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REGIONS-BASED TWO DIMENSIONAL CONTINUA: THE EUCLIDEAN CASE

Abstract. We extend the work presented in [7, 8] to a regions-based, two-dimensional, Euclidean theory. The goal is to recover the classical continuum on a point-free basis. We first derive the Archimedean property for a class of readily postulated orientations of certain special regions, "generalized quadrilaterals" (intended as parallelograms), by which we cover the entire space. Then we generalize this to arbitrary orientations, and then establishing an isomorphism between the space and the usual point-based $\mathbb{R} \times \mathbb{R}$. As in the one-dimensional case, this is done on the basis of axioms which contain no explicit "extremal clause" (to the effect that "these are the only ways of generating regions"), and we have no axiom of induction other than ordinary numerical (mathematical) induction. Finally, having explicitly defined '*point*' and '*line*', we will derive the characteristic Parallel's Postulate (Playfair axiom) from regions-based axioms, and point the way toward deriving key Euclidean metrical properties.

Keywords: mereology, point-free geometry, Euclidean geometry, Tarski, gunk, Archimedean property, points

Introduction

This paper builds on work already presented [7, 8] on recovering the classical one-dimensional continuum on a point-free basis. The next section will summarize that work, omitting proofs. Those already familiar with it may wish to skip that section. Although our approach in the two-dimensional case is similar in certain respects, several new problems and issues arise demanding their own treatment. A well-known precedent for

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such efforts is Tarski's [12] ingenious reconstruction of three-dimensional Euclidean geometry on a point-free basis, using the primitive "sphere" (see also [2, 4]). There, however, once he had defined *point* and *equidistance* of two points from a third, Tarski simply adopted a known system of axioms for three-dimensional Euclidean geometry based on these two primitives.[1] This last step, however, is the very antithesis of "honest toil", better described as "grand larceny". Indeed, Tarski acknowledged that his system was "far from being simple and elegant".[2] Moreover, he wrote that "it seems very likely that this postulate system can be essentially simplified by using intrinsic properties of the geometry of solids".

Here we will present an "honest toil" regions-based reconstruction of two-dimensional continua and Euclidean geometry. To aid the exposition, we will proceed in stages, first deriving the Archimedean property for a class of readily postulated orientations of certain special regions, "generalized quadrilaterals" (intended as parallelograms), by which we cover the entire space. The second stage consists in generalizing this to arbitrary orientations, and then establishing an isomorphism between the space and the usual point-based $\mathbb{R} \times \mathbb{R}$. As in the one-dimensional case, this is done on the basis of axioms which contain no explicit "extremal clause" (to the effect that "these are the only ways of generating regions"), and we have no axiom of induction other than ordinary numer-

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[1] The system is due to Pieri [11].

[2] Tarski here severely understates the problem with his procedure. The usual Nagelian (e.g., [9],[10]) standard for reducing a theory T_2 to another T_1 requires both (1) that the predicates of T_2 be explicitly defined by those of T_1, *and* (2) that the axioms of T_2 be derived as theorems of T_1 using the definitions. Here (2) serves as a substantive constraint on the definitions of (1). But Tarski did not provide axioms for his T_1, beyond those of pure logic and mereology, and some postulates on "solid", defined as the fusion of a nonempty set of "spheres". Tarski simply adopted the translates of axioms of T_2 as new axioms of T_1. By such a procedure, any theory is automatically reducible to any other (with enough predicates of the right-arity), for nothing constrains the "definitions"!

Perhaps this has been missed because, in fact, Tarski's definitions of external and internal tangency and diametricality, concentricity, etc., aren't at all arbitrary but rather qualify as good and ingenious, despite his failure to state regions-based axioms on 'sphere'. Our evidence for this is that, within the point-based theory (T_2), we can derive theorems to the effect that each of the predicates to be defined holds if and only if the proposed *definiens* holds, but where the latter is translated back into the point-based theory. (No doubt, such reasoning is what led Tarski to contrive his definitions in the first place.) However, that reasoning all takes place in the "superstructure", that is in T_2, and so provides no substitute for the Nagelian requirement (2).

See [6] for an in depth treatment of Tarski's definitions and axioms.

ical (mathematical) induction. Finally, having explicitly defined '*point*' and '*line*', we will derive the characteristic Parallel's Postulate (Playfair axiom) from regions-based axioms, and point the way toward deriving key Euclidean metrical properties.

1. The basic theory: an atomless, mereological continuum in one dimension

Our formalism begins with classical first-order logic with identity supplemented with a standard axiom system for second-order logic (or logic of plural quantification, with an unrestricted comprehension axiom for plurals), and with an adaptation of the standard (Tarskian) axioms of mereology.

Axioms of Mereology:

Axioms 1.1a (on $x \leq y$; "x is part of y"): reflexive, anti-symmetric, transitive.

Certain of our axioms and theorems are stated in terms of a binary relation called "overlaps", defined in a standard way:

$$x \circ y \overset{\text{df}}{\Longleftrightarrow} \exists z(z \leq x \;\&\; z \leq y).$$

Axiom 1.1b (on $\leq$ and $\circ$).

$$\forall x \forall y \big(\forall z(z \circ x \to z \circ y) \to x \leq y\big).$$

Theorem 1.1. *Axioms 1a and 1b imply the Extensionality Principle*:

$$x = y \leftrightarrow \forall z(z \circ x \leftrightarrow z \circ y).$$

Thus we adopt an equivalent of a now standard mereology.

Axiom 1.2 (Fusion or whole comprehension).

$$\exists u \Phi(u) \to \big(\exists x \forall y[y \circ x \leftrightarrow \exists z(\Phi(z) \;\&\; z \circ y)]\big),$$

where Φ is a predicate of the second-order language (or language of plurals) lacking free x.

We write $x + y$ for the mereological sum or fusion of x and y, such that

$$\forall z\big(z \circ x + y \leftrightarrow (z \circ x \;\vee\; z \circ y)\big),$$

and we use $\sum_{n=0}^{\infty} x_n$ to designate fusions of infinitely many regions. Also, if $x \circ y$, then we write $x \wedge y$ for the *meet* of x and y, which satisfies

$$\forall z\big(z \le x \wedge y \leftrightarrow (z \le x \ \& \ z \le y)\big).$$

(If x and y have no common part, $x \wedge y$ is undefined.)

For "x is discrete from y" we use the following definition

$$x \mid y \overset{\text{df}}{\iff} \neg\, \exists z(z \le x \ \& \ z \le y)$$

i.e., $x \mid y \iff \neg\, x \circ y$.

Furthermore, if $\exists z(z \circ x \ \& \ z \mid y)$, then $x - y$ is the largest part of x which does not overlap y (and if there is no such part, then $x - y$ is undefined).[3] So

$$\forall z\big(z \le x - y \leftrightarrow (z \le x \ \& \ z \mid y)\big).$$

We define G to be the fusion of all regions. It is the entire space.

We introduce a geometric primitive, $L(x, y)$, to mean "x is (entirely) to the left of y". The axioms for L specify that it is irreflexive, asymmetric, and transitive. And we define '$R(x, y)$', "x is (entirely) to the right of y", as $L(y, x)$.

From this, we define a geometric relation, *betweenness*: $Betw(x, y, z)$ for "y is (entirely) between x and z":

$$Betw(x, y, z) \overset{\text{df}}{\iff} \big(L(x, y) \ \& \ R(z, y)\big) \vee \big(R(x, y) \ \& \ L(z, y)\big)$$

It follows that $Betw(x, y, z) \leftrightarrow Betw(z, y, x)$.

$L(x, y)$ obeys the following axioms:

AXIOM 1.3a. $L(x, y) \to x \mid y$. (Of course, $x \mid y$ implies $x \neq y$.)

It follows that $R(x, y) \to x \mid y$.

AXIOM 1.3b. $L(x, y) \leftrightarrow \forall z, u[z \le x \ \& \ u \le y \to L(z, u)]$.

Now we can define an essential notion, that of a "*connected* part of G". Intuitively, such a part has no gaps. The definition is straightforward:

$$Conn(x) \overset{\text{df}}{\iff} \forall y, z, u(z, u \le x \ \& \ Betw(z, y, u) \to y \le x).$$

[3] The definitions of "meet" and "difference" given in Hellman and Shapiro [7, 8] are incorrect.

In words, x is connected just in case anything lying between any two parts of x is also a part of x.

Furthermore, we can define what it means for a connected part of G to be *bounded*:

$$Bounded(p) \overset{\text{df}}{\Longleftrightarrow} \exists x, y(Conn(x) \;\&\; Conn(y) \;\&\; Betw(x, p, y).$$

In words, a connected region is bounded if it lies wholly between two others.

We call bounded, connected regions "*intervals*" and write '$Int(j)$', etc., when needed. However, note that, lacking points, we cannot describe intervals as either "open", "closed", or "half-open".

Once we establish that G is bi-infinite, i.e. infinite in both directions, and that G is Archimedean, it will follow that boundedness is also sufficient for "finite in extent", relative to any interval.

Using L, we can impose a condition of dichotomy for discrete intervals:

AXIOM 1.4 (Dichotomy axiom).

$$\forall i, j\big(i, j \text{ are two discrete intervals } \rightarrow \; (L(i, j) \vee L(j, i))\big).$$

Now we can prove a linearity condition among intervals:

THEOREM 1.2 (Linearity). *Let x, y, z be any three pairwise discrete intervals. Then exactly one of x, y, z is between the other two.*

To guarantee that arbitrarily small intervals exist everywhere along G, we adopt the following axiom:

AXIOM 1.5. $\forall x \exists j(Int(j) \;\&\; j < x)$.

This is what guarantees that the space is gunky.

An important relation of two intervals is "adjacency", which is defined as follows:

$$Adj(j, k) \overset{\text{df}}{\Longleftrightarrow} j \mid k \;\&\; \nexists m[Betw(j, m, k)].$$

The following equivalence relations on intervals will also prove useful:

"*j and k are left-end equivalent*" just in case

$$\exists p\big(p \leq j \;\&\; p \leq k \;\&\; \nexists q[(q \leq j \;\vee\; q \leq k) \;\&\; L(q, p)]\big).$$

Equivalently, two intervals are left-end equivalent just in any region that is left of one of them is left of the other. "*Right-end equivalent*" is defined analogously.

Left- (Right-) end equivalence means, intuitively, that the intervals "share their left (right) ends, or end-points, in common", but our system does not recognize "ends" or "end-points" as objects entering into mereological relations.

Our final primitive, for *congruence* among intervals, is used to insure that G is infinite in extent and in recovering, in effect, the rational numbers as a countable, dense subset of the (arithmetic) continuum, viz. *congruence*, as a binary relation among intervals. Intuitively, $Cong(i, j)$ is intended to mean "the lengths of intervals i and j are equal". Thus, we adopt the usual first-order axioms specifying that $Cong$ is an equivalence relation.

We will sometimes write this as $|i| = |j|$, but with the understanding that we have not yet given any meaning to '$|i|$' standing alone, but only in certain whole contexts. Similarly, for intervals i, j, we can define, contextually, $|i| < |j|$ as meaning: $\exists j'[j'$ an interval & $j' < j$ & $Cong(i, j')]$; and we may write $|i| > |j|$ as equivalent to $|j| < |i|$.

The next axiom is crucial to our characterization of G:

AXIOM 1.6 (Translation axiom). Given any two intervals, i and j, each is congruent both to a unique left-end-equivalent and to a unique right-end-equivalent of the other.

In effect, this guarantees that a given length can be "transported" (more accurately, instantiated) anywhere along G, and that these instances are unique as congruent and either left- or right-end equivalent to the given length. In particular, we can prove

THEOREM 1.3 (Trichotomy). *For any two intervals, i, j, either $|i| = |j|$ or $|i| < |j|$ or $|i| > |j|$.*

Our final axiom is that congruence respects nominalistic summation of adjacent intervals:

AXIOM 1.7 (Additivity). Given intervals i, j, i', j' such that $Adj(i, j)$, $Adj(i', j')$, $Cong(i, i')$, $Cong(j, j')$, then $Cong(k, k')$, where $k = i + j$ and $k' = i' + j'$.

We now turn to the matter of the bi-infinitude of G. In fact, our axioms already guarantee this, as we can now prove.

Theorem 1.4 (Bi-Infinity of G). *Let any interval i be given. Then there exist exactly two intervals, j, k, such that $Cong(i, j)$, and $Cong(i, k)$, and $Adj(i, j)$, and $Adj(i, k)$, and one of j, k is left of i and the other is right of i.*

Since bi-extension obviously iterates, this already insures that G is "bi-infinite" in the sense of containing, as a part, the fusion of the minimal closure of any interval i under the operation of "*bi-extension*" defined in the theorem. But we can do better and also insure that G is *exhausted* by iterating the process of flanking a given interval by two congruent ones as in Bi-infinity. This is just the Archimedean property:

Theorem 1.5 (Characterization of G). *Let G be the fusion of the objects in the range of the quantifiers of our axioms; and let i be any interval. Let i^* be the fusion of the minimal closure of i under biext. Then $G = i^*$.*

We can also show that G is Dedekind-complete.

Finally, we need a guarantee that any interval has a unique bisection. That can be proved as a theorem:

Theorem 1.6 (Existence and uniqueness of bi-sections). *Given any interval i, there exist intervals j, k such that $j < i$, and $k < i$, and $j \mid k$, and $j + k = i$, and $Cong(j, k)$, and j, k are unique with these properties.*

2. Regions-based two-dimensional continuum: derivation of the Archimedean property (restricted)

As in the one-dimensional case, we begin with first-order logic with $=$, augmented with plural quantification (or second-order logic) and mereology, taking over axioms 1.1a, 1.b, and 1.2 above. The parts of our space, which constitute the range of our first-order variables, we call "regions". Informally, we'll sometimes refer to our space as G^2. We write '$s \leq r$' for "s is part of r"; '$s < r$' for"s is a proper part of r", i.e., $s \leq r \;\&\; s \neq r$; $s \circ r$, for "s overlaps r", i.e., $\exists t(t \leq r \;\&\; t \leq s)$; and '$s \mid r$' for "$s$ is discrete from r", i.e., $\neg\, s \circ r$. Nominalistic summing or fusion is indicated by '+', which applies to arbitrary regions, with

$$r + s = t \Longleftrightarrow \forall u\big(u \circ t \leftrightarrow (u \circ r \vee u \circ s)\big).$$

As before, sums or fusions of all things of any (non-empty) plurality (or satisfying any non-vacuous predicate) are recognized. We use $\sum_{n=0}^{\infty} x_n$

to designate fusions of infinitely many regions. In the case of predicates or formulas, quantification over arbitrary wholes or pluralities is permitted, so that the system is in this sense “impredicative”, as expected in a classical reconstruction.

At this stage, our only other primitives are as follows: First, we have a primitive for *congruence* of regions, $Cong(s, r)$, understood as it usually is, to mean “same shape and size”. We also have a special type of region which we call “generalized quadrilateral” $gq(r)$. Our axioms will in effect delimit these to parallelograms, although we don’t use that term, as the notion of “line” is not yet available.[4] We also introduce four relative “direction” primitives associated with any given gq viz. ‘Up’, ‘Down’, ‘Left’, and ‘Right’ (using upper case first letters as abbreviations), subject to axioms to be introduced below. Now we define a notion of “co-oriented adjacency” of gq’s,

$$Adj(r, s) \text{ iff } r \mid s \;\&\; gq(r + s).$$

Intuitively, this means that r and s share exactly one entire border in common; that either s is U of r and r is D of s, or vice-versa, or r is L of s and s is R of r, or vice-versa; that there are no gaps between r and s; and that they are oriented in the same way, not oppositely, if they are not bilaterally symmetric. (In the latter case, if they are not both rectangles, their sum would not form a gq, despite sharing an entire border in common.) We normally drop the reference to co-orientation.

Our first axiom specifies obvious properties of congruence and the second introduces generalized quadrilaterals:

Axiom 2.0. $Cong(s, r)$ is an equivalence relation.

[4] In the next section we generalize the Archimedean property to a plenum of directions and derive the Euclidean Parallels Postulate. There, we will introduce further primitives pertaining to *angles* (or to angles less than 2π radians), conceived informally as regions shaped as sectors of circles (although we do not need ‘circle’ as a primitive).

Notice, incidentally, that in the one-dimensional theory of the previous section, “congruence” is only defined for intervals; here it is a relation on all regions. In the present section, however, we only apply “congruence” to generalized quadralaterals. In the next section, “congruence” is also applied to “angles”, construed as sectors of circles, and to “triangles”, which are certain parts of gq’s. For other regions, we can informally think of “congruence” as identity, or, for that matter, any other equivalence relation.

AXIOM 2.1. There exist *gq* regions. Any *gq* region, r, has a "*quadra-section*", that is, $r = s_1 + s_2 + s_3 + s_4$, each itself a *gq*, with all the s_i discrete from one another, with $Cong(s_i, s_j)$, and such that for each s_i, there are exactly two s_j such that $Adj(s_i, s_j)$. Furthermore, r has a "*quadra-extension*", that is, there are *exactly* four *gq*'s, r_i, pairwise discrete and discrete from r, with $Cong(r_i, r)$ and $Adj(r_i, r)$, for $i = 1, \ldots, 4$, such that each r_j is adjacent to the fusion of exactly two s_i, and each s_i enters into exactly two such fusions. Finally, r has a "*nona-section*", that is, $r = \sum_{i=1}^{9} s_i$, each s_i a *gq*, with $s_i \mid s_j$, $i \neq j$, and $Cong(s_i, s_j)$. Each extension r_i of r determined by quadra-extension is adjacent to the fusion of exactly three sections s_i of the nona-section; each of exactly four of the s_i enters into exactly two of these triple fusions, and exactly one of the s_i enters into no such triple fusion.

(To aid visualization: The first four nona-sections s_i of the final clause, informally speaking, are "at the corners" of r, and the single s_i not entering into any triple fusion adjacent to an r_j is the "central" one of the nona-section.)

Then repeated quadra-extension results in (among other things) four sums (call them "strings") of congruent regions starting with a central one, each string arranged linearly with each region (beyond the central one) adjacent to its predecessor and its successor, proceeding outward in a particular direction (the "direction of the string") corresponding to those of the principal axes of the central *gq*-region. A *segment* of such a string — equal to the fusion of the initial region and some of its successors and closed under "immediate predecessor" — is called "*bounded*" just in case there is some region discrete from all the members of the segment and beyond all of them in the same direction as that of the string. Our next axiom stipulates that sectioning and extending are inherited by congruents and certain sums thereof, and that these "operations" are indefinitely iterable:[5]

AXIOM 2.2. Any region congruent to a *gq* is itself a *gq*.

Below we will introduce "rectangles" as special kinds of *gq*'s and will adopt as an axiom (Axiom 2.3b) that every region has a rectangle as a part. Thus our axioms guarantee infinite divisibility of the space into *gq*-regions. Furthermore, they guarantee that the space is infinite in

[5] Strictly speaking, we are specifying what regions exist, regardless of application of operations. But speaking of operations is natural and perhaps of heuristic value.

extent in all directions, and that we can find *gq*-regions (intuitively) at arbitrary finite distances from any given one. Of course, we do not yet have that the space is Archimedean.

Note that Axiom 2.1 embodies closure of *gq*'s under fusion of a *gq* and any one of its extensions, by the requirement of adjacency as defined. One further closure condition on *gq*-regions will prove useful in recovering the classical Dedekind-Cantor continuum as superstructure over any "line" in our space, once that notion has been defined. To state this, define a sequence, $\rho = \langle r_j \rangle$, of *gq*'s to be "*sequentially adjacent*" just in case, for each j, $Adj(r_{j+1}, r_j)$, all in the same direction. Also, denote by $\Sigma_j(\langle r_j \rangle)$ the fusion of the *gq*'s, r_j, of such a sequence. Then we stipulate

AXIOM 2.3a. If $\rho = \langle r_j \rangle$ is sequentially adjacent and for some *gq* r, and every j, $r_j \leq r$, then $\Sigma_j(\langle r_j \rangle)$ is a *gq*.

This is, in effect, a Cauchy completeness condition adapted to *gq*-regions, and it will guarantee that, once the relevant terms have been suitably defined, *gq*'s with "borders" of arbitrary real "length" always exist.

Because of the iterability of sectioning, we are also furnished with many nested sequences of gq-regions, and many of these are convergent (in standard terms, to a point).

$$gq\text{-regions } \langle r_i \rangle \textit{ are nested} \text{ iff } r_k < r_j, \text{ for } k > j.$$

For point-wise convergence, however, we need a stronger nesting property:

$$gq\text{-regions } \langle r_i \rangle \text{ are } \textit{properly nested} \text{ iff } r_{k+1} \text{ is one of the sections of quadra-secting or nona-secting } r_k, \text{ for all } k = 1, 2, \ldots.$$

We will also, for convenience, sometimes call properly nested sequences "convergent", although we cannot yet distinguish true "convergence to a point" from "convergence to an infinitesimal region", not yet having proved that our space is Archimedean.

THEOREM 2.1. *There are properly nested gq-regions.*

PROOF. By Axiom 2.1, we may, for instance, take r_1 of our sequence to be, say, the UL, upper left quadrant of a first quadrasection of given gq-region, r. Given r_i, then let r_{i+1} be the DR, down-right quadrant of r_i. This defines a properly nested sequence (intuitively converging to the DR-most corner of the NW quadrant of r, i.e., the center of r). (Alternatively, work with repeated nona-sections to obtain a properly nested sequence converging in the same way, by always choosing the central region of the nona-section at any stage. □

Let r be a gq-region. We now introduce an operation, based on quadra-extension, that provides larger and larger regions, starting with r, growing exponentially without finite bound. Congruents of parts of any of these will then be "finite relative to r". The operation, called "4-fold-extension", is best illustrated with an example: Start with r, with its own relative orientation, labeled 'U', 'R', 'D', 'L'. Quadra-extend r, and designate by r_U the congruent of r adjacent to r and up, and similarly for r_R. Now by r_{R_U} designate the extension of r_R congruent and adjacent to it and up from it (selected to agree with"up" from r). Then the *4-fold-extension of* r — "4-$f(r)$" — is the *gq*-region $r + r_R + r_{R_U} + r_U$. Intuitively, the area of 4-f(r) is four times that of r. By iterating 4-f we specify exponentially larger and larger *gq*'s, intuitively occupying the UR quadrant of our space. (If the space is Archimedean, all of that quadrant will be covered by these regions.)

As indicated above by the choice of names of "directions", these terms are not absolute but are relative to each *gq*. In particular, the various *gq*'s need not be "oriented" the same way (although we take some steps to orient some of them, for purposes of simple exposition).

Say that Left and Right are *opposite* to each other, and also that Up and Down are *opposite* to each other. Say that the other pairs are *orthogonal.* So Up and Down are each orthogonal to Left and Right (and vice versa). In what follows, we'll use meta-variables T, T', T_1, etc. to range over directions (or direction labels). If T is a direction, then T- is its opposite. So if T is Left, then T- is Right, and vice versa, and if T is Up, then T- is Down, and vice versa.

Let s be a *gq* and let T be a direction. Let m be any region. We want some axioms governing the relation of m being T from s. Intuitively, the idea is that m is discrete from s and lies entirely on the T side of s, between lines that form the opposite borders of s (not necessarily strictly between). Of course, we cannot say that officially, since there are no lines (and so no borders). Instead, we give some axioms governing the relation. Of course, in proving the theorems, we rely only on the axioms.

Technically, we have four new binary relations between gq's and regions: *Left*(s, m), *Right*(s, m), *Up*(s, m), and *Down*(s, m). But we will usually speak more informally of a given region m being T of a *gq* s, where T is one of the four directions.

AXIOM 2.4a. Let m be a region, s a *gq*, and T a direction. If m is T from s, then $m \mid s$.

It follows immediately that the relations m *is* T *of* s are irreflexive.

AXIOM 2.4b. Let m, m' be regions, s a *gq*, and T a direction. If m is T of s and $m' \leq m$, then m' is T of s.

AXIOM 2.4c. Let s be *gq*, T a direction, and let M be any set (or plurality) of regions. If every member of M is T of s, then the fusion of M is also T of s.

Suppose that m is T from s. Then Axiom 2.4a says that the two are discrete and Axiom 2.4b says that every part of m is also T from s. Axiom 2.4c is a sort of converse to Axiom 2.4b.

Next are axioms that relate the four directions to the quadra-extensions, quadra-sections, and nona-sections of a given *gq*. Here we coordinate some of the directions.

AXIOM 2.5a. If s is a *gq* and T is a direction, then exactly one of the four quadra-extensions of s is T of s. So we can speak of the T-quadra-extension of s.

AXIOM 2.5b. If t is the T-quadra-extension of s, then s is the T-quadra-extension of t. Moreover, the T-quadra-extension of t is also T of s (and thus the T-quadra-extension of s is T- of t).

AXIOM 2.6a. Let s be a *gq*. Then the four quadra-sections of s can be labeled Up-Left (UL), Up-Right (UR), Down-Left (DL), and Down-Right (DR), such that (i) the UL-quadra-section of s is Left from the UR-quadra-section of s and Up from the DL-quadra-section of s ; the UR-quadra-section of s is Right from the UL-quadra-section of s and Up from the DR-quadra-section; and similarly for the other two. And (ii) the Left quadra-extension of the UL-quadra-section of s is the UR-quadra-section of the Left quadra-extension of s; and the Up quadra-extension of the UL-quadra-section of s is the DL-quadra-section of the Up quadra-extension of s; and similarly for the other three quadra-sections of s.

Axiom 2.6b. Let s be a *gq*. Then the nine nona-sections of s can be labeled Up-Left (UL), Up-Middle (UM), Up-Right (UR), Middle-Left (ML), Middle-Middle (MM), Middle-Right (MR), Down-Left (DL), Down-Middle (DM), and Down-Right (DR), such that (i) the Right quadra-extension of the UL-nona-section of s is the UM-quadra-section of s, the Left quadra-extension of the UL-nona-section of $s$is the UR-nona-section of the Left quadra-extension of s, the Up quadra-section of the UL-nona-section of s is the DL-nona-section of the Up quadra-section of s, and the Down quadra-section of the UL-nona-section of s is the ML-nona-section of s; and similarly for the other eight nona-sections.

Axiom 2.6c. If a region m is UP from the UL-quadra-section, the UR-quadra-section, the UL-nona-section, the UM-nona-section, or the UR-nona-section of a *gq* r, then m is UP from r. And similarly for the other directions.

Define the *spread* of a *gq* r to be the set (or plurality) of all *gq*'s in the minimal closure of $\{r\}$ under quadra-extension, quadra-section, and nona-section. The above axioms (are meant to) make sure that the various directions of *gq*'s in the spread of r are oriented with each other.[6]

Define two *gq*'s p, q to be *Aligned* if there are directions T, T' such that p is T of q and q is T' of p. If p and q are in the same spread, then, by the axioms 2.5, T' will be T-, the opposite direction of T, but that need not hold in general.

For example, let p be a *gq* and let q be the Up quadra-extension of the Up quadra-extension of p. Then p and q are Aligned.

Recall that, intuitively, a region m is, say, Left of a *gq* s just in case m is entirely to the Left of s, and fits between the extensions of the upper and lower boundaries of s. So, intuitively, two gq's are Aligned if they are discrete and the lines formed by two of their opposite boundaries are the same. Of course, this is all at the level of heuristics.

[6] We could extend the coordination a bit. Let q be a *gq* and let r be its Right quadra-extension. Then $q + r$ is itself a *gq*. We can stipulate that the UL-quadra-section of $q + r$ is the fusion of the UL-quadra-section and the UR-quadra-section of q. And so on. But there is no hope of coordinating *all* of the *gq*'s in G^2, for some of them might not even have the same "orientation" as q. Intuitively, some *gq*'s might be at an angle to others — so to speak. So we will have to live with the relativity of the directions.

AXIOM 2.7a (Transitivity). Let p be Aligned with q. So there are directions T, T' such that q is T of p; and p is T' of q. Then any region that is T'- of q is also T of p.

In other words, if p is Aligned with q, then, in effect, the T'- direction from q is the same as the T direction from p; anything that is in the T'-direction from q is also T of p. This also guarantees that our original gq p is not T'- of q. In effect, the Transitivity axiom guarantees that the space is not "closed" or, in other words, no matter how far one goes in the T direction, one will not get back to the gq at which we started. (As already noted, Axiom 2.4a implies that 'T of' is irreflexive.)

AXIOM 2.7b (Dichotomy). Let p be Aligned with q. So there are directions T, T' such that q is T of p; and p is T' of q. Let m be any region that is T' of q and discrete from p. Then either m is T of p, or m is T-of p.

In words, the Dichotomy axiom says if p is Aligned with q, so that there are directions T, T' such that q is T of p; and p is T' of q, then any region that is T' from q and discrete from p is either in between the gq's or on the opposite side of p.

AXIOM 2.8 (gq covering). Let T be a direction, and suppose that a region m is T from a gq r. Then m is a part of the fusion of all gq's that are T of r, congruent with r, and Aligned with r.

In other words, we can cover any region T from r with gq's, all of which are Aligned with r, congruent with r, and T of r. Notice that a converse of Axiom 2.8 follows from Axiom 2.4.

We need to rule out models in which the universe consists of two completely discrete "spaces", where no part of one of them can be "reached" from any part of the other via some combination of the four directions. For example, we should rule out models that consist of two copies of a point-free space, where no region in one of them is Up, Down, Left, or Right from any gq in the other. Our next axiom accomplishes this:

AXIOM 2.9 (Unity of Space). Let r be any gq, and let m be any region. Then there is a gq s such that (i) either $s = r$, or else both s is congruent with r and there is a direction T_1 of s such that r is T_1 of s and Aligned with s; and (ii) there is a direction T_2 of s, such that m overlaps the fusion of s and all gq's that are T_2 of s.

Intuitively, the idea is this: Start with any region r and let m be any region whatsoever. Then we can reach part of m by going in some direction from r, turning either right or left (or up or down), and then going in that direction.

Notice that nothing so far entails that the space is Archimedean. Even with Axiom 2.9, we have not ruled out models that have gq's, r, s where, say, s is Left of r, but infinitely far from r (relative to r, and relative to s).[7] We have also not ruled out gq's that are infinitely large relative to each other. Ruling out such models, via an Archimedean theorem, is the main order of business of this section.

More definitions: Let r and s be gq's that are Aligned with each other, and let m be any region. Say that m is *Between* r and s if and only if both of the following hold:

(i) there is a direction T_1 of r such that m and s are both T_1 from r; and
(ii) there is a direction T_2 of s such that m and r are both T_2 of s.

It follows that if m is Between r and s, then r is Aligned with s.

Let m and n be regions. Say that m and n are *Contiguous* just in case there is no gq t and no direction T of t, such that m is T of t and n is $T-$ of t. This definition is loosely modeled after Aristotle's. He says that two things are contiguous just in case nothing can get between them (e.g., *Physics* 226b21). Think of two adjacent books on a tightly packed shelf or two adjacent houses whose outer walls touch (or overlap). Here, we'd *like* to say that two regions are contiguous if there can be no region that comes between them. But we need at least one gq in order to have an orientation or a "direction", and thus a sense of "between". So we say that two regions are Contiguous if no gq can separate them, so that one of the regions is on one side of the gq, and the other region is on the other side.

Aristotle defines two things to be *continuous* if, when they are contiguous, the boundary between them is absorbed, and they become one

[7] To construct such a model, begin with a two-dimensional non-Archimedean space, such as one constructed from Robinson-style hyper-reals. Let the "regions" be regular open sets (those that are identical to the interior of their closure), and let the gq's be the interiors of parallelograms.

thing.[8] Think of bodies of water or, for that matter, regions of space. If they are contiguous with each other, they are (or become) one.

We can get something similar here, at least for gq's. Let r and s be gq's. Say that r is *Continuous* with s just in case they are Aligned, Contiguous, and there is no region that is between them.

Recall that we have another primitive $Adj(s, r)$, whose intended meaning is that r and s are discrete from each other and share part of a border, but, again, that is at the level of heuristics. It is part of the axiom defining *gq*'s that quadra-extensions are adjacent to one another. We can relate that to the present primitives:

AXIOM 2.10. Every *gq* is Continuous with its quadra-extensions. In general, if r and s are Aligned *gq*'s such that $Adj(r, s)$, then r and s are Continuous.

Notice, however, that it can happen that two *gq*'s are congruent with each other, Aligned, and *Contiguous*, and still there is a region that is between them — so that they are not Continuous. Think of two congruent parallelograms side by side, but oriented in opposite directions. We define rectangles to be *gq*'s where this does not happen:

Let q be a *gq*. Then q is a *Rectangle* just in case for every *gq* r, if r is Aligned with q, Contiguous with q, and congruent with q, then r is Continuous with q (i.e., there is no region that is between r and q). In fact, r is a quadra-extension of q.

AXIOM 2.11a. If a region q is congruent with a rectangle q', then q is a rectangle.

AXIOM 2.11b. If r and r' are Aligned, Contiguous rectangles, then $r + r'$ is a rectangle.

Some properties of rectangles:

AXIOM 2.12a (Decomposition I). Let r, s, t be rectangles such that r is Aligned with both s and t and that $s \circ t$. Then either $s \leq t$, or $t \leq s$, or there are three pairwise discrete rectangles s', t', v, all Aligned with r such that $s = s' + v$ and $t = t' + v$. Moreover, v is between s' and t'.

[8] *Physics*, Book V (227a6): "The continuous is just what is contiguous, but I say that a thing is continuous when the extremities of each at which they are in contact become one and the same and are (as the name implies) contained in each other. Continuity is impossible if these extremities are two. This definition makes it plain that continuity belongs to things that naturally, in virtue of their mutual contact form a unity."

AXIOM 2.12b (Decomposition II). Let q and r be any gq's (or we can restrict this to rectangles). If $q < r$, then $\neg\, Cong(q, r)$.

AXIOM 2.12c (Decomposition III). If q is a rectangle, then so are its quadra-sections and nona-sections.

AXIOM 2.12d (Additivity). Let r, r_1, r_2, s, s_1, and s_2 be rectangles, such that $r = r_1 + r_2$ with r_1 discrete from r_2, and $s = s_1 + s_2$ with s_1 discrete from s_2. If $Cong(r_1, s_1)$ and $Cong(r_2, s_2)$, then $Cong(r, s)$.

Our next axiom guarantees an abundance of rectangles, including arbitrarily small ones (in light of Axiom 2.1), as mentioned above (this axioms also guarantees that the space is gunky):

AXIOM 2.13b (Existence of Rectangles). Let m be any region. Then there is a rectangle r such that $r \leq m$.

Here, then, is our plan for establishing the Archimedean property: Let r be any rectangle and let T be a direction from r. Let r^T be the fusion of all regions that are T of r. Let r_1 be the T quadra-extension of r, and let R_T be the minimal closure of $\{r_1\}$ under the operation of taking the T quadra-extension. Notice that every rectangle in R_T is in the spread of r, and so has the same "orientation", by Axiom 2.6a. Let r_T be the fusion of R_T.

After adding one more definition and one more axiom, we will show that $r_T = r^T$. Then, with the Unity of Space and the Existence of Rectangles axioms, we are able to show that the entire space is Archimedean in the following sense: Let r be any rectangle. Let r^* be the fusion of all regions in the minimal closure of $\{r\}$ under quadra-extension. Then $r^* = G^2$. It also follows that all gq's are finite relative to each other.

Here is our final definition (for this part of the project): Let r be a gq and T a direction, and let a region m be T from r. Say that m is *Bounded to the* T of r if there is a gq s, such that (i) s is T of r and Aligned with r, and (ii) there is a direction T' of s such that r and m are both T' of s. In other words, m is Bounded to the T of r just in case there is a gq s Aligned with r, such that m is Between r and s.

And the final axiom of this section is this:

AXIOM 2.13 (Translation). Let r be a gq and T a direction. Suppose that a region m is T of r. Then, if m is Bounded to the T of r, then there is a gq r' such that (i) r' is congruent to r and Aligned with r; (ii) m is between r and r'; and (iii) m and r' are Contiguous.

Intuitively, the antecedent of the Translation axiom is that m is T from r and there is some gq on the "other side" of m (from r). The consequent of the axiom is that we can get a copy of r on the "other side" of m, Aligned with r and move it so close to m that no gq can get between them. In a sense, we have this copy of r just touching m.

THEOREM 2.2. *Let r be any rectangle and let T be any direction from r. Let r^T be the fusion of all regions that are T of r. Let r_1 be the T quadra-extension of r, and let R_T be the minimal closure of $\{r_1\}$ under the operation of taking the T quadra-extension. Let r_T be the fusion of R_T. Then $r_T = r^T$.*

PROOF. By transitivity, every member of r_T is T of r, and so, by Axiom 2.4c, r_T is T of r. So $r_T \leq r^T$. Suppose $r_T \neq r^T$. Then $r^T - r_T$ exists. By Axiom 2.4b, $r^T - r_T$ is T of r.

Let s be the set (or plurality) of all of all gq's that are T of r, congruent with r, and Aligned with r. The members of s are all rectangles. By Axiom 2.8 (gq covering), $r^T - r_T$ is part of the fusion of s. So let s be any rectangle in s that overlaps $r^T - r_T$. Since s is Aligned with r, there is a direction T' from s such that r is T' of s.

There are two cases. Case 1: Assume that s overlaps r_T. Then there is a rectangle r_n in r_T such that s overlaps r_n. So we have that s and r_n are both Aligned with r, and all three are rectangles (invoking Axiom 2.11a for r_n). So by Axiom 2.12a, either $s \leq r_n$, $r_n \leq s$, or there are three pairwise discrete rectangles s', r'_n, v, all Aligned with r, such that $s = s' + v$ and $r_n = r'_n + v$, with v between s' and r'_n. But if $s \leq r_n$, then r_n overlaps $r^T - r_T$, which is absurd since clearly $r_n \leq r_T$. If $r_n \leq s$, then we must have $r_n < s$, since s overlaps $r^T - r_T$. So, by Axiom 2.12b, Decomposition II, $\neg\, Cong(r_n, s)$, and so $\neg\, Cong(r, s)$, which is also absurd.

So there are three pairwise discrete rectangles s', r'_n, v, all Aligned with r, such that $s = s' + v$ and $r_n = r'_n + v$, with v between s' and r'_n. Since $v \leq r_T$, we have that s' overlaps $r^T - r_T$. Let r_{n+1} be the T quadra-extension of r_n. Since s is Aligned with r, there is a direction T' from s such that r is T' from s. Let s'' be the T'- quadra-extension of s. Since v is Aligned with r there is a direction T'' from v such that r is T'' from v. By Dichotomy (Axiom 2.7b) it is straightforward to show that s' and s'' are both T''- from v. Indeed, s' is between v and s''. So s' is Bounded to the T''- of v. By Translation (Axiom 2.13), there is a gq v' such that (i) v' is congruent to v and Aligned with v (and so with r, and everything else here); (ii) s' is between v and v'; and (iii) s' and v'

are Contiguous. By Axiom 2.11b, s' and v' are Continuous with each other and $s' + v'$ is a rectangle. By additivity (Axiom 2.12d), $s' + v'$ is congruent with s (which, recall, is $s' + v$). So $s' + v'$ is congruent with r and thus with r_n. It is also a straightforward consequence of Dichotomy (Axiom 2.7b) that $s' + v'$ is adjacent to r_n and T from r_n. So, by the uniqueness of T quadra-extension, we have that $s' + v'$ is r_n. But then $s' \leq r_T$, which contradicts the claim that s' overlaps $r^T - r_T$.

Case 2. Assume that s does not overlap r_T. By Axiom 2.7b (Dichotomy) either r_T is T' of s or r_T is T'– of s. But the latter is absurd. We would have s coming between r and its T quadra-extension r_1. So r_T is T' of s. So r_T is Bounded to the T of r. So, by Axiom 2.13 (Translation), there is a rectangle t such that (i) t is congruent to r and Aligned with r; (ii) r_T is between r and t; and (iii) r_T and t are Contiguous. So there is a direction T'' such that r is T'' of t. Let u be the T'' quadra-extension of t. So u overlaps r_T (by the contiguity of t and r_T). So there is a rectangle r_n in r_T such that u overlaps r_n. If either $r_n \leq u$ or $u \leq r_n$ then $r_n = u$ (by Axiom 2.12b, Decomposition II). But in that case, by the uniqueness of quadra-extension, t would be the T quadra-extension of r_n and so $t \leq r_T$, which contradicts the hypothesis (ii) that r_T is between r and t.

So there are three pairwise discrete rectangles u', r'_n, v, all Aligned with r, such that $u = u' + v$ and $r_n = r'_n + v$, and v is between u' and r'_n. Proceeding similarly to Case 1, note that since v is Aligned with r there is a direction T_1 from v such that r is T_1 from v. By Dichotomy (Axiom 2.7b) it is straightforward to show that u' and t are both T_1-from v. Indeed, u' is between v and t. So u is bounded to the T_1-of v. By Translation (Axiom 2.13), there is a *gq* v' such that (i) v' is congruent to v and Aligned with v (and so with r, and everything else here); (ii) u' is between v and v'; and (iii) u' and v' are Contiguous. By Axiom 2.11b, u' and v' are *Continuous* with each other and $u' + v'$ is a rectangle. By additivity (Axiom 2.12d), $s' + v'$ is congruent with u (which, recall, is $u' + v$). So $u' + v'$ is congruent with r and thus with r_n. It is also a straightforward consequence of Dichotomy (Axiom 2.7b) that $u' + v'$ is adjacent to r_n and T from r_n. So, by the uniqueness of T quadra-extension, we have that $s' + v'$ is r_{n+1}, the T quadra-extension of r_n. So $v' \leq r_T$. By hypothesis, v' is discrete from both v and u', and so from $v + u' = u$. With Dichotomy (Axiom 2.7b), it is straightforward to check that v' is between u and t. But this contradicts the fact that u and t are quadra-extensions of each other. $\square$

Theorem 2.3. *Let r be any rectangle. Let r be the minimal closure of $\{r\}$ under quadra-extension, and let r^* be the fusion of r. Then r^* is G^2, the entire space.*

Proof Sketch. Suppose not. Then, since obviously $r^* \leq G^2$, $G^2 - r^*$ exists. So let m be any region that is discrete from r^*.

By Axiom 2.9, Unity of Space, There is a *gq* s such that (i) either $s = r$, or else both s is congruent with r and there is a direction T_1 of s such that r is T_1 of s and Aligned with s; and (ii) there is a direction T_2 of s, such that m overlaps the fusion of s and all gq's that are T_2 of s.

If $s = r$ then m overlaps the fusion of all gq's that are T_2 of r. Let r_1 be the T_2 quadra-extension of r. We have that m overlaps the fusion of r and all *gq*'s that are T_2 of r. By Theorem 2.2, m overlaps the fusion of r and the fusion of the minimal closure of $\{r_1\}$ under T_2 quadra-extension. Either way, m overlaps r^*, which is a contradiction.

If, instead, $s \neq r$, then, by the definition of Alignment, there is a direction T of r such that s is T from r. As in Theorem 2.2, let r_1 be the T quadra-extension of r; and let r_T be the minimal closure of $\{r_1\}$ under the operation of taking the T quadra-extension. By a construction similar to that used in the proof of Theorem 2.2, there is a member r_n of r_T such that $s \leq r_n + r_{n+1}$, where r_{n+1} is the T quadra-extension of r_n. We have that there is a direction T_2 of s such that m overlaps the fusion of all *gq*'s that are T_2 of s. It is straightforward to check that there is a direction T' of $r_n + r_{n+1}$ such that m overlaps the fusion of all gq's that are T' of $r_n + r_{n+1}$. Applying Theorem 2.2 again, we have that m overlaps the fusion of the minimal closure of $\{r_n + r_{n+1}\}$ under T' quadra-section. It is straightforward to verify that m thus overlaps r^*, a contradiction. □

3. Generalization of the Archimedean property and recovery of the parallels postulate

So far, we have derived the Archimedean property as it pertains to any of the four directions relative to a given *gq*. There is, however, nothing so far that guarantees that gq's exist in arbitrary orientations, as usually understood in terms of lines or classes of parallel lines. In fact, the following describes a model $\mathfrak{M}$ of the axioms thus far. The "regions" of $\mathfrak{M}$ are regular, open sets of pairs of real numbers, with the usual metric and topology.[9] So the entire space G^2 of $\mathfrak{M}$ is $\mathbb{R}^2$. Mereological "parts" are subsets, and the fusion of a set S of regions is the interior of the closure of the union of the members of S. Let i be any open parallelogram, say one whose interior angles are $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ (radians). Say that a regular, open set m is a gq of $\mathfrak{M}$ just in case m is a parallelogram whose sides are parallel to those of i. So, in $\mathfrak{M}$ all gq's are oriented the same way (i.e., they are all oriented the same as i). Let s and r be two regions of $\mathfrak{M}$. Say that $Cong(s, r)$ holds in $\mathfrak{M}$ just in case either s and r are gq's and s is congruent (in $\mathbb{R}^2$) to r, or else neither is a gq and $s = r$ (see note 4 above). It is straightforward to verify that $\mathfrak{M}$ satisfies all of the axioms of the previous section. In $\mathfrak{M}$ all gq's are "rectangles" (even though they are not rectangles in $\mathbb{R}^2$).

In effect, the Unity of Space Axiom 2.9 and the Principle 1.2 of unrestricted fusions guarantees that, in any model, there will be *regions* with all sorts of sizes, shapes, and orientations. For example, there will be regions that are *intuitively* "congruent" to a given gq, but oriented differently, say at a $\frac{\pi}{6}$ angle from it. But, as indicated by the model $\mathfrak{M}$, we do not yet have the means to show that these regions *are* gq's and that they stand in the $Cong$ relation to any gq's (despite Axiom 2.2).

In this section, we add some further primitives on regions and axioms governing them that enable us to rule out models like $\mathfrak{M}$, and, indeed, to establish the full Archimedean property for arbitrary directions. These primitives, it turns out, also suffice for the full, regions-based recovery of Euclidean geometry, to be shown explicitly below in the case of the parallels postulate ("EPP").

The bulk of our task is definitional, introducing enough conceptual machinery to state axioms adequate for generalizing the Archimedean property and recovering Euclidean geometry. Our method employs a notion of "*angle*" as a region, written $Ang(\theta)$. In ordinary terms, we may think of an angle as a sector or a circle, or better, as an equivalence class of such sharing a common vertex, with the angles respectively of concentric circles such that they all share a common vertex and each is an "initial part describing the same angle", i.e. all are are angles as sectors of larger concentric circles, all of the same angle norm (to be defined, below). But this is just heuristics. We will often drop the primitive '*Ang*', reserving θ, φ, ψ, etc. as angle variables (typically for

[9] So $\mathfrak{M}$ is an analogue of one of the topological models of our one dimensional theory ([7, 8]).

generic representatives of equivalence classes). Of course, we don't yet have vertices as points, so we will need some new primitives on such regions and axioms governing them. It helps that we already have *gq*'s, intuitively containing four angles (as regions) at it's "corners". Further, we have the resources to define explicitly the "corners" of a given *gq*. For example, using our relative direction terms, the "top-left corner" of given *gq*, r, can be identified with the nested sequence $\langle r_i \rangle$ of *gq*'s r_i under iterated quadrasection starting with $r_0 = r$, setting $r_{i+1} =$ the top-left quadrasection of r_i (and similarly for the other three corners).[10] Thus we can simply identify the "vertex" of an angle ($< \pi$ radians) with the equivalence class of co-convergent n- and q-sequences defining the relevant "corner" of a *gq* from which it derives. Further, since angles in our sense are regions, we already have applicable our primitive congruence relation, which, in the case of angles, will be written $Cong(\theta, \varphi)$, where, as usual, this means, in ordinary terms, "same size and same shape". Here are the axioms we will use:

Axiom 3.1a. Let r be a *gq*; then r has exactly four angles, $Ang(\theta_i)$, $i = 1, \ldots, 4$, with each $\theta_i < r$, such that the vertex of $\theta_1 =$ the UL corner as defined, and likewise proceeding clockwise. Further, diagonally opposite angles are congruent, $Cong(\theta_1, \theta_3)$ and $Cong(\theta_2, \theta_4)$.

We also want to stipulate that an angle, as part of a *gq* "fills up" a corner of the *gq* in the sense of having "sides" that coincide with segments of two adjacent sides of the *gq*. This can be expressed using the relative direction terms, "to the D of", as in the previous section:

Axiom 3.1b. Let r be a *gq* and let θ be an angle at the FF′ corner of r (where 'FF′' takes the values 'UL', 'UR', 'DL', or 'DR'); then no part of θ lies F of r or F′ of r, and no part of r lies F of θ or F′ of θ.

Based on this, we can re-define *rectangle*(r) as: r is a *gq* all four of whose angles are congruent.[11] Then each such angle, θ, is called "*right*", and its norm, $|\theta| = \frac{\pi}{2}$. By our Axiom 2.13 of the previous section, rectangles exist (in abundance).

[10] The definition of '*point*' in our framework is given below via properly nested sequences of *gq*'s under iterated nona-sectioning, and co-convergence of these with properly nested sequences under quadra-sectioning is also defined.

[11] As indicated by the above model $\mathfrak{M}$, this definition of "rectangle" is not equivalent to that of the previous section. It is straightforward that rectangles, in the present sense, meet the defintion of "rectangle" in the previous section.

Given just this much, we already can *define* a suitable relation of *adjacency* of two angles to one another: $Adj(\varphi, \psi)$ iff $\varphi \mid \psi$ & $Ang(\varphi + \psi)$. It is convenient also to introduce two subrelations of angle adjacency, $+Adj(\varphi, \psi)$ and $-Adj(\varphi, \psi)$, the former defined as $Adj(\varphi, \psi)$ & φ is clockwise of ψ, the latter as the same except substituting 'counterclockwise' for 'clockwise'.

Invoking quadraextensions, we have that each right angle, say, θ_1, of a rectangle is adjacent to two others (one, θ_2, $+Adj$ to θ_1, the other, θ_3, $-Adj$), and those two are adjacent to a fourth, θ_4, completing a *circle* (as a region) $= \Sigma_{i=1}^{4}\theta_i$. All circles may be identified in this way, as an angle.

Our next axiom is one of Euclid's:

Axiom 3.1c. Let θ and ψ be any two right angles; then $Cong(\theta, \psi)$. If C is a circle, its norm, $|C| = 2\pi$.

In general, the (angle) norm function is stipulated to be countably additive with respect to sequentially adjacent angles: $|\Sigma_j\theta_j| = \Sigma_j|\theta_j|$, mod 2π, whenever $Adj(\theta_j, \theta_{j+1})$, each j, with $|\theta_j| \leqslant 2\pi$ (where we have abused notation, using the Σ inside the brackets to denote mereological fusion and the Σ outside to be ordinary addition of real numbers). This is taken to apply to finite sums as well as countably infinite.

Axiom 3.2. Every angle has a (polar) bisection:

$$\forall\theta\exists\varphi, \psi\big(Cong(\varphi, \psi) \;\&\; Adj(\varphi, \psi) \;\&\; \theta = \varphi + \psi\big)$$

('+' in the sense of mereology). Except in the case of $\theta =$ a circle, the bisection is unique.

This already implies that angles are "bilaterally symmetric", and hence that, as it is usually expressed in terms of points and lines, the two "sides" of an angle (as a circle sector) are "of equal length". And since Axiom 3.2 can be iterated any finite number of times, this guarantees that all of the angles obtained under iterated polar bisection have sides of the same length as one another. Our next axiom enables us to extend this to subangles of arbitrary real norm ($\bmod 2\pi$) relative to that of any given angle.

Axiom 3.3. Let $\langle\theta_j\rangle$ be a sequence of angles, each $\theta_j < \theta$, such that $Adj(\theta_j, \theta_{j+1})$, each j; then the fusion, $\Sigma_j\theta_j$, of the θ_j is an angle, θ^j, with $\theta^j \leqslant \theta$.

Angles of a sequence $\langle\theta_j\rangle$ satisfying the hypothesis of this last axiom will (collectively) be called "sequentially adjacent". If all the *Adj* instances are in the same sense, + or −, we call the angles "sequentially +adjacent" and "sequentially −adjacent", respectively. Since angles of irrational norm (mod 2π) can be obtained exactly by fusion of convergent sequentially adjacent angles of binary rational norm, the effect of these last two axioms is to furnish us with a continuum of angles of any given circle, in one-one correspondence with those as usually defined via intersecting half-lines.

Our next axiom insures that fusions of sequentially adjacent angles preserve congruence:

AXIOM 3.4. If the angles of $\langle\theta_j\rangle$ are either sequentially +adjacent or −adjacent and those of $\langle\varphi_j\rangle$ are either sequentially +adjacent or −adjacent, with all the $\theta_j < \theta$ and all the $\varphi_j < \varphi$, and $Cong(\theta_j, \varphi_j)$, all j, then $Cong(\Sigma_j\theta_j, \Sigma_j\varphi_j)$.

Similarly we adopt a kind of "converse" to this:

AXIOM 3.5. Under the same hypotheses as the previous axiom but with $Cong(\theta_j, \varphi_j)$ holding for all j except for one, $j = k$, then if also $Cong(\Sigma_j\theta_j, \Sigma_j\varphi_j)$, then $Cong(\theta_k, \varphi_k)$.

Our next step is to obtain the effect of rotation operators, here understood as rotating a given angle through another given angle. Once that is accomplished, we will be able to state an axiom guaranteeing existence of *gq*'s (or rectangles) oriented in arbitrary directions in space. Then the derivation of the Archimedean property of the previous section will straightforwardly generalize as desired.

In the usual terms of points and lines, any angle (in our sense of region, as well as on the standard conception) uniquely determines a circle, with the vertex as center and either side of the angle as radius. How can we express this function from angles to circles just in terms of regions? One requirement for a given angle θ to be *of* a circle C is that the fusion of θ together with sufficiently (finitely) many copies of θ, sequentially adjacent in the same sense (+ or −), is identical with C.[12] Thus we postulate

[12] It makes no difference if the last copy of θ returns to θ with an overlapping excess: the fusion of the sequence is still just the circle determined by θ.

Axiom 3.6. (Angle biextensions). For any given angle θ ($< 2\pi$), there are exactly two unique angles, θ', θ'', such that $Cong(\theta, \theta')$, $Cong(\theta, \theta'')$, and $+Adj(\theta', \theta)$ and $-Adj(\theta'', \theta)$.

Then we define that θ *is* of C to mean that, for some n, there are n angles, θ_j, each congruent to θ, sequentially +adjacent (or −adjacent), with $\theta_1 = \theta$, such that the fusion, $\Sigma_j \theta_j = C$. (Of course, then $\theta \leqslant C$.)

Now let C and C' be given congruent circles, and let θ be an angle of C, and let φ be an angle of C'. Our next axiom generalizes the last, guaranteeing that copies of either angle exist adjacent to either side of the other.

Axiom 3.7 (Angle translation). Given circles C, C' with $Cong(C, C')$ and angles θ of C and φ of C', each $< 2\pi$, there are unique φ, φ' with $Cong(\varphi', \varphi)$ and $Cong(\varphi'', \varphi)$ and $+Adj(\varphi', \theta)$ and $-Adj(\varphi'', \theta)$.

(Since the circle and angle variables in this are universally quantified, it already implies the statement with the roles of θ, φ, and C, C' reversed.)[13]

Angle translation and the intended interpretation of *Adj* also provide a route to recovering the usual definition of 'circle' via equal-length radii from a fixed point (center, vertex of angles in our sense).

The pieces are now in place for us to define general rotation operators on angles, and hence on *gq*'s. Specifically, for example, we would like to complete a formula that defines: $+R(\theta, \theta', \varphi)$ iff θ', with $Cong(\theta, \theta')$, represents "the result of rotating θ clockwise through angle φ", where $+Adj(\varphi, \theta)$.

This can be done without defining any new relations as follows: first, apply biextension to φ, obtaining φ' with $Cong(\varphi', \varphi)$ and $+Adj(\varphi', \varphi)$; then require that $-Adj(\theta', \varphi')$. Now, in terms of line segments, the same sides of θ and θ' form angles congruent to φ. Formally, we first define the four-place relation,

$$\begin{aligned} +R(\theta, \theta', \varphi, \varphi') \ \text{ iff } \ & Cong(\theta, \theta') \ \& \\ & +Adj(\varphi, \theta) \ \& \ Cong(\varphi, \varphi') \ \& \ +Adj(\varphi', \varphi) \ \& \ -Adj(\theta', \varphi')]. \end{aligned}$$

[13] This axiom also implies the previous one on angle biextensions as a special case. The latter, however, is needed in order to infer that the definition of "θ is of C" is instantiated, as in the axiom of angle translation.

Then we define

$$+R(\theta, \theta', \varphi) \text{ iff } \exists\varphi'[+R(\theta, \theta', \varphi, \varphi')].$$

'$-R(\theta, \theta', \varphi)$', "$\theta'$ results from counterclockwise rotation of θ through φ", is defined analogously.

The last axiom of this group is key to proving our main results of this and the next section, generalization of the Archimedean property to arbitrary directions and recovering the Euclidean parallels postulate:

AXIOM 3.8 (Angles to gq's). Let θ be any angle $< \pi$; then there is a gq, r, with θ as one of its angles.[14]

As a special case, let r be a given rectangle with ψ as one of its right angles, and let φ be an arbitrary angle $< \pi$; then, by Angle translation, there exists ψ' with $Cong(\psi, \psi')$ such that, for some φ' satisfying $Cong(\varphi, \varphi')$, we have that $+R(\psi, \psi', \varphi, \varphi')$ (and similarly for $-R$), i.e., ψ' results from positive (negative) rotation of ψ through angle φ. Then by the "Angles to gq's" axiom, there is a rectangle r' with ψ' as one of its angles. Since φ is arbitrary, this furnishes us with rectangles with arbitrary orientations, as desired. Thus, we have

THEOREM 3.1. Same statement as Theorem 2.2: *Let r be any rectangle and let T be any direction from r. Let r^T be the fusion of all regions that are T of r. Let r_1 be the T quadra-extension of r, and let R_T be the minimal closure of $\{r_1\}$ under the operation of taking the T quadra-extension. Let r_T be the fusion of R_T. Then $r_T = r^T$.*

Here, however, the variable 'r' over rectangles ranges over those of arbitrary orientations. The proof of Theorem 2.2 carries over intact. Similarly with regard to the next theorem and its proof:

THEOREM 3.2. Same statement as Theorem 2.3: *Let r be any rectangle. Let R be the minimal closure of $\{r\}$ under quadra-extension, and let r^* be the fusion of R. Then r^* is G^2, the entire space.*

Now we turn to the recovery of the Euclidean parallels postulate ("EPP"). This breaks down naturally into two tasks: first, we need to

[14] Although angles were first introduced above as deriving from gq's, occupying their "corners", this axiom is not redundant as it applies to arbitrary angles, including all those obtained from given ones via the operations of bisection, biextension, translation, and fusion of sequentially adjacent angles.

introduce '*point*' and '*line*' and enough relations among these to be able to *express* the EPP in our framework. Second, we then will introduce '*triangle*' and the relation of '*similarity*' between triangles and some axioms governing these predicates as a means of recovering the Euclidean proof of *uniqueness* of parallels (through a given point, in relation to a given line). The *existence* of parallels already will follow from our arbitrarily oriented gq's.

Turning to the first task: We will want to introduce "lines" as certain sets or pluralities of points, where points are taken to be sequences of properly nested gq's. For this, we must be assured of distinct points, which translates as "non-co-convergent" sequences. Thus, we need a criterion of co-convergence (intuitively meaning, "to the same point").

Note that two co-convergent sequences (speaking informally here), if obtained by repeated *quadra*-secting, may be entirely discrete from one another, i.e., no region of one sequence overlaps any of the other. It can even be that no region of one is adjacent to any of the other. (For example, take one sequence to be that defined in the proof of Theorem 2.1, above, and take a second, also converging to the "center" of r, to be that which results from switching the roles of 'NW' and 'SE' in the definition of the first sequence). However, if we restrict our definition to sequences of gq-regions obtained always by *nona-secting* the previous region, then the criterion for co-convergence is especially simple, and, moreover, the proof that non-co-convergent sequences exist is made easy. So we define:

Properly nested sequence $\sigma = \langle \sigma_i \rangle$ of gq-regions is an *n-sequence* iff each σ_{i+1} is the central region of the nona-section of σ_i.

Properly nested n-sequences $\rho = \langle \rho_i \rangle$, $\sigma = \langle \sigma_j \rangle$ are *co-convergent* (we write $CoConv(\rho, \sigma)$) iff $\forall i \exists j\ \sigma_j \leq \rho_i\ \&\ \forall j \exists i\ \rho_i \leq \sigma_j$.

It is easily proved that $CoConv$ is an equivalence relation. (This is left to the reader.)

Now we prove that there are many non-co-convergent n-sequences.

Lemma 3.3. *Given any n-sequence σ, there exists an n-sequence τ such that $\neg\, CoConv(\sigma, \tau)$.*

Proof. If the first term σ_1 of σ is part of gq-region r, let τ_1 be any gq-region such that $\tau_1 < r'$, where r' can be any region resulting from applying quadra-extension to r. Then, since τ_1 is discrete from σ_1, in fact all the τ_i are discrete from all the σ_j by the choice of central regions of nona-sections at each stage. $\square$

In light of our previous axioms, this provides many non-co-convergent pairs of convergent sequences, hence many mutually distinct "points" and, as will emerge, distinct "lines" as well.

It will also be useful to invoke properly nested *gq*-sequences obtained by *quadra*-section at each stage, especially when speaking of "corners" and "border points" of *gq*-regions (cf. Axiom 3.1a, above). Such points must be defined by *n*-sequences, but we may say that properly nested sequences via quadra-section — call them "*q-sequences* — are also co-convergent with a fixed *n*-sequence. For this, we may define

Q-sequences ρ, σ are *co-convergent with n-sequence* τ (we write: $CoConv(\rho, \sigma, \tau)$) iff $\forall \tau_i, \rho_j, \sigma_k(\tau_i \circ \rho_j \ \& \ \tau_i \circ \sigma_k)$. Holding *n*-sequence τ fixed, this is also an equivalence relation between the first two terms, provided the relation holds (i.e., reflexivity, $CoConv(\sigma, \sigma, \tau)$, obtains whenever the condition $\tau_i \circ \sigma_j$ for all i, j, is met).[15]

We define "*p is a point*" to mean p is an equivalence class of properly nested co-convergent *n*-sequences. Occasionally, when convenient, we will refer to a generic member of such a class rather than to the class itself.

Our definition of line, one of several equivalents chosen for convenience, follows closely that of classical analysis applied to geometry: Stated informally, "*l is a line*" iff l = the maximal class of "*collinear*" points, where three distinct points — i.e., pairwise non-co-convergent — n, p, q, are defined to be *collinear* (we write $Coll(n, p, q)$) just in case they are all points of a common "border" of a *gq*, r. This in turn is spelled out via q-sequences co-convergent with the *n*-sequences defining the respective points. For example, to be a point of, say, the U ("up") border of *gq* r is to be specified by a q-sequence $\langle \sigma_j \rangle$, where $\sigma_1 = r$ and σ_{j+1} = the "UX-quadrasection of σ_j", where 'X' can vary over relative directions R and L. Similar patterns specify the other "border points" of r, *mutatis mutandis.*

Our next two lemmas establishes the equivalence of our definition of 'line' with the more common one as a pair of distinct points:

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[15] What we call *n*-sequences are closely related to a special case of "representatives of points", as introduced in [5]. See also [3]. The relation of co-convergence is analogous to the relation of mutual covering of sets of regions. Much of this work is inspired by [13].

LEMMA 3.4. *Let p and q be distinct points. Then they both lie on exactly one line.*

PROOF. Invoking rectangles of an n-sequence for p, say, and rotation of one of these if needed, one obtains a rectangle r with p and q as adjacent corners. One of the borders, call it B, of r connects p and q. This determines a line l thus: take the minimal closure of r under bi-extension in the two directions orthogonal to B; l then consists of all the points of the B-borders of these rectangles. Clearly it is a collinearity class, and the Archimedean property insures that it is maximal. To prove uniqueness of l, suppose both p and q also lie on a line $k \neq l$. Since k and l are not parallel, they meet at a point, m, which may be supposed to be p or q or neither. Now it is a property of gq borders that any two points of a border are at a straight angle (of norm π) to each other (the vertex of which is the mid-point between them). (Cf. Lemma 3.7, below.) Suppose m is p; then there is a point n of k such that p is equidistant from q and n (via a gq with segment $\overline{pn}$ of k as a border congruent to a gq with a segment $\overline{pq}$ of l as a border). But the angle qpn formed at $p \neq \pi$ unless the segment $\overline{pn}$ is of line l as well as k. In that case, by an induction, iterating this construction along k indefinitely in both directions would show that $k = l$, contradicting the hypothesis. Thus, we can suppose that $\overline{pn}$ is of k but not of l, and q, p, n are not collinear with n, therefore p and q cannot both lie on k, contradicting the hypothesis. The other two cases, assuming $m = q$ or m distinct from p and q, are argued analogously. This proves the uniqueness of l as the maximal collinearity class determined by the pair, p, q. □

Immediate from Lemma 3.4. we obtain:

COROLLARY. *Two distinct non-parallel lines share exactly one point in common.*

We now have a kind of "transitivity" of the collinearity relation:

LEMMA 3.5. *If $Coll(n, p, q)$ and $Coll(p, q, u)$, then both $Coll(n, p, u)$ and $Coll(n, q, u)$.*[16]

[16] In this and several further lemmas and theorems, we are using vocabulary of point-based geometry. But also such terms are used as defined in our framework of regions, and, furthermore, they do not occur, even as defined, in our axioms (contrast this with [12]). Thus, our procedures are entirely in line with those of a full-fledged reduction of one theory to another.

PROOF. By definition, the hypothesis together with the previous lemma implies that n, p, q, and u all lie on the same line, l, viz. that uniquely determined by the pair p, q, whence the conclusion follows by definition of the *Coll* relation. □

For any *gq*, r, we have the minimal closure, $Cl^{X,Y}(r)$, of $\{r\}$ under bi-extension in two given opposite directions (i.e. $X = \mathrm{U}$, $Y = \mathrm{D}$ or $X = \mathrm{L}$, $Y = \mathrm{R}$). Lines in our sense can be thought of as the common bi-infinite borders of the fusion of all the gq's in any one of these minimal closures. It should be clear that these constructions along with Axiom 3.8 on the "isotropy" of angles and their parent *gq*'s, guarantee that our space has a plenum of lines in 1–1 correspondence with those of ordinary point-based Euclidean geometry, or classical (Cartesian) analytic geometry.[17]

This much already suffices to recover half of the Euclidean Parallels Postulate. Here is an especially easy proof:

THEOREM 3.6 (Existence of parallels). *Given any line l and a point p not on l, there exists a line l' with p on l' such that no point lies on both l and l'.*

PROOF. (sketch) The idea is to start with an arbitrary *gq* r with a segment b of l as one of the borders of r. (By definition of '*line*', there is, for any point q on l, such r with q on a border of r.) Without loss, suppose that p is D and D$'$ of r, where D and D$'$ are two adjacent directions relative to r, e.g., U and R, and where one of these, say it's U, is opposite to the border b. (In favorable cases, one of the directions would suffice, or p could already be "in r", where that means that, for any n-sequence $\sigma_p = \langle \sigma_i \rangle_p$ defining p, for some j and every $m \geq j$, $\sigma_m \leq r$.) Next extend r in these directions a finite number of times in each which are sufficient to result via fusion in a *gq* $r' \geq r$ so that p is in r'. (Here the Archimedean property of our space is used.) Call that fusion F. If p already is a U-border point of r', we're done, as the bi-infinite L-R extension of F then has as its U border a parallel, l', to l. In the general case, it is necessary to subtract a *gq* from an end of F (from the "top" of F in the case where b is the D (down) border of F. (This is always possible by sufficiently finely subdividing F, via quadrasection along the axis determined by U, and taking fusions of Cauchy sequences of *gq*'s (in this instance, strips extending from L to R) as needed.) The result of

[17] In effect, we have just sketched a key part of the recovery of point-based geometry as superstructure over our regions-based space.

such a subtraction is then a *gq*, F', with p lying on its U border. The bi-infinite extension of this then provides the desired parallel l'. The proof that l' so constructed never meets l invokes the fact that any n-sequence defining a point of l' is eventually discrete from any such defining a point of l (in accordance with visualization). □

To recover the uniqueness of parallels, a bit more apparatus is useful, viz. the introduction of *triangles* and the essentially Euclidean phenomenon of ever larger *similar* triangles to a given one. (Indeed, the existence of similars that are non-congruent is just a version of the EPP, inter-derivable with the Playfair version we're recovering. The absence of non-congruent similars is, of course, one of the key elementary features of spaces of non-zero curvature.)

We define "*triangle*" as a diagonal half of a *gq*: $Tr(t)$ iff $\exists r, t'(r$ a *gq* & $r = t+t'$ & "t, t' result from diagonal bisection of r"), where the clause in quotes is defined thus: t shares two adjacent borders with r and t' shares the other two adjacent borders with r & $Cong(t, t')$ & $Adj(t, t')$, where a border of a *gq* is defined via q-sequences as above, and adjacency of such borders is defined via disjunction of relevant pairs of border-labels, proceeding clockwise, i.e., (U,R) ∨ (R,D) ∨ (D,L) ∨ (L,U).

To insure existence of triangles, we add to the group of axioms labeled 3.1 the following

Axiom 3.1d. Let r be a *gq* region. Then r has two diagonal bi-sections, each into two congruent, non-overlapping triangles, t, t' (with $r = t+t'$), each having exactly three angles, θ, φ, ψ of t, θ', φ', ψ' of t', with $Cong(\theta, \theta')$, $Cong(\varphi, \varphi')$, and $Cong(\psi, \psi')$, and satisfying $Adj(\varphi, \psi')$ and $Adj(\psi, \varphi')$ with $\varphi + \psi'$ an angle of r congruent to $\psi + \varphi'$, the diagonally opposite angle of r, and with θ and θ' the remaining two angles of r, respectively.

In view of our axioms on angles and *gq*'s, this guarantees arbitrary triangles existing everywhere and in all orientations ("ubiquity" and "isotropy" of triangles, inherited from those properties of *gq*'s).

Now suppose, as in the hypothesis of the statement of uniqueness of parallels, we are given a line, l, and a point, p, not on l. Suppose, further that we have available also a line, l', through p and parallel to l, as already established. The proof should then proceed to demonstrate that any other line, k, through p eventually meets the given l. That will follow by constructing ever larger similar triangles (say, by doubling segments) formed by segments of l' and segments of k, both proceeding from p (as a vertex common to all the triangles involved) in the direction D or D$-$ such that that half of k "lies on the l-side of l'". Of course, we need to be able to render this in our regions-only language. And we need to be sure that our "lines" as defined behave in accordance with the presupposition, that lines in the roles of l' and k indeed "cross one another" at p so that there are indeed a half-line of k and a direction D such that, to the D (or D$-$) of p, k lies on the l-side of l'.

A couple of definitions:

Given a line l and points p and q of l, point m is *between* p and q, $Betw(p, m, q)$, just in case p and q are corners of a border, b, of a given gq, r, such that $m \neq p$, $m \neq q$, and m is also a point of b.

Let l be a line; then a (finite) *segment s of l is the set or plurality of all points between any two given points, p and q, of l.*

The next lemma states a fact cited above in the proof of Lemma 3.4:

LEMMA 3.7. *Let l be a given line with p an arbitrary point on l. Then p is the vertex of a straight angle (of norm π) formed by a segment of l about p.*

PROOF. Since p is defined by an n-sequence, indeed of rectangles, it is also a "corner" of four rectangles, meeting at p as center of a larger rectangle. By a simultaneous rotation of these rectangles, the same border of two of them adjacent to each other, we obtain a segment of l about p as desired. □

LEMMA 3.8. *Let lines l and l' meet at p. Then segments of them form four angles at p such that the two opposite angles of either of the two pairs of opposite angles are congruent to one another.*

The proof employs Lemma 3.7 repeatedly and is just the familiar one of standard Euclidean geometry, and it is left to the reader.

We will also need an axiom to insure the obvious requirements that congruence of two gq's suffices for congruence of the triangles of one with those of the other, and that congruence of two gq's or of two triangles implies congruence of corresponding angles of those respective figures.

AXIOM 3.9a. Let $Cong(r, r')$ for gq's r and r', and let t and t' be triangles of r and r', respectively. Then $Cong(t, t')$.

AXIOM 3.9b. Let $Cong(r, r')$ for gq's r and r' and let the four angles θ_i of r correspond to the four angles θ'_i of r', as indicated by their indices. Then $Cong(\theta_i, \theta'_i)$, each i.

AXIOM 3.9c. Let $Cong(t, t')$ for triangles t and t' and let the three angles θ_i of t correspond to the three angles θ'_i of t', as indicated by their indices. Then $Cong(\theta_i, \theta'_i)$, each i.

The pieces are now in place to prove the uniqueness part of the Euclidean Parallels Postulate:

THEOREM 3.9 (Uniqueness of parallels). *Given a line l and a point p not on l, there is a unique line l' through p that is parallel to l.*

PROOF. Assume, for a contradiction, that, in addition to a parallel, l', to l through p (constructed as in the proof of Theorem 3.5), there is a second parallel, k, to l through p. By lemmas 3.6, 3.7, and 3.8, k and l' form an angle $\theta < \pi$ on the l side of l'. For ease of exposition, suppose that l is "down" relative to the gq's defining l', and that k proceeds "down" and "left" relative to those gq's. Further, we may assume, without loss, that those gq's are rectangles $\langle r_j \rangle$, $j = \ldots -2, -1, 0, 1, 2, \ldots$, and that it is their "up" borders that define l' (so that they extend toward l). Either one of those rectangles (say r_0) or a rectangle that is a proper part of one of them has point p as its UR corner. Let q_0 be any other point of k down and left of p, chosen so that q_0 lies "up" from given line l. There is a rectangle, s_0, with p as its UR corner, q_0 as its DL corner, with its UL corner a point m on l', and with its DR corner a point p_0 on a line $l_\perp$ through p perpendicular to l' and l. (By bisecting the straight angle centered on p, licensed by Axiom 3.2, a perpendicular to a given line at one of its points always exists and is unique. That a line perpendicular to both l' and l exists is guaranteed by the construction of parallels, as in the proof of Theorem 3.5, where the gq's involved are taken to be rectangles.) By Axiom 3.1d, there is a triangle, t_0, derived from s_0 by diagonal bisection, with corners (vertices) p, q_0, m_0. Now, quadra-extending s_0 down and left and quadra-extending the left one down, construct a larger rectangle, s_1, with s_0 forming its UR quadrant. In addition to p as UR corner of s_1, designate the other corners p_1 (DR), q_1 (DL) and m_1 (UL). Then the diagonal bisection of s_1 defined by a segment of k (from p to q_1), yields as upper triangle t_1 containing t_0 as a proper "initial" part (proceeding down and left from p). Claim: Triangles t_0 and t_1 are similar. Proof of Claim: By Axiom 2.1, the four rectangles making up s_1 are congruent; so, in particular, the two pairs of triangles formed by k as constituting the UR quadrant and the DL quadrant of s_1 are all congruent, whence — by Axiom 3.9c — so are their corresponding angles, in particular the two formed by k and the "verticals" perpendicular to l', proving the claim. It follows that the respective borders of t_1, in particular the verticals just mentioned, are twice the length of those of t_0. (This can be taken as a definition of relative lengths of line segments.) Since, by the Archimedean property, the vertical distance between l' and l is finite, a finite number of doublings of rectangles, of sequence $\langle s_i \rangle$, and their (upper) triangles, $\langle t_i \rangle$, suffices to insure that some t_i overlaps one of the rectangles whose U borders serve to define the given line l.[18] The existence of an intersection point of k and l can now be established as follows. Call the triangle t_1 defined above a "DL-*extension*" of t_0, i.e., with t_0 similar to t_1 and with segment $\overline{p, q_1} = \overline{p, q_0} + \overline{q_0, q_1}$ as hypotenuse of t_1 (and likewise, *mutatis mutandis*, for the arms of t_1. Similarly, the rectangle s_1 from which t_1 is derived is a DL-extension of s_0 from which t_0 is derived. Now we generalize this so that DL-extensions of a triangle or rectangle can be of any size greater than the given figures, but retaining similarity. This is assured by the existence of perpendiculars from arbitrary points of l' and $l_\perp$, which always meet k (by the Archimedean property and the derivation of lines from strings of gq's or rectangles). Now form the fusion F of all rectangles s which are DL-extensions of s_0 such that s is discrete from the half-space — call it H_D — down from given line l. We claim that F is a rectangle whose D border is a segment n of l, and hence a point, call it q^*, of which is also a point of k, in fact the point being the DL corner of F. Suppose not: there are two cases.

Case 1. F overlaps H_D. This is impossible, since by mereology, if a fusion of objects meeting a condition C overlaps an object x, then at least one of the C objects overlaps x, and, in this case, C requires discreteness from H_D.

Case 2. Some region r is down from F but up from H_D (in this sense, "lying between" F and H_D. But this too is impossible, since then there would be a DL-extension of s_0 beyond F and overlapping r yet still discrete from H_D, contradicting the hypothesis that F is the fusion

[18] In a point-based analytic treatment, an analogue of the intermediate value theorem would establish that line k meets line l. That argument, of course, is not yet available in our framework, so we still must establish an intersection point of k and l.

of all DL-extensions of s_0 discrete from H_D. That such a DL-extension of F exists can be seen by taking a rectangular part of r, guaranteed by Axiom 10 of the previous section, re-orienting it (via a rotation) if necessary so that its Up and Down borders are parallel with l, and extending it Left so the DL corner of the extended rectangle meets k, and extending it Right to meet $l_\perp$ as needed to construct a DL-extension of F, contradicting the definition of F. Thus, F is indeed adjacent to a rectangular strip of H_D bounded above by the D-border of F, i.e., that border of F must indeed coincide with a segment of l, which we labeled n. Furthermore, the L-border of F does not overlap the half-space H_L defined as follows: let q be any point of k lying within the half-space H_D (i.e., such that any n-sequence for q eventually is part of H_D), and let v_q be the vertical line through q perpendicular to l'. Let H_{v_q} be the half-space left of v_q, and let H_L be the fusion of all H_{v_q} as q varies over the points of k lying in H_D. By construction of F, it must be discrete from H_L. Furthermore, by an argument exactly analogous to that just given relating the D-border of F to H_D, it follows that the L-border of F is adjacent to a rectangular strip of H_L defined up and down by the parallels l and l', respectively. This enables a definition (via an n-sequence) of the DL corner of F, and this specifies a point q^* as the intersection point of l and k. □

4. Closing Reflections

The above thus recapitulates the essentials of Euclidean geometry in a regions-based framework. In particular, the “points”, defined above, are isomorphic to $\mathbb{R}^2$. As in the one dimensional case, it is also fairly straightforward to construct models of the regions-based theory (or theories) in the more usual punctiform $\mathbb{R}^2$. In one such model, the “regions” are regular, open sets of points, and the “gq’s” are open parallelograms. In another model, the “regions” are regular closed sets of points, and the “gq’s” are closed parallelograms. In both cases, a “region” m is part of a “region” m' if $m \subseteq m'$. The other primitives are straightforward.

It is also fairly straightforward to extend the two-dimensional theory to three or more dimensions. We’d begin with the notion of a “generalized rectangular solid” and go from there. It is perhaps not so straightforward to develop various regions-based non-Euclidean geometries (especially of variable curvature). That is work in progress.

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