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SIMPLE CUT ELIMINATION PROOF FOR HYBRID LOGIC

Abstract. In the paper we present a relatively simple proof of cut elimination theorem for variety of hybrid logics in the language with satisfaction operators and universal modality. The proof is based on the strategy introduced originally in the framework of hypersequent calculi but it works well also for standard sequent calculi. Sequent calculus examined in the paper works on so called satisfaction formulae and cover all logics adequate with respect to classes of frames defined by so called geometric conditions.

Keywords: hybrid logic; cut elimination theorem; hypersequent calculi; sequent calculi; geometric conditions

1. Introduction

Hybrid logic (HL) is an interesting generalization of standard modal logic obtained by enrichment of ordinary modal languages. The first versions of HL were provided by Prior [1957] in 50s but at first this proposal did not find any attention. Modern studies on HL were developed first by so called Sofia School (Gargov and Goranko [1993], Goranko [1994]) and then by Blackburn [1992] and others. We are not going to enter into details of the history, theory and applications of HL; one may find enough information in [Indrzejczak, 2007] or [Braüner, 2014]. However, some basic information is needed to make a text self-contained.

The basic language of HL is obtained by the addition of the second sort of propositional atoms called nominals. Informally, they denote propositions true in exactly one world of a model and may serve as names of these worlds. Additionally one can add several specific operators and binders (if nominals are treated as variables). The most

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important specific constants are so called satisfaction operators indicating that a formula is satisfied in the world denoted by some nominal. This allows to internalize in the language the devices which are used in labelled systems but as extralinguistic additions. What is nice with HL is the fact that changes in the language do not affect seriously the rest of the machinery applied in standard modal logic. In particular, modifications in the relational semantics are minimal. The concept of a frame is the same as in ordinary normal modal (or tense) logics, only on the level of models we have some small changes.

These relatively small modifications of standard modal languages gives us many advantages, like: more expressive language, better behavior in completeness theory, more natural and simpler proof theory. In particular, one may define in HL such frame conditions like irreflexivity, assymetry, trichotomy and other not expressible in standard modal languages. Proof theory of HL, developed in the framework of tableaux or natural deduction offers even more general approach than application of labels popular in proof theory for standard modal logic.[1]

The aim of this paper is to present a uniform cut elimination theorem for a wide class of of hybrid logics formulated in the framework of sequent calculus (SC). Although proof theory of HL is developed in quite satisfying way[2] this problem was not treated extensively so far. There are cut-free SC for some HL due to [Blackburn, 2000; Braüner, 2009; Seligman, 2001], but all of them were obtained indirectly on the basis of other systems like tableaux or natural deduction.

In Section 2 we will recall the basic information concerning HL. In Section 3 we will introduce respective sequent calculi and in Section 4 we will prove cut elimination theorem for them.

2. Hybrid logics

In what follows we are using a monomodal language with denumerable set of propositional variables PROP := $\{p, q, r, p_1, p_2, \ldots\}$ and constants: $\neg$, $\wedge$, $\Diamond$. The *basic hybrid propositional modal language* $\mathbf{L_{H@}}$ is obtained by adding to this modal language:

(a) the second sort of propositional symbols called *nominals*. We assume the denumerable set NOM $= \{i, j, k, i_1, i_2, \ldots\}$; members of NOM

[1] See, for example, a discussion in [Indrzejczak, 2010].

[2] See especially [Braüner, 2009; Zawidzki, 2013], [Indrzejczak, 2010, Chapter 12].

are introduced for naming states of a model domain. Moreover, PROP ∪ NOM =: AT is the set of atomic formulae.

(b) denumerable collection of unary satisfaction operators $@_\imath$ indexed by nominals.

Now we define the set FOR of formulae of $\mathbf{L_{H@}}$ with the new clause for non-atomic formulae. The set FOR is the smallest set that satisfies the following conditions:

- AT ⊆ FOR,
- if $\varphi \in$ FOR, then both $\neg\varphi \in$ FOR and $\Diamond\varphi \in$ FOR,
- if $\varphi, \psi \in$ FOR then $(\varphi \wedge \psi) \in$ FOR,
- if $\varphi \in$ FOR and $\imath \in$ NOM, then $@_\imath\varphi \in$ FOR.

A new formula $@_\imath\varphi$ we read as "formula φ is satisfied in a state $\imath$".

Note two important features of $\mathbf{L_{H@}}$:

- Both nominals and satisfaction operators are genuine language elements not an extra metalinguistic machinery.[3]
- Although nominals are terms they are treated as ordinary sentences. In particular, they can be connected with the help of boolean operators and combined with modal and tense operators. In fact, they play double role:
  - of propositional symbols representing propositions of the form "the name of the actual state is i";
  - of names of states when they occur as indexes of unary satisfaction operators.

The notion of a *frame* is defined as for standard modal logic. A *model* on the frame $\mathfrak{F}$ is any structure $\mathfrak{M} = \langle \mathfrak{F}, V, a\rangle$, where V is a valuation function on propositional symbols, i.e. $V\colon \mathrm{PROP} \to \mathcal{P}(\mathcal{W})$, and a is an assignment function on nominals, i.e. $a\colon \mathrm{NOM} \to W$. Satisfaction of new formulae in states of a model $\mathfrak{M} = \langle \mathfrak{F}, V, a\rangle$ is defined as follows:

$$\mathfrak{M} \vDash_w \imath \quad \text{iff} \quad w = a(\imath)$$

$$\mathfrak{M} \vDash_w @_\imath\varphi \quad \text{iff} \quad \mathfrak{M} \vDash_{a(\imath)} \varphi$$

The concepts of global satisfiability and of validity are the same as for ordinary modal language. Also definitions of consequence relations remain intact. The only difference is that if we say "model" we mean a model in a hybrid sense with a constraint on valuation of nominals.

Let us focus on some consequences of the above definitions. The most important features of $\mathbf{L_{H@}}$ seem to be:

[3] This is the main difference with labelled systems of Fitting or Gabbay.

1. Internalization of local discourse — nominals give direct representation of states in a language (we have an object-language mechanism for storing model data).
2. Possible jumping to already specified states in a model (we have a mechanism for retrieving model data).
3. Internalization of $\vDash$ by sat-formulae $@_\imath\varphi$.
4. Representation of identity theory (for states) by pure formulae of the form $@_\imath\jmath$. Indeed, we have: $\mathfrak{M} \vDash_w @_\imath\jmath$ iff $a(\imath) = a(\jmath)$.
5. Internalization of accessibility relation by pure formulae of the form $@_\imath\Diamond\jmath$. Indeed, we have: $\mathfrak{M} \vDash_w @_\imath\Diamond\jmath$ iff $\langle a(\imath), a(\jmath)\rangle \in R$, where $\mathfrak{F} = \langle W, R\rangle$.

Although the basic hybrid language offers many improvements over standard modal language it has still strong limitations which may be overcome by further strengthenings. In what follows we consider an extensions obtainable by addition of global modality $\mathcal{E}$. Such an extension is very expressive but the logic is still decidable. The semantic clause for this modality looks like this:

$$\mathfrak{M} \vDash_w \mathcal{E}\varphi \quad \text{iff} \quad \mathfrak{M} \vDash_v \varphi, \text{ for some } v \in W$$

3. Cut-free SC for HL

So far various deductive systems were offered as formalizations of different hybrid logics. The most popular, except axiomatic systems, were tableaux systems [Blackburn, 2000] and natural deduction [Braüner, 2004]. One may also find sequent calculi for some hybrid logics. The earliest proposal is that of [Seligman, 2001] which was first formulated in the context of situation theory. One may also find some nonstandard SC of [Demri, 1999] and [Demri and Goré, 1999], but these are of different kind than ours.

The most popular approach in proof theory for HL is to devise so called sat-calculus where each formula is preceded with satisfaction operator. Such a solution corresponds nicely to labelled calculi although one should remember that satisfaction operators are not metalogical devices but elements of the language of HL. Using sat-calculi instead of calculi working with any formulae is justified by the fact that φ holds in (any) HL iff $@_\imath\varphi$ holds, provided a nominal i is not a subformula of φ. So to provide a proof for $@_\imath\varphi$ is the same as providing a proof for φ in some hybrid logic.

One may find several cut-free sat-SC for some HL in slightly different languages independently proposed by Blackburn [2000], Braüner [2004], Bolander and Braüner [2006], Indrzejczak and Zawidzki [2013]. In all these cases, proposed systems are obtained by translation; either from tableau system or [in Braüner, 2004] from normalizable natural deduction system. Hence these systems are cut-free but with no direct syntactical proof for cut elimination. In what follows we will define a sat calculus which is equivalent to Bolander and Braüner's system from [2006] although some of the rules are taken rather from different Braüner's system from [2004]. Our main aim is to provide a cut elimination theorem which holds for this particular sat-SC.

Since the present calculus is sat-SC, sequents are composed from finite multisets of sat-formulae of the form $@_\imath\varphi$. It consists of the following rules.

Structural rules:

(AX) $@_\imath\varphi \Rightarrow @_\imath\varphi$

(Cut) $$\frac{\Gamma \Rightarrow \Delta, @_\imath\varphi \qquad @_\imath\varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

(W⇒) $$\frac{\Gamma \Rightarrow \Delta}{@_\imath\varphi, \Gamma \Rightarrow \Delta}$$ (⇒W) $$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, @_\imath\varphi}$$

(C⇒) $$\frac{@_\imath\varphi, @_\imath\varphi, \Gamma \Rightarrow \Delta}{@_\imath\varphi, \Gamma \Rightarrow \Delta}$$ (⇒C) $$\frac{\Gamma \Rightarrow \Delta, @_\imath\varphi, @_\imath\varphi}{\Gamma \Rightarrow \Delta, @_\imath\varphi}$$

Logical rules for connectives:

(¬⇒) $$\frac{\Gamma \Rightarrow \Delta, @_\imath\varphi}{@_\imath\neg\varphi, \Gamma \Rightarrow \Delta}$$ (⇒¬) $$\frac{@_\imath\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, @_\imath\neg\varphi}$$

(∧⇒) $$\frac{@_\imath\varphi, @_\imath\psi, \Gamma \Rightarrow \Delta}{@_\imath(\varphi \wedge \psi), \Gamma \Rightarrow \Delta}$$ (⇒∧) $$\frac{\Gamma \Rightarrow \Delta, @_\imath\varphi \qquad \Gamma \Rightarrow \Delta, @_\imath\psi}{\Gamma \Rightarrow \Delta, @_\imath(\varphi \wedge \psi)}$$

Modal rules:

(@⇒) $$\frac{@_\imath\varphi, \Gamma \Rightarrow \Delta}{@_\jmath @_\imath\varphi,\ \Gamma \Rightarrow \Delta}$$ (⇒@) $$\frac{\Gamma \Rightarrow \Delta, @_\imath\varphi}{\Gamma \Rightarrow \Delta,\ @_\jmath @_\imath\varphi}$$

$(\Diamond\Rightarrow)^{1,2}$ $$\frac{@_\imath\Diamond\jmath,\ @_\jmath\varphi, \Gamma \Rightarrow \Delta}{@_\imath\Diamond\varphi, \Gamma \Rightarrow \Delta}$$ (⇒◇) $$\frac{\Gamma \Rightarrow \Delta, @_\imath\Diamond\jmath \qquad \Gamma \Rightarrow \Delta, @_\jmath\varphi}{\Gamma \Rightarrow \Delta, @_\imath\Diamond\varphi}$$

$(\mathcal{E}\Rightarrow)^1$ $$\frac{@_\jmath\varphi, \Gamma \Rightarrow \Delta}{@_\imath\mathcal{E}\varphi,\ \Gamma \Rightarrow \Delta}$$ $(\Rightarrow\mathcal{E})$ $$\frac{\Gamma \Rightarrow \Delta, @_\jmath\varphi}{\Gamma \Rightarrow \Delta,\ @_\imath\mathcal{E}\varphi}$$

side condition:

1. $\jmath$ does not occur in the conclusion,
2. φ is not a nominal.

Remark. Strictly speaking the side condition 2 is not necessary since this rule is correct when applicable to nominals. However, there are two important obstacles for allowing nominals in this rule in a role of principal formulae. The first is connected with proof search; uncontrolled application of these rules to nominals leads to unwanted introduction of new nominals in the proof-tree. For our purposes more serious is the fact that if we allow nominals as arguments of $\Diamond$ in this rule we encounter problems in our proof of cut elimination theorem. It will be commented in the proper place. ⊣

Special rules for nominals:

$$(\text{Ref})\quad \frac{@_\imath \imath, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$(\text{Nom}_1{\Rightarrow})\quad \frac{\Gamma \Rightarrow \Delta, @_\imath \jmath \qquad \Gamma \Rightarrow \Delta, @_\imath \varphi}{\Gamma \Rightarrow \Delta, @_\jmath \varphi} \qquad \text{where } \varphi \in \text{AT}$$

$$(\text{Nom}_2{\Rightarrow})\quad \frac{\Gamma \Rightarrow \Delta, @_\imath \jmath \qquad \Gamma \Rightarrow \Delta, @_\imath \Diamond\kappa}{\Gamma \Rightarrow \Delta, @_\jmath \Diamond\kappa}$$

Note that almost all rules are context-insensitive i.e. one may replace any parametric formula with any other formula without loosing correctness. Only rules with side conditions do not satisfy this characteristics. A definition of proof is standard, as well as the notions of principal, side and parametric (context) formulae. A calculus comprising all these rules provides a formalization of the weakest logic with universal modality HL-K which is adequate with respect to the class of all models. One may easily prove:

LEMMA 1 (Validity-preservation). *All rules of* SC-HL-K *are validity-preserving in* HL*; i.e., if all premisses are valid, then the conclusion is valid.*

The next results is more involved:

THEOREM 1 (Weak adequacy). *If $\imath$ is not in φ: $\models \varphi$ iff $\vdash_K \Rightarrow @_\imath \varphi$.*

Suitable completeness proof is provided by Bolander and Braüner [2006] (see also [Braüner, 2009]). However to justify our claim we should demonstrate that our sat-SC is indeed equivalent to their system. One should mention the following differences:

1. Their system is formulated in sequents build from finite sets not multisets. Moreover, they use axioms in general form $\Gamma \Rightarrow \Delta$ with non-empty intersection. In our system there are primitive structural rules of weakening and contraction and axioms in simple form. However, this is a well known fact that such solutions provide provably equivalent forms of SC.

2. Bolander and Braüner's rules $(\Diamond\Rightarrow)$, $(\mathrm{Nom}_1\Rightarrow)$, and $(\mathrm{Nom}_2\Rightarrow)$ are slightly different. One may easily show that their original rules are derivable in our system (the opposite also holds), hence our calculus is also complete. All differences in the shape of rules are forced by the technical demands of our proof of cut elimination theorem.

In order to cover stronger logics adequate with respect to restricted classes of frames one must add some special rules for frame conditions. It may be done in an uniform fashion for many logics by means of standard hybrid translation HT of atomic formulae of the first-order language for standard frames into atomic formulae of the basic hybrid language. Let us assume that individual variables of the former language are just nominals, and that its atomic formulae are of the form: $R\imath\jmath$ or $\imath = \jmath$. Suitable clauses of HT are the following:

$$\mathrm{HT}(R\imath\jmath) = @_\imath\Diamond\jmath$$
$$\mathrm{HT}(\imath = \jmath) = @_\imath\jmath$$

We are interested in frame conditions expressible by means of so called universal implications of the form $\forall\imath_1 \ldots \forall\imath_k(\alpha_1 \wedge \cdots \wedge \alpha_n \rightarrow \beta_1 \vee \cdots \vee \beta_m)$, where all α's and β's are atomic formulae. To each such universal implication, the general schema of SC rule is:

$$\frac{\Gamma \Rightarrow \Delta, \mathrm{HT}(\alpha_1) \quad \ldots \quad \Gamma \Rightarrow \Delta, \mathrm{HT}(\alpha_n)}{\Gamma \Rightarrow \Delta, \mathrm{HT}(\beta_1), \ldots, \mathrm{HT}(\beta_m)}$$

For example, transitivity is expressed by the rule:

$$(\mathrm{Tr}) \quad \frac{\Gamma \Rightarrow \Delta, @_\imath\Diamond\jmath \qquad \Gamma \Rightarrow \Delta, @_\jmath\Diamond\kappa}{\Gamma \Rightarrow \Delta, @_\imath\Diamond\kappa}$$

Although it is sufficient for most of well known and popular modal logic we may strengthen the scope of application of the method following Braüner. For every basic geometric formula of the form:

$$\forall\imath_1 \ldots \forall\imath_k(\alpha_1 \wedge \cdots \wedge \alpha_n \rightarrow \exists\jmath_1 \ldots \exists\jmath_l(\beta_1 \vee \cdots \vee \beta_m)),$$

where $k \geqslant 1$, $l, n, m \geqslant 0$, α's are atomic formulae and β's are either atomic formulae or finite conjunctions of atoms, there corresponds a rule of the following form:

$$\frac{\Gamma \Rightarrow \Delta, \mathrm{HT}(\alpha_1) \ \ldots \ \Gamma \Rightarrow \Delta, \mathrm{HT}(\alpha_n) \quad \Psi_1, \Gamma \Rightarrow \Delta \ \ldots \ \Psi_m, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

where no nominals of $\jmath_1, \ldots, \jmath_l$ occur in Γ, Δ, $\alpha_1, \ldots, \alpha_n$, and for each $i = 1, \ldots m$: Ψ_i is the set of HT-translations of atoms that form the conjunction β_i.

4. Cut elimination

At first it seems that the best solution to our problem is to apply Negri's approach [2005] based on Dragalin's strategy but defined in a way suitable for labelled SC. However, this approach requires all basic rules invertible and this property fails for some rules in SC-HL. That is why we have introduced contraction rules as primitive in our system instead of proving it as (height-preserving) admissible.

We apply the strategy of proof due to [Ciabattoni, Metcalfe, and Montagna, 2010] originally introduced in the framework of hypersequent calculi. As we will see it works well for standard SC as well. The nice thing with this method is that we need only one preliminary result for proving cut elimination (in contrast to Dragalin's strategy).

LEMMA 2 (Substitution). *If* $\vdash_{\mathrm{K}} \Gamma \Rightarrow \Delta$, *then* $\vdash_{\mathrm{K}} (\Gamma \Rightarrow \Delta)[\imath/\jmath]$.

PROOF. By induction on the height of a proof. It is straightforward but tedious exercise. Note that we provided not sheer admissibility but height-preserving admissibility. ⊣

Complexity of formulae is counted in the following way:

- Every nominal is of complexity 0 and every propositional variable of complexity 1.
- For every unary functor or binder (with argument) added to formula of complexity n we have $n + 1$.
- For every binary functor we have $n + m + 1$, where n and m are complexities of arguments.

Thus for every formula of complexity n a corresponding sat-formula has complexity $n+1$. In particular, the nominal atom $@_i j$ is of complexity 1, whereas the Boolean atom $@_i p$ is of complexity 2. Formulae $@_i \neg\varphi$,

$@_i\Diamond\varphi$, and $@_j@_i\varphi$ are of complexity $n+2$, if φ is of complexity n. Formulae $@_i(j \wedge k)$, $@_i(p \wedge j)$, $@_i(p \wedge q)$, and $@_i(@_jk \wedge @_ki)$ are of complexity 2, 3, 4, and 4, respectively.

Again the difference in the length of nominals and propositional variables is needed for the proof of cut elimination to go through induction on cut-degree. Let us define the notions of cut-degree and proof-degree:

1. Cut-degree is the complexity of cut-formula $@_\imath\varphi$ ($d@_\imath\varphi$),
2. Proof-degree is the maximal cut-degree in $\mathcal{D}$ ($d\mathcal{D}$).

Moreover, we assume that all proofs satisfy the condition of regularity: every fresh nominal is fresh in the entire proof. Note that every proof may be systematically transformed into regular proof by Substitution lemma.

The proof of cut elimination theorem is based on two lemmata which make a reduction first on the right and secondly on the left premiss of cut. The general strategy of proof is somewhat similar to Curry's proof [1963] of cut admissibility but simpler in some respects and still based rather on local transformations of proof instead of global ones characteristic for Curry's proof.

Lemma 3 (Right reduction). *Let* $\mathcal{D}_1 \vdash \Gamma \Rightarrow \Delta, @_\imath\varphi$, $\mathcal{D}_2 \vdash @_\imath\varphi^n, \Pi \Rightarrow \Sigma$, $d\mathcal{D}_1, d\mathcal{D}_2 < d@_\imath\varphi$, *and* $@_\imath\varphi$ *principal in* $\Gamma \Rightarrow \Delta, @_\imath\varphi$. *Then we can construct a proof* $\mathcal{D}$ *such that* $\mathcal{D} \vdash \Gamma^n, \Pi \Rightarrow \Delta^n, \Sigma$ *and* $d\mathcal{D} < d@_\imath\varphi$.

Proof. By induction on the height of $\mathcal{D}_2$. The basis is trivial. Induction step requires consideration of all cases of possible derivation of $@_\imath\varphi^n, \Pi \Rightarrow \Sigma$ and the role of cut-formula in the transition. In all cases where all occurrences of $@_\imath\varphi$ are parametric we simply apply the induction hypotheses to premisses of $@_\imath\varphi^n, \Pi \Rightarrow \Sigma$ and then apply to them respective rule — it is essentially due to the context independence of almost all rules and regularity of proofs. In case of troubles with side condition on fresh nominals we must first apply Substitution lemma. In the case one of the occurrence of $@_\imath\varphi$ in the premisse(s) is a side formula of the last rule we must additionally apply weakening to restore the lacking formula before the application of a rule.

Note that this situation covers also applications of rules Ref and Nom and rules for frame conditions since either all have active formulae in succedents or only in antecedents of premisses. In the latter case even if active formula is identical with $@_\imath\varphi$ we need only additional applications of contraction.

In cases where one occurrence of $@_\imath\varphi$ in $@_\imath\varphi^n, \Pi \Rightarrow \Sigma$ is principal we make use of the fact that $@_\imath\varphi$ in the left premiss is principal too (note that for C and W it is trivial).

Let us consider an example with $\varphi = \psi \wedge \chi$. By the induction hypothesis we get $\mathcal{D}'_2 \vdash @_\imath\psi, @_\imath\chi, \Gamma^{n-1}, \Pi \Rightarrow \Delta^{n-1}, \Sigma$ with $d\mathcal{D}'_2 < d@_\imath\varphi$. Then we continue with the premisses of $\Gamma \Rightarrow \Delta, @_\imath\varphi$:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, @_\imath\chi \qquad \dfrac{\Gamma \Rightarrow \Delta, @_\imath\psi \qquad @_\imath\psi, @_\imath\chi, \Gamma^{n-1}, \Pi \Rightarrow \Delta^{n-1}, \Sigma}{@_\imath\chi, \Gamma^n, \Pi \Rightarrow \Delta^n, \Sigma}\,(\text{Cut})}{\Gamma^{n+1}, \Pi \Rightarrow \Delta^{n+1}, \Sigma}\,(\text{Cut})}{\Gamma^n, \Pi \Rightarrow \Delta^n, \Sigma}\,(\text{C})$$

This new proof has obviously the degree lower than $d@_\imath\varphi$.

In case the last rule in $\mathcal{D}_2$ is $(\Diamond\Rightarrow)$ we have the following situation:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, @_\imath\Diamond\jmath \qquad \Gamma \Rightarrow \Delta, @_\jmath\varphi}{\Gamma \Rightarrow \Delta @_\imath\Diamond\varphi}\,(\Rightarrow\Diamond) \qquad \dfrac{@_\imath\Diamond\kappa, @_\kappa\varphi, @_\imath\Diamond\varphi^{n-1}\Pi \Rightarrow \Sigma}{@_\imath\Diamond\varphi^n, \Pi \Rightarrow \Sigma}\,(\Diamond\Rightarrow)}{\Gamma^n, \Pi \Rightarrow \Delta^n, \Sigma}\,(\text{Cut})$$

By the induction hypothesis and Substitution lemma (note that its application does not affect the left premiss since κ is fresh in it) we obtain $@_\imath\Diamond\jmath, @_\jmath\varphi, \Gamma^{n-1}, \Pi \Rightarrow \Delta^{n-1}, \Sigma$ and continue:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, @_\jmath\varphi \qquad \dfrac{\Gamma \Rightarrow \Delta, @_\imath\Diamond\jmath \qquad @_\imath\Diamond\jmath, @_\jmath\varphi, \Gamma^{n-1}, \Pi \Rightarrow \Delta^{n-1}, \Sigma}{@_\jmath\varphi, \Gamma^n, \Pi \Rightarrow \Delta^n, \Sigma}\,(\text{Cut})}{\Gamma^{n+1}, \Pi \Rightarrow \Delta^{n+1}, \Sigma}\,(\text{Cut})}{\Gamma^n, \Pi \Rightarrow \Delta^n, \Sigma}\,(\text{C})$$

A new proof has obviously the degree lower than $d@_\imath\varphi$ even if φ is a propositional variable. It cannot be a nominal since it is excluded by the restriction 2 on $(\Diamond\Rightarrow)$. Although it is allowed for $(\Rightarrow\Diamond)$ to have a nominal as an argument in principal formula it does not make harm since (all occurrences of) cut-formula on the right may be only parametric and this is tackled easily by the induction hypothesis. ⊣

We make an analogous transformation on the left premiss but this time we do not need to assume that cut-formula in the left premiss is principal.

Lemma 4 (Left reduction). *Let $\mathcal{D}_1 \vdash \Gamma \Rightarrow \Delta, @_\imath\varphi^n$, $\mathcal{D}_2 \vdash @_\imath\varphi, \Pi \Rightarrow \Sigma$, and $d\mathcal{D}_1, d\mathcal{D}_2 < d@_\imath\varphi$. Then we can construct a proof $\mathcal{D}$ such that $\mathcal{D} \vdash \Gamma, \Pi^n \Rightarrow \Delta, \Sigma^n$ and $d\mathcal{D} < d@_\imath\varphi$.*

Proof. Similarly, now by induction on the height of $\mathcal{D}_1$. Note that we do not require $@_\imath\varphi$ principal in $@_\imath\varphi, \Pi \Rightarrow \Sigma$. Now in cases where one occurrence of $@_\imath\varphi$ in $\Gamma \Rightarrow \Delta, @_\imath\varphi^n$ is principal we make use of Right reduction lemma. For example: let $\varphi = \psi \vee \chi$, then by the induction hypothesis we get $\mathcal{D}'_1 \vdash \Gamma, \Pi^{n-1} \Rightarrow \Delta, \Sigma^{n-1}, @_\imath\psi, @_\imath\chi$ with $d\mathcal{D}'_1 < d@_\imath\varphi$, and by $(\Rightarrow\vee)$ we get $\Gamma, \Pi^{n-1} \Rightarrow \Delta, \Sigma^{n-1}, @_\imath\varphi$. Since $@_\imath\varphi$ is principal in this sequent, the Right reduction lemma applies to it and to $\mathcal{D}_2$ and we obtain $\mathcal{D} \vdash \Gamma, \Pi^n \Rightarrow \Delta, \Sigma^n$ with $d\mathcal{D} < d@_\imath\varphi$.

Additionally we must consider now the cases of the application of rules for frame conditions being universal implications. Let us consider the situation with the application of (Tr). We have the following:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, @_\imath\Diamond\kappa^{n-1}, @_\imath\Diamond\jmath \qquad \Gamma \Rightarrow \Delta, @_\imath\Diamond\kappa^{n-1}, @_\jmath\Diamond\kappa}{\Gamma \Rightarrow \Delta, @_\imath\Diamond\kappa^n}\,(\mathrm{Tr}) \qquad @_\imath\Diamond\kappa, \Pi \Rightarrow \Sigma}{\Gamma, \Pi^n \Rightarrow \Delta, \Sigma^n}\,(\mathrm{Cut})$$

which by the induction hypothesis is transformed into

$$\dfrac{\dfrac{\Gamma, \Pi^{n-1} \Rightarrow \Delta, \Sigma^{n-1}, @_\imath\Diamond\jmath \qquad \Gamma, \Pi^{n-1} \Rightarrow \Delta, \Sigma^{n-1}, @_\jmath\Diamond\kappa}{\Gamma, \Pi^{n-1} \Rightarrow \Delta, \Sigma^{n-1}, @_\imath\Diamond\kappa}\,(\mathrm{Tr}) \qquad @_\imath\Diamond\kappa, \Pi \Rightarrow \Sigma}{\Gamma, \Pi^n \Rightarrow \Delta, \Sigma^n}\,(\mathrm{Cut})$$

where the last application of cut is justified by the Right reduction lemma. ⊣

Now we can prove:

Theorem 2. *Every proof may be transformed into cut-free proof.*

Proof. By double induction: primary on d and subsidiary on the number of maximal cuts (in the basis and in the inductive step of the primary induction; in the basis cut elimination is trivial, in the inductive step we refer to Left reduction lemma). We always take the topmost maximal cut, hence we have the following situation:

$$(\mathrm{Cut})\ \dfrac{\Gamma \Rightarrow \Delta, @_\imath\varphi \qquad @_\imath\varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

If $@_\imath\varphi$ is not atomic, then by Left reduction lemma we obtain a proof of $\Gamma, \Pi^k \Rightarrow \Delta_{@_\imath\varphi}, \Sigma^k$ of lower degree. This, by applications of $(\Rightarrow\mathrm{W})$ and (C) (if necessary) yields $\Gamma, \Pi \Rightarrow \Delta, \Sigma$. By successive repetition of this procedure we diminish either the degree of a proof or the number of maximal cuts in it. ⊣

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