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Andrzej Pietruszczak

CLASSICAL MEREOLOGY IS NOT ELEMENTARILY AXIOMATIZABLE

Abstract. By *the classical mereology* I mean a theory of mereological structures in the sense of [10]. In [7] I proved that the class of these structures is not elementarily axiomatizable. In this paper a new version of this result is presented, which according to my knowledge is the first such presentation in English. A relation of this result to a certain Hsing-chien Tsai's theorem from [13] is emphasized.

Keywords: classical mereology; mereological structures; the absence of elementary definability of classical mereology

1. Mereological structures

By *a mereological structure* (*in Tarski sense* [10]) we mean any relational structure of the form $\langle M, \sqsubseteq\rangle$, with a non-empty set M and a transitive relation $\sqsubseteq$ in M,[1] satisfying the following condition:[2]

$$\forall_{S\in 2^M\setminus\{\emptyset\}}\exists^1_{x\in M}\ x \,\mathsf{sum}\, S\,, \tag{$\exists^1$sum}$$

where sum is the following binary relation in $M \times 2^M$:

$$x \,\mathsf{sum}\, S \iff \forall_{y\in S}\ y \sqsubseteq x\ \wedge \\ \forall_{z\in M}\big(z \sqsubseteq x \Rightarrow \exists_{y\in S}\exists_{u\in M}(u \sqsubseteq y \wedge u \sqsubseteq z)\big). \tag{df sum}$$

[1] I.e., the relation $\sqsubseteq$ in M satisfies the condition ($\mathrm{t}_{\sqsubseteq}$) being a special case of (t_R) given in Appendix B, where $R := \sqsubseteq$ and $U := M$ (p. 495).

[2] A formula of the form $\ulcorner\exists^1_{x\in X}\ \varphi(x)\urcorner$ says that in a set X there exists exactly one object x such that $\varphi(x)$. This formula is an abbreviation of $\ulcorner\exists_{x\in X}\ \varphi(x) \wedge \forall_{x,y\in X}(\varphi(x) \wedge \varphi(x/y) \Rightarrow x = y)\urcorner$.

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The class of all mereological structures will be denoted by 'MS'. Following Leśniewski [4], we call $\sqsubseteq$ an *ingrediens relation* and in the case of $x \sqsubseteq y$ we say that x *is ingrediens of* y (i.e., x is (proper) part of y or $x = y$; see ($\star$)). Moreover, in the case of $x \;\mathsf{sum}\; S$ we say that an object x is a *mereological sum* (or a *collective class*) of all members of a (distributive) set S. The axioms ($\mathrm{t}_\sqsubseteq$) and ($\exists^1\mathsf{sum}$) say, respectively, that the relation $\sqsubseteq$ is transitive in M and that for every non-empty subset S of M there exists exactly one mereological sum of all members of S.

For any structure $\langle M, \sqsubseteq\rangle$ from the class MS we obtain that $\sqsubseteq$ is a separative partial order, i.e., $\sqsubseteq$ is also reflexive, antisymmetrical and separative, i.e., $\sqsubseteq$ satisfies the conditions ($\mathrm{r}_\sqsubseteq$), ($\mathrm{antis}_\sqsubseteq$), and ($\mathrm{sep}_\sqsubseteq$) (see [6, 7, 8, 10]).[3]

From ($\mathrm{r}_\sqsubseteq$) we obtain that sum is included in $M \times 2^M \setminus \{\emptyset\}$, that is:

$$\forall_{S \in 2^M}(\exists_{x\in M}\, x \;\mathsf{sum}\; S \Longrightarrow S \neq \emptyset),$$

so, in the light of ($\exists^1\mathsf{sum}$), we have:

$$\forall_{S\in 2^M\setminus\{\emptyset\}}\exists_{x\in M}\, x \;\mathsf{sum}\; S\,, \tag{$\exists\mathsf{sum}$}$$

$$\forall_{S\in 2^M}\forall_{x,y\in M}(x \;\mathsf{sum}\; S \;\wedge\; y \;\mathsf{sum}\; S \Longrightarrow x = y), \tag{fun-sum}$$

i.e., the relation sum is a (partial) function of the second argument.

By ($\exists^1\mathsf{sum}$), there exists the unity $\mathsf{1}$ of this structure, since $M \neq \emptyset$:[4]

$$\mathsf{1} := (\iota\, z)\; z \;\mathsf{sum}\; M\,, \tag{df 1}$$

$$\mathsf{1} = (\iota\, z)\; \forall_{y\in M}\; y \sqsubseteq z\,.$$

Moreover, we can introduce a unary (partial) operation on $2^M \setminus \{\emptyset\}$ of *being of the mereological sum of all members of a given non-empty set*:

$$S \neq \emptyset \;\Longrightarrow\; \bigsqcup S := (\iota\, z)\; z \;\mathsf{sum}\; S\,. \tag{df$\bigsqcup$}$$

Thus, $\mathsf{1} = \bigsqcup M$ and we can introduce the following binary operation in M:

$$x \sqcup y := \bigsqcup\{x, y\}\,. \tag{df$\sqcup$}$$

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[3] See the conditions (r_R), (antis_R), and (sep_R) from Appendix B for $R := \sqsubseteq$ and $U := M$ (pp. 494–495).

[4] The Greek letter 'ι' stands for the standard description operator. The expression $\ulcorner(\iota x)\,\varphi(x)\urcorner$ is read "the only object x which satisfies the condition $\varphi(x)$". Before using it, first we have to prove that there exists exactly one object x such that $\varphi(x)$, i.e., $\exists^1_x\, \varphi(x)$.

Of course, $\sqcup$ is idempotent and commutative, and we obtain:

$$x \sqcup y = \bigsqcup\{u \in M : u \sqsubseteq x \ \vee \ u \sqsubseteq y\}.$$
$$x \sqsubseteq y \iff y = x \sqcup y\,.$$

For any mereological structure $\langle M, \sqsubseteq\rangle$ we introduce three auxiliary binary relations in M: *of being (proper) part*, *of overlapping* and *of being exterior to*:

$$x \sqsubset y \iff x \sqsubseteq y \ \wedge \ x \neq y, \tag{df $\sqsubset$}$$
$$x \bigcirc y \iff \exists_{z \in M}(z \sqsubseteq x \ \wedge \ z \sqsubseteq y), \tag{df $\bigcirc$}$$
$$x \wr y \iff \neg\, x \bigcirc y\,. \tag{df $\wr$}$$

If $x \sqsubset y$ (resp. $x \bigcirc y$; $x \wr y$), then we say that: x *is* (*proper*) *part of* y (resp. x *overlaps* y; x *is exterior to* y). Of course, $\bigcirc$ and $\wr$ are symmetric. By ($\mathrm{r}_\sqsubseteq$), $\bigcirc$ is reflexive, $\wr$ is irreflexive, $\sqsubseteq$ is included in $\bigcirc$ (so $\wr$ is disjoint from $\sqsubseteq$ and $\sqsubset$). The relation $\sqsubset$ is irreflexive, asymmetric, and transitive. Thus, we have the following conditions: ($\mathrm{irr}_\sqsubset$), ($\mathrm{as}_\sqsubset$), ($\mathrm{t}_\sqsubset$), ($\mathrm{r}_\bigcirc$), ($\mathrm{s}_\bigcirc$), ($\mathrm{irr}_\wr$), and ($\mathrm{s}_\wr$).[5] Moreover, all mereological structures satisfy the so-called *Weak Supplementation Principle*:

$$\forall_{x,y \in M}(x \sqsubset y \implies \exists_{z \in M}(z \sqsubset y \ \wedge \ z \wr x)). \tag{WSP}$$

The aforementioned formula ($\mathrm{sep}_\sqsubseteq$) is called *Strong Supplementation Principle*.

By ($\mathrm{r}_\sqsubseteq$) and ($\mathrm{antis}_\sqsubseteq$), we also obtain:

$$\forall_{x,y \in M}(x \sqsubseteq y \iff x \sqsubset y \ \vee \ x = y), \tag{$\star$}$$
$$\forall_{x,y \in M}(x \sqsubset y \iff x \sqsubseteq y \ \wedge \ y \not\sqsubseteq x),$$

We say that a mereological structure $\langle M, \sqsubseteq\rangle$ is *non-trivial* iff M has at least two members. It is equivalent to the fact that M has at least two members which are exterior to each other and to the fact that in M there is no smallest element, that is:

$$|M| > 1 \iff \exists_{x,y \in M}\, x \wr y \iff \neg\, \exists_{x \in M} \forall_{y \in M}\, x \sqsubseteq y, \tag{\#}$$

where $|M|$ is the cardinality of M.

By ($\mathrm{r}_\sqsubseteq$), we have $\{\langle x, y\rangle \in M \times M : x \bigcirc y\} \neq \emptyset$. So, by ($\exists^1$sum), we can introduce the following partial binary operation $\sqcap$: $\{\langle x, y\rangle \in M \times M : x \bigcirc y\} \to M$:

$$x \bigcirc y \implies x \sqcap y := \bigsqcup\{u \in M : u \sqsubseteq x \ \wedge \ u \sqsubseteq y\}. \tag{df $\sqcap$}$$

[5] Again, see the conditions (irr_R), (as_R), (t_R), (r_R), and (s_R) from Appendix B for $U := M$ and $R := \sqsubset, \bigcirc, \wr$, respectively (pp. 494–495).

The object $x \sqcap y$ is called the (*mereological*) *product* of two overlapping objects x and y. For the operations $\sqcup$ and $\sqcap$ we obtain:

$$x \bigcirc y \implies (x = x \sqcap y \Leftrightarrow y = x \sqcup y),$$
$$x \bigcirc y \implies \forall_{u \in M}(u \sqsubseteq x \sqcap y \Leftrightarrow u \sqsubseteq x \wedge u \sqsubseteq y).$$

Notice that we can prove the following equivalence (see e.g. [6, 7, 8]):

$$\forall_{S \in 2^M} \forall_{x \in M}\big(x \,\mathsf{sum}\, S \iff \forall_{z \in M}(z \bigcirc x \Leftrightarrow \exists_{y \in S}\, y \bigcirc z)\big). \qquad (\%)$$

All members of M overlap $\mathsf{1}$, so in the light of (WSP) we have:

$$\forall_{x \in M}\big(x \neq \mathsf{1} \iff \exists_{y \in M}\, y \wr x\big).$$

Hence, for any $x \neq \mathsf{1}$ we have $\{u \in M : u \wr x\} \neq \emptyset$ and by (%) we obtain $\bigsqcup\{u \in M : u \wr x\} \neq \mathsf{1}$. Thus, in non-trivial mereological structures we can introduce the following unary operation $-\colon M \setminus \{\mathsf{1}\} \to M \setminus \{\mathsf{1}\}$:

$$x \neq \mathsf{1} \implies -x := \bigsqcup\{u \in M : u \wr x\}. \qquad (\mathrm{df}\,-)$$

The object $-x$ will be called the (*mereological*) *complement of* x. The following hold in all mereological structures (cf. e.g. [6, 7, 8]):

$$\forall_{x \in M \setminus \{\mathsf{1}\}}\, x = --x,$$
$$\forall_{x \in M \setminus \{\mathsf{1}\}}\, x \wr -x,$$
$$\forall_{x \in M \setminus \{\mathsf{1}\}}\, x \sqcup -x = \mathsf{1},$$
$$\forall_{x,y \in M \setminus \{\mathsf{1}\}}(-x = -y \iff x = y),$$
$$\forall_{x,y \in M \setminus \{\mathsf{1}\}}(x \sqsubseteq y \iff -y \sqsubseteq -x),$$
$$\forall_{x,y \in M \setminus \{\mathsf{1}\}}(x \sqsubset y \iff -y \sqsubset -x),$$
$$\forall_{x,y \in M}(x \wr y \iff y \neq \mathsf{1} \wedge x \sqsubseteq -y),$$
$$\forall_{x,y \in M}(x \not\sqsubseteq y \iff y \neq \mathsf{1} \wedge x \bigcirc -y).$$

For every structure $\langle M, \sqsubseteq \rangle$ from **MS** we obtain:

$$\forall_{S \in 2^M} \forall_{x \in M}(x \,\mathsf{sum}\, S \iff S \neq \emptyset \wedge x \,\mathsf{sup}_{\sqsubseteq}\, S).$$
$$\forall_{S \in 2^M \setminus \{\emptyset\}}\big(\textstyle\bigsqcup S = \sup_{\sqsubseteq} S\big)$$

Thus, by (#): $\langle M, \sqsubseteq \rangle$ is non-trivial iff there is no z such that $z \,\mathsf{sup}_{\sqsubseteq}\, \emptyset$ iff sum and $\mathsf{sup}_{\sqsubseteq}$ are equal:

$$|M| > 1 \iff \forall_{S \in 2^M} \forall_{z \in M}(z \,\mathsf{sum}\, S \Leftrightarrow z \,\mathsf{sup}_{\sqsubseteq}\, S).$$

Of course: $x \sqcup y = \sup_{\sqsubseteq}\{x, y\}$. Moreover, we have:

$$x \bigcirc y \implies x \sqcap y = \inf_{\sqsubseteq}\{x, y\}.$$

In the light of (%), and after Leśniewski [5, Chapter X], we can choose a different explication of the concept of a *collective set*. In [3] Leonard and Goodman expressed this concept in the language of set theory, as the relation of *being a fusion of* all elements of a given distributive set. This relation is designated by 'fu' and for all $x \in M$ and $S \subseteq M$ we put:

$$x \mathsf{\ fu\ } S \iff \forall_{z\in M}(z \bigcirc x \Leftrightarrow \exists_{y\in S}\, y \bigcirc z). \tag{df fu}$$

Thus, by (%), in all mereological structures $\mathsf{fu} = \mathsf{sum}$.

We have the following equivalent axiomatizations of the class **MS**:

THEOREM 1.1 ([6, 7, 8]). *For any non-empty set M and any binary relation $\sqsubseteq$ in M the following conditions are equivalent (relations $\sqsubset$, $\bigcirc$, sum, and fu are defined as above):*

1. *$\langle M, \sqsubseteq\rangle$ is a member of* **MS**.
2. *$\langle M, \sqsubseteq\rangle$ satisfies* ($\mathrm{t}_{\sqsubseteq}$), (fun-sum) *and* (∃sum).
3. *$\langle M, \sqsubseteq\rangle$ satisfies* ($\mathrm{t}_{\sqsubseteq}$), ($\mathrm{antis}_{\sqsubseteq}$), ($\mathrm{sep}_{\sqsubseteq}$) *and* (∃sum).
4. *$\langle M, \sqsubseteq\rangle$ satisfies* ($\mathrm{t}_{\sqsubseteq}$), (WSP), *and* (∃sum).
5. *$\langle M, \sqsubseteq\rangle$ satisfies* ($\mathrm{t}_{\sqsubseteq}$), ($\mathrm{antis}_{\sqsubseteq}$), ($\mathrm{sep}_{\sqsubseteq}$), *and*

$$\forall_{S\in 2^M\setminus\{\emptyset\}}\exists_{x\in M}\, x \mathsf{\ fu\ } S\,. \tag{∃fu}$$

6. *$\langle M, \sqsubseteq\rangle$ satisfies* ($\mathrm{t}_{\sqsubseteq}$), ($\mathrm{antis}_{\sqsubseteq}$), (∃sum), *and*

$$\forall_{S\in 2^M}\forall_{x,y\in M}(x \mathsf{\ fu\ } S \wedge y \mathsf{\ fu\ } S \implies x = y). \tag{fun-fu}$$

2. The connection between mereological structures and complete Boolean lattices (complete Boolean algebras)

The following theorems[6] reveal some essential dependencies between mereological structures and complete Boolean lattices (resp. algebras).

THEOREM 2.1 (cf. e.g. [11, 7]). *Let $\langle B, \leq, 0, 1\rangle$ be a non-trivial complete Boolean lattice. We put $M := B \setminus \{0\}$ and $\sqsubseteq := \leq|_M := \leq \cap (M \times M)$. Then $\langle M, \sqsubseteq\rangle$ is a mereological structure, 1 is the unity of $\langle M, \sqsubseteq\rangle$, and:*

$$\forall_{S\in 2^M\setminus\{\emptyset\}}\ \sup_{\leq} S = \sup_{\sqsubseteq} S = \bigsqcup S\,.$$

[6] Concerning these theorems see footnote 1 in [11, pp. 333–334].

For any Boolean algebra $\langle A, +, *, -, 0, 1\rangle$ and for the relation $\leq$, which is defined by (df$\leq$), p. 495, the structure $\langle A, \leq, 0, 1\rangle$ is a Boolean lattice. Thus Theorem 2.1 also holds for any non-trivial complete Boolean algebra with $\leq$.

THEOREM 2.2 (cf. e.g. [11, 7]). *Let $\langle M, \sqsubseteq\rangle$ be any mereological structure and 0 be an arbitrary object such that $0 \notin M$. We put $M^0 := M \cup \{0\}$ and $\sqsubseteq^0 := \sqsubseteq \cup (\{0\} \times M^0)$, i.e., for any $x, y \in M^0$: $x \sqsubseteq^0 y \iff x \sqsubseteq y \vee x = 0$. Then $\langle M^0, \sqsubseteq^0, 0, 1\rangle$ (where 1 is the unity of $\langle M, \sqsubseteq\rangle$) is a non-trivial complete Boolean lattice such that:*

$$\forall_{S \in 2^M \setminus \{\emptyset\}} \sup_{\sqsubseteq^0} S = \sup_{\sqsubseteq} S = \bigsqcup S\,. \tag{†}$$

Moreover, for any $x, y \in M^0$ we have:

$$x + y = \begin{cases} x \sqcup y & \text{if } x, y \in M \\ x & \text{if } y = 0 \\ y & \text{if } x = 0 \end{cases} \qquad x \cdot y = \begin{cases} x \sqcap y & \text{if } x \bigcirc y \\ 0 & \text{otherwise} \end{cases}$$

$$\backsim x = \begin{cases} -x & \text{if } x \in M \setminus \{1\} \\ 0 & \text{if } x = 1 \\ 1 & \text{if } x = 0 \end{cases}$$

where the operations $+$, $\cdot$ and $\backsim$ are defined by (df$+$), (df$\cdot$), *and* (df$\backsim$), *respectively* (pp. 495–496). *So $\langle M^0, +, \cdot, \backsim, 0, 1\rangle$ is a complete Boolean algebra such that the relation $\leq$, introduced by* (df$\leq$), *is equal to $\sqsubseteq^0$.*

In the light of theorems 2.1 and 2.2 we obtain the following theorem.

THEOREM 2.3 (cf. e.g. [9]). *For any non-empty set M and for any binary relation $\sqsubseteq$ in M the following conditions are equivalent.*

(i) *$\langle M, \sqsubseteq\rangle$ belongs to* **MS**.
(ii) *For some (equivalently: any) $0 \notin M$, for $M^0 := M \cup \{0\}$ and for $\sqsubseteq^0 := \sqsubseteq \cup (\{0\} \times M^0)$ the structure $\langle M^0, \sqsubseteq^0, 0, 1\rangle$ (where 1 is the unity of $\langle M, \sqsubseteq\rangle$) is a non-trivial complete Boolean lattice.*
(iii) *For some non-trivial complete Boolean lattice $\langle B, \leq, \mathrm{o}, \mathrm{i}\rangle$ we have $M = B \setminus \{\mathrm{o}\}$, $\sqsubseteq\, = \,\leq|_M$, and $1 = \mathrm{i}$.*
(iv) *For some non-trivial complete Boolean algebra $\langle A, +, *, -, 0, 1\rangle$ we have $M = A \setminus \{0\}$, $1 = 1$, and $\sqsubseteq\, = \,\leq|_M$, where $\leq$ is defined by* (df$\leq$).

PROOF. "(i)⇒(ii)" By Theorem 2.2.

"(ii)⇒(iii)" We put $B := M^{\mathit{0}}$, $\leq\ := \sqsubseteq^{\mathit{0}}$, $\mathrm{o} := \mathit{0}$, and $\mathrm{1} := 1$. Then $M = B \setminus \{\mathrm{o}\}$ and $\sqsubseteq\ = \ \leq\!|_M$.

"(ii)⇒(iv)" In a non-trivial complete Boolean lattice $\langle M^{\mathit{0}}, \sqsubseteq^{\mathit{0}}, \mathit{0}, 1\rangle$ by means of (df +), (df ·) and (df ∽) we define the operations $+$, $\cdot$ and $\backsim$, respectively. So $\langle M^{\mathit{0}}, +, \cdot, \backsim, \mathit{0}, 1\rangle$ is a complete Boolean algebra and — by Theorem 2.2 — the relation ≤, introduced by (df ≤), is equal to $\sqsubseteq^{\mathit{0}}$. So $\sqsubseteq\ = \ \leq\!|_M$.

"(iii)⇒(i)" By Theorem 2.1.

"(iv)⇒(i)" By the relationship between complete Boolean algebras and complete Boolean lattices, and Theorem 2.1 (see p. 490). □

3. The main result

For mereological structures we use the first-order language $\mathrm{L}_{\sqsubseteq}$ with equality which has only one binary predicate '$\sqsubseteq$'. Of course, all mereological structures are $\mathrm{L}_{\sqsubseteq}$-structures.

First, we introduce the following $\mathrm{L}_{\sqsubseteq}$-structures: $\mathfrak{P}_\omega := \langle 2^\omega \setminus \{\emptyset\}, \subseteq\rangle$ and $\mathfrak{FC}_\omega := \langle \mathrm{FC}(\omega) \setminus \{\emptyset\}, \subseteq\rangle$, where $\mathrm{FC}(\omega)$ is the set of all finite and all co-finite subsets of ω. In [7] we noticed:

- By Theorem 2.1, $\mathfrak{P}_\omega$ is a mereological structure, since the Boolean lattice $\mathfrak{B}_1 := \langle 2^\omega, \subseteq, \emptyset, \omega\rangle$ is complete (see p. 497).
- By Theorem 2.2, $\mathfrak{FC}_\omega$ is not a mereological structure, because the Boolean lattice $\mathfrak{B}_2 := \langle \mathrm{FC}(\omega), \subseteq, \emptyset, \omega\rangle$ is not complete (see p. 497).

Second, in [7] we proved:

FACT 3.1. *The* $\mathrm{L}_{\sqsubseteq}$*-structures* $\mathfrak{P}_\omega$ *and* $\mathfrak{FC}_\omega$ *are elementarily equivalent, i.e.,* $\mathrm{Th}(\mathfrak{P}_\omega) = \mathrm{Th}(\mathfrak{FC}_\omega)$.

THE PROOF FROM [7]. We use Corollary B.4 and the following fact:

CLAIM. *We assign to an arbitrary* $\mathrm{L}_{\sqsubseteq}$*-structure* $\mathfrak{A} = \langle A, \sqsubseteq\rangle$ *an arbitrary* $\mathit{0} \notin A$ *along with the structure* $\mathfrak{A}^{\mathit{0}} = \langle A^{\mathit{0}}, \sqsubseteq^{\mathit{0}}\rangle$ *defined as in Theorem 2.2. We connect this structure with the first-order language* $\mathrm{L}^{\mathrm{o}}_{\leq}$ *with identity and two specific constants: the binary predicate '*$\leq$*' and the individual constant '*o*', which are interpreted with the help of* $\sqsubseteq^{\mathit{0}}$ *and* $\mathit{0}$*, respectively.*

Let σ *be an arbitrary* $\mathrm{L}_{\sqsubseteq}$*-sentence. We turn* σ *into a* $\mathrm{L}^{\mathrm{o}}_{\leq}$*-sentence* σ^* *with the help of the following transformation: in place of the predicate '*$\sqsubseteq$*' we substitute the predicate '*$\leq$*'; we exchange an arbitrary quantifier* *binding* x_i *with a quantifier limited by the condition*: $\neg\, x_i = \mathrm{o}$.[7] *Then*: $\mathfrak{A} \models \sigma$ *iff* $\mathfrak{A}^0 \models \sigma^*$.

So for any $\mathrm{L}_{\sqsubseteq}$-sentence σ we have:

$$\begin{aligned} \sigma \in \mathrm{Th}(\mathfrak{P}_\omega) \text{ (by Claim) } &\text{ iff } \sigma^* \in \mathrm{Th}(\mathfrak{B}_1) \text{ (by Corollary B.4)} \\ &\text{ iff } \sigma^* \in \mathrm{Th}(\mathfrak{B}_2) \text{ (by Claim)} \\ &\text{ iff } \sigma \in \mathrm{Th}(\mathfrak{F}\mathfrak{C}_\omega)\,. \end{aligned}$$ □

Another proof based on some result of [12]. In [12] Tsai proved that $\mathfrak{P}_\omega$ and $\mathfrak{F}\mathfrak{C}_\omega$ are models of some complete first-order $\mathrm{L}_{\sqsubseteq}$-theory. So these models are elementarily equivalent. □

Finally, considering the structures $\mathfrak{P}_\omega$ and $\mathfrak{F}\mathfrak{C}_\omega$, by Fact 3.1 and Fact A.1 from Appendix A, we obtain:

Theorem 3.2 ([7]). *The class* **MS** *of all mereological structures is not elementarily axiomatizable.*

4. A comment on some result of [13]

In [13] Tsai considers a certain first-order $\mathrm{L}_{\sqsubseteq}$-theory **CEM** + (G) with equality ('P' is used instead of '$\sqsubseteq$'). This theory has the following specific axioms: $(\mathrm{r}_{\sqsubseteq})$, $(\mathrm{antis}_{\sqsubseteq})$, $(\mathrm{t}_{\sqsubseteq})$ and $(\mathrm{sep}_{\sqsubseteq})$[8], and the axioms of "finite sum", "finite product" and "the greatest member":

$$\forall_x \forall_y (\exists_u (x \sqsubseteq u \wedge y \sqsubseteq u) \implies \exists_z \forall_w (w \circ z \Leftrightarrow (w \circ x \vee w \circ y))) \tag{FS}$$

$$\forall_x \forall_y (x \circ y \implies \exists_z \forall_w (w \sqsubseteq z \Leftrightarrow (w \sqsubseteq z \wedge w \sqsubseteq y))) \tag{FP}$$

$$\exists_x \forall_y\, y \sqsubseteq x\,. \tag{G}$$

We put $\mathrm{AxT} := \{(\mathrm{r}_{\sqsubseteq}), (\mathrm{antis}_{\sqsubseteq}), (\mathrm{t}_{\sqsubseteq}), (\mathrm{sep}_{\sqsubseteq}), (\mathrm{FS}), (\mathrm{FP}), (\mathrm{G})\}$.

All models of the theory **CEM** + (G) (i.e., all $\mathrm{L}_{\sqsubseteq}$-structures from Mod(AxT)) Tsai calls "mereological structures". Moreover, Tsai says that a structure $\langle M, \sqsubseteq\rangle$ from Mod(AxT) is "*complete*" iff for any non-empty subset S of M, there is $x \in M$ such that $x \,\mathsf{fu}\, S$, where fu is the binary relation defined by (df fu). That is, a given structure from Mod(AxT) is "complete" iff it satisfies the condition ($\exists$fu). We denoted

[7] Formally: after exchanging the predicate '$\sqsubseteq$', instead of $\ulcorner\forall_{x_i}\varphi\urcorner$ and $\ulcorner\exists_{x_i}\varphi\urcorner$ we take $\ulcorner\forall_{x_i}(\neg\, x_i = \mathrm{o} \to \varphi)\urcorner$ and $\ulcorner\exists_{x_i}(\neg\, x_i = \mathrm{o} \wedge \varphi)\urcorner$, respectively.

[8] In [13] these are the formulas: (P1)–(P3), and (SSP), respectively

the class of "complete" structures from Mod(AxT) by cMod(AxT). We have: cMod(AxT) $\subsetneq$ Mod(AxT).

By Theorem 1.1 we see that the class of all $\mathrm{L}_{\sqsubseteq}$-structures which satisfy the conditions ($\mathrm{t}_{\sqsubseteq}$), ($\mathrm{antis}_{\sqsubseteq}$), ($\mathrm{sep}_{\sqsubseteq}$), ($\exists$fu) is equal to **MS**. Moreover, in the light of Section 1, all structures from **MS** satisfy the conditions (FS), (FP), (G). Thus, we have: cMod(AxT) = **MS**.

In [13, the proof of Claim 1] the following meta-sentence:

(C) *'Being a complete mereological structure' is first-order definable*

means that "there is such a sentence α in the mereological language [i.e. $\mathrm{L}_{\sqsubseteq}$] which defines the completeness of a mereological structure [in author's sense], that is, for any mereological structure M, M is complete if and only if $M \models \alpha$". Thus — in our terminology — the meta-sentence (C) has the following meaning:

- for some sentence α in $\mathrm{L}_{\sqsubseteq}$, for any $\mathrm{L}_{\sqsubseteq}$-structure $\mathfrak{A}$ from Mod(AxT): $\mathfrak{A} \in$ cMod(AxT) iff $\mathfrak{A} \models \alpha$.

In other words,

- for some sentence α in $\mathrm{L}_{\sqsubseteq}$, for any $\mathrm{L}_{\sqsubseteq}$-structure $\mathfrak{A}$: $\mathfrak{A} \in$ cMod(AxT) iff $\mathfrak{A} \in$ Mod(AxT $\cup \{\alpha\}$).

So (C) says that

(C′) for some sentence α in $\mathrm{L}_{\sqsubseteq}$, Mod(AxT $\cup \{\alpha\}$) = cMod(AxT) = **MS**.

Thus, (C) says that the class **MS** is finitely elementarily axiomatizable[9], since instead of any finite set $\{\sigma_1, \ldots, \sigma_n\}$ of sentences we can use $\ulcorner \sigma_1 \wedge \cdots \wedge \sigma_n \urcorner$. Tsai proves that (C) is not true (see [13, Claim 1]). So — in our terminology — he proves that the class **MS** is not finitely elementarily axiomatizable. Our Theorem 3.2 gives the stronger result: **MS** is not elementarily axiomatizable.

A. Appendix: Elementarily axiomatizable classes of structures

***L*-structures. Models.** Let L be any first-order language (with or without equality). An *L-structure* is an ordered pair of the form $\langle U, \Im \rangle$, where U is a non-empty set (*the universe of structure*) and $\Im$ is a set-theoretical interpretation of non-logical symbols of L.

[9] See Appendix A, p. 494

If an L-formula φ is true in an L-structure $\mathfrak{A}$, then we write $\mathfrak{A} \vDash \varphi$. All L-formulas without free variables are called *L-sentences*. For any L-sentence φ and any L-structure $\mathfrak{A}$: φ is true in $\mathfrak{A}$ iff $\mathfrak{A}$ satisfies φ.

For any set Φ of L-formulas, a *model of* Φ is any L-structure $\mathfrak{A}$ such that for any $\varphi \in \Phi$ we have $\mathfrak{A} \vDash \varphi$, i.e., all formulas of Φ are true in $\mathfrak{A}$ (we write: $\mathfrak{A} \vDash \Phi$). Let $\mathrm{Mod}(\Phi)$ be the class of all models of Φ. Of course, for any sets of L-formulas Φ and Ψ: if $\Phi \subseteq \Psi$ then $\mathrm{Mod}(\Psi) \subseteq \mathrm{Mod}(\Phi)$.

Elementarily equivalent structures. A *theory* of an L-structure $\mathfrak{A}$ is the set of all L-sentences which are true in $\mathfrak{A}$, that is, the following set:

$$\mathrm{Th}(\mathfrak{A}) := \{\varphi : \varphi \text{ is an } L\text{-sentence and } \mathfrak{A} \vDash \varphi\}.$$

L-structures $\mathfrak{A}$ and $\mathfrak{B}$ are *elementarily equivalent* iff $\mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{B})$, i.e., $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same L-sentences.

Elementarily axiomatizable class of structures. Let $\boldsymbol{K}$ be any class of L-structures. We say that $\boldsymbol{K}$ is *elementarily axiomatizable* (or *elementary in the wider sense*) iff there is a set Σ of L-sentences such that $\boldsymbol{K} = \mathrm{Mod}(\Sigma)$. If additionally the set Σ is finite, then we say that $\boldsymbol{K}$ is *finitely elementarily axiomatizable* (or *elementary in the narrow sense*).

Directly from definitions we obtain:

FACT A.1. *Every elementarily axiomatizable class of L-structures is closed under elementary equivalence. In other words, for any class $\boldsymbol{K}$ of L-structures and any L-structures $\mathfrak{A}$ and $\mathfrak{B}$: if $\boldsymbol{K}$ is an elementarily axiomatizable, $\mathfrak{A} \in \boldsymbol{K}$ and $\mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{B})$, then $\mathfrak{B} \in \boldsymbol{K}$.*

B. Appendix: Some facts about binary relations, Boolean algebras, and Boolean lattices

Some types of binary relations. Let U be any non-empty set. All subsets of $U \times U$ are called *binary relations* on U. A binary relation R is called, respectively, *reflexive*, *irreflexive*, *symmetric*, *asymmetric*, *antisymmetric*, *transitive*, *separative* iff R fulfills respective condition from the following set:

$$\forall_{x \in U}\ x\,R\,x\,, \tag{r_R}$$

$$\forall_{x \in U}\ \neg\ x\,R\,x\,, \tag{irr_R}$$

$$\forall_{x,y \in U}(x\,R\,y \ \Rightarrow\ y\,R\,x), \tag{s_R}$$

$$\forall_{x,y \in U}\ \neg(x\,R\,y \ \wedge\ y\,R\,x), \tag{as_R}$$

$$\forall_{x,y\in U}(x\,R\,y \ \wedge\ y\,R\,x \Longrightarrow x = y), \tag{antis$_R$}$$
$$\forall_{x,y,z\in U}(x\,R\,y \ \wedge\ y\,R\,z \ \Longrightarrow\ x\,R\,z), \tag{t$_R$}$$
$$\forall_{x,y\in U}\big(\neg\, x\,R\,y \Longrightarrow \exists_{z\in U}(z\,R\,x \wedge \neg\,\exists_{u\in U}(u\,R\,y \wedge u\,R\,z))\big). \tag{sep$_R$}$$

Partially ordered sets. A pair $\langle U, R\rangle$ is a *partially ordered set* iff U is non-empty set and R satisfies (r_R), (antis_R), (t_R). Besides, $\langle U, R\rangle$ is *separative* iff it satisfies (sep_R).

In any partially ordered set $\langle U, R\rangle$ we introduce two binary relations sup_R of *being of the least upper bound of* and inf_R of *being of the greatest lower bound of* which are included in $U \times 2^U$:

$$x\ \mathsf{sup}_R\ S \iff \forall_{z\in S}\, z\,R\,x \ \wedge\ \forall_{y\in M}(\forall_{z\in S}\, z\,R\,y \ \Rightarrow\ y\,R\,x), \tag{df sup_R}$$
$$x\ \mathsf{inf}_R\ S \iff \forall_{z\in S}\, x\,R\,z \ \wedge\ \forall_{y\in M}(\forall_{z\in S}\, y\,R\,z \ \Rightarrow\ x\,R\,y). \tag{df inf_R}$$

By (antis_R), sup_R and inf_R are (partial) functions of the second argument:

$$\forall_{S\in 2^U}\forall_{x,y\in U}(x\ \mathsf{sup}_R\ S \ \wedge\ y\ \mathsf{sup}_R\ S \Longrightarrow x = y), \tag{fun-sup_R}$$
$$\forall_{S\in 2^M}\forall_{x,y\in U}(x\ \mathsf{inf}_R\ S \ \wedge\ y\ \mathsf{inf}_R\ S \Longrightarrow x = y). \tag{fun-inf_R}$$

So if $x\ \mathsf{sup}_R\ S$ (resp. $x\ \mathsf{inf}_R\ S$), then we also write $x = \sup_R S$ (resp. $x = \inf_R S$).

A partially ordered set $\langle U, R\rangle$ is called *complete* iff it fulfils the following condition: $\forall_{S\in 2^U}\exists_{x\in U}\ x\ \mathsf{sup}_R\ S$ (equivalently, $\forall_{S\in 2^U}\exists_{x\in U}\ x\ \mathsf{inf}_R\ S$).

Boolean algebras. An algebraic structure $\langle A, \mathsf{+}, \mathsf{*}, \mathsf{-}, \mathsf{0}, \mathsf{1}\rangle$ is a *Boolean algebra* iff it satisfies certain well-known equalities (cf. e.g. [1]). A Boolean algebra is *non-trivial* iff $|A| > 1$ iff $\mathsf{0} \neq \mathsf{1}$. The binary relation $\mathsf{\le}$ in A defined by

$$x \mathrel{\mathsf{\le}} y \iff y = x \mathsf{+} y \iff x = x \mathsf{*} y \tag{df $\mathsf{\le}$}$$

is a separative partial order.

Lattices. A partially ordered set $\langle L, \leq\rangle$ is a *lattice* iff for any $x, y \in L$ there are the least upper bound and the greatest lower bound of $\{x, y\}$. So we have the following two binary operations on L:

$$x + y := \sup_{\leq}\{x, y\}\,, \tag{df +}$$
$$x \cdot y := \inf_{\leq}\{x, y\}\,. \tag{df $\cdot$}$$

Of course, $+$ and $\cdot$ are idempotent and commutative, and we obtain:

$$x \leq y \iff y = x + y \iff x = x \cdot y\,.$$

A lattice $\langle L, \leq\rangle$ is *bounded* iff it has a least element $\mathtt{0}$ and a greatest element $\mathtt{1}$, i.e., we have: $\forall_{x\in L}\ \mathtt{0} \leq x$ and $\forall_{x\in L}\ x \leq \mathtt{1}$. Then we write $\langle L, \leq, \mathtt{0}, \mathtt{1}\rangle$. A bounded lattice is *non-trivial* iff $\mathtt{0} \neq \mathtt{1}$. Moreover, a bounded lattice $\langle L, \leq, \mathtt{0}, \mathtt{1}\rangle$ is *complemented* iff each element of L has a complement, i.e., we have $\forall_{x\in L}\exists_{y\in L}(x + y = \mathtt{1} \wedge x \cdot y = \mathtt{0})$.

Boolean lattices. A bounded lattice $\langle B, \leq, \mathtt{0}, \mathtt{1}\rangle$ is a *Boolean lattice* iff it is distributive, i.e., for the operations $+$ and $\cdot$ the following condition holds: $\forall_{x,y,z\in B}[x \cdot (y + z) = ((x \cdot y) + (x \cdot z))]$, and complemented (see e.g. [1]). Under these conditions for any $x \in B$ there is the unique complement of x; so we can put

$$\backsim x := (\iota\, z)(x + z = \mathtt{1}\ \wedge\ x \cdot z = \mathtt{0}). \qquad (\mathrm{df}\backsim)$$

We have: $\langle B, +, \cdot, \backsim, \mathtt{0}, \mathtt{1}\rangle$ is a Boolean algebra and $\leq\ =\ \leqslant$, where $\leqslant$ is defined by (df$\leqslant$).

For a Boolean lattice $\mathfrak{B} = \langle B, \leq, \mathtt{0}, \mathtt{1}\rangle$, an element a of B is an *atom* of $\mathfrak{B}$ iff $a \neq \mathtt{0}$ and for any $x \in A$: if $\mathtt{0} \neq x \neq a$, then $x \nleq a$. $\mathfrak{B}$ is *atomic* iff for each $x \in B \setminus \{\mathtt{0}\}$ there is an atom a such that $a \leq x$.

For any (complete) Boolean algebra $\mathfrak{A} = \langle A, +, *, -, 0, 1\rangle$, the structure $\mathfrak{B}_{\mathfrak{A}} := \langle A, \leqslant, 0, 1\rangle$ is a (complete) Boolean lattice and the operations $+$, $*$, and $-$ coincide, respectively, with $+$, $\cdot$, and $\backsim$. Of course, atoms of $\mathfrak{A}$ are exactly atoms of $\mathfrak{B}_{\mathfrak{A}}$. Moreover, $\mathfrak{A}$ is *atomic* iff $\mathfrak{B}_{\mathfrak{A}}$ is atomic.

For all Boolean lattices we can use the first-order language $\mathrm{L}^{\mathtt{0},\mathtt{1}}_{\leq}$ with equality, which has one binary predicate '$\leq$' and two individual constans '$\mathtt{0}$' and '$\mathtt{1}$'. Of course, all Boolean lattices are $\mathrm{L}^{\mathtt{0},\mathtt{1}}_{\leq}$-structures.

Elementary invariants. Let ω be the set of all natural numbers. As in [2, pp. 289–290], to any Boolean lattice $\mathfrak{B}$ we can assign exactly one special triple $\mathrm{inv}(\mathfrak{B}) = \langle \mathrm{inv}_1(\mathfrak{B}), \mathrm{inv}_2(\mathfrak{B}), \mathrm{inv}_3(\mathfrak{B})\rangle$ of *elementary invariants* of $\mathfrak{B}$, where $\mathrm{inv}_1(\mathfrak{B}) \in \{-1\} \cup \omega$, $\mathrm{inv}_2(\mathfrak{B}) \in \{0, 1\}$, and $\mathrm{inv}_3(\mathfrak{B}) \in \omega \cup \{\omega\}$.

Elementary invariants fully characterize Boolean lattices (algebras) with regard to their elementary equivalence (see Appendix A, p. 494). Namely, we have the following theorem:

THEOREM B.1 (cf. e.g. [2]). *Any two Boolean lattices have the same elementary invariants iff they are elementarily equivalent.*

Moreover, notice that the following facts hold:

Lemma B.2 (cf. e.g. [7]). *For any Boolean lattice* $\mathfrak{B}$:

1. $\mathfrak{B}$ *is atomic iff* $\mathrm{inv}_1(\mathfrak{B}) = 0 = \mathrm{inv}_2(\mathfrak{B})$.
2. *If* $\mathfrak{B}$ *is atomic and has infinitely many atoms, then* $\mathrm{inv}_3(\mathfrak{B}) = \omega$.

Applications. We put $\mathfrak{B}_1 := \langle 2^\omega, \subseteq, \emptyset, \omega\rangle$ and $\mathfrak{B}_2 := \langle \mathrm{FC}(\omega), \subseteq\rangle$, where $\mathrm{FC}(\omega)$ is the set of all finite and all co-finite subsets of ω. It is well known that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are atomic non-trivial Boolean lattices, which have infinitely many atoms. Moreover, $\mathfrak{B}_1$ is complete, but $\mathfrak{B}_2$ is not complete. So, in the light Lemma B.2, we obtain:

Corollary B.3. $\mathrm{inv}(\mathfrak{B}_1) = \langle 0, 0, \omega\rangle = \mathrm{inv}(\mathfrak{B}_2)$.

Thus, from the above lemma and Theorem B.1, we have:

Corollary B.4. *The Boolean lattices* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *are elementarily equivalent, i.e.,* $\mathrm{Th}(\mathfrak{B}_1) = \mathrm{Th}(\mathfrak{B}_2)$.

Finally, by the above corollary and Fact A.1, we get:

Theorem B.5. *The class of all complete Boolean lattices* (*resp. algebras*) *is not elementarily axiomatizable.*

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Andrzej Pietruszczak
Department of Logic
Faculty of Humanities
Nicolaus Copernicus University in Toruń, Poland
pietrusz@umk.pl