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MEREOLOGICAL FOUNDATIONS OF POINT-FREE GEOMETRY VIA MULTI-VALUED LOGIC

Abstract. We suggest possible approaches to point-free geometry based on multi-valued logic. The idea is to assume as primitives the notion of *a region* together with suitable vague predicates whose meaning is geometrical in nature, e.g. 'close', 'small', 'contained'. Accordingly, some first-order multi-valued theories are proposed. We show that, given a multi-valued model of one of these theories, by a suitable definition of *point* and *distance* we can construct a metrical space in a natural way. Taking into account that interesting metrical approaches to geometry exist, this looks to be promising for a point-free foundation of the notion of *space*. We hope also that this way to face point-free geometry provides a tool to illustrate the passage from a naïve and 'qualitative' approach to geometry to the 'quantitative' approach of advanced science.

Keywords: point-free geometry; multi-valued logic; fuzzy logic; continuous logic; metric geometry; mereology; naïve science

1. Introduction

Łukasiewicz's many-valued logic (see [19]), Chang and Keisler's continuous logic [6], and Pavelka's fuzzy logic [21] all form a very interesting chapter of formal logic. Recently, under the name 'continuous logic', these work has been re-examined with a view to extending model theory to important classes of structures that cannot be defined in classical first-order logic [30], namely, all the structures assuming as a primitive

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a real-valued function (metric spaces, measure spaces, normed spaces and probabilities are typical examples). The idea is that it is possible to reinterpret the real numbers as truth values and the real-valued functions as vague predicates in a first-order multi-valued logic.

In this paper we investigate the possibility of applying this idea to point-free geometry (see also [8, 9]). The starting point is proposals for a metric-based point-free geometry already existing in literature ([10, 11, 22, 23, 17, 14, 15, 16]). In each of these proposals, together with the inclusion relation, distances and diameters are also considered and a system of axioms T is proposed. Moreover, it can be shown that, given a model of T, it is possible to give a suitable definition of *point* and *distance* and thence to obtain a metric space. Then, in accordance with the ideas of the metrical approaches to geometry (see [2]), it is possible to define the notion of *an alignment of points* and therefore all the basic notions of geometry. Notice that these approaches are in some way connected with the one presented by Tarski in [25] in which one assumes the notion of *ball* as a primitive while the points and the relation of *equidistance* are defined (see also [18]).

The next step, in accordance with continuous logic, is to show that it is possible to associate with each theory T based on real-valued functions a theory T^* in a multi-valued logic based on vague predicates that are geometrical in nature (such as 'small', 'close', 'contained in', etc.). By a sort of duality principle every model of T^* is associated with a model of T and therefore with a metric space. This gives a basis for the foundation of Euclidean geometry.

We emphasize that one of the motivations is to give a mathematical model of the transition from the naïve predicate-based theory of space, which is qualitative in nature, to the modern real-number-based theory, quantitative in nature. In this sense, fuzzy logic seems a significant tool for the analysis of the *pre-theoretic scientific beliefs* of ordinary man (see [3, 24, 20, 5]). In turn, we are convinced that our research could be useful when it comes to understanding the scientific view of the world children have and how best to teach them science.

2. Preliminaries: algebra of the truth values

We consider multi-valued logics in which the set of truth values is the real interval $[0,1]$ and the conjunction connective is interpreted by a continuous triangular norm.

Definition 1. A *continuous triangular norm* (briefly *t-norm*) is a continuous binary operation $\otimes$ on $[0,1]$ such that, for all x, y, $z \in [0,1]$:

- $x \otimes y = y \otimes x$ (commutativity)
- $(x \otimes y) \otimes z = x \otimes (y \otimes z)$ (associativity)
- $x \leqslant y \Rightarrow x \otimes z \leqslant y \otimes z$ (isotonicity)
- $1 \otimes x = x$ and $0 \otimes x = 0$ (boundary conditions).

Once a t-norm is fixed, we are able to define a further operation to interpret the implication $\Rightarrow$.

Definition 2. Given a t-norm, the *residuation* associated with $\otimes$ is the operation $\rightarrow$ defined by

$$x \rightarrow y := \sup\{z : x \otimes z \leqslant y\}.$$

The following proposition summarizes the main properties of $\rightarrow$.

Proposition 1. *If $\otimes$ is a t-norm and $\rightarrow$ the associated residuation, then for all $x, y, z \in [0,1]$:*

(i) $x \otimes z \leqslant y \iff z \leqslant x \rightarrow y$,
(ii) $(x \rightarrow y) \otimes (y \rightarrow z) \leqslant x \rightarrow z$,
(iii) $x \rightarrow y = 1$ *and* $y \rightarrow x = 1 \Rightarrow x = y$,
(iv) $x \rightarrow y = 1 \iff x \leqslant y$,
(v) $(z \rightarrow y) \otimes z \leqslant y$.

Important examples of continuous t-norms are:

- *Gödel t-norm*: $x \otimes y := \min\{x, y\}$,
- *Goguen t-norm*: $x \otimes y := x \cdot y$ (usual product of real numbers),
- *Łukasiewicz t-norm*: $x \otimes y := \max\{0, x + y - 1\}$.

The corresponding residuations are defined by setting $x \rightarrow y := 1$, if $x \leqslant y$ and, otherwise:

- *Gödel residuation*: $x \rightarrow y := y$,
- *Goguen residuation*: $x \rightarrow y := \frac{y}{x}$,
- *Łukasiewicz residuation*: $x \rightarrow y := x + y - 1$.

We are interested in a particular class of continuous t-norms, the *Archimedean t-norms*.

Definition 3. A continuous t-norm $\otimes$ is called *Archimedean* if for any $x, y \in [0,1]$, $y \neq 0$, there is a natural number n such that $x^{(n)} < y$, where $x^{(n)}$ is defined by: $x^{(1)} := x$ and $x^{(n+1)} := x^{(n)} \otimes x$.

The usual product and Łukasiewicz t-norm are examples of Archimedean continuous t-norms, while the minimum is an example of continuous t-norm that is not Archimedean. There is a general way to obtain a continuous Archimedean norm which is based on the notion of continuous generator.

Definition 4. An *additive generator* is a continuous strictly decreasing function $f\colon [0,1] \to [0,\infty]$ such that $f(1) = 0$. The *pseudoinverse* $f^{[-1]}\colon [0,\infty] \to [0,1]$ of f is defined by setting:

$$f^{[-1]}(y) := \begin{cases} f^{-1}(y) & \text{if } y \in f([0,1]) \\ 0 & \text{otherwise.} \end{cases}$$

We list some properties of the pseudoinverse whose proofs are trivial.

Proposition 2. *Let f be an additive generator. Then*:

(i) $f^{[-1]}$ *is order-reversing,*
(ii) $f^{[-1]}(0) = 1$ *and* $f^{[-1]}(\infty) = 0$,
(iii) $f([0,1]) = [0, f(0)]$,
(iv) $f^{[-1]}(f(x)) = x$, *for any* $x \in [0,1]$,
(v) $f(f^{[-1]}(x)) = f(0)$, *if* $x \leqslant f(0)$,
(vi) $f(f^{[-1]}(x)) = f(0)$, *if* $x > f(0)$,
(vii) $f(f^{[-1]}(x)) \leqslant x$.

Definition 5. Let $f\colon [0,1] \to [0,\infty]$ be an additive generator and define the operation $\otimes$ by setting for all $x, y \in [0,1]$:

$$x \otimes y := f^{[-1]}(f(x) + f(y)). \tag{$\otimes$}$$

Then we say that f is an *additive generator* of $\otimes$.

Proposition 3. *An operation $\otimes$ is a continuous Archimedean t-norm iff it is has an additive generator. In such case the residuation is defined for all $x, y \in [0,1]$ by:*

$$x \to y := f^{[-1]}(f(y) - f(x)).$$

For example, the additive generator of the Goguen t-norm is $f(x) := -\ln(x)$ (where ln is the natural logarithm) and the additive generator of the Łukasiewicz t-norm is $f(x) := 1 - x$.

3. Preliminaries: first-order multi-valued logic

The languages of the first-order multi-valued logic we will consider contain:

- the logical connectives: $\wedge$, $\Rightarrow$, Ct,
- the quantifiers: $\forall$, $\exists$,
- two logical constants: $\underline{0}$, $\underline{1}$,
- predicate symbols,
- constant and operation symbols.

We interpret the logical connectives '$\wedge$' and '$\Rightarrow$' by a t-norm and the related residuum, and the logical connective 'Ct' by the function $ct\colon [0,1] \to [0,1]$ defined by setting $ct(x) := 1$, if $x = 1$, and $ct(x) := 0$, otherwise. Given a formula α, the intended meaning of $\mathrm{Ct}(\alpha)$ is that α is completely true. The quantifiers '$\forall$' and '$\exists$' are interpreted as the greatest lower bound and the least upper bound, respectively. The logical constants '$\underline{0}$' and '$\underline{1}$' as the truth values 0 and 1. Given a non-empty set D, an *n-ary fuzzy relation* in D is a map $r\colon D^n \to [0,1]$. We call *crisp* a fuzzy relation whose only values are 0 and 1, and we identify a classical relation $\mathcal{R} \subseteq D^n$ with the crisp relation $c_{\mathcal{R}}\colon D^n \to [0,1]$ defined by setting $c_{\mathcal{R}}(d_1, \ldots, d_n) := 1$, if $(d_1, \ldots, d_n) \in \mathcal{R}$, and $c_{\mathcal{R}}(d_1, \ldots, d_n) := 0$, otherwise. In other words, we can identify $\mathcal{R}$ with its characteristic function $c_{\mathcal{R}}$.

Definition 6. A *multi-valued interpretation* (D, I) of a multi-valued logic consists of a nonempty domain D and a function I associating every constant c with an element $I(c) \in D$, every n-ary operation symbol with an n-ary operation in D, and every n-ary relation symbol $\underline{r}$ with an n-ary fuzzy relation $r = I(\underline{r})$, i.e. a map $r\colon D^n \to [0,1]$.

Given a multi-valued interpretation (D, I), the interpretation $I(t)\colon D^n \to D$ of a term t is defined as in classical logic. The valuation of a sentence is defined in a truth-functional way as follows (if $\underline{\bullet}$ is a unary connective, $\bullet\colon [0,1] \to [0,1]$ denotes its interpretation, and similarly for binary connectives).

Definition 7. Given a multi-valued interpretation (D, I), a formula α whose variables are among $x_1, \ldots, x_n$, and $d_1, \ldots, d_n$ in D, the value $\mathrm{Val}(\alpha, d_1, \ldots, d_n)$ is defined by recursion on the complexity of α:

$$\mathrm{Val}(\underline{0}, d_1, \ldots, d_n) := 0 \text{ and } \mathrm{Val}(\underline{1}, d_1, \ldots, d_n) := 1,$$
$$\mathrm{Val}(\underline{r}(t_1, \ldots, t_p), d_1, \ldots, d_n) := I(\underline{r})(I(t_1)(d_1, \ldots, d_n), \ldots, I(t_p)(d_1, \ldots, d_n)),$$

$$\mathrm{Val}(\alpha_1 \underline{\diamond} \alpha_2, d_1, \ldots, d_n) := \mathrm{Val}(\alpha_1, d_1, \ldots, d_n) \diamond \mathrm{Val}(\alpha_q, d_1, \ldots, d_n),$$
$$\mathrm{Val}(\underline{\bullet}\alpha, d_1, \ldots, d_n) := \bullet(\mathrm{Val}(\alpha, d_1, \ldots, d_n)),$$
$$\mathrm{Val}(\forall x_h \beta, d_1, \ldots, d_n) := \inf\{\mathrm{Val}(\beta, d_1, \ldots, d_{h-1}, d, d_{h+1}, \ldots, d_n) : d \in D\},$$
$$\mathrm{Val}(\exists x_h \beta, d_1, \ldots, d_n) := \sup\{\mathrm{Val}(\beta, d_1, \ldots, d_{h-1}, d, d_{h+1}, \ldots, d_n) : d \in D\}.$$

If α is a closed formula, then $\mathrm{Val}(\alpha, d_1, \ldots, d_n)$ does not depend on the elements $d_1, \ldots, d_n$ and we simply write $\mathrm{Val}(\alpha)$. In case there are free variables in α, we write $\mathrm{Val}(\alpha)$ to denote $\mathrm{Val}(\forall x_1 \ldots \forall x_n(\alpha))$, where $\forall x_1 \ldots \forall x_n(\alpha)$ is the universal closure of α.

Definition 8. Given a multi-valued interpretation (D, I), we say that a formula α *is satisfied by* (D, I) if $\mathrm{Val}(\alpha) = 1$. Given a theory T, i.e. a set of formulas, if every formula in T is satisfied by (D, I) we say that (D, I) *is a multi-valued model of* T.

The multi-valued logic that we have just defined is quite expressive. For example, if $\underline{r}$ is an n-ary relation symbol, then the formula:

$$\forall x_1 \ldots \forall x_n(\mathrm{Ct}(\underline{r}(x_1, \ldots, x_n)) \Leftrightarrow \underline{r}(x_1, \ldots, x_n))$$

is satisfied if and only if $\underline{r}$ is interpreted by a crisp relation. Indeed, it is sufficient to observe that this formula is satisfied if and only if $ct(r(d_1, \ldots, d_n)) = r(d_1, \ldots, d_n)$, for all $d_1, \ldots, d_n$ in D. In other words, ‘to be crisp’ is a first-order property of the multi-valued logic. Accordingly, every classical notion we can define in first-order classical logic is definable also in the multi-valued logic. In particular, we can give the following definition.

Definition 9. Given a language with the relation symbol $\leq$, we denote by ‘Order($\leq$)’ the claim that the interpretation of $\leq$ is a crisp order relation.

4. Mereometrology and multi-valued logic: a general schema

The first two steps toward point-free geometry are:

- *mereology*: the theory whose only primitive notion is the binary “part of” relation,
- *mereotopology*: based on the binary “part of” relation and some additional notions topological in nature.

These two steps suggest the possibility of a further step we call *mereometrology*, its objective being the founding of a point-free geometry on notions which are metrical in nature. Several explorations of this area have been made in the literature. In this paper we will examine them and introduce some minor modifications in order to emphasize their common ideas and to unify the language.

The basic schema of the aforementioned theories may be summarized in the following way.

1. One considers a theory T whose intended models are ordered sets with elements called *regions* and the order called *inclusion*. This theory involves further primitives that allow for the definitions of the notions of *the diameter* of a region and *the distance* between two regions. T is obtained by isolating some significant properties of a prototypical model. Usually this model is defined in a subclass Re of the class RC of regular closed subsets of the Euclidean space. Recall that a closed subset is called *regular* if it is equal to the closure of its interior and that in the Euclidean case RC is a complete and atomless Boolean algebra. In this paper we set Re equal to the class of nonempty and bounded elements of RC.

Equivalently, we can consider the class of open regular subsets, i.e., the subsets which coincide with the interior of their closure.

2. Given a model of T, AP denotes the class of the order-reversing sequences $\langle p_n \rangle_{n\in\mathbb{N}}$ of regions such that

$$\lim_{n\to\infty} |p_n| = 0,$$

where $|x|$ is the diameter of a region x. We call *abstraction processes* the elements in AP.

3. Next, in AP the function $d\colon \mathsf{AP}\times\mathsf{AP} \to [0,\infty)$ is defined by putting

$$d(\langle p_n \rangle_{n\in\mathbb{N}}, \langle q_n \rangle_{n\in\mathbb{N}}) := \lim_{n\to\infty} \delta(p_n, q_n), \tag{$\dagger$}$$

where $\delta(x, y)$ is the distance between regions x and y.

4. One proves that the pair (AP, d) is a pseudo-metric space and therefore its *quotient* $(\underline{\mathsf{AP}}, \underline{d})$ is a metric space. Then every model of T is associated with a metric space $(\underline{\mathsf{AP}}, \underline{d})$ such that:

- elements of $\underline{\mathsf{AP}}$ (*points*) are complete equivalence classes $[\langle p_n \rangle_{n\in\mathbb{N}}]$,
- the *distance* between two points $[\langle p_n \rangle_{n\in\mathbb{N}}]$ and $[\langle q_n \rangle_{n\in\mathbb{N}}]$ is defined by setting

$$\underline{d}([\langle p_n \rangle_{n\in\mathbb{N}}], [\langle q_n \rangle_{n\in\mathbb{N}}]) := \lim_{n\to\infty} \delta(p_n, q_n). \tag{$\ddagger$}$$

The fact that every model of T is associated with a metric space is important. Indeed if we add to T any system of axioms for a point-based foundation of Euclidean geometry metrical in nature (see e.g. [2]), then we obtain a metric point-free foundation for Euclidean geometry. Obviously, in this system points and distance are not primitive but defined notions.

There is no difficulty connecting multi-valued logic with mereometrology in accordance with the ideas of continuous logic. This is done by using an additive generator $f\colon [0,1] \to [0,\infty]$ to establish a sort of duality.

5. We show that for every theory T in mereometrology there is a theory $\underline{T}$ in first-order multi-valued logic such that every model of $\underline{T}$ is transformed into a model of T via the function f. As a consequence, every model of $\underline{T}$ is associated with a metric space. $\underline{T}$ involves vague predicate for regions metrical in nature.

Observe that there is a different way to define the points and therefore the associated metric space. Call *Cauchy sequence* a sequence $\langle p_n \rangle_{n\in\mathbb{N}}$ of regions such that:

$$\lim_{n\to\infty} |p_n| = 0 \quad \text{and} \quad (\forall \epsilon > 0)(\exists m \in \mathbb{N})(\forall h, k \geq m)\ d(p_h, p_k) < \epsilon\,.$$

Denote by CS the class of Cauchy sequences and define d as in (†). Then one obtains a pseudo metric space (CS, d) and therefore a metric space $(\underline{\mathsf{CS}}, \underline{d})$. It can be proven that every representing sequence is a Cauchy sequence and therefore that (CS, d) is an extension of (AP, d). An important fact is that $(\underline{\mathsf{CS}}, \underline{d})$ is the completion of $(\underline{\mathsf{AP}}, \underline{d})$ (see [10]).

5. Point-free geometry based on closeness and smallness

The first example of the metrical approach we will consider is obtained by assuming as primitives *region*, *inclusion*, *distance* and *diameter*. The *prototypical model* is defined in the class Re, where two functions $\delta\colon \mathsf{Re}\times\mathsf{Re} \to [0,\infty)$ and $|\cdot| : \mathsf{Re} \to [0,\infty)$ are defined by setting for all $x, y \in \mathsf{Re}$:

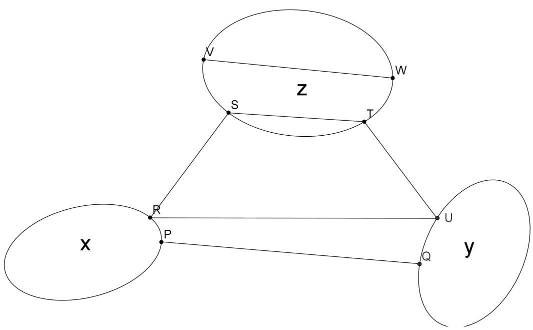
$$\delta(x,y) := \inf\{d(P,Q) : P \in x, Q \in y\},$$
$$|x| := \sup\{d(P,Q) : P, Q \in x\}.$$

It is immediate that δ is order-reversing, $|\cdot|$ is order-preserving, $\delta(x,x) = 0$, and $\delta(x,y) = \delta(y,x)$, for all $x, y \in \mathsf{Re}$. A more interesting property is the following *generalized triangle inequality*, for all $x, y, z \in \mathsf{Re}$:

$$\delta(x,y) \leq \delta(x,z) + \delta(z,y) + |z|\ .$$

We can prove it by observing that it is not restrictive to assume that sets x, y, and z are closed and therefore that there are points P, Q, R, S, V, W, T, and U such that:

$$\underline{PQ} = \delta(x,y), \quad \underline{RS} = \delta(x,z), \quad \underline{VW} = |z|\,, \quad \underline{TU} = \delta(z,y)\,,$$
$$\underline{PQ} \leq \underline{RU} \leq \underline{RS} + \underline{ST} + \underline{TU} \leq \underline{RS} + \underline{VW} + \underline{TU}\,.$$



Finally, taking into account the properties of the regular subsets of a metric space, we also have that:

$$\forall x \forall n \exists z (z \leq x \wedge |z| \leqslant \tfrac{1}{n}).$$

This prototypical structure suggests the following definition.

Definition 10. A *pointless pseudo-metric space* (briefly: ppm-space) is a structure $(\mathcal{R}, \leq, \delta, |\cdot|)$, such that $(\mathcal{R}, \leq)$ is an ordered set, $\delta \colon \mathcal{R} \times \mathcal{R} \to [0, \infty)$ is order-reversing, $|\cdot| : \mathcal{R} \to [0, \infty]$ is order-preserving, and for all $x, y, z \in \mathcal{R}$ the following axioms hold:

$$\delta(x,x) = 0 \tag{ppm1}$$
$$\delta(x,y) = \delta(y,x) \tag{ppm2}$$
$$\delta(x,y) \leqslant \delta(x,z) + \delta(z,y) + |z| \tag{ppm3}$$
$$(\forall n \in \mathbb{N})(\exists z \leq x)\ \ |z| \leqslant \tfrac{1}{n}\,. \tag{ppm4}$$

The elements of $\mathcal{R}$ are called *regions*; the relation $\leq$ — *inclusion*; the number $\delta(x,y)$ — *distance* between the regions x and y; the number $|x|$ — the *diameter* of x. A region x is *bounded* if its diameter $|x|$ is finite.

It is easy to control that the defined prototypical structure is a ppm-space.

PROPOSITION 4. *For all $x, y \in \mathcal{R}$, if there is $z \in \mathcal{R}$ such that $z \leq x$ and $z \leq y$, then $\delta(x, y) = 0$. Consequently, if there is a minimum in $\mathcal{R}$, then δ is constantly equal to* 0.

PROOF. If $z \leq x$, then $\delta(z, x) \leq \delta(z, z) = 0$. So, if $z \leq x$ and $z \leq y$ we have that $\delta(x, y) \leq \delta(x, z) + \delta(z, y) + |z| = |z|$. With (ppm4) this entails that $\delta(x, y) = 0$. □

In accordance with the above proposition in the following we assume that no minimum in $\mathcal{R}$ exists. Recall that AP is the class of abstraction processes and $d \colon \mathsf{AP} \times \mathsf{AP} \to [0, \infty)$ is defined by (†).

Before stating the next theorem we introduce the notion of *a pseudo-metric space.*

DEFINITION 11. *A pseudo-metric space* is a structure (R, δ) such that R is a non-empty set and $\delta : R \times R \to [0, \infty)$ is a mapping such that, for all $x, y, z \in R$:

$$\delta(x, x) = 0 \tag{d1}$$

$$\delta(x, y) = \delta(y, x) \tag{d2}$$

$$\delta(x, y) \leq \delta(x, z) + \delta(z, y)\,. \tag{d3}$$

THEOREM 1. (AP, d) *is a pseudo-metric space.*

PROOF. To prove that AP is non-empty we observe that, by (ppm4), for any region x we can define an abstraction process $\langle p_n \rangle_{n \in \mathbb{N}}$ by setting $p_1 = x$ and p_n equal to some region which is contained in p_{n-1} and such that $|p_n| \leq 1/n$. Also, the existence of a finite limit in (†) stems from the fact that the sequence $\langle \delta(p_n, q_n) \rangle_{n \in \mathbb{N}}$ is order-preserving and:

$$\begin{aligned} \delta(p_n, q_n) &\leqslant \delta(p_n, p_1) + \delta(p_1, q_1) + \delta(q_1, q_n) + |p_1| + |q_1| \\ &= \delta(p_1, q_1) + |p_1| + |q_1|\,. \end{aligned}$$

It remains to prove (d1), (d2) and (d3). Now, (d1) and (d2) are evident. To prove (d3) we observe that for any abstraction processes $\langle p_n \rangle_{n \in \mathbb{N}}$, $\langle q_n \rangle_{n \in \mathbb{N}}$, and $\langle r_n \rangle_{n \in \mathbb{N}}$:

$$\begin{aligned} & d(\langle p_n \rangle_{n \in \mathbb{N}}, \langle q_n \rangle_{n \in \mathbb{N}}) \\ &\quad = \lim_{n \to \infty} \delta(p_n, q_n) \leqslant \lim_{n \to \infty} \delta(p_n, r_n) + \delta(r_n, q_n) + |r_n| \\ &\quad = \lim_{n \to \infty} \delta(p_n, r_n) + \lim_{n \to \infty} \delta(r_n, q_n) + \lim_{n \to \infty} |r_n| \\ &\quad = d(\langle p_n \rangle_{n \in \mathbb{N}}, \langle r_n \rangle_{n \in \mathbb{N}}) + d(\langle r_n \rangle_{n \in \mathbb{N}}, \langle q_n \rangle_{n \in \mathbb{N}}). \end{aligned}$$

□

Definition 12. By *a metric space associated with* $(\mathcal{R}, \leq, \delta, |\cdot|)$ we understand the quotient $(\underline{\mathsf{AP}}, \underline{d})$ of (AP, d). By *a point* we mean every element in $\underline{\mathsf{AP}}$.

Then a point of $(\mathcal{R}, \leq, \delta, |\cdot|)$ is a complete equivalence class $[\langle p_n \rangle_{n \in \mathbb{N}}]$ and the distance between two points is defined by (‡).

We are now ready to transform the metrical approach to point-free geometry furnished by the ppm-spaces into a multi-valued approach.

Definition 13. Consider a language with three predicate symbols '$\leq$', 'Close', and 'Small'. Then we call a *point-free theory based on closeness and smallness* (in brief *c-s-theory*) the following theory:

(O)	$\mathrm{Order}(\leq)$
(S1)	$\forall x \forall y (x \leq y \ \wedge\ \mathrm{Small}(y) \ \Rightarrow\ \mathrm{Small}(x))$
(S2)	$\forall x \exists z (z \leq x \ \wedge\ \mathrm{Small}(z))$
(C1)	$\forall x \forall y (x \leq y \ \wedge\ \mathrm{Close}(x, z) \ \Rightarrow\ \mathrm{Close}(y, z)))$
(C2)	$\forall x\ \mathrm{Close}(x, x)$
(C3)	$\forall x \forall y (\mathrm{Close}(x, y) \Rightarrow \mathrm{Close}(y, x))$
(C4)	$\forall x \forall y \forall z (\mathrm{Close}(x, z) \wedge \mathrm{Close}(z, y) \wedge \mathrm{Small}(z) \Rightarrow \mathrm{Close}(x, y))$

We call a *c-s-structure* a model of this theory.

Notice that (C4) claims that 'Close' is a transitive relation as long as we consider only small regions. In [13] this system of axioms is used to give a solution of Poincaré's paradox of indiscernibility.

A *c-s-structure* is a quadruple $(\mathcal{R}, \leq, \mathit{close}, \mathit{small})$ such that $\leq$ is an order relation, *close* is order-preserving, *small* is order-reversing and:

- $\mathit{close}(x, x) = 1$,
- $\mathit{close}(x, y) = \mathit{close}(x, y)$,
- $(\mathit{close}(x, z) \otimes \mathit{close}(z, y)) \otimes \mathit{small}(z) \leqslant \mathit{close}(x, y)$[1],
- for every $x \in \mathcal{R}$ and $n \in \mathbb{N}$ there is $z \leq x$ such that $\mathit{small}(z) \geqslant 1 - \frac{1}{n}$.

Theorem 2. *Let $\otimes$ be any Archimedean t-norm. Then every c-s-structure is associated with some ppm-space.*

Proof. Let $f\colon [0, 1] \to [0, \infty]$ be an additive generator of $\otimes$ and define δ and $|\cdot|$ by setting:

$$\delta(x, y) := f(\mathit{close}(x, y)) \text{ and } |x| := f(\mathit{small}(x)).$$

[1] Recall that $\otimes$ is a continuous t-norm (see Definition 1).

Then it is evident that δ is order-reversing, $|\cdot|$ is order-preserving, and that (ppm1) and (ppm2) are satisfied. To prove (ppm3), we observe that, applying ($\otimes$) to (C4): $f^{[-1]}[f(f^{[-1]}(f(close(x,z)+f(close(z,y))))+f(small(z))]\leqslant close(x,y)$. So: $f(f^{[-1]}(f(close(x,z)+f(close(z,y))))+f(small(z))\geqslant f(close(x,y))$, i.e.: $f(f^{[-1]}(\delta(x,z)+\delta(z,y)))+|z|\geqslant\delta(x,y)$. Thus, by Proposition 2(vii), we obtain (ppm3).

Finally, to prove (ppm4), we observe that given $x\in\mathcal{R}$, by (S2), $\sup\{small(r):r\leq x\}=1$. Since $f^{[-1]}(\frac{1}{n})<f^{[-1]}(0)=1$, we have that for every $n\in\mathbb{N}$ there is $r\leq x$ such that $small(r)\geq f^{[-1]}(\frac{1}{n})$ and therefore such that $|r_n|\leqslant f(f^{[-1]}(\frac{1}{n}))\leqslant\frac{1}{n}$. □

Corollary 1. *Let $\otimes$ be any Archimedean t-norm. Then every c-s-structure is associated with some metric space.*

Proof. It is sufficient to associate the c-s-structure with the related ppm-space and such ppm-space with the related metric space. □

In light of this, in a metric space associated with a c-s-structure $(\mathcal{R},\leq, close, small)$, *a point* is a complete equivalence class defined by an order-reversing sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ of regions such that $\lim_{n\to\infty} small(x_n)=1$. The *distance between two points* $[\langle x_n\rangle_{n\in\mathbb{N}}]$ and $[\langle y_n\rangle_{n\in\mathbb{N}}]$ is defined by setting $d([\langle x_n\rangle_{n\in\mathbb{N}}],[\langle y_n\rangle_{n\in\mathbb{N}}])=f(\lim_{n\to\infty} close(x_n,y_n))$.

6. Point-free geometry based on smallness

In the literature there are also metric approaches to point-free geometry based only on the notion of diameter (see e.g. [22, 23, 1, 16]). In this section we consider the system proposed in [16] where in a partially ordered set we define the *overlapping relation* O by setting xOy iff there is an element z which is not the minimum and such that $z\leq x$ and $z\leq y$.

Definition 14. A *diametric poset* is a structure $(\mathcal{R},\leq,|\cdot|)$, where $(\mathcal{R},\leq)$ is a poset without a minimum and the $|\cdot|:\mathcal{R}\to[0,\infty]$ is an order-preserving *diameter function* such that:

(D_1) xOy entails that there is r such that $x\leq r$, $y\leq r$ and $|r|\leq|x|+|y|$,
(D_2) for each x and y there is a bounded region z such that zOx and zOy,
(D_3) given x, for every $n>0$ there is $z\leq x$ such that $|z|\leq\frac{1}{n}$.

As in Section 3, we call *regions* elements of $\mathcal{R}$; *inclusion* the relation $\leq$.

The property expressed by (D_1) extends to a finite number of regions.

PROPOSITION 5. *If a_1, a_2, ..., a_n are regions such that $a_1 O a_2$, ..., $a_{n-1} O a_n$, then there is a region r including all a_1, ..., a_n and such that*

$$|r| \leqslant |a_1| + \cdots + |a_n|\,.$$

PROOF. We prove this proposition by induction on n. In the case $n = 1$ it is sufficient to put $r = a_1$. Assume $n > 1$ and that $a_1 O a_2$, ..., $a_{n-1} O a_n$. Then by the induction hypothesis there is an r_n such that:

$$a_1 \leqslant r_n, \ldots, a_{n-1} \leqslant r_n \text{ and } |r_n| \leqslant |a_1| + \cdots + |a_{n-1}|\,.$$

Since $a_{n-1} \leqslant r_n$ entails $r_n O a_n$, by (D_1), an upper bound r of both r_n and a_n exists such that $|r| \leqslant |r_n| + |a_n|$. Hence r is an upper bound of a_1, ..., a_n such that $|r| \leqslant |a_1| + \cdots + |a_n|$. □

We now define a notion of *lower distance* between two regions x and y which is based on the idea of the length of a "bridge" z between x and y.

DEFINITION 15. We call *a lower distance* the function $\delta : \mathcal{R}^2 \to [0, \infty)$ defined by:

$$\delta(x, y) := \inf\{|z| : zOx \text{ and } zOy\}\,.$$

THEOREM 3. *Let $(\mathcal{R}, \leq, |\cdot|)$ be any diametric poset. Then the structure $(\mathcal{R}, \leq, \delta, |\cdot|)$ is a ppm-space. So, it is possible to associate every diametric poset with a metric space.*

PROOF. To prove (ppm1) we observe that $\delta(x, x) \leqslant \inf\{|z| : z \leq x\}$ and we apply (D_3). To prove (ppm3), assume that x, y and z are regions. Then if the diameter of z is infinite (ppm3) is evident. Otherwise, let u be a region such that uOx, uOz and v be a region such that vOz and vOy. Since uOz and vOz, a region r exists such that $r \geq u$, $r \geq v$ and $r \geq z$ and $|r| \leqslant |u| + |v| + |z|$. Since rOx and rOy, we have that $\delta(x, y) \leqslant |r| \leqslant |u| + |v| + |z|$. Thus, $\delta(x, y) \leqslant \inf\{|u| : uOx, uOz\} + \inf\{|v| : vOz, vOy\} + |z| = \delta(x, z) + \delta(z, y) + |z|$. It is evident that δ assumes finite values and that δ is order-reversing. □

The following first-order theory is obtained by adding a suitable axiom to the axioms concerning $\leq$ and Small in Definition 13.

DEFINITION 16. Consider a first-order language with the predicate symbols '$\leq$' and 'Small'. Then *a point-free theory based on smallness* (in brief: *s-theory*) is the theory whose axioms are (O), (S1), (S2), and:

(S3) $xOy \Rightarrow \exists r(x \leq r \wedge y \leq r \wedge \mathrm{Ct}(\mathrm{Small}(x) \wedge \mathrm{Small}(y) \Rightarrow \mathrm{Small}(r)))$

By an *s-structure* we mean a model of this theory.

Theorem 4. *If $\otimes$ is an Archimedean t-norm, then every s-structure is associated with a diametric poset and therefore with a metric space.*

Proof. Let $(\mathcal{R}, \leq,)$ be an s-structure and f be a continuous generator of $\otimes$. If we put $|x| := f(small(x))$, then $|\cdot|$ is an order-preserving function. To prove that $|\cdot|$ satisfies (D_1) assume that xOy. Then, by (S3), the formula $\exists r(x \leq r \wedge y \leq r \wedge \mathrm{Ct}(\mathrm{Small}(x) \wedge \mathrm{Small}(y) \Rightarrow \mathrm{Small}(r)))$ assumes the value 1 and therefore, given x and y in $\mathcal{R}$:

$$\sup\{ct(small(x) \otimes small(y) \to small(r)) : x \leq r \text{ and } y \leq r\} = 1.$$

This entails that there is r such that $r \geq x$ and $r \geq y$ and $small(x) \otimes small(y) \leq small(r)$, i.e., $f^{[-1]}(f(small(x)) + f(small(y))) \leqslant small(r)$. Consequently: $f(f^{[-1]}(f(small(x)) + f(small(y)))) \geqslant f(small(r))$, and therefore: $|x| + |y| = f(small(x)) + f(small(y)) \geqslant f(small(y)) = |r|$. $\square$

7. Point-free geometry by graded inclusions (fuzzy mereology)

Another possible metric approach to point-free geometry is obtained by considering a graded inclusion between regions. In this case we refer to quasi-metrics, i.e. "distances" in which the symmetric property is not required (see [10]). The prototypical example is furnished by the *excess* measure of a subset x with respect to a subset y, upon which the definition of the famous Hausdorff distance is founded.

Definition 17. Given a metric space (S, d), the *excess measure* is the function $e_d \colon \mathcal{P}(S) \times \mathcal{P}(S) \to [0, \infty]$ defined, for every pair x and y of non-empty subsets of S, by setting

$$e_d(x, y) := \sup\{d(P, y) : P \in x\}$$

where, in turn, $d(P, y)$ is the *distance of the point P from the subset y* defined by setting

$$d(P, y) := \inf\{d(P, Q) : Q \in y\}.$$

Since the elements of Re are bounded, in the prototypical model the value $e_d(x, y)$ is finite. It is immediate to see that e_d is not symmetric, that $e_d(x, x) = 0$ and that the triangle inequality holds. This suggests reference to the following class of structures.

Definition 18. A *quasi-metric space* is a structure $(\mathcal{R}, \delta)$ such that $\mathcal{R}$ is a non-empty set and $\delta\colon \mathcal{R} \times \mathcal{R} \to [0, \infty)$ is a mapping such that, for all x, y, $z \in \mathcal{R}$:

$$\delta(x, x) = 0 \tag{d1}$$

$$\delta(x, y) \leq \delta(x, z) + \delta(z, y)\,. \tag{d3}$$

Then quasi-metric space theory is obtained from metric space theory by leaving out the symmetry of δ and the axiom claiming that $\delta(x, y) = 0$ entails $x = y$. Every quasi-metric space is associated with a pre-order in the following way.

Proposition 6. *Let $(\mathcal{R}, \delta)$ be a quasi-metric space, then the relation $\leq$ defined by setting*:

$$x \leq y \overset{df}{\iff} \delta(x, y) = 0$$

is a pre-order. Moreover, the diameter of any $z \in \mathcal{R}$ is the number:

$$|z| = \sup\{\delta(x, y) : x \leq z \text{ and } y \leq z\}.$$

In the prototypical model the associated pre-order is the usual set theoretical inclusion and the diameter is the usual diameter. Observe that for all x and y such that $y \leq x$ we have $|x| \geqslant \delta(x, y)$. This entails that x is an atom if and only if $|x| = 0$. Indeed, if $|x| = 0$ then for any y, if $y \leq x$, then $\delta(x, y) = 0$ since, by definition, $\delta(y, x) = 0$. By (d2), we have that $y = x$ and this proves that x is an atom. Conversely, it is evident that if x is an atom, then $|x| = 0$.

The following proposition emphasizes the fact that, differently from the case of the *ppm*-spaces, δ is not order-reversing.

Proposition 7. *The function δ in a quasi-metric space $(\mathcal{R}, \delta)$ is order-preserving with respect to the first variable and order-reversing with respect to the second variable. Also, the diameter $|\cdot| : \mathcal{R} \to [0, \infty]$ is order-preserving.*

We are ready to give the following basic definition where, given a real number r, $\|r\|$ denotes the absolute value of r.

Definition 19. A *quasi-metric space of regions* is a quasi-metric space $(\mathcal{R}, \delta)$ satisfying the following axioms for all $x, y \in \mathcal{R}$:

(d3) $\|\delta(x, y) - \delta(y, x)\| \leq |x| + |y|$,
(d4) $(\forall n \in \mathbb{N})(\exists z \leq x)\ |z| \leq \frac{1}{n}$.

Axiom (d3) says that if we confine ourselves to the class of "small" regions, then the map δ is approximately symmetric and therefore is a metric (approximately). In any quasi-metric space of regions we can define (AP, d) and $(\underline{\mathsf{AP}}, \underline{d})$ as in the previous cases and we can prove the following theorem.

Theorem 5. *Let* $(\mathcal{R}, \delta)$ *be a quasi-metric space of regions. Then* (AP, d) *is a pseudo-metric space and therefore* $(\underline{\mathsf{AP}}, \underline{d})$ *is a metric space.*

Proof. We observe only that, since $\delta(p_n, q_n) \leqslant \delta(q_n, p_n) + |p_n| + |q_n|$,

$$\begin{aligned} d(\langle p_n \rangle_{n\in\mathbb{N}}, \langle q_n \rangle_{n\in\mathbb{N}}) &\leqslant \lim_{n\to\infty} \delta(q_n, p_n) + \lim_{n\to\infty} |p_n| + \lim_{n\to\infty} |q_n| \\ &= \lim_{n\to\infty} \delta(q_n, p_n) = d(\langle q_n \rangle_{n\in\mathbb{N}}, \langle p_n \rangle_{n\in\mathbb{N}}). \qquad \square \end{aligned}$$

In order to transform this metrical approach into a multi-valued approach, let us consider a first-order language with a binary relation symbol 'Incl' whose intended interpretation is a graded inclusion. An interpretation of such a language is defined by a pair $(\mathcal{R}, incl)$ where $\mathcal{R}$ is a non-empty set and $incl\colon \mathcal{R} \times \mathcal{R} \to [0,1]$ is a fuzzy binary relation. We write '$x \leq y$' to denote the formula '$\mathrm{Ct}(\mathrm{Incl}(x,y))$' and '$\mathrm{Eq}(x,y)$' to denote the formula '$\mathrm{Incl}(x,y) \wedge \mathrm{Incl}(y,x)$'. The intended meaning is that '$\leq$' is the ordinary inclusion and 'Eq' a graded equality. Also, we denote by '$\mathrm{Pl}(x)$' the formula $\forall z(z \leq x \to \mathrm{Eq}(x,z))$. This formula represents the graded version of the Euclidean definition of a point as a geometric element which has no part, i.e., an element x such that $x' \leq x$ entails $x' = x$. So, if x satisfies '$\mathrm{Pl}(x)$' we say also that x is a *point-like region.*

Definition 20. By *a point-free theory based on a graded inclusion* we mean the following system of axioms

(A1) $\forall x(\mathrm{Incl}(x,x))$
(A2) $\forall x \forall y \forall z(\mathrm{Incl}(x,z) \wedge Incl(z,y) \Rightarrow Incl(x,y))$
(A3) $\forall x \forall y(\mathrm{Pl}(x) \wedge \mathrm{Pl}(y) \wedge \mathrm{Incl}(x,y) \Rightarrow \mathrm{Incl}(y,x))$
(A4) $\forall x \exists z(z \leq x \wedge \mathrm{Pl}(z))$

We call a *graded inclusion space* every model of (A1)–(A4).

Axioms (A1) and (A2) say that 'Incl' is a graded pre-order, (A3) says that this pre-order is symmetric for point-like regions and therefore that 'Incl' is a graded equivalence in the class of these regions.

Theorem 6. *Let* $(\mathcal{R}, incl)$ *be any model of the point-free theory based on a graded inclusion. Then the model is associated with a quasi-metric space of regions and therefore with a metric space.*

Proof. Let f be a continuous generator of the triangular norm $\otimes$ and define δ by setting:

$$\delta(x, y) := f(incl(x, y)).$$

Then it is evident that $(\mathcal{R}, \delta)$ satisfies (d1). Moreover, in accordance with (A2), $incl(x, z) \otimes incl(z, y) \leqslant incl(x, y)$ and therefore $f^{[-1]}(f(incl(x, z)) + f(incl(z, y))) \leqslant incl(x, y)$. Since f is order-reversing, so $f(incl(x, z)) + f(incl(z, y))) \geqslant f(f^{[-1]}(f(incl(x, z)) + f(incl(z, y)))) \geqslant f(incl(x, y))$, by Proposition 2(vii). Thus, δ satisfies (d2).

To prove that δ satisfies (d3), first observe that:

$$\begin{aligned} f(pl(x)) &= f(\inf\{incl(x, z) : z \leq x\}) = \sup\{f(incl(x, y) : z \leq x\} \\ &= \sup\{\delta(x, y) : z \leq x\} = |x| \,. \end{aligned}$$

Moreover, in the case $\delta(y, x) \geqslant \delta(x, y)$, i.e. $f(incl(y, x)) \geqslant f(incl(x, y))$, since $f(incl(y, x)) - f(incl(x, y)) \leqslant f(incl(y, x)) \leqslant f(0)$, we have:

$$f(f^{[-1]}(f(incl(y, x)) - f(incl(x, y))) = f(incl(y, x)) - f(incl(x, y)).$$

Now, in accordance with (A3), we have:

$$pl(x) \otimes pl(y) \leqslant (incl(x, y) \to incl(y, x))$$

and therefore:

$$f^{[-1]}(f(pl(x)) + f(pl(y))) \leqslant f^{[-1]}(f(incl(y, x)) - f(incl(x, y))).$$

By applying f to both the sides of this inequality, we obtain:

$$\begin{aligned} f(f^{[-1]}(f(pl(x)) + f(pl(y)))) &\geqslant f(f^{[-1]}(f(incl(y, x)) - f(incl(x, y)))) \\ &= f(incl(y, x)) - f(incl(x, y)). \end{aligned}$$

Thus, this proves (d3):

$$\begin{aligned} |x| + |y| &= f(pl(x)) + f(pl(y))) \geqslant f(f^{[-1]}(f(pl(x)) + f(pl(y)))) \\ &\geqslant \| f(incl(y, x)) - f(incl(x, y)) \| \,. \end{aligned}$$

Finally, to prove (d4) observe that, by (A4), for every x we have $\sup\{pl(z) : z \leq x\} = 1$. So $\inf\{f(pl(z)) : z \leq x\} = f(\sup\{pl(z) : z \leq x\})$ $= f(1) = 0$. $\square$

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