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SEQUENTS FOR NON-WELLFOUNDED MEREOLOGY

Abstract. The paper explores the proof theory of non-wellfounded mereology with binary fusions and provides a cut-free sequent calculus equivalent to the standard axiomatic system.

Keywords: mereology; sequent calculi; proof theory

1. Introduction

Non-wellfounded mereology has been introduced in [1] to provide a formal account of genuine parthood circularity, namely cases in which an individual is a proper part of itself. Since any wellfounded parthood relation $\sqsubset$ rules out such cases, as well as those in which two individuals are proper part of each other, in non-wellfounded mereology parthood is not assumed to be a strict partial order as $\sqsubset$ is neither irreflexive nor asymmetric. In non-wellfounded mereology $\sqsubset$ is just a transitive relation. The question as to whether and how other mereological notions are affected by $\sqsubset$ being just transitive is addressed in [1]. In particular, authors argue that the principle of extensionality is invalid in non-wellfounded mereology. While in extensional mereologies[1] there cannot be two distinct but indistinguishable individuals, non-wellfounded mereology accounts for situations in which, for instance, two individuals are composed of exactly the same parts without being identical.

[1] See [9] for a survey of extensional mereologies and [8] for a philosophical defense of the extensionality principles.

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The formal theory of non-wellfounded mereology as developed in [1] is axiomatic. In this paper, which is mainly methodological, we present a sequent calculus approach. The first and most serious obstacle to such an approach is that in general an axiomatic theory cannot be systematized in sequent calculus without jeopardizing the fundamental cut-elimination theorem. Cut elimination asserts the redundancy of the rule of cut: no theorem of the theory under consideration depends on cut to be derived.

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Phi \Rightarrow \Psi}{\Gamma, \Phi \Rightarrow \Psi, \Delta} \text{ cut}$$

The importance of cut elimination is that it reduces the indeterminacy of cut. When determining whether a sequent $\Gamma, \Phi \Rightarrow \Psi, \Delta$ is derivable in a system where cut elimination fails, we could always try to reduce the task into $\Gamma \Rightarrow \Delta, A$ and $A, \Phi \Rightarrow \Psi$, where A is an arbitrary new formula, with no end. Because of this lack of determinism introduced by cut, the main task of structural proof theory is to prove that it is redundant in a given system of rules. Cut elimination holds in systems for predicate logic like $\mathsf{G3c}$ from [7], but is not generally valid when proper axioms are added on top it. One way to recover cut elimination is to express axioms as rules of inference. Previous work by S. Negri and J. von Plato [6] has shown how to develop cut-free sequent systems for a wide range of mathematical theories, including lattice and order theory, aparteness, and geometry, whereas [4] has made clear how to develop similar methods for extension mereology. Recently, R. Dyckhoff and S. Negri, building on a early work by T. Skolem, proposed in [3] a new proof-theoretic methodology to convert into an inference rule *any* first-order theory (not just specific theories in some fragment of first-order language). This paper is mainly concerned with applying such new methodology to non-wellfounded mereology.

The paper is organized as follows. In §2 we introduce the language and the axioms system for non-wellfounded mereology with binary fusions and provide an equivalent system (given in the appendix) in which all the axioms are of a special syntactic form, called coherent implications. In §3 we show how to obtain a sequent calculus from the given axiom system and in §4 we prove cut elimination and the admissibility of the structural rules. In §5 we show that our sequent system and the axiomatic system are deductively equivalent. We leave §6 to discuss the existing literature and future work.

2. Coherent non-wellfounded mereology

The language $\mathscr{L}$ of non-wellfounded mereology contains:

- denumerably many individual variables: $x_0, x_1, x_2, \ldots$;
- the symbol $\bot$ (falsehood);
- the connectives $\wedge$ (and), $\vee$ (or), $\rightarrow$ (if … then);
- the quantifiers $\forall$ (for all) and $\exists$ (exists);
- the predicate constants $=$ (identity) and $\sqsubset$ (proper parthood);
- the auxiliary symbols (and) (parentheses).

Atomic formulas (or just atoms) are all the expressions of the form: $x_i = x_j$ and $x_i \sqsubset x_j$. The set of formulas is the smallest set which contains the atoms as well as $\bot$, and is closed under connectives and quantifiers. Sequents are multisets (lists without order) of formulas indicated as Γ, Δ and separated by the symbol $\Rightarrow$. A sequent $\Gamma \Rightarrow \Delta$ is interpreted in the standard way as $\bigwedge \Gamma \rightarrow \bigvee \Delta$. We agree to use:

- x, y, z as metavariables for individual variables;
- P, Q, R as metavariables for atomic formulas;
- A, B, C as metavariables for formulas;
- $\neg A$ as an abbreviation for $A \rightarrow \bot$;
- $\top$ as an abbreviation for $\neg\bot$;
- $A \leftrightarrow B$ as an abbreviation for $(A \rightarrow B) \wedge (B \rightarrow A)$.

Substitutions, derivations, derivation height and derivability are defined in the standard way as in [4]. We only recall that $\vdash_{\mathsf{S}} \Gamma \Rightarrow \Delta$ and $\vdash^{n}_{\mathsf{S}} \Gamma \Rightarrow \Delta$ are understood as 'the sequent $\Gamma \Rightarrow \Delta$ is derivable in the system S' and 'the sequent $\Gamma \Rightarrow \Delta$ is derivable in the system S with a derivation height n'. The subscript S will be often omitted.

In axiomatizing mereology it is convenient to introduce the notions of (improper) parthood $\sqsubseteq$, overlap $\circ$, disjointness $\wr$ and fusion Fus. Such notions are usually defined as metalinguistic abbreviations, but for our purposes it is better to use *definitional equivalences*. Recall from [3, Definition 4.1] that a definitional equivalence is a formula with no free variables of the form $\forall x(M(x) \leftrightarrow A(x))$, where $A(x)$ is a formula in $\mathscr{L}$ and M is a new predicate constant. The result of adding M, along with its definitional equivalence, to an axiomatic theory is called in [3, Definition 4.2] an immediate definitional extension of that theory.

We shall refer to the axiomatic theory of non-wellfounded mereology with binary fusions as ANW (*Axiomatic Non-Wellfounded* mereology).

$\mathtt{ANW1}$ $\quad \forall x \forall y(x \sqsubseteq y \leftrightarrow x \sqsubset y \vee x = y)$

ANW2 $\forall x \forall y(x \circ y \leftrightarrow \exists z(z \sqsubseteq x \wedge z \sqsubseteq y))$

ANW3 $\forall x \forall y(x \wr y \leftrightarrow \neg x \circ y)$

ANW4 $\forall x \forall y \forall z(\mathsf{Fus}(x,y,z) \leftrightarrow x \sqsubseteq z \wedge y \sqsubseteq z \wedge \forall u(x \sqsubseteq u \wedge y \sqsubseteq u \rightarrow z \sqsubseteq u))$

ANW5 $\forall x \forall y(x \sqsubset y \wedge y \sqsubset z \rightarrow x \sqsubset z)$

ANW6 $\forall x \forall y(y \not\sqsubseteq x \rightarrow \exists z \forall u(u \sqsubseteq z \leftrightarrow u \sqsubseteq y \wedge u \wr x))$

ANW7 $\forall x \forall y \exists z \ \mathsf{Fus}(x,y,z)$

Axioms ANW1–ANW4 are the definitional equivalences of improper parthood, overlap, disjointness and fusion. Thus, while $x \sqsubseteq y$ means that x is part of y or coincides with it, and $x \circ y$ (resp. $x \wr y$) means that x and y have (resp. do not have) common improper parts. The ternary predicate $\mathsf{Fus}(x,y,z)$ indicates that z is the fusion of x and y. According to ANW4, fusion is understood a minimal upper bound (with respect to $\sqsubseteq$) of x and y. Notice that ANW4 is weaker than the corresponding axiom Fu_3 of [1], where fusion is a higher-order predicate $\mathsf{Fus}(z, A)$ that applies to an individual z and a first-order formula A; it follows that our ANW is weaker that the system presented in [1]. The axiom ANW5 is the transitivity of the proper parthood relation. The axiom ANW6 is known as the principle of complementation and asserts that for any two individuals x and y such that y is not part of x there is a remainder of the two, namely a third individual which is composed of exactly those individuals that are at the same time part of y and disjoint from x. The axiom ANW7 ensures that any two individuals have a fusion. Notice that due to the lack of antisymmetry of $\sqsubseteq$ that characterize extensional mereologies, fusions exists but are not in general unique. This is why we refer to fusions as 'minimal upper bounds,' instead of 'least upper bounds'.

The sequent calculus GNW (Gentzen Non-Wellfounded mereology) is obtained by converting the axiomatic theory ANW into a set of sequent rules. Such a conversion requires the axioms to be in a certain syntactic form. In particular, the axiomatic theory of non-wellfounded mereology has to be a *coherent theory* in order for its axioms to be treated as inference rules of a sequent calculus. An axiomatic theory is coherent when its axioms are *coherent implications*, namely formulas of the form $\forall \bar{x}(A \rightarrow B)$, where A and B are built from atoms and $\perp$ using only conjunction, disjunction and existential quantification (see [3, definitions 2.1 and 2.4]). As a matter of fact, ANW is not coherent as only ANW5 (proper parthood transitivity) is coherent. In this section we show how to obtain a coherent theory for non-wellfounded mereology from the non-coherent ANW.

Some axioms of ANW can be easily thought of as coherent implications juts applying principles of first order logic: this is the case for axioms ANW1–ANW3 and ANW7. However, ANW4 and ANW6 require an extra work.

We start from ANW7. This axiom, asserting the existence of binary fusions, is not coherent but can be made coherent with an obvious application of the principle *verum ad quodlibet*: $\forall x \forall y(\top \to \exists z \mathsf{Fus}(x, y, z))$. Analogously the axioms ANW1–ANW3 (i.e., the definition equivalences of improper parthood, overlap and disjointness) can be easily transformed into coherent implications using other well-known principles of first-order logic. First, ANW1 is equivalent to the conjunction of two coherent implications cANW1.1 and cANW1.2:

$$\texttt{ANW1} \leftrightarrow \underbrace{\forall x \forall y(x \sqsubseteq y \to x \sqsubset y \vee x = y)}_{\texttt{cANW1.1}} \wedge \underbrace{\forall x \forall y(x \sqsubset y \vee x = y \to x \sqsubseteq y)}_{\mathsf{cANW1.2}}$$

Second, ANW2 can equivalently expressed as cANW2.1 and cANW2.2:

$$\texttt{ANW2} \leftrightarrow \underbrace{\forall x \forall y(x \circ y \to \exists z(z \sqsubseteq x \wedge z \sqsubseteq y))}_{\texttt{cANW2.1}} \wedge \underbrace{\forall x \forall y(\exists z(z \sqsubseteq x \wedge z \sqsubseteq y) \to x \circ y)}_{\mathsf{cANW2.2}}$$

Finally, ANW3 is equivalent to cANW3.1 and cANW3.2:

$$\texttt{ANW3} \leftrightarrow \underbrace{\forall x \forall y(x \wr y \wedge x \circ y \to \bot)}_{\texttt{cANW3.1}} \wedge \underbrace{\forall x \forall y(\top \to x \wr y \vee x \circ y)}_{\texttt{cANW3.2}}$$

Now we move to the axioms that requires a more sophisticated methodology. Indeed, ANW4 and ANW5 are not reducible to coherent implications using just first-order logic. Nevertheless, it is possible to transform them them into coherent implications by applying the strategy of [3], inspired by an early work by Skolem.

We consider first the axiom ANW4. Since this axiom in not coherent as it contains $\forall u(x \sqsubseteq u \wedge y \sqsubseteq u \to z \sqsubseteq u)$, we introduce a new ternary predicate constant M such that $M(x, y, z)$ is syntactically identical to $\forall u(x \sqsubseteq u \wedge y \sqsubseteq u \to z \sqsubseteq u)$. Thus, ANW4 can be written as:

$$\forall x \forall y \forall z(\mathsf{Fus}(x, y, z) \leftrightarrow x \sqsubseteq z \wedge y \sqsubseteq z \wedge M(x, y, z)) \tag{1}$$

The formula (1) is clearly equivalent to the conjunction of the two following coherent implications:

$$\forall x \forall y \forall z(\mathsf{Fus}(x, y, z) \to x \sqsubseteq z \wedge y \sqsubseteq z \wedge M(x, y, z)) \tag{2a}$$

$$\forall x \forall y \forall z(x \sqsubseteq z \wedge y \sqsubseteq z \wedge M(x, y, z) \to \mathsf{Fus}(x, y, z)) \tag{2b}$$

Then we consider the definitional equivalence of $M(x, y, z)$:

$$\forall x \forall y \forall z (M(x, y, z) \leftrightarrow \forall u (x \sqsubseteq u \wedge y \sqsubseteq u \rightarrow z \sqsubseteq u)) \tag{3}$$

The formula (3) is equivalent to the conjunction of the two following formulas:

$$\forall x \forall y \forall z \forall u (M(x, y, z) \wedge x \sqsubseteq u \wedge y \sqsubseteq u \rightarrow z \sqsubseteq u) \tag{4a}$$

$$\forall x \forall y \forall z (\exists u (x \sqsubseteq u \wedge y \sqsubseteq u \wedge z \not\sqsubseteq u) \vee M(x, y, z)) \tag{4b}$$

While the former is a coherent implication, the latter is not as it contains $z \not\sqsubseteq u$. The procedure above can be iterated by introducing a new binary predicate constant N and setting $N(z, u)$ to be identical to $z \not\sqsubseteq u$. Thus, we obtain from (4b) the following coherent formula:

$$\forall x \forall y \forall z (\top \rightarrow \exists u (x \sqsubseteq u \wedge y \sqsubseteq u \wedge N(z, u)) \vee M(x, y, z)) \tag{5}$$

Finally, we consider the definitional equivalence of $N(z, u)$, namely:

$$\forall z \forall u (N(z, u) \leftrightarrow z \not\sqsubseteq u) \tag{6}$$

The latter is equivalent to the two following coherent implications:

$$\forall z \forall u (N(z, u) \wedge z \sqsubseteq u \rightarrow \bot) \tag{7a}$$

$$\forall z \forall u (\top \rightarrow N(z, u) \vee z \sqsubseteq u) \tag{7b}$$

Thus, the axiom `ANW4` characterizing binary fusions are minimal upper bounds can be equivalently replaced by the conjunction of the following six coherent implications, i.e., we have:

$$\mathsf{ANW4} \leftrightarrow (2a) \wedge (2b) \wedge (4a) \wedge (5) \wedge (7a) \wedge (7b)$$

The case of the complementation principle `ANW6` is easier and we can deal with that using a weaker form of its definitional equivalence known as *positive semidefinitional implication*. Recall from [3, Definition 4.4.] that a positive semidefinitional implication is a sentence $\forall x (M(x) \rightarrow A(x))$, where A is a formula of $\mathscr{L}$, whereas M is a new predicate constant. Consider now `ANW6` in its disjunctive (rather than implicative) formulation:

$$\forall x \forall y (y \sqsubseteq x \vee \exists z \forall u (u \sqsubseteq z \leftrightarrow u \sqsubseteq y \wedge u \wr x)) \tag{8}$$

This formula is not coherent as it contains the subformula $\forall u (u \sqsubseteq z \leftrightarrow u \sqsubseteq y \wedge u \wr x)$. We now introduce a new ternary predicate constant R

such that $R(x,y,z)$ is defined as $\forall u(u \sqsubseteq z \leftrightarrow u \sqsubseteq y \wedge u \wr x)$. Then (8) becomes a coherent implication:

$$\forall x\forall y(\top \to y \sqsubseteq x \vee \exists z\, R(x,y,z)) \tag{9}$$

Instead of considering the definitional equivalence of $R(x,y,z)$, we take its positive semidefinitional equivalence, namely

$$\forall x\forall y\forall z(R(x,y,z) \to \forall u(u \sqsubseteq z \leftrightarrow u \sqsubseteq y \wedge u \wr x)) \tag{10}$$

Although (10) is not a coherent implication it can be transformed into two coherent implications by applying principles of first-order logic:

$$\forall x\forall y\forall z\forall u(R(x,y,z) \wedge u \sqsubseteq z \to u \sqsubseteq y \wedge u \wr x) \tag{11a}$$
$$\forall x\forall y\forall z\forall u(R(x,y,z) \wedge u \sqsubseteq y \wedge u \wr x \to u \sqsubseteq z) \tag{11b}$$

In this way, the axiom of complementation `ANW6` can be replaced by the conjunction of three coherent implications as follows, i.e., we have:

$$\texttt{ANW6} \leftrightarrow (9) \wedge (11a) \wedge (11b)$$

Coherent implications can be further simplified by removing inessential disjunctions and existential quantifications in the antecedent using the following first-order principles:

$$((A \vee B) \to C) \leftrightarrow ((A \to C) \wedge (B \to C))$$
$$(\exists x\, A \to B) \leftrightarrow \forall x(A \to B),$$

provided that x is not among the free variables of B. We apply these principles to `ANW1.2` and `ANW2.2`.

We obtain a coherent theory for non-wellfounded mereology if we replace in ANW all the non-coherent implications with their coherent counterparts. We shall call the resulting system cANW and it is given in Appendix.

Axioms of a coherent theory can be transformed into inference rules of a sequent calculus. The conversion can be defined in full generality and the reader interested is referred to [4]. We shall only give here some example to provide an intuitive idea of how the method of axioms-as-rules works. A coherent implication of the form $\forall x\forall y(P(x) \wedge Q(x,y) \to R(y) \vee S(y,x))$ – known as universal axiom – corresponds to the following inference rule of the form:

$$\frac{R(y), P(x), Q(x,y), \Gamma \Rightarrow \Delta \quad S(y,x), P(x), Q(x,y), \Gamma \Rightarrow \Delta}{P(x), Q(x,y), \Gamma \Rightarrow \Delta}$$

A coherent implication of the form $\forall x \forall y(P(x) \wedge Q(x,y) \to \exists z(T(z) \vee U(z,x)))$ — known as geometric axiom — corresponds to the following inference rule of the form:

$$\frac{T(z'), P(x), Q(x,y), \Gamma \Rightarrow \Delta \quad U(z',x), P(x), Q(x,y), \Gamma \Rightarrow \Delta}{P(x), Q(x,y), \Gamma \Rightarrow \Delta}$$

where $T(z')$ and $U(z',x)$ are obtained from $T(z)$ and $U(z,x)$, respectively, by replacing the existentially quantified variable z by z', where z' (called the *eigenvariable*) must not occur free in Γ and Δ.

3. Sequents for non-wellfounded mereology

The system GNW consists of G3c from [7] by adding the following universal and geometric rules.

Identity $=$

$$\frac{P(y), x = y, P(x), \Gamma \Rightarrow \Delta}{x = y, P(x), \Gamma \Rightarrow \Delta}\ \texttt{GNW0.1} \qquad \frac{x = x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \texttt{GNW0.2}$$

Parthood $\sqsubseteq$

$$\frac{x \sqsubset y, x \sqsubseteq y, \Gamma \Rightarrow \Delta \qquad x = y, x \sqsubseteq y, \Gamma \Rightarrow \Delta}{x \sqsubseteq y, \Gamma \Rightarrow \Delta}\ \texttt{GNW1.1}$$

$$\frac{x \sqsubseteq y, x \sqsubset y, \Gamma \Rightarrow \Delta}{x \sqsubset y, \Gamma \Rightarrow \Delta}\ \texttt{GNW1.2} \qquad \frac{x \sqsubseteq y, x = y, \Gamma \Rightarrow \Delta}{x = y, \Gamma \Rightarrow \Delta}\ \texttt{GNW1.3}$$

Overlap $\circ$ (z not free in the conclusion of GNW2.1)

$$\frac{z \sqsubseteq x, z \sqsubseteq y, x \circ y, \Gamma \Rightarrow \Delta}{x \circ y, \Gamma \Rightarrow \Delta}\ \texttt{GNW2.1} \qquad \frac{x \circ y, z \sqsubseteq x, z \sqsubseteq y, \Gamma \Rightarrow \Delta}{z \sqsubseteq x, z \sqsubseteq y, \Gamma \Rightarrow \Delta}\ \texttt{GNW2.2}$$

Disjointness $\wr$

$$\frac{}{x \wr y, x \circ y, \Gamma \Rightarrow \Delta}\ \texttt{GNW3.1} \qquad \frac{x \wr y, \top, \Gamma \Rightarrow \Delta \qquad x \circ y, \top, \Gamma \Rightarrow \Delta}{\top, \Gamma \Rightarrow \Delta}\ \texttt{GNW3.2}$$

Fusion Fus (u not free in the conclusion of GNW4.4)

$$\frac{x \sqsubseteq z, y \sqsubseteq z, M(x,y,z), \mathsf{Fus}(x,y,z), \Gamma \Rightarrow \Delta}{\mathsf{Fus}(x,y,z), \Gamma \Rightarrow \Delta}\ \texttt{GNW4.1}$$

$$\frac{\mathsf{Fus}(x,y,z), x \sqsubseteq z, y \sqsubseteq z, M(x,y,z), \Gamma \Rightarrow \Delta}{x \sqsubseteq z, y \sqsubseteq z, M(x,y,z), \Gamma \Rightarrow \Delta}\ \texttt{GNW4.2}$$

$$\frac{z \sqsubseteq u, M(x,y,z), x \sqsubseteq u, y \sqsubseteq u, \Gamma \Rightarrow \Delta}{M(x,y,z), x \sqsubseteq u, y \sqsubseteq u, \Gamma \Rightarrow \Delta}\ \texttt{GNW4.3}$$

$$\frac{x \sqsubseteq u, y \sqsubseteq u, N(z,u), \top, \Gamma \Rightarrow \Delta \qquad M(x,y,z), \top, \Gamma \Rightarrow \Delta}{\top, \Gamma \Rightarrow \Delta}\ \texttt{GNW4.4}$$

$$\frac{}{z \sqsubseteq u, N(z,u), \Gamma \Rightarrow \Delta}\ \texttt{GNW4.5}$$

$$\frac{N(z,u), \top, \Gamma \Rightarrow \Delta \qquad z \sqsubseteq u, \top, \Gamma \Rightarrow \Delta}{\top, \Gamma \Rightarrow \Delta}\ \texttt{GNW4.6}$$

Proper parthood $\sqsubset$

$$\frac{x \sqsubset z, x \sqsubset y, y \sqsubset z, \Gamma \Rightarrow \Delta}{x \sqsubset y, y \sqsubset z, \Gamma \Rightarrow \Delta}\ \texttt{GNW5}$$

Complementation principle (z not free in the conclusion of `GNW6.1`)

$$\frac{R(x,y,z), \top, \Gamma \Rightarrow \Delta \qquad y \sqsubseteq x, \top, \Gamma \Rightarrow \Delta}{\top, \Gamma \Rightarrow \Delta}\ \texttt{GNW6.1}$$

$$\frac{u \sqsubseteq z, R(x,y,z), u \sqsubseteq y, u \wr x, \Gamma \Rightarrow \Delta}{R(x,y,z), u \sqsubseteq y, u \wr x, \Gamma \Rightarrow \Delta}\ \texttt{GNW6.2}$$

$$\frac{u \sqsubseteq y, u \wr x, R(x,y,z), u \sqsubseteq z, \Gamma \Rightarrow \Delta}{R(x,y,z), u \sqsubseteq z, \Gamma \Rightarrow \Delta}\ \texttt{GNW6.3}$$

Existence of fusion (z not free in the conclusion of `GNW7`)

$$\frac{\mathsf{Fus}(x,y,z), \top, \Gamma \Rightarrow \Delta}{\top, \Gamma \Rightarrow \Delta}\ \texttt{GNW7}$$

We shall also use the following rule of inference, clearly derivable in $\mathsf{G3c}$:

$$\frac{\top, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \top$$

4. Structural properties

In this section, we prove that GNW is cut-free (Theorem 3). Cut elimination is an important property of sequent calculi as it allows an explicit control on the structure of the formal proof. In other words, without cut we are sure that what is contained in the derivation under consideration is all there is to know about that derivation, with no need to look outside of it. Furthermore, derivation without cut are open to proof-search. As we shall see, in cut-free systems it is possible to start derivations from the sequent to be derived and apply systematically all the rules in order to check whether the sequent under consideration is actually derivable. In this respect, contraction admissibility (Theorem 2) is as much as important of cut admissibility. Therefore, we shall prove the admissibility of the following structural rules.

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ \text{ctr} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ \text{ctr}$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Phi \Rightarrow \Psi}{\Gamma, \Phi \Rightarrow \Psi, \Delta}\ \text{cut}$$

However, it easy to see that GNW does not satisfy the subformula property, though it is cut-free. Thus, the question naturally arises as to whether the procedure of proof-search is adversely affected by the fact subformula property is not a corollary of cut elimination, as usually is in first-order logic. It should be noted that, in spite of such a limitation, the rules still offer a heuristics for finding proofs that can be used systematically, if not automatically[2].

We need some preliminary results, in particular the admissibility of the rules of weakening.

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ \text{weak} \qquad\qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ \text{weak}$$

Like contraction, weakening is admissible with the preservation of derivation height. This is almost immediate in **G**-like systems where initial sequents $P, \Gamma \Rightarrow \Delta, P$ are formulated with arbitrary contexts Γ and Δ. Finally, we have to show that all the logical rules, including the mereological ones, are height-preserving invertible, i.e., they can be applied backwards without changing the derivation height (Theorem 1.4).

[2] Another possibility to improve proof search is to abandon cut elimination altogether and consider restricted form of cut as in [2].

More specifically, a rule is invertible when the rule obtained from it by inverting the conclusion with the premises is admissible. Beside the immediate application of proof-search, height-preserving invertibility is of pivotal importance in the proof of contraction admissibility.

Similar to Theorem 1 from [4] we obtain:

Theorem 1. *The following hold in* GNW:

1. *If* $\vdash^n \Gamma \Rightarrow \Delta, \bot$, *then* $\vdash^n \Gamma \Rightarrow \Delta$.
2. *If* $\vdash^n \Gamma \Rightarrow \Delta$, *then* $\vdash^n \Gamma(x/y) \Rightarrow \Delta(x/y)$, *for* y *free for* x *in* $\Gamma \cup \Delta$.
3. *If* $\vdash^n \Gamma \Rightarrow \Delta$, *then* $\vdash^n A, \Gamma \Rightarrow \Delta$ *and* $\vdash^n \Gamma \Rightarrow \Delta, A$.
4. *All rules are height-preserving invertible.*

By mutual induction on n we obtain:

Theorem 2. *Contraction is height-preserving admissible in* GNW, *i.e., for all* Γ, Δ, A, *and* $n \geqslant 0$:

1. *If* $\vdash^n A, A, \Gamma \Rightarrow \Delta$, *then* $\vdash^n A, \Gamma \Rightarrow \Delta$.
2. *If* $\vdash^n \Gamma \Rightarrow \Delta, A, A$, *then* $\vdash^n \Gamma \Rightarrow \Delta, A$.

Proof. For $n = 0$. Regarding the claim 1 we have the following cases:

- $P \in \Gamma \cap \Delta$, for some P. Hence $\vdash^0 A, \Gamma \Rightarrow \Delta$;
- A is P and $P \in \Delta$, for some P. Hence $\vdash^0 A, \Gamma \Rightarrow \Delta$;
- A is $\bot$. Hence $\vdash^0 A, \Gamma \Rightarrow \Delta$;
- $\bot \in \Gamma$. Hence $\vdash^0 A, \Gamma \Rightarrow \Delta$;
- $x \wr y, x \circ y \in \Gamma$, i.e., $A, A, \Gamma \Rightarrow \Delta$, by `GNW3.1`. Hence $\vdash^0 A, \Gamma \Rightarrow \Delta$;
- $z \sqsubseteq u, N(z,u) \in \Gamma$, i.e., $A, A, \Gamma \Rightarrow \Delta$, by `GNW4.5`. Hence $\vdash^0 A, \Gamma \Rightarrow \Delta$;
- A is $x \wr y$ and $x \circ y \in \Gamma$, i.e., $A, A, \Gamma \Rightarrow \Delta$, by `GNW3.1`. So $\vdash^0 A, \Gamma \Rightarrow \Delta$;
- A is $x \circ y$ and $x \wr y \in \Gamma$, i.e., $A, A, \Gamma \Rightarrow \Delta$, by `GNW3.1`. So $\vdash^0 A, \Gamma \Rightarrow \Delta$;
- A is $z \sqsubseteq u$ and $N(z,u) \in \Gamma$, i.e., $A, A, \Gamma \Rightarrow \Delta$, by `GNW4.5`. Hence $\vdash^0 A, \Gamma \Rightarrow \Delta$;
- A is $N(z,u)$ and $z \sqsubseteq u \in \Gamma$, i.e., $A, A, \Gamma \Rightarrow \Delta$, by `GNW4.5`. Hence $\vdash^0 \Gamma \Rightarrow \Delta$;.

The proof of the claim 2 is analogous.

For $n = k+1$ we consider two cases. First, some $B \in \Gamma$ other than A is principal. By cases on the last rule R applied. If R is a one-premise rule its premise is $\vdash^k A, A, \Gamma' \Rightarrow \Delta'$. By IH $\vdash^k A, \Gamma' \Rightarrow \Delta'$, hence by R again $\vdash^{k+1} A, \Gamma \Rightarrow \Delta$. The case of a two-premise rule is similar.

Second, A is principal. Once again, by cases on the last rule R applied. If R is a mereological rule, we first apply IH on its premise(s) — this is always possible as the contraction formulas are repeated into the premise(s); then via another application of R we obtain the conclusion. If R is either a propositional or a quantifier rule we first need to apply the Lemma 1.4 (height-preserving invertibility) on the premise(s), then IH and finally R again to get the conclusion. □

THEOREM 3. *Cut is admissible in* GNW, *i.e., for all* Γ, Δ, Φ, Ψ, *and* A:

$$\text{if } \vdash \Gamma \Rightarrow \Delta, A \text{ and } \vdash A, \Phi \Rightarrow \Psi, \text{ then } \vdash \Gamma, \Phi \Rightarrow \Psi, \Delta$$

PROOF. By induction on the *rank* of cut, i.e., the size of the cut formula and the sum of the derivation heights of the premises of cut. Let $\mathscr{D}$ ($\mathscr{E}$) be the derivation of $\Gamma \Rightarrow \Delta, A$ (resp. of $A, \Phi \Rightarrow \Psi$).

1. Either $\mathscr{D}$ or $\mathscr{E}$ is initial or the conclusion of a 0-ary inference rule.

(a) $P \in \Gamma \cap \Delta$, for some P. Hence $\vdash \Gamma, \Phi \Rightarrow \Psi, \Delta$;
(b) $P \in \Phi \cap \Psi$, for some P. Hence $\vdash \Gamma, \Phi \Rightarrow \Psi, \Delta$;
(c) A is P and $P \in \Gamma$, for some P. By Theorem 1.3 on $\mathscr{E}$;
(d) A is P and $P \in \Psi$, for some P. By Theorem 1.3 on $\mathscr{D}$;
(e) $\bot \in \Gamma$. Hence $\vdash \Gamma, \Phi \Rightarrow \Psi, \Delta$;
(f) A is $\bot$. By Theorem 1.1;
(g) $x \wr y, x \circ y \in \Gamma$, i.e., $\mathscr{D}$ is by GNW3.1. Hence $\vdash \Gamma, \Phi \Rightarrow \Psi, \Delta$;
(h) $z \sqsubseteq u, N(z, u) \in \Gamma$, i.e., $\mathscr{D}$ is by GNW4.5. Hence $\vdash \Gamma, \Phi \Rightarrow \Psi, \Delta$;
(i) $x \wr y, x \circ y \in \Phi$, i.e., $\mathscr{E}$ is by GNW3.1. Hence $\vdash \Gamma, \Phi \Rightarrow \Psi, \Delta$;
(j) $z \sqsubseteq u, N(z, u) \in \Phi$, i.e., $\mathscr{E}$ is by GNW4.5. Hence $\vdash \Gamma, \Phi \Rightarrow \Psi, \Delta$;
(k) A is $x \wr y$ and $x \circ y \in \Phi$, i.e., $\mathscr{E}$ is by GNW3.1. Take $\mathscr{D}$. If it is initial or the conclusion of a 0-ary inference rule:
   (i) $P \in \Gamma \cap \Delta$, for some P. As in (a);
   (ii) $x \wr y \in \Gamma$. As in (c);
   (iii) $\bot \in \Gamma$. As in (e);
   (iv) $z \wr u, z \circ u \in \Gamma$, i.e., $\mathscr{D}$ is by GNW3.1. As in (g);
   (v) $N(z, u), z \sqsubseteq u \in \Gamma$, i.e., $\mathscr{D}$ is by GNW4.5. As in (h);

   Otherwise, if $\mathscr{D}$ has been concluded by some R then $x \wr y$ must be not principal and cut is permuted. We consider the case in which R is GNW7 (with z eigenvariable), namely:

$$\dfrac{\dfrac{\mathsf{Fus}(x', y', z), \top, \Gamma' \Rightarrow \Delta, x \wr y}{\top, \Gamma' \Rightarrow \Delta, x \wr y}\,\texttt{GNW7} \qquad \dfrac{}{x \wr y, x \circ y, \Phi' \Rightarrow \Psi}\,\texttt{GNW4.5}}{\top, \Gamma', x \circ y, \Phi' \Rightarrow \Psi, \Delta}\,\text{cut}$$

First, we replace every occurrence of z in the premise of `GNW7` with a completely new variable z'. Notice that since z' is the eigenvariable, the substitution does not affect the contexts. Then cut is permuted upwards in the usual way:

$$\dfrac{\dfrac{\mathsf{Fus}(x',y',z'),\top,\Gamma' \Rightarrow \Delta, x \wr y \qquad \dfrac{}{x \wr y, x\circ y, \Phi' \Rightarrow \Psi}\,\texttt{GNW4.5}}{\mathsf{Fus}(x',y',z'),\top,\Gamma', x\circ y, \Phi' \Rightarrow \Psi, \Delta}\,\text{cut}}{\top,\Gamma', x\circ y, \Phi' \Rightarrow \Psi, \Delta}\,\texttt{GNW7}$$

(l) A is $x \circ y$ and $x \wr y \in \Phi$, i.e., $\mathscr{E}$ is by `GNW3.1`. As in (k);
(m) A is $z \sqsubseteq u$ and $N(z,u) \in \Phi$, i.e., $\mathscr{E}$ is by `GNW4.5`. As in (k);
(n) A is $N(z,u)$ and $z \sqsubseteq u \in \Phi$, i.e., $\mathscr{E}$ is by `GNW4.5`. As in (k);

2. Neither $\mathscr{D}$ nor $\mathscr{E}$ is initial or the conclusion of a 0-ary inference rule. Let R (resp. S) be the last rule applied in $\mathscr{D}$ (resp. $\mathscr{E}$).

(a) A is not principal in R. Cut is permuted upward with S as above.
(b) A is principal in R only. Cut is permuted upward with S as above.
(c) A is principal in both R and S. This case involves no mereological rule as they only have atomic formulas as principal. Thus there is no new case with respect to cut elimination for first order logic. □

5. Correspondence with the axiomatic system

In this section we show that the axiomatic system ANW and the sequent calculus GNW coincide, i.e., that all the axioms ANW are derivable in GNW and that also that the rules of GNW are admissible in ANW.

THEOREM 4. *For any A: $\vdash_{\mathsf{ANW}} A$ implies $\vdash_{\mathsf{GNW}} \Rightarrow A$.*

PROOF. Double inference line denote multiple applications of rules. Rule labels of G3c rules will be omitted. First, notice that Modus Ponens, the only rule of ANW, is admissible via cut elimination. Secondly, consider the following cases:

Axiom `ANW1` (improper parthood)

$$\cfrac{\dfrac{x \sqsubset y, x \sqsubseteq y \Rightarrow x \sqsubset y, x = y \qquad x = y, x \sqsubseteq y \Rightarrow x \sqsubset y, x = y}{x \sqsubseteq y \Rightarrow x \sqsubset y, x = y}\,\texttt{GNW1.1}}{\Rightarrow \forall x \forall y (x \sqsubseteq y \leftrightarrow x \sqsubset y \vee x = y)}\,\mathscr{D}$$

where the subderivation $\mathscr{D}$ is:

$$\dfrac{\dfrac{x \sqsubseteq y, x \sqsubset y \Rightarrow x \sqsubseteq y}{x \sqsubset y \Rightarrow x \sqsubseteq y}\,\texttt{GNW1.2} \quad \dfrac{x \sqsubseteq y, x = y \Rightarrow x \sqsubseteq y}{x = y \Rightarrow x \sqsubseteq y}\,\texttt{GNW1.3}}{x \sqsubset y \vee x = y \Rightarrow x \sqsubseteq y}$$

Axiom ANW2 (overlap)

$$\dfrac{\dfrac{\dfrac{\dfrac{z \sqsubseteq x, z \sqsubseteq y, x \circ y \Rightarrow A, z \sqsubseteq x \quad z \sqsubseteq x, z \sqsubseteq y, x \circ y \Rightarrow A, z \sqsubseteq y}{z \sqsubseteq x, z \sqsubseteq y, x \circ y \Rightarrow A, z \sqsubseteq x \wedge z \sqsubseteq y}}{z \sqsubseteq x, z \sqsubseteq y, x \circ y \Rightarrow A}\,\texttt{GNW2.1}}{x \circ y \Rightarrow A} \quad \mathscr{D}}{\Rightarrow \forall x \forall y (x \circ y \leftrightarrow \underbrace{\exists z (z \sqsubseteq x \wedge z \sqsubseteq y)}_{A})}$$

where the subderivation $\mathscr{D}$ is

$$\dfrac{\dfrac{x \circ y, z \sqsubseteq x, z \sqsubseteq y \Rightarrow x \circ y}{z \sqsubseteq x, z \sqsubseteq y \Rightarrow x \circ y}\,\texttt{GNW2.2}}{A \Rightarrow x \circ y}$$

Axiom ANW3 (disjointness)

$$\dfrac{\dfrac{}{x \wr y, x \circ y \Rightarrow}\,\texttt{GNW3.1} \quad \dfrac{\dfrac{x \wr y, \top \Rightarrow x \wr y, x \circ y \quad x \circ y, \top \Rightarrow x \wr y, x \circ y}{\top \Rightarrow x \wr y, x \circ y}\,\texttt{GNW3.2}}{\Rightarrow x \wr y, x \circ y}\,\texttt{GNW3.3}}{\Rightarrow \forall x \forall y (x \wr y \leftrightarrow \neg x \circ y)}$$

Axiom ANW4 (fusion)

$$\dfrac{\dfrac{\dfrac{\dfrac{x \sqsubseteq z, y \sqsubseteq z, M(x,y,z), \mathsf{Fus}(x,y,z) \Rightarrow x \sqsubseteq z}{\mathsf{Fus}(x,y,z) \Rightarrow x \sqsubseteq z}\,\texttt{GNW4.1} \quad \mathscr{D}}{\mathsf{Fus}(x,y,z) \Rightarrow x \sqsubseteq z \wedge y \sqsubseteq z} \quad \mathscr{E}}{\mathsf{Fus}(x,y,z) \Rightarrow x \sqsubseteq z \wedge y \sqsubseteq z \wedge A} \quad \mathscr{F}}{\Rightarrow \forall x \forall y \forall z (\mathsf{Fus}(x,y,z) \leftrightarrow x \sqsubseteq z \wedge y \sqsubseteq z \wedge \underbrace{\forall u (x \sqsubseteq u \wedge y \sqsubseteq u \to z \sqsubseteq u)}_{A})}$$

where, first, the subderivation $\mathscr{D}$ is

$$\dfrac{x \sqsubseteq z, y \sqsubseteq z, M(x,y,z), \mathsf{Fus}(x,y,z) \Rightarrow y \sqsubseteq z}{\mathsf{Fus}(x,y,z) \Rightarrow y \sqsubseteq z}\,\texttt{GNW4.1}$$

Second, the subderivation $\mathscr{E}$ is

$$\dfrac{\dfrac{\dfrac{z \sqsubseteq u, x \sqsubseteq u, y \sqsubseteq u, x \sqsubseteq z, y \sqsubseteq z, M(x,y,z), \mathsf{Fus}(x,y,z) \Rightarrow z \sqsubseteq u}{x \sqsubseteq u, y \sqsubseteq u, x \sqsubseteq z, y \sqsubseteq z, M(x,y,z), \mathsf{Fus}(x,y,z) \Rightarrow z \sqsubseteq u}\ \texttt{GNW4.3}}{x \sqsubseteq z, y \sqsubseteq z, M(x,y,z), \mathsf{Fus}(x,y,z) \Rightarrow A}}{\mathsf{Fus}(x,y,z) \Rightarrow A}\ \texttt{GNW4.1}$$

Finally, the subderivation $\mathscr{F}$ is

$$\dfrac{\dfrac{\dfrac{\overset{\mathscr{F}_0}{\top, x \sqsubseteq z, y \sqsubseteq z, A \Rightarrow B, x \sqsubseteq u} \quad \overset{\mathscr{F}_1}{\top, x \sqsubseteq z, y \sqsubseteq z, A \Rightarrow B, y \sqsubseteq u}}{\top, x \sqsubseteq z, y \sqsubseteq z, A \Rightarrow B, x \sqsubseteq u \wedge y \sqsubseteq u}}{x \sqsubseteq z, y \sqsubseteq z, A \Rightarrow B, x \sqsubseteq u \wedge y \sqsubseteq u}\ \top \qquad \overset{\mathscr{F}_2}{x \sqsubseteq z, y \sqsubseteq z, A, z \sqsubseteq u \Rightarrow B}}{x \sqsubseteq z \wedge y \sqsubseteq z \wedge A \Rightarrow \underbrace{\mathsf{Fus}(x,y,z)}_{B}}$$

where, first, the subderivation $\mathscr{F}_0$ is

$$\dfrac{x \sqsubseteq u, y \sqsubseteq u, N(z,u), C, A \Rightarrow B, x \sqsubseteq u \qquad \dfrac{\mathsf{Fus}(x,y,z), M(x,y,z), C, A \Rightarrow B, x \sqsubseteq u}{M(x,y,z), C, A \Rightarrow B, x \sqsubseteq u}\ \texttt{GNW4.2}}{\underbrace{\top, x \sqsubseteq z, y \sqsubseteq z}_{C}, A \Rightarrow B, x \sqsubseteq u}\ \texttt{GNW4.4}$$

Second, the derivation $\mathscr{F}_1$ is similar to $\mathscr{F}_0$, while derivation $\mathscr{F}_2$ is

$$\dfrac{\dfrac{\dfrac{}{x \sqsubseteq u, y \sqsubseteq u, N(z,u), C, A, z \sqsubseteq u \Rightarrow B}\ \texttt{GNW4.5} \qquad \dfrac{\mathsf{Fus}(x,y,z), M(x,y,z), C, A, z \sqsubseteq u \Rightarrow B}{M(x,y,z), C, A, z \sqsubseteq u \Rightarrow B}\ \texttt{GNW4.2}}{C, A, z \sqsubseteq u \Rightarrow B}\ \texttt{GNW4.4}}{\underbrace{\top, x \sqsubseteq z, y \sqsubseteq z}_{C}, A, z \sqsubseteq u \Rightarrow B}\ \top$$

Axiom ANW5 (parthood)

$$\dfrac{\dfrac{x \sqsubset z, x \sqsubset y, y \sqsubset z \Rightarrow x \sqsubset z}{x \sqsubset y, y \sqsubset z \Rightarrow x \sqsubset z}\ \texttt{GNW5}}{\Rightarrow \forall x \forall y (x \sqsubset y \wedge y \sqsubset z \rightarrow x \sqsubset z)}$$

Axiom ANW6 (complementation)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{u \sqsubseteq y, u \wr x, u \sqsubseteq z, R(x,y,z), \top \Rightarrow y \sqsubseteq x, A, u \sqsubseteq y \quad \mathscr{D}}{u \sqsubseteq y, u \wr x, u \sqsubseteq z, R(x,y,z), \top \Rightarrow y \sqsubseteq x, A, u \sqsubseteq y \wedge u \wr x}}{u \sqsubseteq z, R(x,y,z), \top \Rightarrow y \sqsubseteq x, A, u \sqsubseteq y \wedge u \wr x}\ \texttt{GNW6.2} \quad \mathscr{E}}{R(x,y,z), \top \Rightarrow y \sqsubseteq x, A \qquad \mathscr{F}}}{\top \Rightarrow y \sqsubseteq x, A}\ \texttt{GNW6.1}}{\Rightarrow \forall x \forall y (y \not\sqsubseteq x \rightarrow \underbrace{\exists z \forall u (u \sqsubseteq z \leftrightarrow u \sqsubseteq y \wedge u \wr x)}_{A})}$$

where, first, the subderivation $\mathscr{D}$ is

$$u \sqsubseteq y, u \wr x, u \sqsubseteq z, R(x,y,z), \top \Rightarrow y \sqsubseteq x, A, u \wr x$$

Second, the subderivation $\mathscr{E}$ is

$$\dfrac{\dfrac{u \sqsubseteq z, u \sqsubseteq y, u \wr x, R(x,y,z), \top \Rightarrow y \sqsubseteq x, A, u \sqsubseteq z}{u \sqsubseteq y, u \wr x, R(x,y,z), \top \Rightarrow y \sqsubseteq x, A, u \sqsubseteq z}\ \texttt{GNW6.3}}{u \sqsubseteq y \wedge u \wr x, R(x,y,z), \top \Rightarrow y \sqsubseteq x, A, u \sqsubseteq z}$$

Thirdly, the subderivation $\mathscr{F}$ is

$$y \sqsubseteq x, \top \Rightarrow y \sqsubseteq x, A$$

Axiom ANW7 (fusion existence)

$$\dfrac{\dfrac{\dfrac{\mathsf{Fus}(x,y,z), \top \Rightarrow \exists z \mathsf{Fus}(x,y,z), \mathsf{Fus}(x,y,z)}{\mathsf{Fus}(x,y,z), \top \Rightarrow \exists z \mathsf{Fus}(x,y,z)}}{\top \Rightarrow \exists z \mathsf{Fus}(x,y,z)}}{\Rightarrow \forall x \forall y \exists z \mathsf{Fus}(x,y,z)}$$

□

THEOREM 5. *For any A: $\vdash_{\mathsf{GNW}} \Rightarrow A$ implies $\vdash_{\mathsf{ANW}} A$.*

PROOF. We consider only the case of the rules for parthood relation, as other rules are completely analogous. Inferential steps labelled by *Inv* are legitimate by Theorem 1 and correspond to the invertibility of the rules of GNW.

Rule GNW1.1

$$\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \forall x \forall y (x \sqsubseteq y \to x \sqsubset y \vee x = y)}{x \sqsubseteq y, \Rightarrow x \sqsubset y, x = y}\ \text{Inv} \qquad x \sqsubset y, x \sqsubseteq y, \Gamma \Rightarrow \Delta}{x \sqsubseteq y, x \sqsubseteq y, \Gamma \Rightarrow \Delta, x = y}\ \text{cut} \qquad x = y, x \sqsubseteq y, \Gamma \Rightarrow \Delta}{x \sqsubseteq y, x \sqsubseteq y, x \sqsubseteq y, \Gamma, \Gamma \Rightarrow \Delta, \Delta}\ \text{cut}}{x \sqsubseteq y, \Gamma \Rightarrow \Delta}\ \text{ctr}$$

Rule GNW1.2

$$\dfrac{\dfrac{\dfrac{\Rightarrow \forall x \forall y (x \sqsubset y \to x \sqsubseteq y)}{x \sqsubset y \Rightarrow x \sqsubseteq y}\ \text{Inv} \qquad x \sqsubseteq y, x \sqsubset y, \Gamma \Rightarrow \Delta}{x \sqsubset y, x \sqsubset y, \Gamma \Rightarrow \Delta}\ \text{cut}}{x \sqsubset y, \Gamma \Rightarrow \Delta}\ \text{ctr}$$

Rule $\texttt{GNW1.3}$

$$\dfrac{\dfrac{\Rightarrow \forall x \forall y (x = y \to x \sqsubseteq y)}{x = y \Rightarrow x \sqsubseteq y}\ \text{Inv} \qquad x \sqsubseteq y, x = y, \Gamma \Rightarrow \Delta}{\dfrac{x = y, x = y, \Gamma \Rightarrow \Delta}{x = y, \Gamma \Rightarrow \Delta}\ \text{ctr}}\ \text{cut}$$

□

6. Conclusions

In this paper we have presented a sequent calculus GNW for non-wellfounded mereology (with binary fusion) which is equivalent to the standard axiomatic system and satisfies cut elimination. In this section we discuss the relation to the existing literature, in particular to [4] and [1]. First, in [4] the axioms extensional mereology were converted into systems of rules, a proof-theoretic methodology introduced in [5] to provide cut-free sequent calculi for a type of axiomatic systems, called generalized geometric theories. A system of rules is a collection of rules that must be applied in a certain order. For instance, one direction of the definitional equivalence of binary fusion, namely the following axiom:

$$\forall x \forall y \forall z (\mathsf{Fus}(x, y, z) \to x \sqsubseteq z \wedge y \sqsubseteq z \wedge \forall u (x \sqsubseteq u \wedge y \sqsubseteq u \to z \sqsubseteq u))$$

can be thought of as a system of rule of the following form:

$$\dfrac{z \sqsubseteq u, x \sqsubseteq u, y \sqsubseteq u, \Phi \Rightarrow \Psi}{x \sqsubseteq u, y \sqsubseteq u, \Phi \Rightarrow \Psi}\ \texttt{GNW4}''$$

$$\vdots$$

$$\mathscr{D}$$

$$\vdots$$

$$\dfrac{x \sqsubseteq z, y \sqsubseteq z, \mathsf{Fus}(x, y, z), \Gamma \Rightarrow \Delta}{\mathsf{Fus}(x, y, z), \Gamma \Rightarrow \Delta}\ \texttt{GNW4}'$$

The system of rules is subjected to the condition that the application of $\texttt{GNW4}'$ must always be below $\texttt{GNW4}''$. In contrast to [4], the system GNW has no system of rules, just ordinary rules. The advantage of system of rules is that there is no need to change the language introducing new predicate constants. On the other hand, systems of rules are less general than the approach proposed in this paper and introduced recently in [3]: while systems of rules cover only generalized geometric theories, using definitional equivalences *any* first-order axiom can be converted into an inference rule of a sequent calculus. For instance, the other direction of the definitional equivalence of binary fusion, namely the formula $\forall x \forall y \forall z(x \sqsubseteq z \wedge y \sqsubseteq z \wedge \forall u(x \sqsubseteq u \wedge y \sqsubseteq u \rightarrow z \sqsubseteq u) \rightarrow \mathsf{Fus}(x, y, z))$ is not a generalized geometric axiom and hence there is no system of rules for it. The price to pay for using definitional equivalences is the introduction of new predicate constants. Secondly, the theory ANW, as well as its coherent counterpart cANW, are limited to binary fusion, whereas the standard system of [1] covers also arbitrary fusions. Although nothing seems to prevent us from extending our approach to arbitrary fusion we have deliberately focused on binary ones to illustrate the methodology of definitional equivalences. We leave the treatment of more general notions of fusions to future work.

Appendix

The system cANW consists of the axioms of predicate logic with equality, plus the following specific axioms:

`cANW1.1`	$\forall x \forall y(x \sqsubseteq y \rightarrow x \sqsubset y \vee x = y)$
`cANW1.2`	$\forall x \forall y(x \sqsubset y \rightarrow x \sqsubseteq y)$
`cANW1.3`	$\forall x \forall y(x = y \rightarrow x \sqsubseteq y)$
`cANW2.1`	$\forall x \forall y(x \circ y \rightarrow \exists z(z \sqsubseteq x \wedge z \sqsubseteq y))$
`cANW2.1`	$\forall x \forall y \forall z(z \sqsubseteq x \wedge z \sqsubseteq y \rightarrow x \circ y)$
`cANW3.1`	$\forall x \forall y(x \wr y \wedge x \circ y \rightarrow \bot)$
`cANW3.2`	$\forall x \forall y(\top \rightarrow x \wr y \vee x \circ y)$
`cANW4.1`	$\forall x \forall y \forall z(\mathsf{Fus}(x, y, z) \rightarrow x \sqsubseteq z \wedge y \sqsubseteq z \wedge M(x, y, z))$
`cANW4.2`	$\forall x \forall y \forall z(x \sqsubseteq z \wedge y \sqsubseteq z \wedge M(x, y, z) \rightarrow \mathsf{Fus}(x, y, z))$
`cANW4.3`	$\forall x \forall y \forall z \forall u(M(x, y, z) \wedge x \sqsubseteq u \wedge y \sqsubseteq u \rightarrow z \sqsubseteq u)$
`cANW4.4`	$\forall x \forall y \forall z(\top \rightarrow \exists u(x \sqsubseteq u \wedge y \sqsubseteq u \wedge N(z, u)) \vee M(x, y, z))$
`cANW4.5`	$\forall z \forall u(N(z, u) \wedge z \sqsubseteq u \rightarrow \bot)$
`cANW4.6`	$\forall z \forall u(\top \rightarrow N(z, u) \vee z \sqsubseteq u)$
`cANW5`	$\forall x \forall y(x \sqsubset y \wedge y \sqsubset z \rightarrow x \sqsubset z)$
`cANW6.1`	$\forall x \forall y(\top \rightarrow y \sqsubseteq x \vee \exists z R(x, y, z))$
`cANW6.2`	$\forall x \forall y \forall z \forall u(R(x, y, z) \wedge u \sqsubseteq z \rightarrow u \sqsubseteq y \wedge u \wr x)$

cANW6.3 $\forall x \forall y \forall z \forall u (R(x, y, z) \wedge u \sqsubseteq y \wedge u \wr x \to u \sqsubseteq z)$

cANW7 $\forall x \forall y (\top \to \exists z \mathsf{Fus}(x, y, z))$

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