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NOTES ON MODELS OF FIRST-ORDER MEREOLOGICAL THEORIES

Abstract. This paper will consider some interesting mereological models and, by looking into them carefully, will clarify some important metalogical issues, such as definability, atomicity and decidability. More precisely, this paper will inquire into what kind of subsets can be defined in certain mereological models, what kind of axioms can guarantee that any member is composed of atoms and what kind of axioms are crucial, by regulating the models in a certain way, for an axiomatized mereological theory to be decidable.

Keywords: mereology; models; first-order mereological theories

1. A quick background

This paper will consider some interesting mereological structures and, by looking into them carefully, will clarify some important metalogical issues.

As usual, we should start with a brief introduction to the mereological language, axioms and theories. The first-order mereological language contains only one binary predicate P (except the equality sign, which is a logical symbol), whose intended meaning is "being a part of". Three additional predicates are defined as follows.

Proper Part: $PPxy := Pxy \wedge \neg Pyx$
Overlap: $Oxy := \exists z(Pzx \wedge Pzy)$
Underlap: $Uxy := \exists z(Pxz \wedge Pyz)$

We can see in the literature the following mereological axioms ([6, 1]).

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(P1) Pxx (reflexivity)
(P2) $(Pxy \wedge Pyx) \to x = y$ (anti-symmetry)
(P3) $(Pxy \wedge Pyz) \to Pxz$ (transitivity)
(EP) $\forall x \forall y(\exists z PPzx \to (\forall z(PPzx \leftrightarrow PPzy) \to x = y))$ (extensionality)
(WSP) $\forall x \forall y(PPxy \to \exists z(PPzy \wedge \neg Ozx))$ (weak supplementation)
(SSP) $\forall x \forall y(\neg Pyx \to \exists z(Pzy \wedge \neg Ozx))$ (strong supplementation)
(FS) $\forall x \forall y(Uxy \to \exists z \forall w(Owz \leftrightarrow (Owx \vee Owy)))$ (finite sum)
(FP) $\forall x \forall y(Oxy \to \exists z \forall w(Pwz \leftrightarrow (Pwx \wedge Pwy)))$ (finite product)
(A) $\forall x \exists y(Pyx \wedge \forall z \neg PPzy)$ (atomicity)
($\overline{\text{A}}$) $\forall x \exists y PPyx$ (atomlessness)
(G) $\exists x \forall y Pyx$ (the greatest member)
(C) $\forall x(\neg \forall z Pzx \to \exists z \forall w(Pwz \leftrightarrow \neg Owx))$ (complementation)

Moreover, for any formula α in which x is free but z and y do not occur free (α might have free variables other than x):

(UF) $\exists x \alpha(x) \to \exists z \forall y(Oyz \leftrightarrow \exists x(\alpha(x) \wedge Oyx))$ (unrestricted fusion axiom schema)

Here we will consider one additional axiom which we will call "local complementation":

(LC) $\forall x \forall y((PPxy \wedge \neg \forall z Pzy) \to \exists z(PPzy \wedge \neg Ozx \wedge \forall w(Owy \leftrightarrow (Owx \vee Owz))))$

We will also use (Infinity) to stand for the list of infinitely many axioms each of which says that there are at least n members, starting from $n = 2$. It should be clear how to formalize each of them.

Any consistent combination of the axioms listed above will axiomatize a mereological theory. The following mereological theories can be found in the literature.[1]

Ground Mereology: **GM** = (P1) + (P2) + (P3)
Minimal Mereology: **MM** = **GM** + (WSP)
Extensional Mereology: **EM** = **GM** + (SSP)
Closure Mereology: **CM** = **GM** + (FS) + (FP)
Minimal Extensional Mereology: **MEM** = **MM** + (FP)

[1] The nomenclature here mainly comes from [1]. It has been argued that the first three axioms (P1), (P2) and (P3) constitute part of the meaning of "part" [6] and hence must be included in any mereological theory. We can see from this that any mereological model must be a partial ordering.

Minimal Closure Mereology: **CMM** = **MM** + (FS) + (FP)
Extensional Closure Mereology: **CEM** = **EM** + (FS) + (FP)
General Extensional Mereology: **GEM** = **EM** + (UF)

It is not difficult to check that (EP), (WSP), (FS), (FP), (G), (C) and (LC) are theorems of **GEM** and therefore **GEM** is the strongest theory on this list. It is also easy to see that both **GEM** + (A) and **GEM** + ($\bar{\text{A}}$) are consistent and that they are the two strongest incompatible mereological theories which can be formed by using the mereological axioms listed above.

If the theory considered is at least as strong as **CEM**, the following two additional binary function symbols can defined.

(Sum) $x+y=z$ iff $(Uxy \wedge \forall w(Owz \leftrightarrow (Owx \vee Owy))) \vee (\neg Uxy \wedge x=z)$.
(Product) $x \times y = z$ iff

$$(Oxy \wedge \forall w(Pwz \leftrightarrow (Pwx \wedge Pwy))) \vee (\neg Oxy \wedge x = z).$$

Moreover, if the theory considered is at least as strong as **GEM**, the following unary meta-function symbol can be defined, for any formula α in which x is free but z and y do not occur free:[2]

$$\sigma x(\alpha(x)) = z \quad \text{iff}$$
$$(\exists x \alpha(x) \wedge \forall y(Oyz \leftrightarrow \exists x(\alpha(x) \wedge Oyx))) \vee (\neg \exists x \alpha(x) \wedge \forall y Pyz)$$

One more note is that for any structure M, we will use Dom(M) to stand for the domain of M.

[2] Observe that **CEM** has (FS) and (FP) as axioms and has $\forall z(Oxz \leftrightarrow Oyz) \to x = y$ as a theorem. Therefore it can guarantee the existence and the uniqueness of the sum or the product of any two members if they are underlapped or overlapped. However, in order to make the interpretations of $+$ and $\times$ total functions, we have to specify what the output is if two input arguments are not underlapped or not overlapped. An anonymous referee correctly points out that $+$ and $\times$ defined above are not commutative. To remove this peculiarity, "$x = z$" in both definitions can be replaced by "$\forall w Pwz$". Nonetheless, since such a peculiarity does not cause any problem to my results and I have already given the same definitions in a previous paper, I will stick to them here. Also observe that **GEM** has (UF) and is stronger than **CEM**. Hence it can guarantee the existence and the uniqueness of the fusion of members of the subset defined by a satisfiable formula α. But, again, we have to specify what the output is if α is not satisfiable. By the way, note that σx maps a formula to a member in the domain. Hence it cannot be thought of as a definable symbol in the object language and that is why we call it a "meta-function symbol".

2. Definability

In the following, we will be concerned only with mereological structures each of which satisfies at least **CEM**+(G). Call a mereological structure M "complete" if for any nonempty subset S of Dom(M), there is some $a \in \mathrm{Dom}(M)$ such that for any member $b \in \mathrm{Dom}(M)$, a overlaps b if and only if some member in S overlaps b. Consider a mereological structure M whose domain has all the nonempty subsets of R^2 and where P is interpreted as the set inclusion, that is, $M \vDash Pxy[a, b]$ if and only if $a \subseteq b$. It is easy to see that M satisfies **GEM** and that it is complete. Intuitively, the completeness of M is not definable since Dom(M) is uncountable while the mereological language is countable, that is, there must be some subset of Dom(M) which cannot be defined by any formula in the mereological language. However, the completeness defined here is a property of a mereological structure and therefore might be captured by a sentence in the mereological language. But by using the fact that **CEM** + (C) + (G) + (A) + (Infinity) is a complete theory [8], we can easily argue that no such sentence exists.

Claim 1. *"Being a complete mereological structure" is not first-order definable.*

Proof. Suppose on the contrary there is such a sentence α in the mereological language which defines the completeness of a mereological structure, that is, for any mereological structure M, M is complete if and only if $M \vDash \alpha$. Consider two atomic mereological structures M_1 and M_2, where both Dom(M_1) and Dom(M_2) have exactly ω-many atoms, but Dom(M_1) contains only members each of which is composed of a nonempty finite subset or a cofinite subset of the atoms in Dom(M_1) while Dom(M_2) contains all the members each of which is composed of a nonempty subset of the atoms in Dom(M_2). Obviously, M_1 is not complete but M_2 is. However, it can be easily checked that both M_1 and M_2 are models of **CEM** + (C) + (G) + (A) + (Infinity), which, as mentioned above, is a complete theory. Therefore, M_1 and M_2 are elementarily equivalent, but they cannot both satisfy α, which leads to a contradiction. Hence the completeness of a mereological structure is not first-order definable. $\square$

What kinds of subsets are definable is also an important issue. Consider the following simple mereological structure M whose domain has all the nonempty subsets of R and in which P is as usual interpreted as the set inclusion. Consider an arbitrary formula $\alpha(x)$ with exactly one free variable. It can be shown that the cardinality of the subset defined by $\alpha(x)$ is either 0 or 1 or uncountable.

Claim 2. *Let M be the mereological structure whose domain has all the nonempty subsets of R and in which P is interpreted as the set inclusion. For any formula $\alpha(x)$ with exactly one free variable, the subset of Dom(M) defined by $\alpha(x)$ is uncountable if it has more than one member.*

Proof. Of course, if M does not satisfy $\alpha(x)$, the subset which it defines has cardinality 0. If M satisfies $\alpha(x)$ only with R, for instance, $\alpha(x)$ is, say, $\forall yPyx$, then the subset which it defines has cardinality 1. Otherwise, M satisfies $\alpha(x)$ with some member $d \neq R$. In that case, call the subset defined by $\alpha(x)$ S and there are three possibilities. (1) For some $d \in S$, d or its complement is bounded at least in one direction. Then for any $r \in R$, $f(x) = x + r$ will induce a homeomorphism on R and in turn an automorphism on M, and this implies that there are in $\mathrm{Dom}(M)$ uncountably many members which can satisfy $\alpha(x)$. For example, if $(-\infty, 1)$ satisfy $\alpha(x)$, then for any positive r, $(-\infty, 1 + r)$ will also satisfy $\alpha(x)$ and since there are uncountably many distinct positive real numbers, there will be in $\mathrm{Dom}(M)$ uncountably many distinct members which can satisfy $\alpha(x)$, that is, S is uncountable. Also note that $\exists y(\alpha(y) \wedge \forall z(Pzx \leftrightarrow \neg Ozy))$ defines the subset each of whose members is the complement of some member of the subset defined by $\alpha(x)$, and it is obvious that these two subsets have the same cardinality if one of them has at least two members. (2) For some $d \in S$, both d and its complement are unbounded but not both dense. Then we can argue in a way similar to the one used in the previous case. Assume without loss of generality that d is not dense. Then we can find a segment s which is not included in d. Let $f(x) = x + r$, where $|r| <$ the length of s. It is obvious that there are uncountably many such $f(x)$ and by the argument given in the previous case, we can see that this implies that S must be uncountable. (3) For any member $d \in S$, both d and its complement are dense. Suppose S is countable and let's enumerate its members as d_0, d_1, d_2, ... (if S is finite, then $d_n = d_{n+1}$, for any $n \in \omega$ such that $n \geqslant i$, for some fixed $i \in \omega$). By induction on $n \in \omega$, we can choose ω many pairwise disjoint bounded open intervals as follows. Step 0: First choose a nonempty bounded open interval I_0. Then choose in I_0 a point $p_0 \in d_0$ and a point $p'_0 \in {\sim} d_0$ (we use ${\sim} x$ to stand for the complement of x). This can be done since both d_0 and ${\sim} d_0$ are dense. Step $n + 1$: First choose a nonempty bounded open interval I_{n+1} such that it is disjoint from any of the open intervals chosen in the previous n steps. Then choose in I_{n+1} a point $p_{n+1} \in d_0$ and a point $p'_{n+1} \in \sim d_{n+1}$. Again, this can be done since both d_0 and $\sim d_{n+1}$ are dense. Now it is not difficult to see that we can define a homeomorphism f on R which maps p_n to p'_n in each I_n. Then $f[d_0]$, that is, $\{x : \exists y \in d_0, f(y) = x\}$, will be in S too, since it is easy to see that $f[\ ]$ is an automorphism on M. However, by the choices of p_n and p'_n, $f[d_0] \neq d_n$, for any $n \in \omega$, which means that $f[d_0] \notin S$. This leads to a contradiction. Therefore, S cannot be countable.[3] □

A similar trick can be played on R^2. Consider the mereological structure M whose domain is $\{x : x$ is a nonempty regular open subset of $R^2\}$ and in which P is as usual interpreted as the set inclusion. It is easy to see that M satisfies **GEM**. Now let $p_0, p_1, p_2, \ldots$ enumerate all the rational ordered pairs of R^2 (a rational ordered pair is an ordered pair whose coordinates are rational numbers). This can be done since there are only countable such pairs. For each $i \in \omega$, let $L(p_i)$ be the first coordinate of p_i and $R(p_i)$ be the second coordinate of p_i. Also let $d_i = \{(x, y) : (x - L(p_i))^2 + (y - R(p_i))^2 < 2^{-2i}\}$, that is, d_i is the open disk which is centered at p_i and whose radius is 2^{-i}.[4] Finally, let $D = \bigcup_{i\in\omega} d_i$. It should be clear that the area of D has a finite upper bound and hence $D \neq R^2$. Besides, it is not difficult to see that for any $x \in \mathrm{Dom}(M)$, x overlaps D if and only if x overlaps R^2. But it can also be easily proved that $x = y \leftrightarrow \forall z(Oxz \leftrightarrow Oyz)$ is a theorem of **GEM**. Therefore, we can conclude that $D \notin \mathrm{Dom}(M)$, that is, D is not regular open. On the other hand, if $\{d_i : i \in \omega\}$ is a definable subset of $\mathrm{Dom}(M)$, the fusion of its members will actually be the whole R^2. However, by an argument similar to the proof of Claim 2, it can be shown that $\{d_i : i \in \omega\}$ cannot be defined by any formula with exactly one free variable.[5]

[3] A similar idea has been used to prove that any nonempty definable (by a formula with exactly one free variable) subset of the mereotopological structure based on R^2 must be a singleton or be uncountable (see [10]).

[4] The definition of such a subset of open disks is owing to [2].

[5] I guess that such a result can be generalized as follows. For any formula $\alpha(x, y_1, \ldots, y_k)$ with exactly $k+1$ free variables, and for any $b_1, \ldots, b_k \in \mathrm{Dom}(M)$, the cardinality of the subset $\{a \in \mathrm{Dom}(M) \colon M \models \alpha(x, y_1, \ldots, y_k)[a, b_1, \ldots, b_k]\}$ will either be finite or uncountable.

To extend our discussion a little bit, let's expand the mereological structure M considered in Claim 2 to a mereotopological structure M' in which the predicate C is interpreted as $M' \vDash Cxy[a, b]$ iff $cl(a) \cap cl(b)$ is nonempty, where cl is the closure operator induced by the standard topology on R.[6] Consider the following axiom schema where $\phi(x)$ has exactly one free variable x.

$$(*)\ \ \exists x\phi(x) \rightarrow (z = \sigma x(\phi(x)) \rightarrow \forall y(Cyz \rightarrow \exists x(\phi(x) \wedge Cyx)))$$

where $z = \sigma x(\phi(x))$ means that z is the fusion of all x such that $\phi(x)$.

Intuitively, such a schema says that the fusion formed by a definable subset contacts something only if some member of that definable subset contacts that thing.

Now let's define the following closed intervals of R. $A_n = [1/n, 1 - 1/n]$, for all $n > 1$. Then the fusion of those A_n is equal to the open interval $(0, 1)$. However, $(0, 1)$ contacts $[-1, 0]$, while none of the intervals A_n does. Of course, those A_n and their fusion, that is, $(0, 1)$, are all in $\mathrm{Dom}(M')$. Does this show that M' does not satisfy $(*)$?[7]

The answer is negative. In order to show that $(*)$ is false in M', one has to find a formula $\phi(x)$ satisfiable in M' such that the fusion formed by members of the subset defined by $\phi(x)$ contacts something but no member of such a subset contacts that thing. The problem is that as shown in Claim 2, no formula in the formal mereotopological language can define a subset of $\mathrm{Dom}(M')$ whose cardinality is ω, and therefore the closed intervals A_n cannot constitute a counterexample to $(*)$ in M' (note that any mapping mentioned in the proof of Claim 2 is a homeomorphism on R and hence won't change topological properties). In short, it is not clear whether M' can satisfy $(*)$ or not, but it is certain that any alleged counterexample must involve uncountably many members in the domain.

[6] The formal language of mereotopology contains one more binary predicate C, which stands for "contact" or "being connected to". For more information, see [1].

[7] The axiom schema $(*)$ has been given by [13] (here we have a simplified case, since the original version allows free variables other than x). This intended counterexample to $(*)$ is owing to Marion Haemmerli.

3. Atomicity

The axiom of atomicity (A) is intended to guarantee that every member in the domain is composed of atoms. However, this axiom alone cannot do the job unless it works with a background theory which is strong enough. It has been shown that **EM** is inadequate by a counterexample and now with a little bit revision on the previous construction, we will see that **CEM** is not good either.[8]

Let S be a nonempty closed interval of R and U be a set of ω many points of R such that $S \cap U$ is empty. Define a mereological structure M as follows.

First define ω many sets M_0, M_1, ..., M_n, ... by induction on n.

Step 0: $S \cup U \in M_0$ and for any nonzero finitely many k points p_1, p_2, ..., p_k from U, $\{p_1, p_2, \ldots, p_k\} \in M_0$.

Step $n+1$: Suppose $x \in M_n$ is of the form $S' \cup U'$ where S' is a nonempty closed interval of R and U' is a set of ω many points of R. Then let S'_1 and S'_2 be two nonempty closed intervals generated by cutting through the middle point of S' and let U'_1 and U'_2 be two disjoint sets of ω many points of R such that $U'_1 \cup U'_2 = U'$ (this can be done with the basic set theory). Let $M_{n+1} \supseteq \{S'_1 \cup U'_1, S'_2 \cup U'_2\}$; that is, if there are k members of the form $S' \cup U'$ in M_n, M_{n+1} will contain exactly $2k$ members each of which is obtained by splitting in the aforementioned way a member of the form $S' \cup U'$ in M_n.

Now let $M_\omega = \bigcup_{n \in \omega} M_n$ and let Dom(M)=the closure of finite union on M_ω. Also let $P^M = \{(x, y) \in \mathrm{Dom}(M) \times \mathrm{Dom}(M)\colon x \subseteq y\}$, that is, P is as usual interpreted as the set inclusion.

M satisfies (P1), (P2) and (P3) since P^M is a partial ordering.

M satisfies (A) since any $x \in \mathrm{Dom}(M)$ must have some points from U.

M satisfies (SSP), since for any distinct $x, y \in \mathrm{Dom}(M)$, if $x \not\subset y, x$ must have some points from U which are not in y and hence some singleton u will be such that $u \subseteq x$ but $u \not\subset y$.

M satisfies (FS) since M has a greatest member $S \cup U$ and its domain is closed under finite union.

M satisfies (FP) since "roughly speaking", M_ω has already been closed under finite intersection (note that no point in S is in the domain,

[8] The model to be constructed here is very similar to the one given previously in [Tsai and Varzi, 2014], where it has been shown that **EM** + (A) cannot exclude members with gunky parts.

which means that two members will be interpreted as non-overlapping if they intersect at a point of S).

In the foregoing model M, any member of the form $S' \cup U'$ can be divided indefinitely into parts. Intuitively, such a thing is composed of a gunk (a nonempty closed interval of R) and of ω many atoms (ω many points of R). However, such a kind of mixed creatures will be ruled out if we take **GEM** as the background theory (this has been pointed out by some informal remarks in [11], but here a more formal explanation will be given). This is because "gunk" can be defined by "$\forall y(Pyx \to \exists zPPzy)$", hence by (UF), the fusion of the gunky parts of a mixed creature must be in the domain, but then (A) won't be satisfied since such a fusion is itself a gunk. Formally, $\forall x(\exists y(Pyx \wedge \forall u(Puy \to \exists zPPzu)) \to \exists z\forall u(Ouz \leftrightarrow \exists y(Pyx \wedge \forall u(Puy \to \exists zPPzu) \wedge Ouy)))$ is a theorem of **GEM**. So if x has gunky parts, there will be a fusion of all such parts which x has.

Now if one wants to make (A) work properly but refuses to accept (UF) due to some kind of philosophical concerns,[9] one might adopt instead a restricted version of fusion axiom schema which is at least strong enough so as to guarantee the existence of the fusion of the gunky parts of a mixed creature (then such a kind of creatures will be ruled out by (A)). If one thinks that such a solution is somewhat ad hoc, one might have to find another axiom which is as intuitive as (A) but strictly stronger and which can do the job with a weaker background theory.

4. Decidability

There are several kinds of neatly formed atomic mereological structures which are worth looking into. Here we will pay special attention to the decidability issues concerning them.

It is known that **GEM** + (A) + (Infinity) is a decidable theory [Tsai, 2013 (1)]. However, **CEM** + (A) + (G) + (Infinity) is undecidable [Tsai, 2009]. At first sight, **GEM** + (A) + (Infinity) is much more complicated than **CEM** + (A) + (G) + (Infinity), for the former has the unrestricted fusion schema, which actually defines infinitely many axioms at once, while the latter does not. But in effect the difference between them is only one axiom, the axiom of complement (C), since **GEM** + (A) +

[9] There are quite a few philosophical writings, for example, [4, 12], on whether "composition" should be restricted.

(Infinity) is equivalent to **CEM** + (A) + (G) + (C) + (Infinity) (recall that the latter is a complete theory).

It is interesting to see why here (C) plays a crucial role for decidability. The "intended" countable model of **GEM** + (A) + (Infinity) is the model which has ω many atoms and each of whose members is composed of nonzero finitely many or co-finitely many atoms. Other countable models will have some "weird" members each of which is composed of infinitely many but not co-finitely many atoms (henceforth we will be concerned only with models each of whose domains has exactly ω many atoms and each of which is countable). Since **GEM** + (A) + (Infinity) is complete, there is no way to exclude those weird members by adding first-order axioms. On the other hand, by the completeness of **GEM** + (A) + (Infinity), a model which has weird members is elementarily equivalent to the intended one. This means that for any finite back-and-forth game played between the intended model and a model with weird members, Duplicator will always win (Note: this does not mean that, for any k, Duplicator can always extend a winning game with length k to a winning game with length $k+1$, for this will imply that Duplicator can win a game with length ω and then will in turn imply that the intended model and a model with weird members are isomorphic, which is impossible).[10]

Suppose a back-and-forth game with length k is being played between the intended model and a model with weird members. The weird members won't pose a threat to Duplicator, since if Spoiler chooses a weird member in the i^{th} move, for $i \leq k$, Duplicator can reply with a member which is composed of at least $k^{(k+1)-i}$ atoms [Tsai, 2013 (2)]. The idea is that since the length of the game has an upper bound, that is, k, Duplicator can use a member composed of finitely many but enough atoms to "imitate" a member composed of infinitely many atoms. However, if the length of a game is unbounded, the same idea won't work since a member composed of finitely many atoms will be exhausted by finitely many steps while a member composed of infinitely many atoms won't.

Now let's turn to **CEM** + (A) + (G) + (Infinity). Consider the model which has ω many atoms and each of whose members is composed of nonzero finitely many atoms, except the greatest one which is composed of all atoms. It should be clear that no member has a complement in such a model. Also consider another model which has some members each of

[10] For definitions of notions concerning finite back-and-forth games (see [3]).

which has a complement (it is obvious that we can indeed find such a kind of models of **CEM** + (A) + (G) + (Infinity)). Playing finite games between those two models, Spoiler will win any game whose length is at least 3. The winning strategy is as follows. Spoiler chooses as the first move from the second model a member which has a complement, then chooses as the second move from the second model again the complement of the member chosen in the first move, and finally chooses an atom from the first model which does not belong to any of the two members chosen by Duplicator in the first two moves. Of course, the fact that there are two models which are not elementarily equivalent only shows that **CEM** + (A) + (G) + (Infinity) is incomplete, and we cannot conclude its undecidability from such a fact. But with the help of (C), Duplicator can turn the table around so as to come up with a winning strategy and this will imply the completeness of and hence the decidability of **CEM** + (A) + (G) + (C) + (Infinity) (it is easy to see that an axiomatized complete theory is decidable; "axiomatized" here means "recursively axiomatized"; same below).

Then how about adding the following axiom (NC) which says that no member has a complement?

(NC) $\forall x \neg \exists y \forall z (Pzy \leftrightarrow \neg Oxz)$

It has been shown by a back-and-forth argument that **CEM** + (A) + (G) + (NC) + (LC) is a complete and hence decidable theory [9]. This also shows that (C) is not a necessary condition for an axiomatized mereological theory to be decidable, since it is obvious that (NC) and (C) are incompatible in any mereological structure with more than one member. However, the same back-and-forth argument cannot apply to the case of **CEM** + (A) + (G) + (NC) (note that any model of **CEM** + (A) + (G) + (NC) must be infinite and hence (Infinity) is implied by such a theory), for otherwise **CEM** + (A) + (G) + (NC) will be equivalent to **CEM** + (A) + (G) + (NC) + (LC), which is not the case. Let S be an infinite but not co-finite subset of ω. Consider the mereological structure M whose domain is $\{x \subseteq \omega : x = \omega$ or x is finite but nonempty or x is the union of S and a finite subset, which can be empty, of $\omega\}$ and in which P is as usual interpreted as the set inclusion. Obviously, M satisfies **CEM** + (A) + (G) + (NC), but not (LC).

Now if **CEM** + (A) + (G) + (NC) + ¬(LC) is decidable, **CEM** + (A) + (G) + (NC) will be decidable too. (In general, if both T + α and T + $\neg\alpha$ are decidable, T is decidable too). However, we can find a strongly undecidable model which satisfy **CEM** + (A) + (G) + (NC) + ¬(LC) and this will imply the undecidability of any theory weaker than it, in particular, **CEM** + (A) + (G) + (NC).

Let (ω, R) be a strongly undecidable countable irreflexive symmetric ordering, that is, $\forall x \neg Rxx$ and $\forall x \forall y(Rxy \to Ryx)$ are satisfied by (ω, R) (see [5, pp. 141–142]). R must be countable and must contain infinitely many pairs (otherwise the theory on such an ordering will be decidable). Let p_0, p_1, ... enumerate unordered pairs $\{x, y\}$ for each (x, y) in R ($x \neq y$ since R is irreflexive and each $\{x, y\}$ will be counted exactly once even though R is symmetric). Also let S_{01}, S_{02}, S_{11}, S_{12}, ... be pairwise disjoint infinite but not co-finite subsets of ω (it should be clear that we can find those sets). Define $D = \{x \subseteq \omega : x = \omega$ or x is a singleton or $x = S_{i1} \cup p_i$ or $x = S_{i2} \cup p_i$, for some $i \in \omega\}$. Let D' be the closure of D under finite union. Consider the mereological structure M whose domain is D' and in which P is again interpreted as the set inclusion.

Note that $\{S_{ij} \cup p_i \subseteq \omega : i \in \omega$ and $j = 1$ or $2\}$ is definable in M by $\neg\forall y Pyx \wedge \exists y PPyx \wedge \forall y(PPyx \to \neg\exists z(PPzx \wedge \neg Ozy \wedge x = y + z))$. Furthermore, the set of singletons of natural numbers $\{\{0\}, \{1\}, \{2\}, \ldots\}$ can be defined in M by $\neg\exists y PPyx$. We can use such a set to stand for ω, which means that the domain of (ω, R) is definable in M. Finally, R can be defined in M by $x \neq y \wedge \neg\exists z PPzx \wedge \neg\exists z PPzy \wedge \exists u \exists t(\neg\forall y Pyu \wedge \exists y PPyu \wedge \forall y(PPyu \to \neg\exists z(PPzu \wedge \neg Ozy \wedge u = y + z)) \wedge \neg\forall y Pyt \wedge \exists y PPyt \wedge \forall y(PPyt \to \neg\exists z(PPzt \wedge \neg Ozy \wedge t = y + z)) \wedge x + y = u \times t)$. So the strongly undecidable ordering (ω, R) can be defined in M, which means that M is also strongly undecidable.[11] But M is a model of **CEM**+(A)+(G)+(NC)+¬(LC). Therefore such a theory is undecidable.

The moral of the foregoing example is that if a mereological theory which is axiomatized by some axioms listed in the first section has atomic models but does not have (LC) as a theorem, then it must be undecidable.

5. Concluding remarks

Consider Claim 2 and the model M defined there. It can be easily seen from the proof that for any formula with exactly one free variable, if it

[11] In general, if the language of a structure A has only finitely many non-logical symbols and A is strongly undecidable, then any structure B in which A can be defined will also be strongly undecidable (see [5, p. 136]).

defines in M a subset S with cardinality 1, then the only member of S must be R. That is to say, such a formula will be equivalent in M to $\forall y Pyx$. For any other formula satisfiable in M, the cardinality of the subset that it defines will jump to be uncountable. This in a sense shows the limitation of the expressiveness of the formal language of mereology (the same trick can be played on many other models).

The axiom of atomicity can exclude a pure gunk but cannot exclude a mixture of a gunk and some atoms from the domain unless it works with a sufficiently strong background theory, such as **GEM**. But note that the counterexample given above is a mathematical one whose construction relies on at least some set-theoretical assumptions.

Local complementation is a necessary requirement for an axiomatized mereological theory which has atomic models to be decidable (again, we are only concerned with the axioms listed in the first section). In view of back-and-forth games, this means that Duplicator does not necessarily have a winning strategy when playing a finite game between atomic models which are not regulated by (LC), since for any axiomatized theory, "being complete" implies "being decidable".

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