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Abstract. In classical extensional mereology, composition is idempotent: if x is part of y , then the sum of x and y is identical to y . In this paper, I provide a systematic and coherent formal mereology for which idempotence fails. I first discuss a number of purported counterexamples to idempotence that have been put forward in the literature. I then discuss two recent attempts at sketching non-idempotent formal mereology due to Karen Bennett and Kit Fine. I argue that these attempts are incomplete, however, and there are many open issues left unresolved. I then construct a class of models of a non-idempotent mereology using multiset theory, consider their algebraic structure, and show how these models can shed light on the open issues left from the previous approaches.

Keywords: universalism; extensionality; supplementation; antisymmetry; mereology; parthood; composition

Introduction

What is the difference between the words types ‘stared’ and ‘starred’? They obviously express different concepts: one is what I sometimes do blankly into space, sitting at my desk; the other is what movie actors do in films. Orthographically, they are composed of different letters — well ‘different’ in the sense of token instances, but in another sense the letter types are the same. A natural thought is that word types are *composed* of letter types, where composition is mereological. But according to the industry-standard *classical extensional mereology* (CEM), they couldn’t be.

Mereological composition (or summation) in CEM satisfies a principle that rules out cases like this (and others to be considered later). Let us call this feature the *idempotence* of composition.

Idempotence If x is part of y , then the sum of x and y is identical to y .

I note that this is a slightly different formulation from the more usual property called ‘idempotence’: the sum of x with itself is just x ; that is, taking the sum of something twice-over gives you no more than taking the sum of it once-over. Of course, we have this in CEM as well: given the reflexivity of parthood, we have it as an instance of the principle above. (The more general formulation will be useful throughout.) If words types are *composed* of their letter types, then idempotence identifies the words ‘stared’ and ‘starred’. After all ‘r’ is part of ‘stared’, and hence their sum would just be ‘stared’.

Effingham and Robson [12] discuss a closely related feature of CEM, called the ‘Parts Just Once Principle’ (PJO).

PJO For every composite x , x cannot have y as a part many times over.

Strictly speaking, PJO is not expressible in CEM due to the occurrence of ‘many-times over’. The idempotence property, however, is expressible and a theorem of CEM. Both principles seem plausible.¹ But the case of ‘stared’ and ‘starred’ seems to provide a counterexample, and recently authors have suggested that there may be other cases where it is possible that an object be part-related to another multiple times.² Bennett explains,

Two people are cousins twice over, or ‘double cousins’, as they are called, just in case they are the children of pairs of siblings. Similarly,

¹ For example, Varzi [24, § 6.2] argues that any part-whole structures which fail to satisfy them are not properly called ‘mereological’, due to the corresponding failures of Weak Supplementation.

² It is important, however, to note that we should *not* misunderstand the ‘many times over’ in PJO as telling us that a composite x cannot have *distinct* proper parts that are identical. That would be trivially true and immune to counterexamples. This misunderstanding forms the basis of Effingham and Robson’s argument against PJO:

For the Parts Just Once Principle to be false there could exist an x that has n proper parts, the ys , (where $n > 1$) such that the ys are not the same proper part, but are the same object. If there is a whole which has two or more different proper parts, the whole has those proper parts by being part-related to two or more different (i.e. distinct) objects. So for each of the ys , that y is not identical to any of the other ys . Yet it is stipulated that the same object (call it z) is a part n times over. So z is identical to each of the ys — and so by the transitivity of identity each of the ys are identical to one another. A clear contradiction. ([12, 635])

They assume that being part-related many times over is equivalent to being part-related to two or more distinct objects. But of course that assumption is precisely the negation of what the ‘parts many times over’ proponents are claiming.

the *being three feet from* relation can hold multiple times between the same two entities: consider two antipodal points on a sphere, such that the shortest distance between them along the surface is three feet [...] But [...] *parthood*? [...] I argue that [...] we *can* make sense of the idea of an entity's having a part twice — or four times — over.

[3, 83–84]

The controversy over idempotence has been the subject of some recent controversy among mereologists, as it forces us to identify things we might ordinarily regard as distinct — much like the word types 'stared' and 'starred'.

The aim of this paper is to explore and develop a systematic formal mereology in which idempotence fails. In §1 I present a number of purported counterexamples to idempotence that have been put forward in the literature. In §§2–3 I discuss two recent attempts at developing non-idempotent formal mereology due to Karen Bennett [3] and Kit Fine [13]. I argue that these attempts are incomplete, however, and there are many open issues left unresolved. In §4 I construct a class of models of a non-idempotent mereology utilising multiset theory. I also discuss the algebraic structure of this class of models; this allows us to compare these models with the structures of CEM. These models can shed light on the issues left open from the previous approaches, and provides some insight into the options and limitations of a non-idempotent kind of composition.

1. Parts Just Once?

There are a couple of ways of conceiving of how an object might have another as a proper part more than once. For concrete objects, Smith [23] suggests the phenomenon should be understood as x having a proper part y that is multiply located in many distinct subregions of the location of x . Endurantists are no strangers to the idea that an individual might be multiply located at different times. And if an object could be multiply located in different regions at the *same* time, perhaps we could generate violations of idempotence and PJO.

Consider the following example due to Effingham and Robson [12]:³

³ Examples such as these first appeared in print in Effingham and Robson [12] and Gilmore [16]. See also Eagle's response [10], and Gilmore's reply [17]. Likewise, see Smith's response [23], and Effingham's reply [11]. Both discussions are addressing

Assume you are an endurantist. Imagine you are presented with what appears to be one hundred bricks, $\text{Brick}_1, \text{Brick}_2, \dots, \text{Brick}_{100}$, stacked together so as to arrange what appears to be a brick wall. The brick-layer, Marty, asks you whether the wall is a composite object or not. Presumably you will answer positively. However Marty claims that contrary to your intuitions the wall in question is not a composite object, and that he will demonstrate this. To begin his demonstration, Marty demolishes the wall. Let the time of the demolishing be t_{100} . He then takes Brick_1 to a nearby time machine, whereupon you both travel back in time to t_1 . Here Marty takes you to a shop and purchases a normal house brick, which he then places in the region that will be occupied by Brick_1 at t_{100} . Obviously Brick_1 is the brick purchased from the shop. Marty then places the future version of Brick_1 from t_{100} next to the past version of Brick_1 so that it is in the region that will be occupied by Brick_2 at t_{100} . Clearly then, Brick_2 is numerically identical to Brick_1 . You both travel forward a hundred units of time to t_{101} , where Marty takes Brick_2 (which you now know to be Brick_1 also), and then you both return to t_2 where Brick_2 is placed in the location where Brick_3 will be. Travelling forward in time again to t_{102} Marty takes Brick_3 (which you now know to be both Brick_1 and Brick_2) and travels back one hundred units of time to t_3 where the brick is placed in the location reserved for Brick_4 . This process is repeated until an entire wall has been constructed from the same object during the interval between t_1 and t_{100} . [12, 633–4]

The brick wall seems to be composed entirely of a single brick many times over. So it appears to be direct counterexample to PJO. Moreover, consider the brick wall just before I add the final brick — call this object, composed of the brick ninety-nine times over, Wall_{99} . Now, the brick is already part of Wall_{99} . According to idempotence, completing the wall by adding the final (copy of the) brick to Wall_{99} results in Wall_{99} . But intuitively the result of adding the last brick should be Wall_{100} . And Wall_{100} is not identical to Wall_{99} (since Wall_{99} is located in a different region of spacetime, has a different shape, etc.).⁴ Thus, idempotence must fail.

issues in the literature on persistence. More relevant discussions that tackle the mereological implications of multi-location head-on are Donnelly [9] and Kleinschmidt [19].

⁴ Of course, one might try to save idempotence by accepting that things might be multiply located, and be self-discernible regarding shape and other properties. This approach will like need to allow that things can be proper parts of themselves. Cotnoir and Bacon [7] develop a mereology that might be useful to such an attempt.

Of course, there are a number of replies that defenders of PJO and idempotence can give in response to such a scenario. One reply springs to mind: the example is not metaphysically possible as it involves problematic cases of time-travel which are known to raise metaphysical difficulties. Gilmore [16] gives the following example, which he takes to be clearly physically possible, and *a fortiori* metaphysically possible.

The General Theory of Relativity permits the occurrence of what physicists call ‘closed timelike curves’. A timelike curve is a continuous path through spacetime corresponding to the possible life-history of a massive particle. [...] A timelike curve is *closed* just in case it forms a loop, thus ‘ending where it began’ so to speak. A particle that traces out an almost closed timelike curve would, just by lasting long enough and taking the appropriate trajectory, return to its own past and coexist with a younger version of itself. Consider [...] the career of a hydrogen atom, which we shall call ‘Adam’. Adam is spatially bi-located throughout its two-billion-year-long career. For any given moment of external time [...] t in the relevant universe, Adam is present ‘twice over’. [...] Suppose that, at each moment of Adam’s proper time, Adam is chemically bonded to itself at a different moment of its proper time, thus forming a molecule of H_2 , which we shall call ‘Abel’. Abel is spatially mono-located through its career (which is only one billion years long). [...] The distinctness of Adam and Abel can be argued for in a number of ways. Adam, being a mere hydrogen atom, has certain chemical properties that Abel lacks. Abel, being a hydrogen molecule, is more massive than Adam. [16, 185–187]

In this case, Abel is composed of a single multi-located atom, Adam. This violates idempotence: Adam is part of itself, but the sum of Adam twice over is not identical with Adam, but Abel. It is beyond our scope here to determine whether Gilmore’s example is in fact a genuine physical or metaphysical possibility. However, it seems clearly conceivable on *some* notion of parthood or composition, and so we might wish to make the attempt at formalising such notions.

Perhaps a more straightforward example of multi-location involves universals. Insofar as universals are thought to be located in spacetime — i.e. ‘immanent’ universals — they are often said to be “wholly located wherever they are instantiated”. Additionally, some have thought that universals can have other universals as *parts*; the locus classicus being Armstrong [1].

As water is H_2O , the structural universal WATER has the universal HYDROGEN as a component twice over, and the universal OXYGEN as

a component once over (and perhaps we'd like to include two copies of the BONDING universal). By contrast, hydrogen peroxide is H_2O_2 . Hence, the structural universal HYDROGEN PEROXIDE has the component HYDROGEN twice over, and the component OXYGEN twice over (plus, perhaps, BONDING thrice over).

Summing together WATER with an extra OXYGEN (and, if we include BONDING, an extra instance of that too) does not yield WATER as *idempotence* would predict, but yields HYDROGEN PEROXIDE. Hence, either OXYGEN is not part of WATER, or *idempotence* fails. So, if structural universals were structured by CEM, then WATER and HYDROGEN PEROXIDE would be identical. But they aren't.

From this, Lewis [20] famously concluded *pace* Armstrong that structural universals do not exist since they cannot be handled within CEM. And for Lewis, it is either (classical extensional) mereology or 'magic'. As Hawley [18] rightly argues, Lewis's 'mereology or magic' is a false dichotomy. Why not give an account of an alternative mereology which allows one to give a full metaphysical theory of structural universals?⁵

2. Parts and Slots

Bennett [3] provides an interesting attempt to rise to the challenge of providing an alternative mereology that rejects PJO. Bennett's mereology makes creative use of the distinction between a *role* and an *occupant* of that role. Bennett argues that this distinction allows us to have two types of objects in our domain: parthood *slots*, and the *fillers* of those slots. On this view, wholes come 'pre-structured' as it were; they are structural shells waiting to be filled in. Once it is allowed that objects have this slot structure, it is a small step toward allowing that a single object might fill more than one slot in the same whole. It might be thought that a structural universal like WATER, for example, has three (perhaps five) slots, two of which are filled by HYDROGEN and one of which is filled by OXYGEN (and perhaps two slots filled by BONDING).⁶

⁵ See for example Bigelow and Pargetter [4] for an early attempt. See also Bader [2], Forrest [15], and Mormann [22] for more recent efforts.

⁶ But see Fisher[14] for reasons why this application to structural universals might not ultimately be successful. (Similar considerations could be raised to the framework below.)

Bennett's mereology uses two primitive relations: P_sxy means that x is a parthood slot of y ; Fxy means that x fills y . This allows us to define general parthood thus: $P(x, y) := \exists z(P_s z, y \wedge Fxz)$. That is, x is a part of y just if x fills some parthood slot of y . Bennett then outlines a number of axioms which make this parthood relation behave in some expected ways.

- A1 *Only Slots are Filled*: $Fxy \rightarrow \exists z P_s zy$
 A2 *Slots Cannot Fill*: $Fxy \rightarrow \neg \exists z P_s xz$
 A3 *Slots Don't Have Slots*: $P_s xy \rightarrow \neg \exists z P_s zx$
 A4 *Improper Slots*: $\exists y P_s yx \rightarrow \exists z (P_s zx \wedge Fxz)$
 A5 *Slot Inheritance*: $(P_s wy \wedge Fxw \wedge P_s zx) \rightarrow P_s zy$
 A6 *Mutual Occupancy is Identity*: $(P_s wy \wedge Fxw) \wedge (P_s zx \wedge Fyz) \rightarrow x = y$
 A7 *Single Occupancy*: $P_s xy \rightarrow \exists! z Fzx$ ⁷
 A8 *Slot Strong Supplementation*:
 $(\exists z P_s zx \wedge \exists z P_s zy) \wedge \neg (\exists z P_s zx \wedge Fyz) \rightarrow \exists z (P_s zy \wedge \neg P_s zx)$

A1 states that an object is a filler only if there's another object with a slot being filled by it. A2 says that slots cannot play the role of a filler while A3 says that slots cannot play the role of an object having a slot to be filled. A4 states that every object has a slot which it itself fills; this allows us to prove a kind of reflexivity. A5 states that any slots of a filler are thereby slots of the filled; this allows us to prove a kind of transitivity. A6 states that only one object can fill each slot; this allows us to prove that parthood is antisymmetric and that proper parthood is *extensional*. A7 states that every parthood slot has exactly one filler. A8 is related to a class of mereological supplementation principles: the antecedent requires that x and y have slots and y not be part of x , and the consequent stipulates the existence of a slot of y that isn't a slot of x .⁸

Bennett's theory provides a minimal way of thinking about parthood which is compatible with an object being a part twice over. But the introduction of slots as distinct from parts complicates the metaphysical picture. How exactly *are* we to think of slots? One option is to think of slots as locations. If we do so, then the various principles involving them might prove to be objectionable constraints on a theory of locations. For

⁷ In Bennett's version, the consequent actually reads ' $\exists! z Fxz$ ' which seems to be a typographical mistake.

⁸ In fact, A8 might be thought to be too weak in a number of ways: it is in effect a 'slot version' of what Varzi [25] calls *Strong Company*.

example, A6 would rule out the possibility of co-located objects. If we don't see slots as reducible to locations and the like, then they appear to be additional ontological commitments. We might do better to avoid such commitments (if we can).

Unfortunately, Bennett's mereology runs into further trouble with mereological sums. She does not propose any kind of mereological sum or fusion operation on parts. Slots cannot be parts of slots; nor are they parts of the objects they are slots for. As a result, it is very difficult to tell when looking at some parts a and b what their sum should be. We need to know what slot-structure is present in the whole before we can determine what the relevant sum is. How many slots need to be filled? There are many possible answers to this question, each of which determines at least one distinct object, but usually many more. Suppose, for example, the whole has three slots. Then there are six possible sums aab , aba , baa , abb , bab , and bba , assuming it matters to the identity of the object which part fills which slot. Even if we identify objects like aba and baa , we still fail to have it that mereological sum is unique (i.e. the extensionality of sums fails). Perhaps this is to be expected; but it would be nice to have a theory of composition that explained this. However, given the complexities of composition Bennett does not develop any theory of it. These are open questions a fully developed non-idempotent mereology would need to answer.

3. A Plurality of Composition Operators

In Fine's [13] insightful and radical paper, he characterises the intuitive concept of parthood as minimally requiring:

CONTAINMENT Wholes *contain* their parts, in the sense of the parts being *integral* to the whole.

BUILDING Wholes are *built* from their parts, in the sense of the whole *comprising* the parts, the parts *composing* the whole.

REPLACEMENT A whole may *change* by the replacement of its parts.

Fine is primarily concerned with arguing for two key theses regarding parts and wholes.

PLURALISM There is more than one basic sense of part, and likewise more than one basic mode of composition.

OPERATIONALISM The operation of composition, rather than the relation of parthood, should be taken as primitive.

A parthood relation is *basic* in Fine's terms when it is not definable in terms of other ways of being a part, but may be defined from a particular composition relation. Fine provides a number of arguments for OPERATIONALISM, most of which involve expressive power and the adequacy of various definitions. (It should be noted, though, that some of Fine's arguments are only strong assuming a prior commitment to pluralism.) The important thesis for our purposes is PLURALISM. Fine accepts the composition operator of CEM as one candidate among many equally legitimate notions of composition. For our purposes, we are looking for a formal theory of composition which allows for an object to have a part twice over. To find such a theory, we need to delve into Fine's attempt to unify and characterise these different notions of composition under a single formal framework.

We begin with a primitive variably polyadic composition operator Σ and apply it to a number of objects x_1, x_2, \dots ⁹ Already we can then define the following:

COMPONENT x is a *component* of y iff $y = \Sigma(\dots, x, \dots)$.

PARTHOOD x is a *part* of y iff there is a sequence of objects x_1, x_2, \dots, x_n for $n > 0$ for which $x = x_1, y = x_n$ and x_i is a component of x_{i+1} for $1 \leq i \leq n - 1$.

In other words, x is a part of y whenever there is a way of 'building' y from x (and other things perhaps) by repeated applications of the composition operator. On this definition, parthood can be shown to be a pre-order (weak or strict, depending on the composition relation in question). Antisymmetry (or in some cases asymmetry) can be proved from the following assumption:

ACYCLICITY If $x = \Sigma(\dots \Sigma(\dots, x, \dots) \dots)$ then $x = \Sigma(\dots, x, \dots)$.

For suppose $x = \Sigma(\dots \Sigma(\dots, x, \dots) \dots)$ but $x \neq y = \Sigma(\dots, x, \dots)$; then x is a component of y , and hence a part of it, while y is a component of x . This is a violation of antisymmetry – or asymmetry.¹⁰

⁹ My preference would be to say that Σ is a *plural* operator, rather than a variably polyadic one. However, for Fine, it is crucial that Σ be capable of taking *no* arguments, to generate e.g. the empty set.

¹⁰ Of course Fine is simply presupposing that Σ is functional – that outputs of its application are always unique. Notice that acyclicity *does not* entail antisymmetry if Σ isn't a functional in this way – that is, if we don't assume that composition is always unique.

So far, we have supposed almost nothing about composition. But one can list a number of key principles that may or may not hold for each individual kind of composition.¹¹

COLLAPSE $\Sigma(x) = x$

LEVELING $\Sigma(\dots \Sigma(\dots, x, y, z, \dots) \dots \Sigma(\dots, u, v, w, \dots) \dots) = \Sigma(\dots, x, y, z, \dots, u, v, w, \dots)$

ABSORPTION $\Sigma(\dots, x, x, \dots, y, y, \dots) = \Sigma(\dots, x, \dots, y, \dots)$

PERMUTATION $\Sigma(\dots, x, y, z, \dots) = \Sigma(\dots, y, z, x, \dots)$ (and similarly for all other permutations)

This results in a number of different possible composition operators depending on whether or not COLLAPSE (C), LEVELING (L), ABSORPTION (A), and PERMUTATION (P) are satisfied.¹² We write e.g. \overline{CLAP} for the composition relation that satisfies ABSORPTION and PERMUTATION but not COLLAPSE and LEVELING. So, for example the sums of CEM satisfy $CLAP$, while sequences correspond to \overline{CLAP} . Another example would be the set-builder \overline{CLAP} .

Not all these are *basic* composition relations. There are (at least) two ways of deriving new parthood and composition relations from basic ones:

SUBSUMPTION A subsumed parthood relation is (the ancestral of) a restriction of a basic parthood relation; x 's being a subsumed part of y holds in virtue of x 's being a basic part of y .

CHAINING If K is a family of parthood relations, then x is a K -part of y if x and y can be linked by relationships of k -part for k in K .

For SUBSUMPTION, we restrict the range of the relata to some subset of the domain. So, we might have a unary predicate F , and define x is an F -part of y iff x is part of y and $F(x)$. The subsumed parthood relation will be the transitive closure of F -parthood. A parthood relation that is a result of CHAINING is called *hybrid*.¹³ Where K is the

¹¹ Notice that these are all *regular* identity conditions: a condition $s = t$ such that the variables appearing in s and the variables appearing in t are the same.

¹² Fine [13, fn. 12] states that there are only twelve possible variants. This is presumably because some of these constraints are jointly inconsistent, although he does not specify. \overline{CLAP} and $CLAP$ violate ACYCLICITY, and so Fine thinks they should be disallowed.

¹³ It is unclear from the text whether one is permitted 'chain' basic relations together with derived ones. But there clearly are hybrid relations of this sort.

family of *all* parthood relations, the result of CHAINING is called *general* parthood.¹⁴

In addition, there are derived composition relations which are neither SUBSUMPTIONS, nor CHAININGS. For example, for any ‘hierarchical’ composition relation Σ – namely, any composition relation for which LEVELING fails – we may define a ‘flat’ correlate \bigcup_{Σ} as follows:

$$\text{FLATTENING } \bigcup_{\Sigma}(\Sigma(x_1, x_2, \dots), \Sigma(y_1, y_2, \dots)) = \Sigma(x_1, x_2, \dots, y_1, y_2, \dots)$$

In general, this will turn any \overline{L} composition into an L one. So, for example, where Σ_{\in} (namely, \overline{CLAP}) is the set-builder operation, $\bigcup_{\Sigma_{\in}}$ is the set union operation which corresponds to \overline{CLAP} . Similarly for other composition operators.

For our non-idempotent mereology, we are looking for a composition relation like the ‘mere sums’ of $CLAP$ without ABSORPTION. In the vicinity are *multisets* (\overline{CLAP}), and multiset *unions* (i.e. flattenings of multisets) (\overline{CLAP}). But what we are really after corresponds to $CLAP$, since in general we want $\Sigma(x) = x$ without $\Sigma(x, x) = x$.¹⁵

While Fine’s framework is extremely rich and suggestive, it leaves something to be desired. The main worry is that Fine’s four principles C , L , A , and P do not seem to be enough to uniquely specify the relevant

¹⁴ Fine is not explicit about about the distinction between *antisymmetry* of a weak order over against the *asymmetry* of a strict order – he calls both the former. But this matters for general parthood. Remember, we are dealing with sometimes strict (e.g. the ancestral of membership), sometimes weak (e.g. subset) partial orders. What guarantees that when we chain them, we have anything like the relevant behavior? Similar worries hold for *irreflexivity/reflexivity*. Or does Fine think the general parthood relation need not be either reflexive or irreflexive?

¹⁵ We should not conflate the flattenings of multisets with the particular non-idempotent composition operator we are after. Fine insists that “the only identities which hold are the ones which can be shown to hold only on the basis of the defining principles for the operations in question” (p. 580). So for example, Fine thinks that there is nothing to force the mereological sum of any two sets to be identical to any set, contra Lewis [21, p. 580].

Lewis’s view, in my opinion, rests upon conflating the derived form of composition for sets with the mereological operation of sum, and has no intuitive support.

The derived form of composition for sets is $\bigcup_{\Sigma_{\in}}$, namely set union, which corresponds to \overline{CLAP} . Mereological summation corresponds to $CLAP$. It is no wonder then that Lewis had to appeal to a primitive *singleton-forming* operator to make his reduction of mereology to set theory work.

form of composition, nor do they pin down the exact structure of the models for each intended composition. For example, Quinean sets (from ‘New Foundations’ set theory) allow for sets to be their own singletons. The Quinean set builder ($CLAP$) can be ‘flattened’ to Quinean set-theoretic unions which will satisfy $CLAP$ —exactly like the sums of CEM. But it is not obvious that Quinean set-union *is* mereological composition, or even that they provably have the same models. Furthermore, the principles $CLAP$ do not appear to completely characterise CEM; one cannot prove e.g. the weak supplementation principle that stipulates that whenever x is a proper part of y , then a non- x -overlapping part of y exists. Models of CEM are not just supplemented, they are *complemented* in the sense that every object that is not the ‘universe’ has a unique complement composed of all and only those things that do not overlap it. I cannot see a way of proving these complements exist by unrestricted $CLAP$ -composition alone. It should be noted that Fine [13, fn. 13] states that he wants the intended models for a given composition operator to be “isomorphic to a ‘word algebra’ over the ‘generators’ of given elements.” This is important, and might suffice to give the models added structure not provable from $CLAP$ alone; but can it guarantee the existence of complements? This fact is not obvious, and Fine says nothing more about it.

Algebraically speaking, in any complete distributive lattice the least upper bound operator \sqcup will satisfy COLLAPSE (by the *idempotence* of \sqcup), LEVELING (by the *associativity* of \sqcup), ABSORPTION (by definition), and PERMUTATION (by the *commutativity* of \sqcup). To ensure the relevant fact about being isomorphic to a word algebra over generators, we need to consider *free* complete distributive lattices which are not always guaranteed to exist. Moreover, there are well-known free complete distributive lattices (e.g. the real interval $[0, 1]$) which are obviously not complete Boolean algebras (as would be required if they were models of CEM). In any case, more needs to be said on this score. Similarly, it is not obvious that Fine’s framework provides us with a full characterisation of our target via $CLAP$.

4. Multiset Models

In this section, I will attempt to provide a precise class of models for a non-idempotent mereology using multiset theory (see e.g. [5]). Just as powerset models are models of (atomistic) mereology, powermultisets

are models for our mereology. By providing models, we show that a mereology without PJO and idempotence is consistent (relative to ZFC). The models illuminate key choice points in the formal implementation of the target mereology. Two particular philosophical upshots are worth highlighting. First, there are (at least) two viable notions of mereological sum. Second, mereological complementation in such a theory is decidedly non-standard.

We begin with a class D of multisets. A *mset* (or *mset*) is like a set, except that the identity of that mset is sensitive to the number of times something appears as an element. Elements of msets, then, have *multiplicities*: the multiplicity of element x in mset A is just the number of times x is a member of A . We can think of an mset A in the universe as being characterized by a map $m_A : D \rightarrow \mathbb{N}$ from multisets to natural numbers, such that $m_A(x) > 0$ iff $x \in A$. We will assume that the multiplicity of x in A is unique — that m_A is a total function.

We can now introduce some basic mereological notions:

PART $A \sqsubseteq B$ iff $m_A(x) \leq m_B(x)$, for all $x \in D$

PROPER PART $A \sqsubset B$ iff $A \sqsubseteq B$ and $A \neq B$

We can show that \sqsubseteq is by definition *reflexive*, and *transitive*. Since $A = B$ iff for all $x \in D$, $m_A(x) = m_B(x)$, we can easily see that \sqsubseteq is *antisymmetric*. Notice that this amounts to the assumption that multisets are *extensional* in the sense that sets are uniquely defined by giving the multiplicities of their elements.

Multisets contain more notions of parthood than the standard ones, however.

WHOLE PARTS A is a *whole part* of B iff for all $x \in D$, if $m_A(x) > 0$, then $m_A(x) = m_B(x)$

Whole parts contain all the multiplicities of atomic parts of a whole, e.g. $\{a, a\}$ is a whole part of $\{a, a, b\}$.

Moreover, we can define:

ROOT The *root* of an mset A is the set $A^* = \{x \in D \mid m_A(x) > 0\}$

FULL PART A is a *full part* of B iff $A \sqsubseteq B$ and $A^* = B^*$.

Roots are roughly *classical*; that is, they contain all elements of A exactly once, e.g. $\{a, b\} = \{a, a, b\}^*$. We say that two msets are *similar* iff they have the same root. Full parts contain every distinct element of the whole at least once, e.g. $\{a, b\}$ is a full part of $\{a, a, b\}$. So full parts

are just parts that have the same root. That these distinct notions of parthood become available is not without philosophical significance. For example, it would allow us to distinguish that the structural universal WATER is a full part, but not a whole part, of the universal HYDROGEN PEROXIDE.

Now, we are in position to define our first notion of composition.

SUM $A \sqcup B$ is the mset defined by $m_{A \sqcup B}(x) = \max(m_A(x), m_B(x))$

PRODUCT $A \sqcap B$ is the mset defined by $m_{A \sqcap B}(x) = \min(m_A(x), m_B(x))$

Sums are our first notion of composition with product its dual. For example, $\{a, a, b\} \sqcup \{b, b\} = \{a, a, b, b\}$. Similarly $\{a, a, b\} \sqcap \{b, b\} = \{b\}$. This allows us to define a useful notion of mereological overlap: A and B overlap whenever $A \sqcap B \neq \emptyset$.

Our models will have the structure of powermultisets.

POWERMSET The powermset of an mset X , $\tilde{\wp}(X)$, is the multiset of containing (multiplicities of) all parts of X .

Determining the exact multiplicity of a given part of X in $\tilde{\wp}(X)$ is fairly intuitive. For example, let $A = \{x, x, y\}$. Then, $\tilde{\wp}(A) = \{\emptyset, \{x\}, \{x\}, \{x, x\}, \{y\}, \{x, y\}, \{x, y\}, \{x, x, y\}\}$. To obtain the powermset, first imagine distinguishing between each individual ‘instance’ of a given element in X ; second, take the classical powerset; and third, undo all the ‘distinctions’ you made in step one.

THEOREM 1. $\langle \tilde{\wp}(X), \sqcup, \sqcap, \emptyset, X \rangle$ is a bounded distributive lattice in which $A \sqsubseteq B$ iff $A \sqcup B = B$.¹⁶

There are a few things to notice about these models. First, bounded distributive lattices already have a lot of the structure of models of CEM. In the case of finite models of CEM, the only difference is that in the classical case we are guaranteed the existence of Boolean complements whereas here we are not.

Second, these models show that multiset union does *not* correspond to \overline{CLAP} , contra Fine. That is because COLLAPSE and ABSORPTION both hold for \sqcup . In some sense, then, this notion of composition is genuinely *mereological* and shockingly similar to that in CEM, even though the *Parts Just Once* principle clearly fails.

¹⁶ See [6, pp. 6–8]

This leads us to our third important point: *idempotence* of composition (which holds) can come apart from *Parts Just Once* (which fails). This is of some importance philosophically, since it entails that it might be possible for an object to have a part twice over even if mereological composition is idempotent as long as it is possible to start with objects that already violate PJO.

But this way of ‘building in’ failures of PJO from the start might strike some as somewhat contrived. After all, we are after some theory of composition which allows us to ‘build up’ failures of PJO from nothing but ordinary things at the start. And for that we will need a notion of composition for which idempotence fails. Luckily, there is another equally good candidate for modeling mereological composition:

MERGE $A \boxplus B$ is the mset defined by $m_{A \boxplus B}(x) = m_A(x) + m_B(x)$

Recall that for sums $\{a, a, b\} \sqcup \{b, b\} = \{a, a, b, b\}$. By contrast for merges $\{a, a, b\} \boxplus \{b, b\} = \{a, a, b, b, b\}$ Notice that $A \boxplus A \neq A$ (unless $A = \emptyset$). But \boxplus is still associative, commutative, and distributes over \sqcup and \sqcap .¹⁷ In fact, where $D^{\mathbb{N}}$ is the set of all mappings m from D to \mathbb{N} , then $\langle D^{\mathbb{N}}, \boxplus \rangle$ is a free Abelian semigroup. However, we should not generally think of our models as being closed under the operation of \boxplus , since $\tilde{\rho}(X)$ is not always so closed. As a result, while mereological composition by \sqcup is *unrestricted*, mereological composition by \boxplus is not. Why? Recall Wall₁₀₀ composed of a single Brick (one-hundred times over): it may well be that this thought experiment is compatible with mereological universalism before the time travel takes place. If so, then why shouldn’t it be after? There are no new *pluralities* of objects that fail to have a sum. However, we should not think simply because mereological sum (i.e. \sqcup) is unrestricted that this entails that \boxplus is unrestricted as well. Allowing for existence of Wall₁₀₀ does not demand we accept the existence Wall₅₀₀₀ and Wall_{1,234,567,890}. No, this would require the existence of *multiplicities* of Brick in excess of what the story gives us.

Until now, we have avoided any kind of mereological complementation (of the sort usually guaranteed by supplementation principles). Here is a natural approach to relative complements:

COMPLEMENTS If $A \not\sqsubseteq B$ then $A - B$ is the mset defined by $m_{A-B}(x) = m_A(x) - m_{A \sqcap B}(x)$, for all $x \in D$.

¹⁷ See Singh, et al. [8, p. 79].

Clearly, where $B \sqsubseteq A$, then $A - B \sqsubseteq A$. Likewise, $A - \emptyset = A$ and $A - A = \emptyset$. Where \top is the ‘universe’ (i.e. the full powerset) we write $\top - A$ as \overline{A} . However, in this mereology we can see some decidedly non-classical behaviour. The classical ‘law’ $A \sqcup \overline{A} = \top$ sometimes fails. Letting $\top = \{a, a, b, b, b\}$ and $A = \{a, b, b\}$, $\overline{A} = \{\overline{a}, b\}$, note that $A \sqcup \overline{A} = \{a, b, b\} \neq \top$. Similarly, the classical ‘law’ $A \sqcap \overline{A} = \emptyset$ sometimes fails. This means that we are modeling a mereology in which objects and their mereological complements can overlap!

THEOREM 2. $\langle \tilde{\varphi}(X), \sqcup, \sqcap, -, \emptyset, X \rangle$ is not a Boolean algebra, but a De Morgan algebra.¹⁸

This non-classical behavior of the most natural notion of complementation might come as a bit of a surprise. But we do have that $A \boxplus \overline{A} = \top$ always holds, and so some of the Boolean behavior of complementation is present relative to \boxplus our other notion of composition.

5. Conclusion

The class of multiset models for Abelian mereology set out above gives us somewhat of a better handle on what a mereology without PJO or idempotence would look like. Unlike Bennett’s mereology, we have two formally precise notions of mereological composition without the need to resort to any *role/occupant* distinction, nor appeal to any kind of slot structure. (Of course, one might avail oneself to such philosophical interpretations, but it is no part of the model theory.) Unlike Fine’s framework, we have a precise class of models for which this mereology is defined, along with a full algebraic characterisation of them and their relation to models of CEM.

Of course, there are some limitations of the above proposal. The first limitation is that these model-theoretic structures validate *atomism*; the members of X are *atomic* in $\tilde{\varphi}(X)$ in the sense of having no non-empty proper parts. Avoiding atomism is a task for future work. Toward that end, it would be worth finding an axiomatization of these structures, so that we can give the first-order theory of Abelian mereology. A second limitation of the approach above is that we have only been considering

¹⁸ See [6, pp. 6–8] for proof. He uses Rasiowa’s term ‘quasi-Boolean’ algebra, rather than the ‘De Morgan’.

binary mereological sums and merges. We have not generalized to allowing for infinitary fusions $\bigsqcup X$ of the sort allowed in CEM. This is certainly desirable, but would face some obstacles. Multisets with infinite multiplicities are not well understood. Moreover, generalising the definition of sum could easily take us into the transfinite, since for members A_i of X , it might well be that the multiplicities of $m_{A_i}(x)$ are unbounded, and so there is no maximum. In any case, these are interesting but not insurmountable technical challenges.

My aim has been to explore and develop a mereology for which idempotence and *Parts Just Once* can fail. We have shown that such mereologies exist consistently, and that they are genuinely mereological in the sense that they share remarkably strong structural similarities to CEM. Whether the framework set out here is adequate to the philosophical task is for future discussion to decide.¹⁹

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