Logic and **L**ogical **P**hilosophy
Volume 24 (2015), 449–468
DOI: 10.12775/LLP.2015.008

Lech T. Polkowski

MEREOLOGY AND UNCERTAINTY

Many a mickle maks a muckle (Scottish)

Abstract. Mereology as an art of composing complex concepts out of simpler parts is suited well to the task of reasoning under uncertainty: whereas it is most often difficult to ascertain whether a given thing is an element of a concept, it is possible to decide with belief degree close to certainty that the class of things is an ingredient of an other class, which is sufficient for carrying out the reasoning whose conclusions are taken as true under given conditions. We present in this work a scheme for reasoning based on mereology in which mereology in the classical sense is *fuzzified* in analogy to the concept fuzzification in the sense of L. A. Zadeh. In this process, mereology becomes *rough mereology*.

Keywords: knowledge; uncertainty; vaguenes; ambiguity; rough sets; fuzzy sets; mereology; rough mereology; granulation of knowledge; granular logics; spatial reasoning

1. Uncertainty as a philosophical problem

Uncertainty has two sources of origin: logical and physical. The latter is known as the Heisenberg Uncertainty Principle, cf. Heisenberg [8] which states, in essence, that the position and the momentum of a quantum particle cannot be known exactly at the same time, and, it is summed up by a well-known inequality:

$$\Delta q \times \Delta p \geqslant \frac{h}{2\pi},$$

where q denotes the position, p stands for the momentum, and, h is the Planck constant. The Heisenberg argument points to the fundamental uncertainty in our relations with Nature: we cannot observe exactly pro-

Special Issue: Mereology and Beyond (I). Edited by A. C. Varzi and R. Gruszczyński
 Published online April 1, 2015

cesses in the real microscopic world by the very nature of the observation process.

The logical source of uncertainty can be traced first to Gottlob Frege's Basic Law V in its original form (cf. Frege [5]) which demonstrated the insufficiency of intuition and the sense of obviousness in dealing with concepts. The Gödel incompleteness theorem showed that uncertainty is immanent to sufficiently rich systems (cf. [6]). In this way uncertainty has become a permanent ingredient in scientific investigations.

2. Uncertainty of knowledge

There are many sources of uncertainty in our practical analysis of real world phenomena. The real world is observed in order to gather *knowledge*, which according to Bocheński is a set of relations among things and their properties (cf. [2]). Our means of gathering knowledge restrict us to relatively small subsets of the universe of possible things and in consequence our knowledge is incomplete, uncertain, and affected by noise. The resulting uncertainty affects our decisions and the effort in research has been directed at finding means to reduce and evaluate uncertainty.

A pioneering in this search for models of uncertainty was an article by Ludwik Fleck, a medical doctor from Lvov in Poland (cf. [4]) who, appearing at the convention of some society for history of medicine at Lvov, laid foundations for analysis of *vagueness* resulting from uncertainty caused by overlapping of symptoms for various diseases and in consequence necessity for dealing with complexes of illnesses.

Vagueness, characterized in the fregean spirit as the absence of a delineation among concepts, was analyzed by Max Black [1] and Karl Menger [10], the former in logical terms, the latter in the language of probabilistic metric spaces.

Vagueness was characterized as a partial membership by Lotfi A. Zadeh [27]; a vague concept X over a universe U of things is characterized in this approach by means of its *membership function* $\mu_X : U \to [0, 1]$ which does assign to each thing $u \in U$ its *membership degree* $\mu_X(u)$, valued in the interval [0, 1]. The value of 0 means that u certainly falls not into X whereas the value of 1 does assert that u certainly falls in X. The concept $\{u : \mu_X(u) \in (0, 1)\}$ is the *boundary of uncertainty* consisting of things which can be assigned with certainty neither to X nor to the complement $-X$ witnessing vagueness of the concept X. Ambiguity of knowledge is discussed, e.g., in *perception calculus* proposed by Zadeh [28] dedicated to queries oriented toward problems with data insufficient for considering them in traditional frameworks. The fuzzy membership based approach rests at foundations of the *fuzzy set* approach.

Very often, our observations of the reality are collected as *data tables*, in which things observed collected in a universe U are characterized by means of values of selected features, *attributes* from a set A. Each attribute $a \in A$ is formally rendered as a mapping $a\colon U \to V$ from the universe U into the *value set* V. The formula $a(u) = v$ which states that the attribute a takes the value v on the thing u is abstracted to the formula (a, v) which denotes the possibility that a may take a value v and is called a *descriptor*. The descriptor (a, v) can be given *meaning*, $[(a, v)]$, by denoting it in the universe U:

$$[(a, v)] = \{u \in U : a(u) = v\}.$$

The pair (U, A) is commonly called an *information system*. Things which are assigned identical values of all attributes are regarded as *indiscernible* which leads to the partition of the set of observed things into disjoint *indiscernibility classes*. The *indiscernibility relation* IND_A is formally defined as follows:

$$IND_A(u, v) \text{ iff } a(u) = a(v) \text{ for each } a \in A.$$

Equivalently, one may say that $IND_A(u, v)$ holds true if and only if for each descriptor (a, w) the condition:

$$u \in [(a, w)] \;\text{ iff }\; v \in [(a, w)]$$

is satisfied. The symbol $[u]_A$ will denote the indiscernibility class of u with respect to the relation IND_A.

On the information system (U, A) there is usually imposed an external partition by the *decision indiscernibility relation* IND_d into *decision classes* of the form $[u]_d$ which results from an external, *oracle, expert*, knowledge encoded by a *decision attribute* d. As both partitions are independent, *ambiguity* does arise from uncertainty what value of decision $d(u)$ to assign to a given indiscernibility class $[u]_A$.

In the *attribute-value* approach adopted for dealing with data, there are concepts $X \subseteq U$ which are *decidable*, i.e., the decision problem whether a thing u belongs in X has the deterministic solution in terms of yes/no. It is easy to see that a concept X is decidable if and only if it is a union of a family of indiscernibility classes. Deterministic concepts are also called in this formalism *exact concepts*; concepts non-exact are called *rough concepts*.

For a rough concept X, the means to delineate it in terms of descriptors (a, v) is to approximate it from below and from above with indiscernibility classes; the *lower approximation* $\underline{X}$ is then the union of indiscernibility classes contained in X:

$$\underline{X} = \{u \in U : [u]_A \subseteq X\}.$$

The *upper approximation* $\overline{X}$ is the union of indiscernibility classes which do intersect X:

$$\overline{X} = \{u \in U : [u]_A \cap X \neq \emptyset\}.$$

As for each rough concept X, one obviously has $\underline{X} \subset \overline{X}$, the difference $\overline{X} \setminus \underline{X}$ is the non-empty *boundary region* BdX witnessing *vagueness* of X. This approach is the cornerstone of the *rough set theory* by Zdzisław Pawlak [11].

Summing up the aforesaid, two main aspects of uncertainty, viz., vagueness and ambiguity, are formalized by both fuzzy and rough set theories. Both influence the development of our approach to uncertainty called *rough mereology*.

3. Rough mereology

Rough mereology stems from mereology in its classical form due to Stanisław Leśniewski [9]. This variant of mereological theory begins with the notion of a *part*, $\pi(u, v)$ in symbols, read *'u is a part of v'* which is a binary relation on things, irreflexive and transitive. Irreflexivity is alleviated by the notion of an *ingredient relation*, $ingr_\pi(u, v)$ in symbols, which is the union of the part relation with the identity relation, and as such is a partial ordering on things:

$$ingr_\pi(u, v) \text{ iff } \pi(u, v) \text{ or } u = v.$$

The important relations derived from the part relation are: the *overlap* relation, $Ov(u, v)$ in symbols which holds if and only if u and v have an ingredient in common; the negation of the overlap relation is the *disjointness* relation $dis = \neg Ov$:

$$Ov(u, v) \text{ iff there exists } w \text{ such that } ingr_\pi(w, u) \text{ and } ingr_\pi(w, v),$$
$$dis(u, v) \text{ iff } \neg Ov(u, v).$$

The most important construct in mereology of Leśniewski is that of the class. It is defined for any non-vacuous property Ψ of things as the unique thing $X = Cls\,\Psi$ such that:

(Cls 1) If $\Psi(u)$ then $ingr_\pi(u, X)$.

(Cls 2) If $ingr_\pi(u, X)$ then there exist things v, w such that $ingr_\pi(v, u)$, $ingr_\pi(v, w)$, $\Psi(w)$ hold.

The class of all things disjoint to a thing x is the *complement* $-x$ to x.

The reader may consult the monograph by Simons [21] for details; an exposition of mereology with spatial reasoning in view is given in Casati and Varzi [3]; a discussion is also given in Polkowski [12].

Both part and ingredient relations impose an exact hierarchy on things but in applied cases it is not often, due to uncertainty, that such a hierarchy can be ascertained. Our work on rough sets and decision making by tools of this paradigm, has caused us to extend the mereological scheme by the process analogical to that of Zadeh's, viz., by considering the ternary relation of a *part to a degree*, $\overline{\pi}(u, v, r)$, read *'u is a part of v to a degree of at least r'* induced by a part relation π (cf. [12]).

As with the part relation, the relation $\overline{\pi}$ is bound to observe certain postulates which reflect the most general properties of the partial containment. For a given relation π and the associated relation $ingr_\pi$, we demand of the relation $\overline{\pi}$ the following:

(P1) $$\overline{\pi}(u, v, 1) \text{ iff } ingr_\pi(u, v)$$

This postulate asserts that parts to degree of 1 are ingredients, thus, the relation $\overline{\pi}(u, v, 1)$ is identical with the relation $ingr(u, v)$.

(P2) If $\overline{\pi}(u, v, 1)$ then $\overline{\pi}(w, u, r)$ implies $\overline{\pi}(w, v, r)$ for every thing w and each $r \in [0, 1]$.

This postulate does express a feature of partial containment that a 'bigger' thing contains a given thing 'more' than a 'smaller' thing. It can be called a *monotonicity condition.*

(P3) If $\overline{\pi}(u, v, r)$ and $s < r$ then $\overline{\pi}(u, v, s)$.

The generic term of *rough inclusion* was proposed in Polkowski and Skowron [20] for $\overline{\pi}$. We propose now to follow paths initiated by Max Black and, respectively, by Karl Menger in logical, respectively, geometric, analysis of uncertainty.

4. Syntax and semantics of decision rules

We address the ambiguity aspect of uncertainty related to reasoning with data. Our context is provided by a *decision system* (U, A, d), i.e., an information system (U, A) endowed with a decision attribute d. As in Section 2, in the triple (U, A, d), U is a set of things, A is a set of attributes, and, d is the decision attribute. Knowledge in logical form is encoded in the decision system by means of the set of *decision rules*.

In order to represent decision rules, a language should be chosen for representing relations among attribute values and things. Commonly, it is the language of descriptors. A *descriptor*, cf. Section 2, is a pair (a, v) where a is an attribute symbol and v a value symbol. Descriptors can be combined by means of propositional functors $\vee$,$\wedge$, $\neg$, $\rightarrow$, to form meaningful formulas which are interpreted in the set U of things as follows, where $[\phi]$ stands for the meaning of the formula ϕ:

- $[(a, v)]=\{u \in U : a(u) = v\}$;
- $[(a, v) \vee (b, w)]=[(a, v)] \cup [(b, w)]$;
- $[(a, v) \wedge (b, w)]=[(a, v)] \cap [(b, w)]$;
- $[\neg(a, v)] = U \setminus [(a, v)]$;
- $[(a, v) \rightarrow (b, w)] = (U \setminus [(a, v)]) \cup [(b, w)]$.

A *decision rule* is a formula of the form:

$$r: \bigwedge_{a\in B} (a, v_a) \rightarrow (d, w),$$

were $B \subseteq A$ is a set of attributes, and, v_a is a value of the attribute a for each $a \in B$. The rule r is said to be *true* in case $[r] = U$ which is equivalent to the condition:

$$\bigcap_{a\in B} [(a, v_a)] \subseteq [(d, w)]. \tag{1}$$

To see this, please note the meaning of the premise:

$$[\bigwedge_{a\in B} (a, v_a)] = \bigcap_{a\in B} [(a, v_a)],$$

hence, in accordance with the adopted meaning of implication, the meaning of r is:

$$[r] = (U \setminus \bigcap_{a\in B} [(a, v_a)]) \cup [(d, w)].$$

It follows that r is true, i.e., $[r] = U$ if and only if the condition (1) is satisfied.

Otherwise, the rule r is *partially true* and the degree of truth is established relative to a rough inclusion $\overline{\pi}$ applied, viz., the rule r is *partially true to a degree of at least* α if and only if the condition:

$$\overline{\pi}(\bigcap_{a\in B}[(a, v_a)], [(d, w)], \alpha)$$

is satisfied.

Partial truth of a formula can be regarded as an *ambiguity condition*: we may say that a decision rule r is *ambiguous to a degree of at least* s if there exist decision values $w_1, \ldots, w_k$ all distinct from the value w and such that:

1. $\overline{\pi}(\bigcap_{a\in B}[(a, v_a)], [(d, w_i)], s_i)$ is satisfied with $s_i > 0$ for each $i = 1, 2, \ldots, k$;
2. $\sum_{i=1}^{k} s_i \geqslant s$.

It is our aim to express the notion of truth for decision rules in logical terms and to this end, we propose a rough mereological granular logic. Its explanation calls first for a notion of a granule of knowledge.

5. Granules of knowledge

Assume that a rough inclusion $\overline{\pi}$ is given along with the associated ingredient relation $ingr_{\overline{\pi}}$, as in postulate (P1). The *granule* $g_{\overline{\pi}}(u, r)$ of the *radius* r about the *center* u is defined as the class, see conditions (Cls 1), (Cls 2) in Section 3, of property $\Phi^{\overline{\pi}}_{u,r}$:

$$\Phi^{\overline{\pi}}_{u,r}(v) \text{ iff } \overline{\pi}(v, u, r).$$

The granule $g_{\overline{\pi}}(u, r)$ is defined as:

$$g_{\overline{\pi}}(u, r) = Cls\, \Phi^{\overline{\pi}}_{u,r}.$$

Properties of granules depend, obviously, on the type of rough inclusion used in their definitions. We consider separate cases, as some features revealed by granules differ from a rough inclusion to a rough inclusion, cf. [12] for a detailed analysis of granule properties.

6. Rough mereological logic for reasoning about uncertainty in decision systems

The idea of a granular rough mereological logic, see Polkowski [13], Polkowski and Semeniuk-Polkowska [17], consists in measuring the meaning of a unary predicate in the model which is a universe of a decision system against a granule defined by means of a rough inclusion. The result can be regarded as the degree of truth (the logical value) of the predicate with respect to the given granule. The obtained logics are intensional as they can be regarded as mappings from the set of granules (possible worlds) to the set of logical values in the interval $[0, 1]$, the value at a given granule regarded as the extension at that granule of the generally defined intension.

Our context requires rough inclusions on finite concepts/sets, hence, for finite sets X, Y, we define the *Łukasiewicz rough inclusion* by letting:

$$\nu_L(X, Y, r) \text{ iff } \frac{|X \cap Y|}{|X|} \geqslant r,$$

where $|X|$ denotes the cardinality of a set X.

Let us observe that ν_L is *regular*, i.e., $\nu_L(X, Y, 1)$ if and only if $X \subseteq Y$ and $\nu_L(X, Y, 0)$ if and only if $X \cap Y = \emptyset$.

The second rough inclusion we may apply is the 3-valued rough inclusion ν_3 defined via the formula:

$$\nu_3(X, Y, r) \text{ iff } \begin{cases} X \subseteq Y \text{ and } r = 1 \\ X \cap Y = \emptyset \text{ and } r = 0 \\ \text{otherwise } r = \frac{1}{2} \end{cases} \tag{2}$$

For a collection of unary predicates Pr, interpreted in the universe U by means of descriptor logic, and for a rough inclusion on finite sets ν, we define the intensional logic GRM_ν by assigning to each predicate ϕ in Pr with the meaning $[\phi]$, see Section 4, its intension $I_\nu(\phi)$ defined by the collection of its extensions $\{I_\nu^\vee(g) : \text{granule } g\}$ at all granules, via:

$$I_\nu^\vee(g)(\phi) \geqslant r \text{ iff } \nu(g, [\phi], r). \tag{3}$$

With respect to the rough inclusion ν_L, the formula (3) becomes:

$$I_{\nu_L}^\vee(g)(\phi) \geqslant r \text{ iff } \frac{|g \cap [\phi]|}{|g|} \geqslant r.$$

The counterpart for ν_3 is specified by definition (2), and it comes down to the following:

$$I^{\vee}_{\nu_3}(g)(\phi) \geqslant r \text{ iff } \begin{cases} g \subseteq [\phi] \text{ and } r = 1 \\ g \cap [\phi] \neq \emptyset \text{ and } r \geqslant \frac{1}{2} \\ g \cap [\phi] = \emptyset \text{ and } r = 0 \end{cases}$$

We say that a formula ϕ interpreted in the universe U of an information system (U, A) is *true at a granule* g with respect to a rough inclusion ν if and only if $I^{\vee}_{\nu}(g)(\phi) = 1$.

Hence, for every regular rough inclusion ν, a formula ϕ interpreted in the universe U, with the meaning $[\phi] = \{u \in U : \phi(u)\}$, is true at a granule g with respect to ν if and only if $g \subseteq [\phi]$.

In particular, for a decision rule $r\colon p \Rightarrow q$ in the descriptor logic,the rule r is true at a granule g with respect to a regular rough inclusion ν if and only if $g \cap [p] \subseteq [q]$. We state these facts in the following

PROPOSITION 1. *For every regular rough inclusion ν, a formula ϕ interpreted in the universe U, with the meaning $[\phi]$, is true at a granule g with respect to ν if and only if $g \subseteq [\phi]$. In particular, for a decision rule $r\colon p \Rightarrow q$ in the descriptor logic, the rule r is true at a granule g with respect to a regular rough inclusion ν if and only if $g \cap [p] \subseteq [q]$.*

PROOF. Truth of ϕ at g means that $\nu(g, [\phi], 1)$ which in turn, by regularity of ν is equivalent to the inclusion $g \subseteq [\phi]$. ⊣

We will say that a formula ϕ is a *tautology* of our intensional logic if and only if ϕ is true at every world g.

The preceding proposition implies that:

PROPOSITION 2. *For every regular rough inclusion ν, a formula ϕ is a tautology if and only if $Cls(G) \subseteq [\phi]$, where G is the property of being a granule; in the case when granules considered cover the universe U this condition simplifies to $[\phi] = U$. This means for a decision rule $p \Rightarrow q$ that it is a tautology if and only if $[p] \subseteq [q]$.*

Hence, the condition for truth of decision rules in the logic GRM_ν is the same as the truth of an implication in descriptor logic, under caveat that granules considered cover the universe U of objects.

In Polkowski [13], relations between granular logics and Łukasiewicz multi-valued logics were examined, some results are as follows.

PROPOSITION 3. *Each tautology of 3-valued Łukasiewicz logic is a tautology of rough mereological granular logic in case of a regular rough inclusion on sets.*

PROPOSITION 4. *Each tautology of infinite-valued Łukasiewicz logic is a tautology of rough mereological granular logic in case of a regular rough inclusion on sets.*

It follows from Proposition 4 that all tautologies of *Basic logic*, cf. Hájek [7], i.e., logic which is the intersection of all many-valued logics with implications evaluated semantically by residual implications of continuous t-norms are tautologies of rough mereological granular logic for each regular rough inclusion ν.

Modalities can be defined in the rough mereological universe by means of the relation of an ingredient as an accessibility relation. Thus, we adopt the collection of granules induced by a rough inclusion $\overline{\pi}$ as the set W of possible worlds, and the accessibility relation $R(g,h)$ will be defined as the ingredient relation $ingr_{\pi}(h,g)$:

$$R(g,h) \text{ iff } ingr_{\pi}(h,g),$$

i.e., a world (granule) h is accessible via R from a world (granule) g if and only if h is an ingredient of g.

For a predicate ψ, and a granule g, it follows that ψ is *necessarily true at g* if and only if ψ is true at every ingredient h of g, which means that, for a regular rough inclusion ν on sets, $h \subseteq [\psi]$ for each ingredient h of g. Possibility of ψ at g means that there exists a granule h with $ingr_{\pi}(h,g)$ such that ψ is true at h, i.e., $h \subseteq [\psi]$.

As the ingredient relation is reflexive and transitive, see sect. 3, we have:

PROPOSITION 5. *The modal logic obtained by taking the ingredient relation on granules as the accessibility relation R and granules of knowledge as possible worlds W satisfies requirements for the system $S4$ in the frame (W,R).*

Possibility and necessity are expressed in rough set theory of concepts by means of approximations, the upper and the lower, respectively, see Section 2. Both are defined with the help of indiscernibility relations, see, loc.cit. We recall that for an information system (U,A), the *indiscernibility over A* is the relation:

$$IND_A(u,v) \text{ iff } a(u) = a(v), \text{ for each } a \in A.$$

For a concept $X \subseteq U$, the *lower*, resp. the *upper approximation* is the set $\underline{X} = \{u \in U : [u]_A \subseteq X\}$, resp., $\overline{X} = \{u \in U : [u]_A \cap X \neq \emptyset\}$, see Section 2. Logical rendering of these modalities in rough mereological logics exploits the approximations. We define two modal operators: M (possibility) and L (necessity). To this end, we let:

$$I_\nu^\vee(g)(M\alpha) \geqslant r \text{ iff } \nu(g, \overline{[\alpha]}, r)$$
$$I_\nu^\vee(g)(L\alpha) \geqslant r \text{ iff } \nu(g, \underline{[\alpha]}, r)$$

Then we have the following criteria for necessarily or possibly true formulas.

A formula α is *necessarily true at a granule* g if and only if $g \subseteq \underline{[\alpha]}$; α is *possibly true at* g if and only if $g \subseteq \overline{[\alpha]}$. Then one has, cf. [12, Ch. 7]:

Proposition 6. *For each regular rough inclusion on finite concepts ν, the modal logic induced by lower and upper approximations is the modal logic* S5.

6.1. An example of reasoning in rough mereological logic: the calculus of perceptions

Calculus of perceptions, posed as an idea by Zadeh [28], intends to solve problems which are definitely ill-posed by standards of, e.g., probability calculus, nevertheless, those problems occur at many practical tasks of our everyday life, hence, they deserve attention and an attempt at solving them in some way. This calculus does address ambiguous statements and queries, and answers them with ambiguous decisions. Here, we apply granular mereological logics toward this problem. To this end, we would like to borrow a part of a complex percept, i.e., a set of vague statements, from [28] and interpret it in terms of granular logic. This example is taken from [12]. Please observe how in it, vagueness and ambiguity are intertwined: we need data to base our reasoning on, and, we need vague concepts related to terms in the query, and, finally, we need a mechanism for reasoning in order to produce an ambiguous decision.

The percept is: (i) *Carol has two children: Robert who is in mid-twenties and Helen who is in mid-thirties* with a query (ii) *how old is Carol.* To interpret it, we begin with Table 1 in which a training decision system *Age* is given with attributes n — the number of children, a_i — the age of the i-th child for $i \leqslant 3$, and with the decision *age* — the age of the mother.

object	n	age_1	age_2	age_3	*Age*
1	3	15	22	30	58
2	3	10	12	16	42
3	2	6	10	–	30
4	2	24	33	–	56
5	2	28	35	–	62
6	3	22	33	40	67
7	2	18	25	–	60
8	2	26	35	–	63
9	2	22	38	–	70
10	3	8	12	16	38
11	2	22	32	–	58
12	3	24	36	40	63
13	2	28	34	–	60
14	3	26	30	35	65
15	3	18	25	35	60
16	3	6	12	16	40
17	3	22	30	35	65
18	2	24	34	–	60
19	3	22	30	34	58
20	2	24	35	–	62

Figure 1. Decision system *Age*

We define for a fuzzy concept X represented by the fuzzy membership function μ_X on the domain D_X, the *c-cut* where $c \in [0,1]$ as the concept $X_c = \{x \in D_X : \mu_X(x) \geqslant c\}$; the value x_s such that $\mu_X(x_s) = c$ is the *support of* c, $supp_c$ in symbols. Concepts *in mid-twenties*, *in mid-thirties* are represented by fuzzy membership functions, μ_{20}, μ_{30}, respectively:

$$\mu_{20}(x) = \begin{cases} 0.25(x-20), \ x \in [20,24] \\ 1, \ x \in [24,26] \\ 1-0.25(x-26), \ x \in [26,30] \end{cases}$$

$$\mu_{30}(x) = \begin{cases} 0.25(x-30), \ x \in [30,24] \\ 1, \ x \in [34,36] \\ 1-0.25(x-36), \ x \in [36,40] \end{cases}$$

The concept *old* is interpreted as:

$$\mu_{old}(x) = \begin{cases} 0.02(x-30), \ x \in [30,60] \\ 0.04(x-60)+0.6, \ x \in [60,70] \\ 0 \text{ else.} \end{cases}$$

We interpret our query by a function $q\colon [0,1]^3 \to [0,1]$, where $f(u,v,w) = t$ would mean that for cut levels u, v, w, respectively for *old*, *in mid-twenties*, *in mid thirties*, t is the truth value in the statement *Carol is at least supp*$_u$ *old to the degree of* t *with respect to* v, w.

In our example, letting $u = 0.6$, we obtain $old_{0.6} = [60,70]$; letting $v = 0.5 = w$, we obtain *in mid twenties*$_{0.5} = [23,27]$ and *in mid thirties*$_{0.5} = [33,37]$. In order to evaluate the truth degree t, we refer to the world knowledge of Table 1 and we find the set of objects $\Lambda_{v,w}$ with two children of ages respectively in the intervals $[23,27], [33,37]$ corresponding to values of v, w as well as the set of objects Γ_u having the value of *Age* in old_u. In our case we have $\Lambda_{0.5,0.5} = \{4, 8, 12, 18, 20\}$ and $\Gamma_{0.6} = \{5, 6, 7, 8, 9, 12, 13, 14, 15, 17, 18, 20\}$.
Finally, we evaluate the truth degree of the predicate *in* $old_{0.6}(x)$ represented in the universe of Table 1 by the set $\Gamma_{0.6}$ with respect to the granule $\Lambda_{0.5,0.5}$. We obtain, by applying the Łukasiewicz rough inclusion ν_L, that:

$$(I^{\nu_L}_{\Lambda_{0.5,0.5}})^{\vee}(old_{0.6}(x)) = \frac{|\Lambda_{0.5,0.5} \cap \Gamma_{0.6}|}{|\Lambda_{0.5,0.5}|} = 0.8$$

The result is the statement: *Carol is over 60 years old to the degree of 0.8 under the assumed interpretation of in mid twenties*$_{0.5}$, *in mid thirties*$_{0.5}$ *with respect to knowledge in* Table 1. The remaining degree of 0.2 is split among other decision values for the age of Carol, in accordance with our notion of an ambiguous decision rule.

7. Mereogeometry induced by rough inclusions

We now address the geometric aspect of uncertainty. For this section, we modify the rough inclusion ν_L to the form:

$$\nu_L(X, Y, r) \text{ iff } \frac{\|X \cap Y\|}{\|X\|} \geqslant r,$$

where X, Y are bounded regions in Euclidean space of finite number of dimensions and $\|X\|$ denotes the area of X. The *mereological distance* $\kappa(X, Y)$ between X and Y is defined as:

$$\kappa(X, Y) := \min\{\max\{r : \nu_L(X, Y, r)\}, \max\{s : \nu_L(Y, X, s)\}\}.$$

We also remind that the mereological distance $\kappa(X, Y)$ takes on the value 1 when $X = Y$ and the minimal value of 0 means that $X \cap Y \subseteq$

$Bd(X) \cap Bd(Y)$, where Bd stands for the 'boundary'. For details of this approach, please consult [12, 14, 15, 16].

Alfred Tarski in His lectures at Warsaw University in 1929 introduced a system of axioms for planar Euclidean geometry based on predicates of equidistance and betweenness (cf. [25]). Modifications to those predicates and some new ones, were introduced by van Benthem (cf. [26]). A new predicate of *nearness*, $N(X, U, V)$ in symbols, redefined in the mereological context, is as follows:

$$N(X, U, V) \text{ iff } \kappa(X, U) > \kappa(V, U).$$

The formula $N(X, U, V)$ reads 'X is closer to U than V is to U'.

The *betweenness* relation in the van Benthem sense, T_B in symbols, is defined as:

(4) $T_B(Z, U, V)$ iff for each W: $Z = W$ or $N(Z, U, W)$ or $N(Z, V, W)$.

The formula $T_B(Z, U, V)$ reads 'Z is between U and V'.

Uncertainty in geometry terms means uncertainty of localization, position estimation, which bear on, e.g., planning motion for autonomous agents. Mereological geometry based on the distance function κ can be applied in control of autonomous mobile agents and their formations. For instance, when our agents are spatially represented as rectangles with sides parallel to the cartesian coordinate axes, the definition (4) comes down to the following (see [16]).

PROPOSITION 7. *$T_B(Z, U, V)$ holds if and only if Z is contained in the smallest rectangle containing U and V.*

For a collection of agents $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$, a *formation on $\mathcal{A}$* is $\mathcal{A}$ with a relation T_B on $\mathcal{A}$. Flexibility of this definition of a formation, in contrast to rigid constraints imposed on formations in, e.g., traditional intelligent robotics, allows agents to travel in difficult environments without violating the formation conditions, cf. [16]. The following Figure 2, due to Paul Ośmiałowski, cf. [14]–[16], shows trails of a team of intelligent agents (mobile robots) in a formation, navigating in an environment.

8. Mereology in approximate spatial reasoning

Mereology allows for topological notions, cf. e.g. [3, 21, 12]. Rough mereology provides a richer topology, albeit granulation radii force it to be

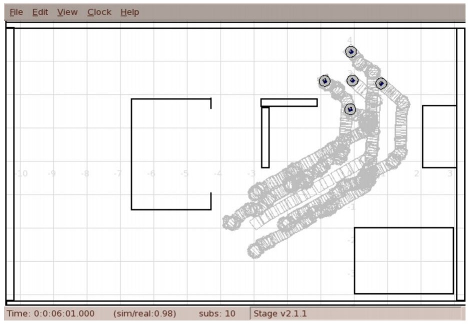

Figure 2. Trails of robots arranged in the cross formation

of a quasi-metric in type. To introduce the rough mereological topology, we assume a rough inclusion $\overline{\pi}$ and we refer to the notion of a granule $g_{\overline{\pi}}(u, r)$, cf. Section 5.

The granule collection $\{g_r^{\overline{\pi}}(x) : x \in U; r \in (0, 1)\}$ on a set of things U can be taken as a neighborhood base for a topology on U that may be called the *granular topology*. We assume that the rough inclusion $\overline{\pi}$ in question is *transitive* which means that:

$$\text{If } \overline{\pi}(x, y, r) \text{ and } \overline{\pi}(y, z, s), \text{ then } \overline{\pi}(x, z, \phi(r, s)),$$

where ϕ in this case is a *t-norm function*, i.e., a function $T \colon [0, 1] \times [0, 1] \to [0, 1]$ which is symmetric, associative, non-decreasing in each coordinate and such that $T(x, 0) = 0$; $T(x, 1) = x$ for each $x \in [0, 1]$. For this notion see e.g. [7].

Open base for a topology is then the collection of sets of the form:

$$N(x, r) = Cls(\psi_{r,x}^{\overline{\pi}}),$$

where:

$$\psi_{r,x}^{\overline{\pi}}(y) \text{ iff there is } s > r \text{ such that } \overline{\pi}(y, x, s).$$

We declare the system $\{N(x, r) : x \in U;\ r \in (0, 1)\}$ to be a neighborhood basis for a topology $\theta_{\overline{\pi}}$. This is justified by the following:

PROPOSITION 8 ([12, Ch. 7]). *Properties of the system $\mathcal{N} = \{N(x, r) : x \in U;\ r \in (0, 1)\}$ are as follows:*

1. *For all $x, y \in U$ and $t \in (0, 1)$: if $ingr_{\pi}(y, N(x, r))$, then there is $\delta > 0$ such that $ingr_{\pi}(N(y, \delta), N(x, r))$.*

2. *For all* $x \in U$ *and* $r, s \in (0,1)$: *if* $s > r$ *then* $ingr_\pi(N(x,s), N(x,r))$.
3. *For all* $x, y, z \in U$ *and* $r, s \in (0,1)$: *if* $ingr_\pi(z, N(x,r))$ *and* $ingr(z, N(y,s))$, *then there is* $\delta > 0$ *such that* $ingr_\pi(N(z,\delta), N(x,r))$ *and* $ingr_\pi(N(z,\delta), N(y,s))$.

This topology can be applied in, e.g., boundary characterizations in approximate reasoning (cf. [18, 19]).

We begin an analysis of the notion of a *boundary*, an important notion both theoretically as well as from the point of view of applications (cf. e.g. [22, 23, 18, 18]) with a definition of an open set as a collection of neighborhoods; the predicate $open(F)$ is therefore defined as:

$$open(F) \text{ iff for each } z\text{: if } z \in F, \text{ then } z \text{ is } N(x,r), \text{ for some } x, r.$$

It is now possible to define open things as classes of open collections:

$$open(x) \text{ iff there is } F \text{ such that } open(F) \text{ and } x \text{ is } Cls(F).$$

Closed things are defined as complements to open things, where the complement $-x$ to a thing x is the class of things disjoint to x, i.e., having no common ingredient with x, see sect. 3:

$$closed(x) \text{ iff } open(-x).$$

We may need as well the notion of a closed collection, as the complement to an open collection:

$$closed(F) \text{ iff } open(-F),$$

where, clearly, the complement $-F$ is the collection obtained by applying the mereological complement $-$ to each member of F. To give an account for the notion of a boundary, we consider a thing x in our mereological universe U along with its complement $-x$. On the family $\mathcal{N}$ of basic open sets, we consider the collection of all non-empty sub-families with the part relation $\subset$ of the strict containment, the resulting ingredient relation $\subseteq$ of the containment, and, the induced class operation of the union of sets. We define the *interior open collection of* x, $INT(x, \mathcal{N})$ as:

$$N(x,r) \in INT(x, \mathcal{N}) \text{ iff } ingr_\pi(N(x,r), x).$$

In the same way, we define the *complementary open collection of* x, $INT(-x, \mathcal{N})$ as:

$$N(x,r) \in INT(x, \mathcal{N}) \text{ iff } ingr_\pi(N(x,r), -x).$$

We denote by the symbol Cls_π the class operator associated with the part relation π, in order to discern between it and the union class operator on $\mathcal{N}$. We define the *boundary of x with respect to the family $\mathcal{N}$*, symbolically $Bd(x, \mathcal{N})$, as:

$$Bd(x, \mathcal{N}) = -[Cls_\pi\, INT(x, \mathcal{N}) + Cls_\pi\, INT(-x, \mathcal{N})],$$

where the addition operator $+$ is the addition operator in the Tarski mereological algebra (see [24]), defined as:

$$\text{for every } t\text{: } ingr_\pi(t, x + y) \text{ iff } ingr_\pi(t, x) \text{ or } ingr_\pi(t, y).$$

Informally, one may see that the boundary defined in that way comprises things whose each neighborhood neither is contained in x nor misses it. Let us make this statement precise.

PROPOSITION 9. *If $ingr_\pi(y, Bd(x, \mathcal{N}))$, then there exists z such that $Ov(y, z)$ and $dis(z, Cls_\pi[INT(x, \mathcal{N}) + Cls_\pi\, INT(-x, \mathcal{N})])$. A fortiori, each neighborhood of y is neither an ingredient of x nor an ingredient of $-x$.*

PROOF. Assume that $ingr_\pi(y, Bd(x, \mathcal{N}))$. By the class property (Cls 2) (see Section 3) there exist w and z such that $ingr_\pi(w, y)$, $ingr_\pi(w, z)$, and $dis(z, Cls_\pi\, INT(x, \mathcal{N}) + Cls_\pi\, INT(-x, \mathcal{N}))$. Hence, $dis(w, Cls_\pi\, INT(x, \mathcal{N}) + Cls_\pi\, INT(-x, \mathcal{N}))$. It follows that y has an ingredient w, which obviously is an ingredient in every neighborhood of y, disjoint with classes of neighborhoods contained in x or $-x$. $\dashv$

Clearly, the class $Cls_\pi\, INT(x, \mathcal{N})$ can be regarded as the *interior* of x, and, the class $Cls_\pi\, INT(-x, \mathcal{N})$ can be regarded as the *interior* of $-x$ which does coincide with the complement to the *closure of x*.

As the collection $INT(x, \mathcal{N}) \cup INT(-x, \mathcal{N})$ is open, $Cls_\pi\, INT(x, \mathcal{N}) + Cls_\pi\, INT(-x, \mathcal{N})$ is an open thing, hence, the boundary $Bd(x, \mathcal{N})$ is closed.

This boundary renders a mereotopological witness of vagueness of x with respect to the induced topology.

Acknowledgement. The author is indebted to Editors for the invitation to take part in this very interesting project and to Referees of this work for their detailed remarks on the manuscript. It comes tacitly that the author is responsible for all faults of this work.

References

[1] Black, M., "Vagueness. An exercise in logical analysis", *Philosophy of Science*, 4, 4 (1937): 427–455.

[2] Bocheński, I. M., *Die Zeitgenossischen Denkmethoden*, A. Francke AG, Bern, 1954.

[3] Casati, R., and A. C. Varzi, *Parts and Places. The Structures of Spatial Representations*, MIT Press, Cambridge MA, 1999.

[4] Fleck, L., "O niektórych swoistych cechach myślenia lekarskiego" ["On some specific features of the medical way of thinking" in Polish], *Archiwum Historji Medycyny i Historji Nauk Przyrodniczych*, 6 (1927): 55–64. In English: pages 39–46 in R. S. Cohen and Th. Schnelle (eds.), *Cognition and fact: Materials on Ludwik Fleck*, Reidel, Dordrecht, 1986. DOI: 10.1007/978-94-009-4498-5

[5] Frege, G., *Grundgesetze der Arithmetik*, Band II, Jena, Verlag Hermann Pohle, 1903. In English: Ph. A. Ebert and M. Rossberg (eds.), *Basic Laws of Arithmetic*, Oxford University Press, 2013.

[6] Gödel, K., "Über formal unentscheidbare Sätze der *Principia Mathematica* und verwandter Systeme I", *Monatshefte für Mathematik und Physik*, 38, 1 (1931): 173–198. In English: pages 144–195 in S. Feferman (ed.), "On formally undecidable propositions of *Principia Mathematica* and related systems I", *Kurt Gödel Collected works*, vol. I., Oxford University Press, 1986.

[7] Hájek, P., *Metamathematics of Fuzzy Logic*, Kluwer, Dordrecht, 1998. DOI: 10.1007/978-94-011-5300-3

[8] Heisenberg, W., "Ueber den anschaulichen Inhalt der quantentheoretischen Kinematik and Mechanik", *Zeitschrift fuer Physik*, 43 (1927): 172–198. In English: pages 62–84 in J. A. Wheeler and W. H. Zurek (eds.), *Quantum Theory and Measurement*, Princeton University Press, Princeton NJ, 2014. DOI: 10.1515/9781400854554

[9] Leśniewski, S., "Podstawy Ogólnej Teoryi Mnogości I" ["Foundations of General Set Theory I" in Polish], *Prace Polskiego Koła Naukowego w Moskwie*, Sekcya Matematyczno-przyrodnicza, No. 2, Moscow, 1916. In English: "On the foundations of mathematics", Vito F. Sinis (transl.), *Topoi*, 2 (1983): 7–52.

[10] Menger, K., "Statistical metrics", *Proceedings of the National Academy of Sciences USA*, 28 (1942): 535–537. DOI: 10.1073/pnas.28.12.535

[11] Pawlak, Z., *Rough Sets: Theoretical Aspects of Reasoning about Data*, Kluwer, Dordrecht, 1991. DOI: 10.1007/978-94-011-3534-4

[12] Polkowski, L., *Approximate Reasoning by Parts. An Introduction to Rough Mereology*, Springer Verlag, Berlin, 2011. DOI: 10.1007/978-3-642-22279-5

[13] Polkowski, L., "A note on 3-valued rough logic accepting decision rules", *Fundamenta Informaticae*, 61 (2004): 37–45.

[14] Polkowski, L., and P. Ośmiałowski, "Spatial reasoning with applications to mobile robotics", pages 433–453 in Jing Xing–Jian, (ed.), *Mobile Robots Motion Planning. New Challenges*, I-Tech Education and Publishing KG, Vienna, 2008. DOI: 10.5772/6002

[15] Polkowski, L., and P. Ośmiałowski, "A framework for multi-agent mobile robotics: Spatial reasoning based on rough mereology in Player/Stage system", pages 142–149 in Chien-Chung Chan, J. W. Grzymala-Busse, and W. P. Ziarko (eds.), *Rough Sets and Current Trends in Computing*, Lecture Notes in Artificial Intelligence vol. 5306, Springer Verlag, Berlin, 2008.

[16] Polkowski, L., and P. Ośmiałowski, "Navigation for mobile autonomous robots and their formations: An application of spatial reasoning induced from rough mereological geometry", pages 329–354 in A. Barrera (ed.), *Mobile Robots Navigation*, InTech, Zagreb, 2010. DOI: 10.5772/209

[17] Polkowski, L., and M. Semeniuk-Polkowska, "On rough set logics based on similarity relations", *Fundamenta Informaticae*, 64 (2005): 379–390.

[18] Polkowski, L., and M. Semeniuk-Polkowska, "Granular mereotopology: A first sketch", pages 322–331 in *Proceedings CS&P 2013 (Concurrency, Specification, Programming)*, 2013. Also as: "On the problem of boundaries from mereology and rough mereology points of view", *Fundamenta Informaticae*, 133, 2–3 (2014): 241–255.

[19] Polkowski, L., and M. Semeniuk-Polkowska, "Boundaries, borders, fences, hedges", *Fundamenta Informaticae*, 129, 1–2 (2014): 149–159.

[20] Polkowski, L., and A. Skowron, "Rough mereology: a new paradigm for approximate reasoning", *International Journal of Approximate Reasoning*, 15, 4 (1996): 333–365. DOI: 10.1016/S0888-613X(96)00072-2

[21] Simons, P., *Parts. A Study in Ontology*, 2nd ed., Clarendon Press, Oxford UK, 2003.

[22] Smith, B., "Mereotopology: A theory of parts and boundaries", *Data and Knowledge Engineering*, 20 (1996): 287–303.

[23] Smith, B., "Boundaries: An essay in mereotopology", pages 534–561 in L. Hahn (ed.), *The Philosophy of Roderick Chisholm*, La Salle, Open Court, 1997.

[24] Tarski, A., "Zur Grundlegung der Booleschen Algebra. I", *Fundamenta Mathematicae*, 24 (1935): 177–198.

[25] Tarski, A., "What is elementary geometry?", pages 16–29 in L. Henkin, P. Suppes, and A. Tarski (eds.), *The Axiomatic Method with Special Reference to Geometry and Physics*, Studies in Logic and Foundations of Mathematics, North-Holland, Amsterdam, 1959. DOI: 10.1016/S0049-237X(09)70017-5

[26] van Benthem, J., *The Logic of Time*, Reidel, Dordrecht, 1983. DOI: 10.1007/978-94-010-9868-7

[27] Zadeh, L. A., "Fuzzy sets", *Information and Control*, 8 (1965): 338–353. DOI: 10.1016/S0019-9958(65)90241-X
[28] Zadeh L. A., "Toward a unified theory of uncertainty", pages 3–4 in *Proceedings of the IPMU the International Conference on Information Processing and Management of Uncertainty 2004, Perugia, Italy*, vol. 1, Editrice Universitá La Sapienza, Rome, 2004.

LECH T. POLKOWSKI
Polish-Japanese Academy of IT
Warszawa, Poland
Department of Mathematics and Computer Science
University of Warmia and Mazury
Olsztyn, Poland
polkow@pjwstk.edu.pl