), and (a10) in the form $\neg(A \wedge B) \rightarrow [(A \wedge B) \vee ((A \wedge B) \rightarrow C)]$. \square

3. Semantics for BN4 and E4

In this section BN4 and E4 are endowed with respective reduced general Routley-Meyer semantics. We will begin by providing reduced models verifying the axioms and rules of Cm.

DEFINITION 3.1 (Cm-models). A Cm-model is a structure $\langle T, K, R, *, \models \rangle$, where K is a set, $T \in K$, R is a ternary relation on K , and $*$ is a unary operation on K subject to the following definitions and postulates for all $a, b, c \in K$:

$$a \leq b := RTab \quad (\text{d1})$$

$$a = b := a \leq b \ \& \ b \leq a \quad (\text{d1}')$$

$$R^2abcd := (\exists x \in K)(Rabx \ \& \ Rxcd) \quad (\text{d2})$$

$$a \leq a \quad (\text{P1})$$

$$(a \leq b \ \& \ Rbcd) \Rightarrow Racd \quad (\text{P2})$$

$$R^2abcd \Rightarrow (\exists x \in K)(Racx \ \& \ Rbxd) \quad (\text{P3})$$

$$a = a^{**} \quad (\text{P4})$$

$$Rabc \Rightarrow Rac^*b^* \quad (\text{P5})$$

Finally, \models is a relation from K to the set of all wffs such that the following conditions (clauses) are satisfied for every propositional variable p , wffs A, B , and $a \in K$:

- (i) $(a \leq b \ \& \ a \models p) \Rightarrow b \models p$,
- (ii) $a \models A \wedge B$ iff $a \models A$ and $a \models B$,
- (iii) $a \models A \vee B$ iff $a \models A$ or $a \models B$,
- (iv) $a \models A \rightarrow B$ iff for all $b, c \in K$, $(Rabc$ and $b \models A) \Rightarrow c \models B$,
- (v) $a \models \neg A$ iff $a^* \not\models A$.

Next, the notions of truth in a Cm-model, validity and semantic consequence are defined.

DEFINITION 3.2 (Truth in a Cm-model). A wff A is true in a Cm-model iff $T \models A$ in this model.

DEFINITION 3.3 (Cm-validity). A formula A is Cm-valid (in symbols, $\models_{\text{Cm}} A$) iff $T \models A$ in all Cm-models.

DEFINITION 3.4 (Semantic Cm-consequence). For any set of wffs Γ and wff A : $\Gamma \models_M A$ (A is a consequence of Γ in the Cm-model M) iff $T \models A$ if $T \models \Gamma$ ($T \models \Gamma$ iff $T \models B$ for all $B \in \Gamma$). Then, $\Gamma \models_{\text{Cm}} A$. (A is a consequence of Γ in Cm-semantics) iff $\Gamma \models_M A$ for each Cm-model M .

Models for logics extending Cm are defined simply by adding to (P1)–(P5) the appropriate semantical postulates while at the same time defining ‘truth in a model’, ‘validity’ and ‘semantical consequence’ similarly as in definitions 3.2, 3.3, and 3.4, respectively. Actually, reduced semantics for BN4 and E4 are defined below in this way.

Next, it will be proved that Cm is sound w.r.t. the semantics just defined. In this sense, the following lemmas are useful. (An adequate version of each one of these lemmas is immediate for any extension of Cm, and for BN4 and E4, in particular).

LEMMA 3.5 (Hereditary condition). For any Cm-model, $a, b \in K$ and wff A : $(a \leq b \ \& \ a \models A) \Rightarrow b \models A$.

PROOF. Induction on the length of A . The conditional case is proved with (P2) and the negation case with the postulate $a \leq b \Rightarrow b^* \leq a^*$, immediate by (P5) and (d1). \square

LEMMA 3.6 (Entailment lemma). *For any wffs A, B : $\models_{\text{Cm}} A \rightarrow B$ iff $a \models A \Rightarrow a \models B$, for all $a \in K$ in all Cm-models.*

PROOF. From left to right: by (P1); from right to left: by Lemma 3.5. \square

We can now prove soundness.

THEOREM 3.7 (Soundness of Cm). *For any set of wffs Γ and wff A : if $\Gamma \vdash_{\text{Cm}} A$, then $\Gamma \models A$.*

PROOF. If $A \in \Gamma$ or A is by (Adj), the proof is trivial. If A is an axiom, then A is proved Cm-valid as in [17, Chapter 4]. Then, it remains to prove the case when A has been derived by (MP). Suppose $\Gamma \vdash_{\text{Cm}} B \rightarrow A$ and $\Gamma \vdash_{\text{Cm}} B$ for some wff B . Further, suppose $T \models \Gamma$. Then, (1) $T \models B \rightarrow A$ and $T \models B$. And by (P1), (2) $RTTT$. So, (3) $T \models A$ by applying clause (iv) in Definition 3.1 to (1) and (2). \square

In what follows, BN4-models and E4-models are defined and soundness of BN4 and E4 w.r.t. its respective semantics is proved.

DEFINITION 3.8 (BN4-models). A BN4-model is a structure $\langle T, K, R, *, \models \rangle$, where $T, K, R, *, \models$ are defined exactly as in Cm-models except for the addition of the following semantical postulates to (P1)–(P5):

$$Rabc \Rightarrow (b \leq a \text{ or } b \leq a^*) \quad (\text{P6})$$

$$Rabc \Rightarrow Rbac \quad (\text{P7})$$

$$Rabc \Rightarrow (b \leq a^* \text{ or } a \leq c) \quad (\text{P8})$$

$$RTab \Rightarrow (T^* \leq b \text{ or } a \leq T) \quad (\text{P9})$$

$$R^2Tabc \Rightarrow (b \leq T \text{ or } b \leq a^*) \quad (\text{P10})$$

DEFINITION 3.9 (E4-models). An E4-model is a structure $\langle T, K, R, *, \models \rangle$, where $T, K, R, *, \models$ are defined exactly as in Cm-models except for the addition of the following semantical postulates to (P1)–(P6):

$$Rabc \Rightarrow R^2abbc \quad (\text{P11})$$

$$(\exists x \in Z) Raxa \text{ [} Za \text{ iff for all } b, c \in K, Rabc \Rightarrow RTbc \text{]} \quad (\text{P12})$$

$$(Rabc \ \& \ Ra^*de) \Rightarrow (a \leq c \text{ or } a \leq e \text{ or } b \leq a^* \text{ or } d \leq a^*) \quad (\text{P13})$$

$$Raaa^* \text{ or } Ra^*aa^* \quad (\text{P14})$$

$$RTab \Rightarrow RT^*ab \quad (\text{P15})$$

Now, as pointed out above, ‘truth in a BN4-model’ (‘truth in an E4-model’) and related notions are defined similarly as in the case of Cm (definitions 3.2–3.4). Once this is done, we can prove soundness.

THEOREM 3.10 (Soundness of BN4 and E4). *For any set of wffs Γ and wff A :*

- (1) *If $\Gamma \vdash_{\text{BN4}} A$, then $\Gamma \models_{\text{BN4}} A$.*
- (2) *If $\Gamma \vdash_{\text{E4}} A$, then $\Gamma \models_{\text{E4}} A$.*

PROOF. Given the proof of soundness of Cm, it suffices to prove (1) (A10)–(A14) are BN4-valid and (dMP) preserves BN4-validity; and (2) (A10) and (A15)–(A19) are E4-valid. Now, (A11) and (A15) are proved as in [17, Chapter 4], and (A16) is proved as in [2, pp. 171–172]; on the other hand, (dMP) is shown to preserve BN4-validity by using *RTTT* (by (P1)), similarly as in the proof that (MP) preserves Cm-validity (Theorem 3.7). So, it remains to prove the validity of (A10)–(A14), (A17)–(A19). (We use the Entailment Lemma – Lemma 3.6 – and proceed by reductio ad absurdum.)

(A10) $\neg A \rightarrow [A \vee (A \rightarrow B)]$ is BN4-valid. Suppose that there are $a \in K$ in some BN4-model and wffs A, B such that (1) $a \models \neg A$ but (2) $a \not\models A \vee (A \rightarrow B)$. By clause (iii), (3) $a \not\models A$ and $a \not\models A \rightarrow B$. By clause (v) and (1), (4) $a^* \not\models A$ and by clause (iv), (5) $b \models A, c \not\models B$ for $b, c \in K$ such that $Rabc$. Now, (6) $b \leq a$ or $b \leq a^*$ follows by (P6). Then, (7) $a \models A$ or $a^* \models A$, by applying Lemma 3.5 to (5) and (6). But (7) contradicts (3) and (4).

It is proved that (A10) is E4-valid in a similar way.

(A12) $(\neg A \wedge B) \rightarrow (A \rightarrow B)$ is BN4-valid. Suppose that there are $a \in K$ in some BN4-model and wffs A, B such that (1) $a \models \neg A$ and (2) $a \models B$ but (3) $a \not\models A \rightarrow B$. Then, (4) $b \models A, c \not\models B$ for $b, c \in K$ such that $Rabc$. By (1) and clause (v), (5) $a^* \not\models A$. By (P8), (6) $b \leq a^*$ or $a \leq c$. Thus, (7) $a^* \models A$ or $c \models B$ by (2), (4), (6) and Lemma 3.5, contradicting (4) and (5).

(A13) $(A \vee \neg B) \vee (A \rightarrow B)$ is BN4-valid. Suppose that there is some BN4-model and wffs A, B such that (1) $T \not\models (A \vee \neg B) \vee (A \rightarrow B)$. Then, (2) $T \not\models A, T \not\models \neg B$ (i.e., $T^* \models B$) and $T \models A \rightarrow B$. By clause (iv), (3) $RTab, a \models A$ and $b \not\models B$, for some $a, b \in K$ in this model. By (P9), (4) $T^* \leq b$ or $a \leq T$, whence by (2) and (3), (5) $b \models B$ or $T \models A$, a contradiction.

(A14) $A \vee [\neg A \rightarrow (A \rightarrow B)]$ is BN4-valid. Suppose that there is some BN4-model and wffs A, B such that (1) $T \not\models A \vee [\neg A \rightarrow (A \rightarrow B)]$.

Then, (2) $T \not\models A$, $T \not\models \neg A \rightarrow (A \rightarrow B)$. By clause (iv), (3) $RTab$, $a \models \neg A$ (i.e., $a^* \not\models A$) and $b \not\models A \rightarrow B$ for $a, b \in K$. Again, by clause (iv), (4) $Rbcd$, $c \models A$ and $d \not\models B$, for $c, d \in K$. By (d2), (5) R^2Tacd , and by (P10), (6) $c \leq T$ or $c \leq a^*$ whence by (4), $T \models A$ or $a^* \models A$, a contradiction.

(A17) $[\neg(A \rightarrow B) \wedge (\neg A \wedge B)] \rightarrow (A \rightarrow B)$ is E_4 -valid. Suppose that there is some E4-model and wffs A, B such that (1) $a \models \neg(A \rightarrow B)$ (i.e., $a^* \not\models A \rightarrow B$), $a \models \neg A$ (i.e., $a^* \not\models A$) and $a \models B$ but (2) $a \not\models A \rightarrow B$. By clause (iv) and (2), (3) $Rabc$, $b \models A$ and $c \not\models B$ for some $b, c \in K$. By clause (iv) and (1), (4) Ra^*de , $d \models A$ and $e \not\models B$ for some $d, e \in K$. By (3), (4) and (P13), (5) $a \leq c$ or $a \leq e$ or $b \leq a^*$ or $d \leq a^*$ whence by (1) ($a \models B$), (3) ($b \models A$) and (4) ($d \models A$), $c \models B$ or $e \models B$ or $a^* \models A$, contradicting (1) ($a^* \not\models A$), (3) ($c \not\models B$) and (4) ($e \not\models B$).

(A18) $[(A \rightarrow B) \wedge (A \wedge \neg B)] \rightarrow \neg(A \rightarrow B)$ is E_4 -valid. Suppose that there is some E4-model and wffs A, B such that (1) $a \models A \rightarrow B$, $a \models A$, $a \models \neg B$ (i.e., $a^* \not\models B$) but (2) $a \not\models \neg(A \rightarrow B)$ (i.e., $a^* \models A \rightarrow B$). By (P14), (3) $Raaa^*$ or (4) Ra^*aa^* . By clause (iv), (1) and (3), (4) $a^* \models B$. By clause (iv), (1), (2) and (4), (5) $a^* \models B$. But both (4) and (5) contradict (1).

(A19) $(A \rightarrow B) \vee \neg(A \rightarrow B)$ is E_4 -valid. Suppose that there is some E4-model and wffs A, B such that (1) $T \not\models (A \rightarrow B) \vee \neg(A \rightarrow B)$. Then, (2) $T \not\models A \rightarrow B$ and $T \not\models \neg(A \rightarrow B)$ (i.e., $T^* \models A \rightarrow B$). By clause (iv) and (2), (3) $RTab$, $a \models A$ and $b \not\models B$ for some $a, b \in K$ in this model. By (P15) and (3), (4) RT^*ab , whence by clause (iv), (2) and (3), (5) $b \models B$, contradicting (3). \square

We end this section by noting that b4-models can be defined similarly as Cm-models save for the addition of P6 and the following postulate that can be used in validating (a11):

$$(Rabc \ \& \ Rcde) \Rightarrow (a \leq c \text{ or } b \leq c \text{ or } c^* \leq c \text{ or } d \leq c \text{ or } b \leq e) \quad (\text{Pa11})$$

4. Extension and primeness lemmas

As pointed out above, by Eb4, we refer to any extension of b4. The aim of this section is to show how to extend Eb4-theories (the notion is defined below) to *prime* Eb4-theories by using a first *extension lemma* based upon “disjunctive Eb4-implicability”. This extension lemma is used essentially in the proof of the preliminary lemmas to the completeness theorems in the following section, as well as in the strong completeness

theorem for E4. We follow Routley *et al.*'s formulation and proof of this lemma (cf. [17, Chapter 4, pp. 307, ff.]).

We begin by defining the notion of a theory and the classes of theories that are important in this paper.

DEFINITION 4.1 (Eb4-theories). An Eb4-theory is a set of formulas closed under Adjunction (Adj) and Eb4-entailment (Eb4-ent). That is, a is a Eb4-theory if whenever $A, B \in a$, then $A \wedge B \in a$; and if whenever $A \rightarrow B$ is a theorem of Eb4 and $A \in a$, $B \in a$.

DEFINITION 4.2 (Classes of Eb4-theories). Let a be an Eb4-theory. We set:

- (1) a is prime iff whenever $A \vee B \in a$, then $A \in a$ or $B \in a$;
- (2) a is empty iff it contains no wffs;
- (3) a is regular iff a contains all theorems of Eb4;
- (4) a is trivial iff every wff belongs to it;
- (5) a is a-consistent (consistent in an absolute sense) iff a is not trivial.

Next, we note a couple of preliminary definitions before proving the extension and primeness lemmas.

DEFINITION 4.3 (Disjunctive Eb4-implicability). For any set of wffs Γ , Θ , Γ disjunctively implies Θ in Eb4 (in symbols $\Gamma \xrightarrow[\text{Eb4}]{d} \Theta$) iff $\vdash_{\text{EB4}} (A_1 \wedge \dots \wedge A_n) \rightarrow (B_1 \vee \dots \vee B_m)$, for some wffs $A_1, \dots, A_n \in \Gamma$ and $B_1, \dots, B_m \in \Theta$. By $\Gamma \not\xrightarrow[\text{Eb4}]{d} \Theta$ is denoted that Θ is not disjunctively implicated by Γ in Eb4.

DEFINITION 4.4 (Eb4 maximal sets). Γ is an Eb4 maximal set of wffs iff $\Gamma \xrightarrow[\text{Eb4}]{d} \bar{\Gamma}$ ($\bar{\Gamma}$ is the complement of Γ).

LEMMA 4.5 (Extension to Eb4 maximal sets). Let Γ, Θ be sets of wffs such that $\Gamma \not\xrightarrow[\text{Eb4}]{d} \Theta$. Then there are sets of wff Γ', Θ' such that $\Gamma \subseteq \Gamma'$, $\Theta \subseteq \Theta'$, $\Theta' = \bar{\Gamma}'$ and $\Gamma' \xrightarrow[\text{Eb4}]{d} \Theta'$ (that is, Γ' is an EB4 maximal set such that $\Gamma' \xrightarrow[\text{Eb4}]{d} \Theta'$).

PROOF. Let A_1, \dots, A_n, \dots be an enumeration of the wffs. The sets Γ' and Θ' are defined as follows: $\Gamma' = \bigcup_{k \in \mathbb{N}} \Gamma_k$, $\Theta' = \bigcup_{k \in \mathbb{N}} \Theta_k$, where $\Gamma_0 = \Gamma$, $\Theta_0 = \Theta$ and for each $k \in \mathbb{N}$, Γ_{k+1} and Θ_{k+1} are defined as follows:

- (i) If $\Gamma_k \cup \{A_{k+1}\} \xrightarrow[\text{Eb4}]{\text{d}} \Theta_k$, then $\Gamma_{k+1} = \Gamma_k$ and $\Theta_{k+1} = \Theta \cup \{A_{k+1}\}$.
- (ii) If $\Gamma_k \cup \{A_{k+1}\} \xrightarrow[\text{Eb4}]{\text{d}} \Theta_k$, then $\Gamma_{k+1} = \Gamma_k \cup \{A_{k+1}\}$ and $\Theta_{k+1} = \Theta_k$.

Notice that $\Gamma \subseteq \Gamma'$, $\Theta \subseteq \Theta'$ and $\Gamma' \cup \Theta' = \mathcal{F}$ (the set of all wffs). We prove: (I) $\Gamma_k \xrightarrow[\text{Eb4}]{\text{d}} \Theta_k$ for all $k \in \mathbb{N}$. We proceed by *reductio ad absurdum*.

So, suppose that for some $i \in \mathbb{N}$, (II) $\Gamma_i \xrightarrow[\text{Eb4}]{\text{d}} \Theta_i$ but $\Gamma_{i+1} \xrightarrow[\text{Eb4}]{\text{d}} \Theta_{i+1}$. We then consider the two possibilities (i) and (ii) above according to which Γ_{k+1} and Θ_{k+1} are defined.

- (a) $\Gamma_i \cup \{A_{i+1}\} \xrightarrow[\text{Eb4}]{\text{d}} \Theta_i$. By (ii), $\Gamma_{i+1} = \Gamma_i \cup \{A_{i+1}\}$ and $\Theta_{i+1} = \Theta_i$.

By the reductio hypothesis (II), $\Gamma_i \cup \{A_{i+1}\} \xrightarrow[\text{Eb4}]{\text{d}} \Theta_i$, a contradiction.

- (b) $\Gamma_i \cup \{A_{i+1}\} \xrightarrow[\text{Eb4}]{\text{d}} \Theta_i$. By (i), $\Gamma_{i+1} = \Gamma_i$ and $\Theta_{i+1} = \Theta_i \cup \{A_{i+1}\}$.

By the reductio hypothesis (II),

1. $\Gamma_i \xrightarrow[\text{Eb4}]{\text{d}} \Theta_i \cup \{A_{i+1}\}$.

Now, let the formulas of Γ_i and Θ_i in this derivation be B_1, \dots, B_m and C_1, \dots, C_n , respectively, and let us refer by B to B_1, \dots, B_m and by C to C_1, \dots, C_n . Then, (1) can be rephrased as follows:

2. $\vdash_{\text{Eb4}} B \rightarrow (C \vee A_{i+1})$.

On the other hand, given the hypothesis (b), there is a conjunction B' of elements of Γ_i and some disjunction C' of elements of Θ_i such that

3. $\vdash_{\text{Eb4}} (B' \wedge A_{i+1}) \rightarrow C'$

Let us now refer by B'' to $B \wedge B'$ and by C'' to $C \vee C'$; we will show (III) $\vdash_{\text{Eb4}} B'' \rightarrow C''$, whence $\Gamma_i \xrightarrow[\text{Eb4}]{\text{d}} \Theta_i$, contradicting the reductio hypothesis, and thus proving (I) (we use Proposition 2.12). By (r1), we have

4. $\vdash_{\text{Eb4}} (B' \wedge B') \rightarrow [(C \vee C') \vee A_{i+1}]$.

from (2) and

5. $\vdash_{\text{Eb4}} [(B' \wedge B') \wedge A_{i+1}] \rightarrow (C \vee C')$

from (3), by (r2). By applying (r3) to (4) and (5), we obtain

6. $\vdash_{\text{Eb4}} (B' \wedge B') \rightarrow (C \vee C')$

But (6) is (III) $\vdash_{\text{Eb4}} B'' \rightarrow C''$, whence as pointed out above, $\Gamma_i \xrightarrow[\text{Eb4}]{d} \Theta_i$, contradicting the reductio hypothesis. Consequently, (I) $\Gamma_k \xrightarrow[\text{Eb4}]{d} \Theta_k$, for all $k \in \mathbb{N}$ is proved. Thus, we have sets of wffs Γ', Θ' such that $\Gamma \subseteq \Gamma', \Theta \subseteq \Theta', \Gamma' \xrightarrow[\text{Eb4}]{d} \Theta'$ (since $\Gamma_k \xrightarrow[\text{Eb4}]{d} \Theta_k$, for all $k \in \mathbb{N}$) and $\Theta' = \overline{\Gamma'}$ (since $\Gamma' \cap \Theta' = \emptyset$ — otherwise $\Gamma_i \xrightarrow[\text{Eb4}]{d} \Theta_i$, for some $i \in \mathbb{N}$ — and $\Gamma' \cup \Theta' = \mathcal{F}$), as it was required. Finally, notice that Γ' is maximal (since $\Gamma' \xrightarrow[\text{Eb4}]{d} \overline{\Gamma'}$). \square

LEMMA 4.6 (Maximal sets are prime Eb4-theories). *If Γ is a maximal set, then it is a prime Eb4-theory.*

PROOF. (1) Γ is closed under (Adj). Suppose that there are wffs A, B such that $A \in \Gamma, B \in \Gamma$ but $A \wedge B \notin \Gamma$. By (A1), $\vdash_{\text{Eb4}} (A \wedge B) \rightarrow (A \wedge B)$ contradicting the maximality of Γ . (2) Γ is closed under (Eb4-ent). Suppose $\vdash_{\text{Eb4}} A \rightarrow B$ and $A \in \Gamma$. If $B \notin \Gamma$, then the maximality of Γ is contradicted. (3) Γ is prime. It is proved similarly as case (1) by using now (A1) in the form $(A \vee B) \rightarrow (A \vee B)$. \square

5. Preliminary lemmas to the completeness theorem

In this section we prove a series of preliminary lemmas to be used in the completeness proofs for both BN4 and E4. We follow Routley *et al.* (cf. [17, Chapter 4]). Given an extension of b4, Eb4, we begin by defining the concept of a \mathcal{T} -theory.

DEFINITION 5.1 (\mathcal{T} -theories). Let Eb4 be an extension of b4 and \mathcal{T} be a regular and prime Eb4-theory (that is, a set of wffs closed under (Adj) and (Eb4-ent); cf. Definition 4.1). A \mathcal{T} -theory is a set of formulas closed under Adjunction (Adj) and \mathcal{T} -entailment (\mathcal{T} -ent). That is, a is a \mathcal{T} -theory if whenever $A, B \in a$, then $A \wedge B \in a$; and if whenever $A \rightarrow B \in \mathcal{T}$ and $A \in a$, then $B \in a$.

Notice that any \mathcal{T} -theory is an Eb4-theory (cf. Definition 4.1): since \mathcal{T} is regular, if $\vdash_{\text{Eb4}} A \rightarrow B$, then $A \rightarrow B \in \mathcal{T}$. So, if $\vdash_{\text{Eb4}} A \rightarrow B$ and $A \in a$, then $B \in a$ as a is closed under \mathcal{T} -ent.

Next, we define some relations on sets of \mathcal{T} -theories.

DEFINITION 5.2 (The sets K^T, K^C). Let \mathcal{T} be a regular and prime Eb4-theory. K^T is the set of all \mathcal{T} -theories, and K^C is the set of all a-consistent non-empty and prime \mathcal{T} -theories (cf. Definition 4.2).

DEFINITION 5.3 (The relations R^T, R^C and \models^C). Let \mathcal{T} be a regular and prime \mathcal{T} -theory and K^T and K^C be defined as in Definition 5.2. R^T is defined on K^T as follows: for all $a, b, c \in K^T$, $R^T abc$ iff for all wffs A, B , $(A \rightarrow B \in a \ \& \ A \in b) \Rightarrow B \in c$. Next, R^C is the restriction of R^T to K^C . On the other hand, \models^C is defined as follows: for any $a \in K^C$ and wff A , $a \models^C A$ iff $A \in a$.

Finally, we define a unary operation on K^C .

DEFINITION 5.4 (The operation $*^C$). The unary operation $*^C$ is defined on K^C as follows: for each $a \in K^C$, $a^* = \{A \mid \neg A \notin a\}$.

Given an extension of b4, Eb4, we use the Extension Lemma to build a regular and prime Eb4-theory \mathcal{T} (cf. Proposition 6.3 below) and define upon \mathcal{T} the notions $K^C, R^C, *^C$ and \models^C as indicated above. Then, the structure $\langle \mathcal{T}, K^C, R^C, *^C, \models^C \rangle$, called the canonical Eb4-model, is shown to be an Eb4-model by means of which non-theorems of Eb4 are falsified.

In the rest of this section, a series of lemmas is proved. These lemmas shall be used in the completeness proofs. We suppose that we are given a regular and prime Eb4-theory \mathcal{T} upon which the items $K^T, K^C, R^C, *^C, \models^C$ are defined as shown above. We begin by investigating the relations R^T and R^C .

LEMMA 5.5 (Defining x for a, b in R^T). *Let a, b be non-empty \mathcal{T} -theories. The set $x = \{B \mid \exists A(A \rightarrow B \in a \ \& \ A \in b)\}$ is a non-empty \mathcal{T} -theory such that $R^T abx$.*

PROOF. It is easy to show that x is a \mathcal{T} -theory (use (a2) – Definition 2.8 – to prove that x is closed under \mathcal{T} -ent). Next, $R^T abx$ is immediate by definition of R^T . Finally, x is non-empty: let $A \in a, B \in b$. By a11 and $R^T abx$, $[(A \vee B) \vee \neg(A \vee B)] \vee (A \rightarrow B) \in x$. \square

LEMMA 5.6 (Extending b in $R^T abc$ to a member in K^C). *Let a and b be non-empty \mathcal{T} -theories, and a and c be a -consistent, prime \mathcal{T} -theories such that $R^T abc$. Then, there is an a -consistent (and non-empty) prime \mathcal{T} -theory x such that $b \subseteq x$ and $R^T axc$.*

PROOF. By using the Extension Lemma or Kuratowski-Zorn's Lemma, b is extended to a prime theory x such that $b \subseteq x$ and $R^T axc$ (cf. [17, pp. 309, ff.]). Next, it is shown that x is a -consistent. Suppose it is not. (We use Proposition 2.12.) Let $A \in a$ and B be a wff belonging to neither a nor c . Then, by (T10) and (MP), we have $\neg A \vee (\neg A \rightarrow B) \in a$.

Hence, by primeness of a , either (1) $\neg A \in a$ or (2) $\neg A \rightarrow B \in a$. Let us consider the case 2. As x is supposed to be trivial, $\neg A \in x$. But then, definition of R^T , we have $B \in c$, since $R^T a x c$, $\neg A \rightarrow B \in a$, and $\neg A \in x$ but this contradicts our hypothesis. Let us now examine the case 1. By (T11) and (MP), for arbitrary C we have $B \vee ((A \wedge B) \rightarrow C) \in a$. Whence $(A \wedge B) \rightarrow C \in a$, since $B \notin a$. Now notice that $A \wedge B \in x$, since x is not a-consistent. Thus, $C \in c$, since $(A \wedge B) \rightarrow C \in a$, $A \wedge B \in x$, and $R^T a x c$ but this contradicts the a-consistency of c . \square

LEMMA 5.7 (Extending a in $R^T abc$ to a member in K^C). *Let a, b be non-empty \mathcal{T} -theories and c be an a-consistent, prime \mathcal{T} -theory such that $R^T abc$. Then there is an a-consistent (and non-empty) prime \mathcal{T} -theory x such that $a \subseteq x$ and $R^T xbc$.*

PROOF. As in the previous lemma, it is shown that there is a prime theory x such that $a \subseteq x$ and $R^T xbc$. Next, it is shown that x is a-consistent. Suppose it is not and let $A \in b$ and B be an arbitrary wff. As x is supposed to be trivial, $A \rightarrow B \in x$. Then, by definition of R^T , we have $B \in c$, since $R^T xbc$, $A \rightarrow B \in x$, and $A \in b$. It is contradicting the a-consistency of c . \square

Consider now the following definition.

DEFINITION 5.8 (The relation \leq^C). For any $a, b \in K^C$: $a \leq^C b$ iff $R^C \mathcal{T} ab$.

The following lemma shows that the relation \leq^C is just set inclusion between a-consistent and non-empty prime \mathcal{T} -theories.

LEMMA 5.9 (\leq^C and \subseteq are coextensive). *For any $a, b \in K^C$: $a \leq^C b$ iff $a \subseteq b$.*

PROOF. From left to right, it is immediate by using a1 in Definition 2.8. Suppose now $a \subseteq b$, for $a, b \in K^C$. Clearly $R^C \mathcal{T} aa$ (cf. definitions 5.1 and 5.3). By the hypothesis, $R \mathcal{T} ab$, i.e., $a \leq^C b$, by Definition 5.8. \square

LEMMA 5.10 (Extension to prime \mathcal{T} -theories). *Let a be a \mathcal{T} -theory and A a wff such that $A \notin a$. Then there is a prime \mathcal{T} -theory x such that $a \subseteq x$ and $A \notin x$.*

PROOF. By direct application of Kuratowski-Zorn's Lemma as in [17, Chapter 4, pp. 310–311]. \square

In what follows, we investigate the operation $*^C$.

LEMMA 5.11 (Primeness of $*$ -images). *Let a be a prime \mathcal{T} -theory. Then*

- (1) a^* is a prime \mathcal{T} -theory as well,
- (2) for any wff A , $\neg A \in a^*$ iff $A \notin a$.

PROOF. (1) a^* is closed under \mathcal{T} -ent, by (T9); a^* is closed under (Adj), by (T5); a^* is prime, by (T4). (2) By (T7) and (T8). \square

LEMMA 5.12 ($*^C$ is an operation on K^C). *Let a be an a -consistent and non-empty prime \mathcal{T} -theory. Then a^* is an a -consistent and non-empty \mathcal{T} -theory as well.*

PROOF. By Lemma 5.11, a^* is a prime \mathcal{T} -theory. Next, it is shown that if a is a -consistent and non-empty, then a^* is also a -consistent and non-empty. (1) a^* is a -consistent. As a is non-empty, there is some wff A such that $A \in a$. Then, $\neg A \notin a^*$, by Lemma 5.11(2). (2) a^* is non-empty. As a is a -consistent, there is some wff A such that $A \notin a$. Then, $\neg A \in a^*$ by Lemma 5.11(2). \square

Finally, it is proved that the relation \models^C obeys requirements (clauses) (i)–(v) in the definition of an Eb4-model (a Cm-model; cf. Definition 3.1).

LEMMA 5.13 (\models^C and clauses (i)–(v)). *For all $a, b, c \in K^C$ and wffs A, B :*

- (i) $(a \leq^C b \ \& \ a \models^C p) \Rightarrow b \models^C p$
- (ii) $a \models^C A \wedge B$ iff $a \models^C A$ and $a \models^C B$
- (iii) $a \models^C A \vee B$ iff $a \models^C A$ or $a \models^C B$
- (iv) $a \models^C A \rightarrow B$ iff for all $b, c \in K^C$, $(R^C abc$ and $b \models^C A) \Rightarrow c \models^C B$
- (v) $a \models^C \neg A$ iff $a^* \not\models^C A$

PROOF. (i) is immediate by Lemma 5.9; (ii) follows by (a3) and closure of a under (Adj); (iii) is proved by (a5) and primeness of a ; and (v) and (iv) (from left to right) are immediate by Definition 5.4 and Definition 5.3, respectively. So, let us prove (iv) from right to left. For wffs A, B and $a \in K^C$, suppose $A \rightarrow B \notin a$ (i.e., $a \not\models^C A \rightarrow B$). We prove that there are $x, y \in K^C$ such that $R^C axy$, $A \in x$ (i.e., $x \models^C A$) and $B \notin y$ (i.e., $y \not\models^C B$). Consider the sets $z = \{C \mid A \rightarrow B \in \mathcal{T}\}$ and $u = \{C \mid \exists D(D \rightarrow C \in a \ \& \ D \in z)\}$. Firstly, notice that $A \in z$, since $A \rightarrow A \in \mathcal{T}$ by (a1) (cf. Definition 2.8). Then, z and u are easily shown \mathcal{T} -theories such that $R^T azu$. Now, $B \notin u$ (if $B \in u$, then $A \rightarrow B \in a$ contradicting the hypothesis). Moreover, u is not empty, by Lemma 5.5. Then, by Lemma 5.10, there is a (a -inconsistent and non-empty) prime \mathcal{T} -theory y such that $u \subseteq y$ and $B \notin y$. Clearly, $R^T azy$ (cf. Defini-

tion 5.3). Next, by using Lemma 5.6, z is extended to an a-consistent, non-empty and prime \mathcal{T} -theory x such that $z \subseteq x$ and $R^C axy$. Clearly, $A \in x$. Therefore, we have a-consistent and non-empty prime \mathcal{T} -theories x, y such that $A \in x, B \notin y$ and $R^C axy$, as was to be proved. \square

6. Completeness of E4

In this and the following section, we shall prove strong completeness theorems for E4 and BN4. Since E4 and BN4 are extensions of b4, we can use the results on Eb4 logics obtained in sections 4 and 5. In the present section, we prove strong completeness of E4 w.r.t. the semantics defined in section 3. The standard concept of “set of consequences of a set of wffs” will be useful.

DEFINITION 6.1 (The set $\text{Cn } \Gamma[\text{E4}]$). The set of consequences in E4 of a set of wffs Γ (in symbols $\text{Cn } \Gamma[\text{E4}]$) is defined as follows: $\text{Cn } \Gamma[\text{E4}] := \{A \mid \Gamma \vdash_{\text{E4}} A\}$ (cf. definitions 2.2 and 2.11).

We note the following remarks (cf. definitions 4.1 and 4.2).

Remark 6.2 ($\text{Cn } \Gamma[\text{E4}]$ is a regular E4-theory). It is obvious that for any Γ , the set $\text{Cn } \Gamma[\text{E4}]$ is a regular theory since it is closed under (Adj) and (MP) and, consequently, under (E4-ent) (the set $\text{Cn } \Gamma[\text{E4}]$ contains all theorems of E4).

Now we have:

PROPOSITION 6.3 (The building of \mathcal{T}). *Let Γ be a set of wffs and A a wff such that $\Gamma \not\vdash_{\text{E4}} A$. Then, there is a regular, a-consistent and prime E4-theory \mathcal{T} such that $\Gamma \subseteq \mathcal{T}$ and $A \notin \mathcal{T}$.*

PROOF. Assuming the hypothesis of Proposition 6.3, suppose $\Gamma \not\vdash_{\text{E4}} A$. Then, $A \notin \text{Cn } \Gamma[\text{E4}]$ and so $\text{Cn } \Gamma[\text{E4}] \xrightarrow[\text{Eb4}]{\text{d}} \{A\}$: otherwise $\vdash_{\text{E4}} (B_1 \wedge \dots \wedge B_n) \rightarrow A$, for some $B_1, \dots, B_n \in \Gamma$, whence A would be in $\text{Cn } \Gamma[\text{E4}]$ after all. Next, lemmas 4.5 and 4.6 apply and there is some (regular and a-consistent) prime E4-theory \mathcal{T} such that $\Gamma \subseteq \mathcal{T}$ (since $\Gamma \subseteq \text{Cn } \Gamma[\text{E4}]$) and $A \notin \mathcal{T}$. \square

Leaning on this theory \mathcal{T} , the canonical E4-model is defined and $\Gamma \not\vdash_{\text{E4}} A$ is proved.

DEFINITION 6.4 (The canonical E4-model). The canonical model is the structure $\langle \mathcal{T}, K^C, R^C, *^C, \models^C \rangle$, where $K^C, R^C, *^C, \models^C$ are defined upon the E4-theory \mathcal{T} as indicated in definitions 5.2–5.4.

Once proved that the canonical E4-model is an E4-model, Proposition 6.3 is used to show $\Gamma \not\models^C A$ in the canonical E4-model, i.e., to show that A is not a semantic E4-consequence of Γ (cf. Definition 3.4).

We proceed to the proof that the canonical E4-model is an E4-model. Firstly, we remark the following important fact.

PROPOSITION 6.5 (E4-theories are closed under MP). *If a is an E4-theory, then a is closed under (MP).*

PROOF. The axiom Modus Ponens, i.e., the thesis $[A \wedge (A \rightarrow B)] \rightarrow B$ is provable in E4 (actually, it is provable in Routley and Meyer's basic positive logic B_+ (cf. [17]) supplemented with the axiom contraction — i.e., (A15) in Definition 2.11). Then, Proposition 6.5 follows by closure of a under (Adj) and (E4-ent). \square

Next, it is shown that the semantical postulates hold canonically and then that the canonical E4-model is an E4-model.

LEMMA 6.6 (E4 postulates hold canonically). *The semantical postulates (P1)–(P6), (P11)–(P15) hold in the canonical E4-model.*

PROOF. The proof is greatly simplified by using Lemma 5.9. By leaning on this lemma (P1)–(P5) and (P11) are proved as in [17, Chapter 4]; and (P12), as in [2, pp. 171–172]. So, let us prove (P6), (P13)–(P15). We proceed by *reductio ad absurdum*.

(P6) ($R^C abc \Rightarrow (b \leq^C a \text{ or } b \leq^C a^*)$) holds in the canonical E4-model: Suppose that there are $a, b, c \in K^C$ and wffs A, B such that (1) $R^C abc$ but (2) $A \in b, A \notin a, B \in b$ and $B \notin a^*$ (i.e., $\neg B \in a$). By (A10), (3) $\neg B \rightarrow [B \vee (B \rightarrow C)]$, for arbitrary wff C . Then, (4) $B \vee (B \rightarrow C) \in a$ (by 2, 3) whence (5) $B \in a$ or (6) $B \rightarrow C \in a$. Let us consider the second alternative, 6. By applying the definition of R^C to 1, 2 ($B \in b$) and 6, we have (7) $C \in c$, contradicting the a-consistency of c . So, let us consider the first alternative, 5. By (T11), (a2), and (a3), we have (8) $(B \wedge \neg B) \rightarrow [A \vee [(A \wedge B) \rightarrow C]]$, for arbitrary C , and by 2, 5, (9) $B \wedge \neg B \in a$. Thus, (10) $A \vee [(A \wedge B) \rightarrow C] \in a$ whence we get (11) $(A \wedge B) \rightarrow C \in a$ by 2 ($A \notin a$). But (12) $A \wedge B \in b$ by 2. So, finally, $C \in c$ (by 1, 11 and 12), contradicting the a-consistency of c .

(P13) ($(R^C abc \ \& \ R^C a^* de) \Rightarrow (a \leq^C c \text{ or } a \leq^C e \text{ or } b \leq^C a^* \text{ or } d \leq^C a^*)$) holds in the canonical E4-model: Suppose that there are $a, b, c, d \in K^C$ and wffs A, B, C such that (1) $R^C abc$ and $R^C a^* de$ but (2) $A \in a, A \notin c, B \in a, B \notin e, C \in b, C \notin a^*$ (i.e., $\neg C \in a$),

$D \in d$ and $D \notin a^*$ (i.e., $\neg D \in a$). By 2, (3) $C \vee D \in d$ and $A \wedge B \notin e$ whence by 1 ($R^C a^* de$), (4) $(C \vee D) \rightarrow (A \wedge B) \notin a^*$. Hence (5) $\neg[(C \vee D) \rightarrow (A \wedge B)] \in a$. On the other hand, by using 2 again, we have (6) $A \wedge B \in a$ and $\neg(C \vee D) \in a$ (by T5 in Proposition 2.12 since $\neg C \in a$ and $\neg D \in a$). Now, (7) $[\neg[(C \vee D) \rightarrow (A \wedge B)] \wedge \neg[(C \vee D) \wedge (A \wedge B)]] \rightarrow [(C \vee D) \rightarrow (A \wedge B)]$ follows by (A17). So, by 5, 6 and 7, we have (8) $(C \vee D) \rightarrow (A \wedge B) \in a$. But (9) $C \vee D \in b$ (by 2). So, (10) $A \wedge B \in e$ by 1 ($R^C abc$), 8 and 9. But 10 contradicts 2 ($A \notin e$).

(P14) ($R^C aaa^*$ or $R^C a^* aa^*$) holds in the canonical E4-model: Suppose that there is $a \in K^C$ and wffs A, B such that (1) $A \rightarrow B \in a$, (2) $A' \rightarrow B' \in a^*$, (3) $A \in a$ and $A' \in a$, but (4) $B \notin a^*$ and $B' \notin a^*$ (i.e., $\neg B \in a$ and $\neg B' \in a$). By T4 we have (5) $(A \rightarrow B) \rightarrow [(A \wedge A') \rightarrow (B \vee B')]$. So, (6) $(A \wedge A') \rightarrow (B \vee B') \in a$ by 1 and 5. By (A18), we have (7) $[(A \wedge A') \rightarrow (B \vee B')] \wedge [(A \wedge A') \wedge \neg(B \vee B')] \rightarrow \neg[(A \wedge A') \rightarrow (B \vee B')]$. Thus, by 3, 4, 6 and 7, we get (8) $\neg[(A \wedge A') \rightarrow (B \vee B')] \in a$, whence (9) $(A \wedge A') \rightarrow (B \vee B') \notin a^*$ follows. But by applying again (T4) (together with some elementary properties of \wedge and \vee), we get (10) $(A' \rightarrow B') \rightarrow [(A \wedge A') \rightarrow (B \vee B')]$, whence (11) $(A \wedge A') \rightarrow (B \vee B') \in a^*$ follows by 2. But 9 and 11 contradict each other.

(P15) ($R^C \mathcal{T} ab \Rightarrow R^C \mathcal{T}^* ab$) holds in the canonical E4-model: Suppose that there are $a, b \in K^C$ and wffs A, B such that (1) $R^C \mathcal{T} ab$, (2) $A \rightarrow B \in \mathcal{T}^*$ (i.e., $\neg(A \rightarrow B) \notin \mathcal{T}$), (3) $A \in a$ but (4) $B \notin b$. By (A19), (5) $(A \rightarrow B) \vee \neg(A \rightarrow B) \in \mathcal{T}$. Thus, by 2 and 5, (6) $A \rightarrow B \in \mathcal{T}$, whence by 1 and 3, (7) $B \in b$ follows, contradicting 4. \square

Now, we can prove the adequacy of the canonical E4-model.

PROPOSITION 6.7 (The canonical E4-model is an E4-model). *The canonical E4-model is indeed an E4-model.*

PROOF. Given Definition 6.4 and Proposition 6.3, the proof follows by Lemma 5.12 ($*^C$ is an operation on K^C), Lemma 5.13 (Adequacy of the canonical clauses) and Lemma 6.6 (The postulates hold canonically). \square

Finally, we prove completeness.

THEOREM 6.8 (Strong completeness of E4). *For any set of wffs Γ and wff A : if $\Gamma \vDash_{E4} A$, then $\Gamma \vdash_{E4} A$.*

PROOF. Suppose $\Gamma \not\vdash_{E4} A$ for some set of wffs Γ and wff A . By Proposition 6.3, there is a regular, a-consistent and prime E4-theory \mathcal{T} such that $\Gamma \subseteq \mathcal{T}$ and $A \notin \mathcal{T}$. Then, the canonical E4-model is defined upon

\mathcal{T} as indicated in Definition 6.4. By Proposition 6.7, the canonical E4-model is an E4-model. Then, $\Gamma \not\vdash^C A$ since $\mathcal{T} \models^C \Gamma$, but $\mathcal{T} \not\models^C A$. Thus, $\Gamma \not\vdash_{E4} A$ by Definition 3.4. \square

7. Completeness of BN4

In this section a strong completeness theorem for BN4 is proved. Let us begin by commenting on the theory \mathcal{T} upon which the canonical E4-model is built. This theory is required to be a-consistent, regular and closed under (Adj) and (E4-ent). But undoubtedly it has been remarked that \mathcal{T} has in addition to be closed under (MP): the postulate $RTTT$ is required in order to verify the rule (MP) (cf. Theorem 3.10). The theory \mathcal{T} is built by using the Extension Lemma (Lemma 4.5) which is proved by using the notion of Eb4-d-implicability. In the case of E4, this notion suffices for proving that any maximal set Γ' is closed under the rules of E4 since it is trivial to show that Γ' is closed under (Adj) and (E4-ent) (cf. Lemma 4.6) and, on the other hand, Γ' is proved closed under (MP) in virtue of the *Axiom Modus Ponens* (cf. Proposition 6.5): if $A \in \Gamma'$ and $A \rightarrow B \in \Gamma'$, then $B \in \Gamma'$, given $\vdash_{E4} [A \wedge (A \rightarrow B)] \rightarrow B$. Unfortunately, BN4 lacks the *Axiom Modus Ponens* and then it is possible to have $A \rightarrow B \in \Gamma'$, $A \in \Gamma'$ and $B \notin \Gamma'$, for some wffs A, B without break of Γ' -maximality in the case of BN4. Consequently, we need a new Extension Lemma defined on a different notion from d-Eb4-implicability, although the rest of the completeness proof for BN4 follows similar lines to that of E4. In order to define the new Extension Lemma, we introduce the notions of d-BN4-derivability and d-BN4-maximality (maximality w.r.t. d-BN4-derivability). We follow Routley *et al.* [17, Chapter 4, pp. 336, ff.].

DEFINITION 7.1 (Disjunctive BN4-derivability). For any sets of wffs Γ, Θ , Θ is disjunctively derivable from Γ in BN4 (in symbols, $\Gamma \vdash_{BN4}^d \Theta$) iff $A_1 \wedge \dots \wedge A_n \vdash_{BN4} B_1 \vee \dots \vee B_m$, for some wffs $A_1, \dots, A_n \in \Gamma$ and $B_1, \dots, B_m \in \Theta$.

Next, we prove a lemma which is essential for proving the extension to maximal sets w.r.t. d-BN4-derivability (in the couple of lemmas to follow the subscript BN4 is, in general, dropped from \vdash_{BN4} , since no confusion can arise as these lemmas are proved only for the logic BN4). We also need the following definition.

DEFINITION 7.2 (Full regularity). A BN4-theory is fully regular if it is a regular BN4-theory (cf. definitions 4.1 and 4.2) which is closed under (MP) and (dMP). That is, a is fully regular if a is a regular BN4-theory and for any wffs A, B, C :

- (1) if $A \rightarrow B \in a$ and $A \in a$, then $B \in a$; and
- (2) if $C \vee (A \rightarrow B) \in a$ and $C \vee A \in a$, then $C \vee B \in a$.

LEMMA 7.3 (Main auxiliary lemma). *For any $A, B_1, \dots, B_n \in \mathcal{F}$: if $\{B_1, \dots, B_n\} \vdash_{\text{BN4}} A$, then, for any wff C , $C \vee (B_1 \wedge \dots \wedge B_n) \vdash_{\text{BN4}} C \vee A$.*

PROOF. (Cf. [6, p. 27]) Induction on the length of the proof of A from $\{B_1, \dots, B_n\}$ (H.I abbreviates hypothesis of induction). (1) $A \in \{B_1, \dots, B_n\}$. Let A be B_i ($1 \leq i \leq n$). By elementary properties of \wedge , $\vdash (B_1 \wedge \dots \wedge B_n) \rightarrow B_i$. By (r4), $(A \rightarrow B \Rightarrow (C \vee A) \rightarrow (C \vee B))$, $C \vee (B_1 \wedge \dots \wedge B_n) \vdash C \vee A$. (2) A is an axiom. By (A4), $\vdash C \vee A$. So, $C \vee (B_1 \wedge \dots \wedge B_n) \vdash C \vee A$. (3) A is by (Adj). Then, A is $D \wedge E$ for some wffs D and E . By H.I, $C \vee (B_1 \wedge \dots \wedge B_n) \vdash C \vee D$ and $C \vee (B_1 \wedge \dots \wedge B_n) \vdash C \vee E$ whence $C \vee (B_1 \wedge \dots \wedge B_n) \vdash (C \vee D) \wedge (C \vee E)$, by (Adj). Finally, $C \vee (B_1 \wedge \dots \wedge B_n) \vdash C \vee (D \wedge E)$ by T3 ($[A \vee (B \wedge C)] \leftrightarrow [(A \vee B) \wedge (C \vee D)]$). (4) A is by (MP). By H.I, $C \vee (B_1 \wedge \dots \wedge B_n) \vdash C \vee (D \rightarrow A)$ and $C \vee (B_1 \wedge \dots \wedge B_n) \vdash C \vee D$, for some wff D . So, $C \vee (B_1 \wedge \dots \wedge B_n) \vdash C \vee A$, by (dMP). (5) A is by (dMP). Then, A is $D \vee E$ for some wffs D and E . By H.I, $C \vee (B_1 \wedge \dots \wedge B_n) \vdash C \vee (D \vee F)$ and $C \vee (B_1 \wedge \dots \wedge B_n) \vdash C \vee [D \vee (F \rightarrow E)]$, for some wff F , whence $C \vee (B_1 \wedge \dots \wedge B_n) \vdash (C \vee D) \vee F$ and $C \vee (B_1 \wedge \dots \wedge B_n) \vdash (C \vee D) \vee (F \rightarrow E)$ by T2 ($[A \vee (B \vee C)] \leftrightarrow [(A \vee B) \vee C]$). So, $C \vee (B_1 \wedge \dots \wedge B_n) \vdash (C \vee D) \vee E$, by (dMP) and, finally, $C \vee (B_1 \wedge \dots \wedge B_n) \vdash C \vee (D \vee E)$, by (T2), as it was required in case 5, which ends the proof of Lemma 7.3. \square

Next, we show how to extend sets of wffs to maximal sets. (The proof that follows mirrors that of Lemma 4.5 by using now the notion of disjunctive BN4-derivability instead of that of disjunctive Eb4-implicability.)

LEMMA 7.4 (Extension to d-BN4 maximal sets). *Let Γ, Θ be sets of wffs such that $\Gamma \not\vdash^d \Theta$. Then there are sets of wffs Γ', Θ' such that $\Gamma \subseteq \Gamma'$, $\Theta \subseteq \Theta'$, $\Theta' = \overline{\Gamma'}$ and $\Gamma' \not\vdash^d \Theta'$ (that is, Γ' is a maximal set such that $\Gamma' \not\vdash^d \Theta'$).*

PROOF. Let A_1, \dots, A_n, \dots be an enumeration of the wffs. The sets Γ' and Θ' are defined as follows: $\Gamma' := \bigcup_{k \in \mathbb{N}} \Gamma_k$, $\Theta' := \bigcup_{k \in \mathbb{N}} \Theta_k$, where $\Gamma_0 = \Gamma$, $\Theta_0 = \Theta$ and for each $k \in \mathbb{N}$, Γ_{k+1} and Θ_{k+1} are defined

as follows: (i) if $\Gamma_k \cup \{A_{k+1}\} \vdash^d \Theta_k$, then $\Gamma_{k+1} = \Gamma_k$ and $\Theta_{k+1} = \Theta_k \cup \{A_{k+1}\}$; (ii) if $\Gamma_k \cup \{A_{k+1}\} \not\vdash^d \Theta_k$, then $\Gamma_{k+1} = \Gamma_k \cup \{A_{k+1}\}$ and $\Theta_{k+1} = \Theta_k$. Notice that $\Gamma \subseteq \Gamma'$, $\Theta \subseteq \Theta'$ and that $\Gamma' \cup \Theta' = \mathcal{F}$ (the set of all wffs). We prove (I) $\Gamma_k \not\vdash^d \Theta_k$ for all $k \in \mathbb{N}$. We proceed by reductio ad absurdum. So, suppose that for some $i \in \mathbb{N}$, (II) $\Gamma_i \not\vdash^d \Theta_i$ but $\Gamma_{i+1} \vdash^d \Theta_{i+1}$. We then consider the two possibilities (i) and (ii) above according to which Γ_{i+1} and Θ_{i+1} are defined: (a) $\Gamma_i \cup \{A_{i+1}\} \not\vdash^d \Theta_i$. By (ii), $\Gamma_{i+1} = \Gamma_i \cup \{A_{i+1}\}$ and $\Theta_{i+1} = \Theta_i$. By the reductio hypothesis (II), $\Gamma_i \cup \{A_{i+1}\} \vdash^d \Theta_i$, a contradiction. (b) $\Gamma_i \cup \{A_{i+1}\} \vdash^d \Theta_i$. By (i), $\Gamma_{i+1} = \Gamma_i$ and $\Theta_{i+1} = \Theta_i \cup \{A_{i+1}\}$. By the reductio hypothesis (II), (1) $\Gamma_i \vdash^d \Theta_i \cup \{A_{i+1}\}$. Now, let the formulas of Γ_i and Θ_i in this derivation be B_1, \dots, B_m and C_1, \dots, C_n , respectively, and let us refer by B to $B_1 \wedge \dots \wedge B_m$ and by C to $C_1 \vee \dots \vee C_n$. Then (1) can be rephrased as follows (2) $B \vdash C \vee A_{i+1}$. On the other hand, given the hypothesis (b), there is a conjunction B' of elements of Γ_i and some disjunction C' of elements of Θ_i such that (3) $B' \wedge A_{i+1} \vdash C'$. Let us now refer by B'' to $B \wedge B'$ and by C'' to $C \vee C'$; we will show (III) $B'' \vdash C''$, that is, $\Gamma_i \vdash^d \Theta_i$, contradicting the reductio hypothesis and thus proving (I). By elementary properties of \wedge and \vee , we have (4) $B'' \wedge A_{i+1} \vdash C''$ from (3), and (5) $B'' \vdash C'' \vee A_{i+1}$ from (2). By (5), we get (6) $B'' \vdash C'' \vee (B'' \wedge A_{i+1})$ and by (4) and Lemma 7.3, (7) $C'' \vee (B'' \wedge A_{i+1}) \vdash C'' \vee C''$ whence by (T1) ($A \leftrightarrow (A \vee A)$), we have (8) $C'' \vee (B'' \wedge A_{i+1}) \vdash C''$. By (6) and (8) we get (III) $B'' \vdash C''$, that is, $\Gamma_i \vdash^d \Theta_i$, contradicting the reductio hypothesis. Consequently, (I) ($\Gamma_k \not\vdash^d \Theta_k$ for all $k \in \mathbb{N}$) is proved. Thus, we have sets of wffs Γ' , Θ' such that $\Gamma \subseteq \Gamma'$, $\Theta \subseteq \Theta'$, $\Theta' = \overline{\Gamma'}$ and $\Gamma' \not\vdash^d \Theta'$ (since $\Gamma_k \not\vdash^d \Theta_k$ for all $k \in \mathbb{N}$) and $\Theta' = \overline{\Gamma'}$ (since $\Gamma' \cap \Theta' = \emptyset$ —otherwise $\Gamma_i \vdash^d \Theta_i$, for some $i \in \mathbb{N}$ —and $\Gamma' \cup \Theta' = \mathcal{F}$), as it was required. Finally, notice that Γ' is maximal (since $\Gamma' \not\vdash^d \overline{\Gamma'}$). \square

Before proving the primeness lemma we pause a second to remark the essential role Lemma 7.3 has played in the proof of the extension lemma just given (notice that the rest of syntactical moves required in the said proof can be carried on by leaning on the simple resources of the positive fragment of Anderson and Belnap's First Degree Entailment Logic FDE—cf. [1, §15.2] about this logic; cf. the proof of Proposition 2.12 above).

LEMMA 7.5 (Ext. to prime BN4-th. closed under (MP) and (dMP)). *If Γ is a maximal set, then it is a BN4-theory closed under (MP) and (dMP).*

PROOF. (Cf. Lemma 8 in [6]) (1) Γ is a BN4-theory closed under (MP) and (dMP): It is trivial to prove that Γ is a BN4-theory closed under (MP) and (dMP). For example, let us prove that Γ is closed under (dMP). For reductio, suppose that there are wffs A, B, C such that $C \vee A \in \Gamma$, $C \vee (A \rightarrow B) \in \Gamma$ but $C \vee B \notin \Gamma$. Then, $(C \vee A) \wedge [C \vee (A \rightarrow B)] \vdash C \vee (A \rightarrow B)$ and $(C \vee A) \wedge [C \vee (A \rightarrow B)] \vdash C \vee A$, whence $(C \vee A) \wedge [C \vee (A \rightarrow B)] \vdash C \vee B$ by (dMP), contradicting the maximality of Γ . (2) Γ is prime: If there are some wffs A, B such that $A \vee B \in \Gamma$, but $A \notin \Gamma$ and $B \notin \Gamma$, then Γ is not maximal by virtue of (A1) $((A \vee B) \rightarrow (A \vee B))$. \square

Once these preliminary facts having been stated, the completeness proof is developed similarly as in the case of E4. In the first place, we have:

PROPOSITION 7.6 (The building of \mathcal{T}). *Let Γ be a set of wffs and A a wff such that $\Gamma \not\vdash_{\text{BN4}} A$. Then there is a prime, fully regular and a-consistent BN4-theory \mathcal{T} such that $\Gamma \subseteq \mathcal{T}$ and $A \notin \mathcal{T}$.*

PROOF. Similarly as in E4, we define the notion $\text{Cn } \Gamma[\text{BN4}]$ (cf. Definition 6.1) and note that, for any Γ , $\text{Cn } \Gamma[\text{BN4}]$ is a fully regular theory. Now, suppose $\Gamma \not\vdash_{\text{BN4}} A$. Then, $A \notin \text{Cn } \Gamma[\text{BN4}]$, and thus $\text{Cn } \Gamma[\text{BN4}] \not\vdash_{\text{BN4}}^{\text{d}} \{A\}$: otherwise $B_1 \wedge \dots \wedge B_n \vdash_{\text{BN4}} A$, for some $B_1, \dots, B_n \in \Gamma$ whence A would be in $\text{Cn } \Gamma[\text{BN4}]$ after all. Next, by Lemma 7.4, there is a maximal set \mathcal{T} such that $\text{Cn } \Gamma[\text{BN4}] \subseteq \mathcal{T}$ and $A \notin \mathcal{T}$. By Lemma 7.5, \mathcal{T} is a prime BN4-theory closed under (MP) and (dMP). Moreover, \mathcal{T} is regular since it includes $\text{Cn } \Gamma[\text{BN4}]$, and it is a-consistent since $A \notin \mathcal{T}$. \square

Now, similarly as it was the case with E4, the canonical BN4-model is defined upon \mathcal{T} . Then Lemma 5.12 guarantees that $*^C$ is an operation on K^C and Lemma 5.13, that the clauses hold canonically (exactly as in the case of E4 – cf. Proposition 6.7). Thus, in order to prove that the canonical BN4-model is a BN4-model, it remains to prove that the postulates hold canonically (cf. Lemma 6.6 in the case of E4).

LEMMA 7.7 (BN4-postulates hold canonically). *The semantical postulates (P1)–(P10) hold in the canonical BN4-model.*

PROOF. We use Lemma 5.9 (cf. Lemma 6.6). (P1)–(P5) and (P7) are proved as in [17, Chapter 4], and (P6), similarly as in Lemma 6.6. So, let us prove (P8)–(P10).

(P8) ($R^C abc \Rightarrow (b \leq^C a^* \text{ or } a \leq^C c)$) holds in the canonical BN_4 -model: Suppose that there are $a, b, c \in K^C$ and wffs A, B such that (1) $R^C abc$ and (2) $A \in b, A \notin a^*$ (i.e., $\neg A \in a$), $B \in a$ and $B \notin c$. By (A12), (3) $(\neg A \wedge B) \rightarrow (A \rightarrow B)$. Thus, (4) $A \rightarrow B \in a$, by 1 and 2, whence $B \in c$ (1, 2 and 4), contradicting 2.

(P9) ($R\mathcal{T}ab \Rightarrow (\mathcal{T}^* \leq b \text{ or } a \leq \mathcal{T})$) holds in the canonical BN_4 -model: Suppose that there are $a, b \in K^C$ and wffs A, B such that (1) $R^C\mathcal{T}ab$ and (2) $A \in \mathcal{T}^*$ (i.e., $\neg A \notin \mathcal{T}$), $A \notin b, B \in a$ and $B \notin \mathcal{T}$. By (A13), (3) $(B \vee \neg A) \vee (B \rightarrow A) \in \mathcal{T}$ (\mathcal{T} is regular). So, (4) $B \rightarrow A \in \mathcal{T}$, by 2 and 3. Finally, (5) $B \in b$, by 1, 2 and 4, contradicting 2.

(P10) ($R^{C^2}\mathcal{T}abc \Rightarrow (b \leq^C \mathcal{T} \text{ or } b \leq^C a^*)$) holds in the canonical BN_4 -model: Suppose that there are $a, b, c \in K^C$ and wffs A, B such that (1) $R^{C^2}\mathcal{T}abc$ and (2) $A \in b, A \notin \mathcal{T}, B \in b$ and $B \notin a^*$ (i.e., $\neg B \in a$). By (d2), there is (3) $x \in K^C$ such that $R^C\mathcal{T}ax$ and R^Cxbc . Let C be a wff such that (4) $C \notin c$ (c is a -consistent). Then (5) $(A \wedge B) \rightarrow C \notin x$, since R^Cxbc (by 3) and $A \wedge B \in b$ (by 2). On the other hand, (6) $\neg A \vee \neg B \in a$ by 2, whence (7) $\neg(A \wedge B) \in b$, by (T6) (cf. Proposition 2.12). Thus, (8) $\neg(A \wedge B) \rightarrow [(A \wedge B) \rightarrow C] \notin \mathcal{T}$, by 5 and 7, since $R^C\mathcal{T}ax$ (by 3). By (A14), (9) $(A \wedge B) \vee [\neg(A \wedge B) \rightarrow [(A \wedge B) \rightarrow C]] \in \mathcal{T}$ (\mathcal{T} is regular). So, by 8 and 9, (10) $A \wedge B \in \mathcal{T}$. But 10 and 2 contradict each other. \square

Finally, as it was the case with E4, we have:

PROPOSITION 7.8 (The canonical BN_4 -model is a BN_4 -model). *The canonical BN_4 -model is indeed a BN_4 -model.*

THEOREM 7.9 (Strong completeness of BN_4). *For any set of wffs Γ and wff A : if $\Gamma \vDash_{BN_4} A$, then $\Gamma \vdash_{BN_4} A$.*

We end the paper by noting that Cm and b4 can be proved strongly complete similarly as BN_4 only if both are extended with the rule (dMP).

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