



Katarzyna Pałasińska

## THREE-ELEMENT NON-FINITELY AXIOMATIZABLE MATRICES AND TERM-EQUIVALENCE

**Abstract.** It was shown in [5] that all two-element matrices are finitely based independently of their classification by term equivalence (the Post classification). In particular, each 2-valued matrix is finitely axiomatizable. We show below that for certain two not finitely axiomatizable 3-valued matrices this property is also preserved under term equivalence. The general problem, whether finite axiomatizability of a finite matrix is preserved under term-equivalence, is still open, as well as the related problem as to whether the consequence operation of a finite matrix is finitely based.

**Keywords:** matrix; term-equivalence; finite axiomatization

### 1. Introduction

Recall that a *matrix* is a triple  $\mathfrak{M} = \langle M, F, D \rangle$ , where  $\emptyset \neq D \subseteq M$  and  $F$  is a finite set of finitary operations on  $M$ . The set  $D$  is called the set of *designated values* of  $\mathfrak{M}$ . Let for each  $f \in F$ ,  $\lambda_f$  be a symbol denoting the operation  $f \in F$  of the same arity as  $f$  and let  $\Lambda_F := \{\lambda_f : f \in F\}$ . Let  $V$  be a fixed, countable set of variables,  $V = \{x_1, x_2, \dots\}$ , and let  $x := x_1, y := x_2, z := x_3$ . The set of all terms in variables from  $V$  and operation symbols from  $\Lambda_F$  is denoted by  $\text{Te}_F$ . The set  $\text{Te}_F$  is turned into an algebra  $\mathbf{Te}_F$  in the standard way and the homomorphisms from this algebra into the algebra  $\langle M, F \rangle$  are called *valuations* in  $\mathfrak{M}$ . Functions from  $V$  into  $M$  are called *valuations of variables* in  $M$  and each valuation is induced by a valuation of variables. If the target algebra of a valuation is the algebra of terms then the valuation is called

a *substitution*. For terms  $t, s \in \text{Te}_F$  and a variable  $v \in V$ , we write  $t[s/v]$  for  $\sigma(t)$ , where  $\sigma$  is the substitution such that  $\sigma(v) = s$  and  $\sigma(x_i) = x_i$  for all  $x_i \neq v$ . For a finite set  $X \cup \{\alpha\} \subseteq \text{Te}_F$  the pair  $\langle X, \alpha \rangle$ , written as  $\frac{X}{\alpha}$ , is called a *rule*. This rule is *valid* in  $\mathfrak{M}$  iff for every valuation  $\varphi$ , we have:  $\varphi(X) \subseteq D \Rightarrow \varphi(\alpha) \in D$ . The members of the set  $X$  are called *premisses* and the term  $\alpha$  the *conclusion* of the rule  $\frac{X}{\alpha}$ . A rule with the empty set of premisses is identified with its conclusion and called *axiomatic*. The conclusion of an axiomatic rule valid in  $\mathfrak{M}$  is a *tautology* of  $\mathfrak{M}$ . The set of all tautologies of  $\mathfrak{M}$  is denoted by  $E(\mathfrak{M})$ . The notion of a *derivation* or *proof* of a term by means of a given set of rules is standard. A set of rules by means of which all tautologies of  $\mathfrak{M}$  and nothing more can be derived will be called an *axiomatization* of  $\mathfrak{M}$  (see [9, 10]). If  $E(\mathfrak{M})$  is not empty, then some of the rules in its axiomatization must be axiomatic, but in general, axiomatization may contain some non-axiomatic rules.

A related notion is that of a *basis* of the consequence operation of  $\mathfrak{M}$ : a set  $\mathcal{R}$  of valid rules of  $\mathfrak{M}$  is called a *basis* of  $\mathfrak{M}$  iff all valid rules of  $\mathfrak{M}$  can be derived by means of  $\mathcal{R}$ . Clearly, every basis is an axiomatization. A matrix is *finitely based*, iff it has a finite basis; it is *finitely axiomatizable* iff there exists a finite set of rules that forms its axiomatization. If a matrix is not finitely axiomatizable then it is not finitely based.

The question whether the finite basis property is preserved under term-equivalence was raised by W. Rautenberg. For matrices with the *finite replacement property* defined by B. Herrmann and W. Rautenberg in [5] this, indeed, is true. It was proved there that all 2-element matrices have the finite replacement property and this result was used in [5] to complete the proof that *all* of them (not just the ones resulting explicitly from the Post classification) are finitely based. It is also known that no matrix term-equivalent to the three-element Wroński's matrix of [11] is finitely based ([7]). Wroński's matrix is finitely axiomatizable but three-element non-finitely axiomatizable matrices also exist. In Section 2 below we recall two such matrices and in Section 5 we show that all matrices term-equivalent to them are also non-finitely based.

## 2. Two non-finitely axiomatizable matrices

The two non-finitely axiomatizable matrices that will be discussed in this paper both are defined on the three-element set  $M = \{0, 1, 2\}$ , have one binary operation and one designated value: 2. Let  $\mathfrak{M}_1 = \langle M, \cdot_1, \{2\} \rangle$  and  $\mathfrak{M}_2 = \langle M, \cdot_2, \{2\} \rangle$ , where  $\cdot_1$  and  $\cdot_2$  are presented in Table 1; the two operations differ only in  $0 \cdot 0$ . (In [6] these two matrices were denoted by  $\mathfrak{M}_7$  and  $\mathfrak{M}_8$  and were shown to be nonfinitely axiomatizable.)

$\cdot_1$	0	1	2
0	2	2	2
1	2	2	2
2	1	2	2

$\cdot_2$	0	1	2
0	1	2	2
1	2	2	2
2	1	2	2

Table 1. Tables for  $\cdot_1$  and  $\cdot_2$

We use the binary operation symbol  $\cdot$  as a metavariable to be interpreted as either  $\cdot_1$  or  $\cdot_2$ . When writing terms in  $\text{Te}_{\{\cdot\}}$ , we will omit the symbol  $\cdot$  and adopt the convention of associating to the left. Let  $\mathbf{2}$  be the term  $\mathbf{2} := x(yz)$  and notice that it is a tautology both in  $\mathfrak{M}_1$  and in  $\mathfrak{M}_2$ .

PROPOSITION 1. *Every term  $t \in \text{Te}_{\{\cdot\}}$  is of the form*

$$t = t_1 v_m \cdots v_1, \tag{1}$$

for some  $m \geq 0$ , where  $v_1, \dots, v_m$  are variables and  $t_1$  is either a variable or a substitution instance of  $\mathbf{2}$ .

By convention, if  $m = 0$  then (1) becomes  $t = t_1$ , where  $t_1$  is a substitution of  $\mathbf{2}$  or a variable. It is easy to see that the term  $xx$  is a tautology of  $\mathfrak{M}_1$  and  $xxx$  is a tautology of  $\mathfrak{M}_2$ . In both  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$ , for an element  $a \in \{1, 2\}$ ,  $a00 = a$  and for an element  $a \in \{0, 1, 2\}$ ,  $a000 = a0$ . So it follows that for  $v_1, \dots, v_m \in V$ :

- if the term  $v_m \cdots v_1$  is a tautology of  $\mathfrak{M}_1$ , then  $m$  is even and  $m \geq 2$ .  
If  $v_m \cdots v_1$  is a tautology of  $\mathfrak{M}_2$ , then  $m$  is odd and  $m \geq 3$ ;
- if the term  $\mathbf{2}v_m \cdots v_1$  is a tautology of either  $\mathfrak{M}_1$  or  $\mathfrak{M}_2$ , then  $m$  is even.

Let  $\mathcal{L}$  be the set of all terms of the form (1) such that  $t_1$  is a variable not occurring among  $v_1, \dots, v_m$ . For a term  $t \in \mathcal{L}$  we put:  $l(t) = t_1$  and  $\mathcal{C}(t) = \{v_1, \dots, v_m\}$ ; we call  $l(t)$  the *leading variable* of  $t$ . Notice

that the functions  $l$  and  $\mathcal{C}$  are not defined on the entire set of terms; the domain of each of them is the set  $\mathcal{L} \subset \text{Te}_{\{\cdot\}}$ . This set has the following obvious property.

**PROPOSITION 2.** *Let  $t_1, s \in \text{Te}_{\{\cdot\}}, t_1 \notin V$  and let  $v \in V$  be a variable occurring in  $s$ .*

1. *If  $t = s[t_1/v]$ , then*

$$t \in \mathcal{L} \quad \text{iff} \quad s, t_1 \in \mathcal{L}, v = l(s) \text{ and } l(t_1) \notin \mathcal{C}(s).$$

2. *More generally, let  $\sigma$  be a substitution such that  $\sigma(v) = t_1, \sigma(s) = t$ . Then*

$$t \in \mathcal{L} \quad \text{iff} \quad s, t_1 \in \mathcal{L}, v = l(s), \sigma(\mathcal{C}(s)) \subseteq V \text{ and } l(t_1) \notin \sigma(\mathcal{C}(s)).$$

In Proposition 3 below, we use the expression “ $v$  is not a leading variable of  $t$ ” as an abbreviation for “either  $t$  does not have a leading variable or it has one but this variable is different from  $v$ ”, i.e., “ $t \in \mathcal{L} \Rightarrow l(t) \neq v$ ”. By inspection of the operation tables one gets the following proposition.

**PROPOSITION 3.** *Let  $\mathfrak{M} = \langle M, \cdot, \{2\} \rangle$  be either  $\mathfrak{M}_1$  or  $\mathfrak{M}_2$ . Let  $\varphi$  and  $\psi$  be valuations in  $\mathfrak{M}, v \in V$ .*

1. *Let  $t \in \mathcal{L}, l(t) = v$ . Assume that  $\varphi(w) = \psi(w) = 0$  for every variable  $w \in \mathcal{C}(t)$ , while  $\varphi(v) = 1$  and  $\psi(v) = 2$ . Then  $\varphi(t) \neq \psi(t)$ .*
2. *If  $t$  is of the form (1),  $\varphi, \psi$  valuations such that for some  $i = 1, \dots, m$   $\varphi(v_i), \psi(v_i) \in \{1, 2\}$  and  $\varphi(v_j) = \psi(v_j)$  for all  $j < i$ , then  $\varphi(t) = \psi(t)$ .*
3. *Let  $t$  be any term in  $\text{Te}_{\{\cdot\}}$  such that  $v$  is not a leading variable of  $t$ . Assume that  $\varphi(w) = \psi(w)$  for all variables  $w \neq v$ , while  $\varphi(v), \psi(v) \in \{1, 2\}$ . Then  $\varphi(t) = \psi(t)$ .*

**COROLLARY 4.** *Let  $t \in \text{Te}_{\{\cdot\}}$ , let  $v$  be a variable and  $\varphi, \psi$  valuations. Assume that  $\varphi(w) = \psi(w)$  for all  $w \neq v$ , while  $\varphi(v), \psi(v) \in \{1, 2\}$ . Then if  $\varphi(t) \neq \psi(t)$  then  $t \in \mathcal{L}, v = l(t)$  and  $\mathcal{C}(t) \subseteq \varphi^{-1}(\{0\})$ .*

**PROOF.** Since  $\varphi(t) \neq \psi(t)$ , it follows by item 3 of Proposition 3, that  $t \in \mathcal{L}$  and  $v = l(t)$ . Hence  $t = vv_m \cdots v_1$  for some variables  $v_1, \dots, v_m \in V$  such that  $v \notin \{v_1, \dots, v_m\}$ . By assumption,  $\varphi(v_i) = \psi(v_i)$  for all  $i = 1 \dots, m$ .

Suppose that for some  $i = 1, \dots, m$ ,  $\varphi(v_i) \neq 0$ . So  $\psi(v_i) \neq 0$  as well and for every  $j < i$ ,  $\varphi(v_j) = \psi(v_j)$ . By item 2. of Proposition 3,  $\varphi(t) = \psi(t)$ , a contradiction.  $\square$

Tautologies of  $\mathfrak{M}$  can be characterized as follows.

PROPOSITION 5. A term  $t$  of the form (1) is a tautology of  $\mathfrak{M}_1$  iff the conjunction of the following conditions holds:

1.  $t_1$  is a substitution of **2** and  $m$  is even or  $t_1$  is a variable and  $m$  is odd,
2. if  $t_1$  is a variable, then  $t_1 \in \{v_1, \dots, v_m\}$ ,
3. for each variable  $v \in \{v_1, \dots, v_m\}$  there exists an odd  $i \in \{1, \dots, m\}$  such that  $v = v_i$  and  $v \notin \{v_1, \dots, v_{i-1}\}$ .

Similarly, a term  $t$  of the form (1) is a tautology of  $\mathfrak{M}_2$  iff conditions **2** and **3** hold as well as the following modification of condition **1**:

- 1'.  $t_1$  is a substitution of **2** or a variable and in both cases  $m$  is even.

Notice that condition **2** is equivalent to “ $t \notin \mathcal{L}$ ” and implies that  $t$  is not a variable. Condition **3** says that when scanning a tautology of the form (1) from the right, the first occurrence of every variable is on an odd position.

PROOF. ( $\Rightarrow$ ) If a term  $t \in \mathcal{L}$ , then by Proposition **3**, item **1**, there are two valuations assigning different values to  $t$ , so  $t$  cannot be a tautology. By the contrapositive, if  $t$  is a tautology then  $t \notin \mathcal{L}$  and condition **2** holds. Condition **1** for tautologies of  $\mathfrak{M}_1$  and condition 1' for tautologies of  $\mathfrak{M}_2$  follow from the observations made on page **483**. To see condition **3**, notice that for every term of the form (1), for each variable  $v \in \{v_1, \dots, v_m\}$ , there is  $i \in \{1, \dots, m\}$  such that  $v = v_i$  and for each  $j < i, v_j \neq v$ . We claim that such an index  $i$  must be odd. For assume  $i$  is even and let  $\varphi(v) = 2$ , while  $\varphi(v_j) = 0$  for all  $j < i$ . Then  $\varphi(t) = 20^{i-1} = 1$ , a contradiction. Hence **3** holds.

( $\Leftarrow$ ) For the proof for  $\mathfrak{M}_1$  assume that  $t$  is of the form (1) and that conditions **1–3** hold. Assume that for some valuation  $\varphi$  into  $\mathfrak{M}_1$ ,  $\varphi(t) \neq 2$ . Since  $t$  is not a variable, we have that  $\varphi(t) = 1$ . Then  $\varphi(t_1 v_m \cdots v_2) = 2$  and  $\varphi(v_1) = 0$ . By condition **3**,  $v_2 = v_1$ , so  $\varphi(v_2) = \varphi(v_1) = 0$  and  $\varphi(t_1 v_m \cdots v_3) = 1$ . Proceeding inductively, for each even  $k \leq m$ , we have that  $\varphi(v_k) = \dots = \varphi(v_1) = 0$  and  $\varphi(t_1 v_m \cdots v_{k+1}) = 1$ . If  $m$  is even, we get that  $\varphi(t_1) \neq 2$ , so  $t_1$  is not a substitution of **2**, contradicting condition **1**. Hence  $m$  is odd and  $t_1$  is a variable. So  $\varphi(v_{m-1}) = \dots = \varphi(v_1) = 0$  and  $\varphi(t_1 v_m) = 1$ . Hence  $\varphi(t_1) = 2$  and  $\varphi(v_m) = 0$ . But by condition **2.**,  $t_1 \in \{v_1, \dots, v_m\}$ , a contradiction. This finishes the proof that conditions **1–3** imply that  $t$  is a tautology of  $\mathfrak{M}_1$ .

Now assume that for a term of the form (1), the conditions 1', 2 and 3 hold. We will show that  $t$  is a tautology of  $\mathfrak{M}_2$ . Assume that for some valuation  $\varphi$  into  $\mathfrak{M}_2$ ,

$$\varphi(t) \neq 2. \tag{2}$$

If for all  $i \leq m$ ,  $\varphi(v_i) = 0$  then by conditions 1' and 2,  $\varphi(t) = 2$ . So there is an index  $i \in \{1, \dots, m\}$  such that  $\varphi(v_i) \neq 0$  and for all  $j < i$ ,  $\varphi(v_j) = 0$ . By (2) this  $i$  is even. But then by condition 3 in the assumptions of the proposition, there is  $j < i$  such that  $v_i = v_j$ , a contradiction.  $\square$

It follows from Proposition 5 that for  $k \geq 2$  the term

$$x_{2k}x_{2k-1} \cdots x_i \cdots x_2x_1x_{2k}x_{2k}x_{2k-2}x_{2k-2} \cdots x_2x_2 \tag{3}$$

is a tautology of  $\mathfrak{M}_1$ . Similarly, the term

$$x_{2k+1}x_{2k}x_{2k-1} \cdots x_1x_{2k+1}x_{2k+1}x_{2k}x_{2k}x_{2k-2}x_{2k-2} \cdots x_ix_i \cdots x_2x_2 \tag{4}$$

is a tautology of  $\mathfrak{M}_2$ .

The proof given in [6] that neither  $\mathfrak{M}_1$  nor  $\mathfrak{M}_2$  is finitely axiomatizable was inspired by [4] and by proofs in [9, 10]; we will now outline its idea. The same idea will later be expanded to an argument that no matrix term-equivalent to  $\mathfrak{M}_1$  or  $\mathfrak{M}_2$  is finitely axiomatizable.

Let  $\mathfrak{M} = \langle M, \cdot, \{2\} \rangle$  be either  $\mathfrak{M}_1$  or  $\mathfrak{M}_2$ . For each  $k \geq 1$  one defines a term  $\hat{t}_k$ , where in case of  $\mathfrak{M} = \mathfrak{M}_1$ ,  $\hat{t}_k := x_{2k}x_{2k-1} \cdots x_1$  and in case of  $\mathfrak{M} = \mathfrak{M}_2$ ,  $\hat{t}_k = x_{2k+1}x_{2k} \cdots x_2x_1$ . Let  $G_k$  be the set of terms of the form  $\hat{t}_k v_m \cdots v_1$ , where  $v_1, \dots, v_m$  are variables. For every  $k$ , the set  $E(\mathfrak{M}) \cap G_k$  is nonempty, for consider (3) and (4). The following is a consequence of Proposition 5.

PROPOSITION 6. *Let  $k \geq 1$  and let  $t \in E(\mathfrak{M}) \cap G_k$ . Then*

$$t = \hat{t}_k v_m \cdots v_1$$

for some even  $m$  and some variables  $v_1, \dots, v_m$ . Also, for each  $i = 1, \dots, k$  the variable  $x_{2i}$  occurs among  $v_1, \dots, v_m$ . Finally, for each  $v$  occurring in  $t$ , the first occurrence of  $v$  when counting from the right is on an odd position.

COROLLARY 7. *If  $t \in E(\mathfrak{M}) \cap G_k$  then  $\hat{t}_k$  is a subterm of  $t$  and there are at least  $k$  distinct variables occurring in  $t$  outside of  $\hat{t}_k$ .*

PROOF. By Proposition 6 at least all  $x_{2i}$ , for  $i = 1, \dots, k$  must occur in  $t$  outside of  $\hat{t}_k$ .  $\square$

Let us mention that in the case that  $\mathfrak{M} = \mathfrak{M}_2$  when  $\hat{t}_k$  is defined as  $\hat{t}_k = x_{2k+1}x_{2k} \cdots x_2x_1$ , also the variable  $x_{2k+1}$  must occur in  $t$  outside of  $\hat{t}_k$ , so in this case there are even at least  $k + 1$  variables in  $t$  outside of  $\hat{t}_k$ .

For a finite set  $A$  the symbol  $\sharp(A)$  denotes the number of elements in the set  $A$ . Corollary 7 can be restated as follows:

PROPOSITION 8. *If  $s \in \mathcal{L}$  and  $t \in E(\mathfrak{M}) \cap G_k$  is obtained by substituting  $\hat{t}_k$  for the leading variable in  $s$ , then  $\sharp(\mathcal{C}(s)) \geq k$ .*

Let  $\mathcal{R}_k$  be the set of all rules valid in  $\mathfrak{M}$  with the conclusion not longer than  $k$ . Using Corollary 7 one shows that the set  $E(\mathfrak{M}) \setminus G_k$  is closed under  $\mathcal{R}_k$ . It follows that no tautology from the nonempty set  $G_k \cap E(\mathfrak{M})$  can be derived by means of the rules from  $\mathcal{R}_k$ . Since every finite set of rules is a subset of some  $\mathcal{R}_k$ , it follows that there is no finite axiomatization.

### 3. Term-equivalence

Let  $M$  be a nonempty set and let  $F$  and  $G$  be two sets of operations on  $M$ . Let  $\Lambda_F$  and  $\Lambda_G$  be the corresponding sets of operation symbols and  $\mathbf{Te}_F$  and  $\mathbf{Te}_G$  corresponding algebras of terms in variables  $V$ . For a valuation of variables  $\varphi : V \rightarrow M$ , let  $\varphi_F : \mathbf{Te}_F \rightarrow \langle M, F \rangle$  and  $\varphi_G : \mathbf{Te}_G \rightarrow \langle M, G \rangle$  denote valuations induced by  $\varphi$ .

DEFINITION 9. Let  $\mathfrak{M} = \langle M, F, D \rangle$  and  $\mathfrak{N} = \langle M, G, D \rangle$  be two matrices with the same underlying set  $M$  and the same set of designated values  $D$ . We say that  $\mathfrak{M}$  and  $\mathfrak{N}$  are term-equivalent and write  $\mathfrak{M} \equiv \mathfrak{N}$  if

1. for every  $n$  and every  $n$ -ary operation  $f \in F$  there is a term  $t \in \mathbf{Te}_G$  such that for every  $\varphi : V \rightarrow M$ ,  $\varphi_F(\lambda_f(x_1, \dots, x_n)) = \varphi_G(t)$ ; and
2. for every  $n$  and every  $n$ -ary operation  $g \in G$  there is a term  $s \in \mathbf{Te}_F$  such that for every  $\varphi : V \rightarrow M$ ,  $\varphi_G(\lambda_g(x_1, \dots, x_n)) = \varphi_F(s)$ .

If two terms  $t \in \mathbf{Te}_F, s \in \mathbf{Te}_G$  are such that for every valuation  $\varphi$  of variables  $\varphi_F(t) = \varphi_G(s)$ , then we write  $t \equiv s$  and say that  $t$  and  $s$  are *equivalent relatively to*  $(\mathfrak{M}, \mathfrak{N})$ . In other words, terms  $t \in \mathbf{Te}_F$  and  $s \in \mathbf{Te}_G$  are equivalent iff the equality  $t \approx s$  is an identity of the algebra  $\langle M, F \cup G \rangle$ . When using this notation, we do not assume that the sets  $F$  and  $G$  are distinct; actually, they may be equal. As the set  $D$  does not play any role in this definition, term-equivalence is really the property of pairs of algebras: matrices are term-equivalent iff their base

algebras are. The finite basis property considered in universal algebra is obviously independent of the choice of the operation sets, as long as they are equivalent. More specifically, if there is a finite basis  $\mathcal{E}$  of identities of an algebra  $\mathbf{A}$  and an algebra  $\mathbf{B}$  is term-equivalent to  $\mathbf{A}$  then  $\mathcal{E}$  can easily be transformed into a finite basis of identities of  $\mathbf{B}$ . The same is true for a finite basis of quasi-identities of  $\mathbf{A}$ . There is still another meaning of a finite basis of an algebra that best resembles the finite axiomatizability property of a matrix: one may ask whether there is a finite set of quasi-identities of an algebra  $\mathbf{A}$  from which all identities of  $\mathbf{A}$  can be derived. This finite basis property, too, is preserved under term-equivalence. If a consequence operation  $C_n$  is algebraizable ([2]) with equivalent algebraic semantics quasivariety  $\mathcal{Q}$  then  $C_n$  is finitely axiomatizable iff there is a finite set of quasi-identities of  $\mathcal{Q}$  from which all identities of  $\mathcal{Q}$  can be derived. A similar connection holds between finite basis property of an algebraizable consequence operation and finite basis property for the quasi-identities of its equivalent algebraic semantics quasivariety  $\mathcal{Q}$ .

A weaker property than algebraizability of a consequence operation is sufficient to reproduce the universal algebraic argument that the finite basis and the finite axiomatizability properties are preserved under term-equivalence. Namely, if the consequence operation of  $\mathfrak{M}$  is *congruential* ([3]),  $\mathfrak{M} \equiv \mathfrak{N}$  and  $\mathfrak{M}$  is finitely axiomatizable (or based) then so is  $\mathfrak{N}$ . Now, such a direct general argument cannot be applied to the two non-finitely axiomatizable matrices considered here, as their consequence operations are not congruential. In Proposition 10 we claim that they are not even protoalgebraic ([1]), which is a weaker property than congruential.

**PROPOSITION 10.** *Let  $\mathfrak{M}$  be  $\mathfrak{M}_1$  or  $\mathfrak{M}_2$ . Then the deductive system defined by the set of all rules valid in  $\mathfrak{M}$  is not protoalgebraic.*

**PROOF.** For the proof by contradiction assume that the deductive system determined by  $\mathfrak{M}$  is protoalgebraic. Then there is a set of binary terms  $\Delta(x, y)$  such that all terms in  $\Delta(x, x)$  are tautologies and  $y$  is deducible from  $\{x\} \cup \Delta(x, y)$ , i.e., the rule  $\frac{\{x\} \cup \Delta(x, y)}{y}$  is valid in  $\mathfrak{M}$ . Since  $\Delta(x, x)$  are tautologies of  $\mathfrak{M}$ , none of the terms in  $\Delta(x, y)$  is a single variable. Under the valuation  $\varphi$  such that  $\varphi(x) = 2, \varphi(y) = 1$ , all compound terms take the designated value 2 (see the table of  $\cdot$ ). So the premisses of the rule  $\frac{\{x\} \cup \Delta(x, y)}{y}$  all take the designated value 2, while the conclusion does not.  $\square$



### 4. Auxiliary definitions and lemmas

Let  $M = \{0, 1, 2\}$  and let  $\mathfrak{M}$  be either  $\mathfrak{M}_1$  or  $\mathfrak{M}_2$ . The following Proposition concerning terms equivalent relatively to  $(\mathfrak{M}, \mathfrak{M})$  is a consequence of Proposition 3 and Corollary 4.

PROPOSITION 11. *Let  $t, s \in \text{Te}_{\{\cdot\}}$ , where  $t = t_1 v_m \cdots v_1$ ,  $s = s_1 w_n \cdots w_1$ , for some  $t_1, s_1, v_1, \dots, v_m, w_1, \dots, w_n \in V$ . Assume that  $t \in \mathcal{L}$  and  $t \equiv s$ . Then  $s \in \mathcal{L}$ ,  $t_1 = s_1$  and  $\{v_1, \dots, v_m\} = \{w_1, \dots, w_n\}$ .*

PROOF. Define valuations  $\varphi$  and  $\psi$  such that  $\varphi(w) = \psi(w) = 0$  for all  $w \in \mathcal{C}(t)$ ;  $\varphi(t_1) = 1, \psi(t_1) = 2$  and  $\varphi(w) = \psi(w) = 2$  for all other variables  $w$ . Then by item 1. of Proposition 3,  $\varphi(t) \neq \psi(t)$ . Since  $t \equiv s$ , it follows that  $\varphi(s) \neq \psi(s)$ . By Corollary 4,  $s \in \mathcal{L}$ ,  $t_1 = l(s)$  and  $\mathcal{C}(s) \subseteq \varphi^{-1}(\{0\})$ . So  $s_1 = t_1$  and  $\{w_1, \dots, w_n\} \subseteq \{v_1, \dots, v_m\}$ . Another application of Corollary 4 yields the opposite inclusion  $\{v_1, \dots, v_m\} \subseteq \{w_1, \dots, w_n\}$ .  $\square$

A consequence of this proposition is that a term  $t \in \mathcal{L}$  depends on each of the variables occurring in it.

Now let  $F$  be some set of operations on  $M$  such that  $\mathfrak{N} = \langle M, F, \{2\} \rangle$  is term-equivalent to  $\mathfrak{M}$ . Let  $\equiv$  denote the equivalence relative to  $(\mathfrak{M}, \mathfrak{N})$ . First of all, by term-equivalence, there is a term  $\otimes(x, y) \in \text{Te}_F$  such that  $xy \equiv \otimes(x, y)$ . We will write the symbol  $\otimes$  between its arguments and use the association to the left. For every  $k \geq 1$  we define the term

$$\begin{aligned} \hat{\alpha}_k &:= x_{2k} \otimes x_{2k-1} \otimes \cdots \otimes x_1 && \text{if } \mathfrak{M} = \mathfrak{M}_1 \quad \text{and} \\ \hat{\alpha}_k &:= x_{2k+1} \otimes x_{2k} \otimes \cdots \otimes x_1 && \text{if } \mathfrak{M} = \mathfrak{M}_2. \end{aligned}$$

Clearly,  $\hat{\alpha}_k \equiv \hat{t}_k$ , where  $\hat{t}_k$  is as defined in section 2.

Next, every operation from  $F$  can be expressed by some term  $t \in \text{Te}_{\{\cdot\}}$ , so for every  $n$ , every  $n$ -ary operation symbol  $\lambda \in \Lambda_F$  there is  $t \in \text{Te}_{\{\cdot\}}$  such that:

$$\lambda(x_1, \dots, x_n) \equiv t. \tag{5}$$

If in addition  $t \in \mathcal{L}$ , then by Proposition 11 the variable  $l(t)$  and the set  $\mathcal{C}(t)$  are independent of the choice of the term  $t$  on the right hand side of (5). As was noticed after Proposition 11, the term  $t \in \mathcal{L}$  depends on all variables occurring in it, i.e., on every variable in the set  $\{l(t)\} \cup \mathcal{C}(t)$ . Therefore, in this case also  $\lambda(x_1, \dots, x_n)$  depends on every variable from this set, hence  $\{l(t)\} \cup \mathcal{C}(t) \subseteq \{x_1, \dots, x_n\}$ . The following convention will be used.

CONVENTION. If (5) holds and  $t \in \mathcal{L}$  then we assume that  $x_1 = l(t)$  and  $\mathcal{C}(t) = \{x_2, \dots, x_l\}$  for some  $l \leq n$ . In this case, we write the term  $\lambda(x_1, \dots, x_n)$  as

$$\lambda(x_1, \dots, x_l; x_{l+1}, \dots, x_n)$$

marking with the semicolon the end of the list of variables actually occurring in  $t$ .

The purpose of this convention is to simplify the notation; all arguments below can be rewritten in the general case when the variables on which  $\lambda(x_1, \dots, x_n)$  actually depends are not necessarily the first  $l$  variables on the list. The variables on which a term depends are also called *essential* in this term. If the set of operations  $F$  contains projections, then there are composed terms in  $\text{Te}_F$  that are equivalent to variables.

NOTATION. For  $T \subseteq \text{Te}_F$  we write  $T \subseteq_{\equiv} V$  iff for every  $t \in T$  there exists a variable  $z \in V$  such that  $t \equiv z$ . If  $T \subseteq_{\equiv} V$  and  $Z = \{z \in V : \alpha \equiv z \text{ for some } \alpha \in T\}$  then we write  $T \equiv Z$ .

The set  $\hat{\mathcal{L}}$  of terms that we are now going to define is a generalization of the set  $\mathcal{L}$  considered in Section 2 to the case of an arbitrary set of operations  $F$  instead of  $\{\cdot\}$ , such that  $\langle M, F, \{2\} \rangle$  is term-equivalent to  $\mathfrak{M}$ . Also, the functions  $\mathcal{C}$  and  $l$  defined previously on  $\mathcal{L}$  are now redefined as the functions with the general domain of  $\hat{\mathcal{L}}$ . In the special case when  $F = \{\cdot\}$ , the concepts defined in Definition 12 coincide with the previous ones.

DEFINITION 12. Inductively, we define the set  $\hat{\mathcal{L}} \subseteq \text{Te}_F$ . For a term  $\alpha \in \hat{\mathcal{L}}$  its leading variable  $l(\alpha)$  and the set  $\mathcal{C}(\alpha)$  of its remaining essential variables are also inductively defined.

1. If  $v \in V$ , then  $v \in \hat{\mathcal{L}}$ ,  $l(v) = v$  and  $\mathcal{C}(v) = \emptyset$ .
2. Let  $\alpha_1 \in \hat{\mathcal{L}}$ . Assume that  $\lambda \in \Lambda_F$  and  $t \in \mathcal{L}$  are such that

$$\lambda(x_1, \dots, x_l; x_{l+1}, \dots, x_n) \equiv t.$$

Assume further that  $\alpha_2, \dots, \alpha_n \in \text{Te}_F$  and that for some  $Z \subseteq V$ ,  $\{\alpha_2, \dots, \alpha_l\} \equiv Z$ , while  $l(\alpha_1) \notin Z$ . Then the term

$$\beta := \lambda(\alpha_1, \dots, \alpha_l; \alpha_{l+1}, \dots, \alpha_n)$$

is a member of  $\hat{\mathcal{L}}$ ,  $l(\beta) = l(\alpha_1)$  and  $\mathcal{C}(\beta) = \mathcal{C}(\alpha_1) \cup Z$ .

By Definition 12 all terms equivalent to variables are in  $\hat{\mathcal{L}}$ . It also follows from this definition and Proposition 11 that if (5) holds then

$$t \in \mathcal{L} \text{ iff } \lambda(x_1, \dots, x_l; x_{l+1}, \dots, x_n) \in \hat{\mathcal{L}}.$$

This can be generalized to all terms in  $\hat{\mathcal{L}}$ .

**PROPOSITION 13.** *Let  $\alpha \in \text{Te}_F$ . Then  $\alpha \in \hat{\mathcal{L}}$  iff there exists a term  $t \in \mathcal{L}$  such that  $\alpha \equiv t$ . For such  $\alpha$  and  $t$  as above,  $l(t) = l(\alpha)$  and  $\mathcal{C}(t) = \mathcal{C}(\alpha)$ .*

**PROOF.** If  $\alpha$  is a variable, then the statements of the proposition clearly hold. Assume that  $\alpha \in \hat{\mathcal{L}}$  and suppose that  $\alpha = \lambda(\alpha_1, \dots, \alpha_l; \alpha_{l+1}, \dots, \alpha_n)$ , where  $\alpha_1 \in \hat{\mathcal{L}}, \{\alpha_2, \dots, \alpha_l\} \equiv Z \subseteq V, l(\alpha_1) \notin Z$  and there is  $s \in \mathcal{L}$  such that  $\lambda(x_1, \dots, x_l; x_{l+1}, \dots, x_n) \equiv s$ . Then by Definition 12,  $l(\alpha) = l(\alpha_1)$  and  $\mathcal{C}(\alpha) = \mathcal{C}(\alpha_1) \cup Z$ . Also, by our Convention  $l(s) = x_1$  and  $\mathcal{C}(s) = \{x_2, \dots, x_l\}$ . By the induction hypothesis there is a term  $t_1 \in \mathcal{L}$  with  $l(t_1) = l(\alpha_1)$  and  $\mathcal{C}(t_1) = \mathcal{C}(\alpha_1)$  such that  $t_1 \equiv \alpha_1$ . Let  $t$  be the result of substituting  $t_1$  for  $x_1$  and  $z_j$  for  $x_j$  for  $j = 2, \dots, l$  in  $s$ , where  $z_j \in Z$  is such that  $\alpha_j \equiv z_j$ . Then  $\alpha \equiv t$  and by Proposition 2,  $t \in \mathcal{L}$ . Also  $l(t) = l(t_1) = l(\alpha_1) = l(\alpha)$  and  $\mathcal{C}(t) = \mathcal{C}(t_1) \cup \{z_2, \dots, z_l\} = \mathcal{C}(\alpha_1) \cup \{z_2, \dots, z_l\} = \mathcal{C}(\alpha_1) \cup Z = \mathcal{C}(\alpha)$ .

For the proof in the other direction, assume that  $\alpha \equiv t \in \mathcal{L}$  and that  $\alpha = \lambda(\alpha_1, \dots, \alpha_n)$ , where  $\lambda \in \Lambda_F$  and  $\alpha_1, \dots, \alpha_n \in \text{Te}_F$ . By term-equivalence there exists a term  $s \in \text{Te}_{\{\cdot\}}$  such that  $s \equiv \lambda(x_1, \dots, x_n)$ . Similarly, let  $t_1, t_2, \dots, t_n \in \text{Te}_{\{\cdot\}}$  be such that

$$t_1 \equiv \alpha_1, t_2 \equiv \alpha_2, \dots, t_n \equiv \alpha_n. \quad (6)$$

Let  $\sigma$  be a substitution in  $\text{Te}_{\{\cdot\}}$  such that for  $i = 1, \dots, n, \sigma(x_i) = t_i$ . Then  $\alpha \equiv \sigma(s)$ , so  $\sigma(s) \equiv t$ . By Proposition 11,  $\sigma(s) \in \mathcal{L}$ . If, for all  $i = 1, \dots, l, \sigma(x_i)$  is a variable, then by Definition 12,  $\alpha \in \hat{\mathcal{L}}$  and the condition on variables in Proposition 13 holds. If for some  $i = 1, \dots, l, \sigma(x_i)$  is not a variable, then by Proposition 2,  $s \in \mathcal{L}$  and  $x_i = l(s)$ . Assume, as in our convention, that  $x_1 = l(s)$ , so this  $i = 1$ , and  $\mathcal{C}(s) = \{x_2, \dots, x_l\}$  for some  $l \leq n$ . Also, by Proposition 2,  $t_1 \in \mathcal{L}, t_2, \dots, t_l$  are variables and  $l(t_1) \notin \{t_2, \dots, t_l\}$ . Then  $l(t_1) = l(t)$ . By the induction hypothesis,  $\alpha_1 \in \hat{\mathcal{L}}, l(\alpha_1) = l(t_1)$  and  $\mathcal{C}(\alpha_1) = \mathcal{C}(t_1)$ . By (6)  $\alpha_2, \dots, \alpha_l$  are equivalent to variables and none of these variables coincides with  $l(\alpha_1)$ . So  $\alpha \in \hat{\mathcal{L}}, l(\alpha) = l(\alpha_1) = l(t_1) = l(t)$  and  $\mathcal{C}(\alpha) = \mathcal{C}(\alpha_1) \cup \{t_2, \dots, t_l\} = \mathcal{C}(t_1) \cup \{t_2, \dots, t_l\} = \mathcal{C}(t)$ .  $\square$

By Proposition 13 we get the following corollaries to Proposition 3 and Corollary 4.

**COROLLARY 14.** *Let  $\varphi$  and  $\psi$  be valuations. Let  $\beta \in \hat{\mathcal{L}}, l(\beta) = v, \varphi(v) \neq \psi(v)$ , both  $\varphi(v), \psi(v) \in \{1, 2\}$  and  $\varphi(w) = \psi(w) = 0$  for  $w \in \mathcal{C}(\beta)$ . Then  $\varphi(\beta) \neq \psi(\beta)$ .*

**COROLLARY 15.** *Let  $\varphi$  and  $\psi$  be valuations. Let  $\gamma \in \text{Te}_F, \varphi(v), \psi(v) \in \{1, 2\}$  and  $\varphi(w) = \psi(w)$  for  $w \neq v, w \in V$ . Assume  $\varphi(\alpha) \neq \psi(\alpha)$ . Then  $\gamma \in \hat{\mathcal{L}}, v = l(\gamma)$  and  $\mathcal{C}(\gamma) \subseteq \varphi^{-1}(\{0\})$ .*

**DEFINITION 16.** For  $k \geq 1$  let  $S_k \subseteq \text{Te}_F$  be the smallest set such that:

1.  $\hat{\alpha}_k \in S_k$ ;
2. if  $\lambda \in \Lambda_F$  is an  $n$ -ary operation such that  $\lambda(x_1, \dots, x_l; x_{l+1}, \dots, x_n) \in \hat{\mathcal{L}}$  and if  $\beta \in S_k, \beta_2, \dots, \beta_n \in \text{Te}_F, \{\beta_2, \dots, \beta_n\} \subset_{\equiv} V$ , then the term  $\lambda_f(\beta, \beta_2, \dots, \beta_n) \in S_k$ .

It follows from this definition that if  $\alpha \in S_k$ , then  $\hat{\alpha}_k$  is a subterm of  $\alpha$ . The following proposition follows from Definition 16 by induction.

**PROPOSITION 17.** *Let  $\alpha, \gamma \in \text{Te}_F$ . Then*

1.  $\alpha \in S_k$  iff there is  $\beta \in \hat{\mathcal{L}}$  such that  $\alpha = \beta[\hat{\alpha}_k/l(\beta)]$ .
2. Assume that  $\gamma \in \hat{\mathcal{L}}, \sigma(l(\gamma)) \in S_k$  and that  $\sigma(\mathcal{C}(\gamma)) \subset_{\equiv} V$ . Then  $\sigma(\gamma) \in S_k$ .

Modifying terms (3) and (4) from Section 2 appropriately, we get the following proposition.

**LEMMA 18.** *The set  $S_k \cap E(\mathfrak{N})$  is nonempty.*

**LEMMA 19.** *If  $\alpha \in S_k \cap E(\mathfrak{N})$  and  $\beta \in \hat{\mathcal{L}}$  is such that  $\alpha = \beta[\hat{\alpha}_k/l(\beta)]$  then  $\sharp(\mathcal{C}(\beta)) \geq k$ .*

**PROOF.** Let  $s \in \mathcal{L}$  be such that  $\beta \equiv s$ . The existence of such a term  $s$  is guaranteed by Proposition 13. Then  $\mathcal{C}(s) = \mathcal{C}(\beta)$  and  $l(\beta) = l(s)$ . Since  $\hat{t}_k \equiv \hat{\alpha}_k$ , we get  $\alpha \equiv s[\hat{t}_k/l(s)]$ , so  $s[\hat{t}_k/l(s)] \in E(\mathfrak{N}) \cap G_k$ . By Proposition 8,  $\sharp(\mathcal{C}(s)) \geq k$ , so  $\sharp(\mathcal{C}(\beta)) \geq k$ .  $\square$

Define the length  $|\beta|$  of a term  $\beta \in \text{Te}_F$ : if  $\beta \in V$  or  $\beta$  is a constant, then  $|\beta| = 1$ . If  $\beta = f(\beta_1, \dots, \beta_n)$ , then  $|\beta| = \sum_{i=1}^n |\beta_i|$ .

LEMMA 20. Let  $k \geq 1, \beta \in \text{Te}_F, |\beta| \leq k$ . Let  $\sigma : \text{Te}_F \rightarrow \text{Te}_F$  be a substitution such that  $\sigma(\beta) \in E(\mathfrak{M}) \cap S_k$ . Define  $\beta_1^0 := \beta$ . Then there exists  $p \geq 0$  such that for each  $1 \leq i \leq p$  there exist:  $n_i \geq 1, 1 \leq l_i \leq n_i, \lambda_i \in \Lambda_F$  of arity  $n_i$  and terms  $\beta_1^i, \dots, \beta_{n_i}^i$  such that

$$\lambda_i(x_1, \dots, x_{l_i}; x_{l_i+1}, \dots, x_{n_i}) \in \hat{\mathcal{L}} \quad (7)$$

$$\beta_1^{i-1} = \lambda_i(\beta_1^i, \dots, \beta_{l_i}^i; \beta_{l_i+1}^i, \dots, \beta_{n_i}^i) \quad (8)$$

$$\sigma(\beta_1^i) \in S_k \setminus \{\hat{\alpha}_k\} \quad (9)$$

$$\{\sigma(\beta_j^i) : j = 2, \dots, l_i\} \subset \equiv V \text{ and} \quad (10)$$

$$\beta_1^p \text{ is a variable.} \quad (11)$$

PROOF. Clearly, there is  $p \geq 0$  such that for all  $0 < i \leq p$  there are  $n_i$ 's,  $\lambda_i$ 's and terms  $\beta_1^i, \dots, \beta_{n_i}^i$  such that (8) holds and  $\beta_1^p$  is either a variable or a constant. The following inductive argument shows that for each  $i = 0, 1, \dots, p$  also (9) holds and if  $i \geq 1$  then (7) and (10) hold. From (9) for  $i = p$  it then follows that  $\beta_1^p$  is not a constant and therefore (11) is true.

Notice that  $\sigma(\beta_1^0) = \sigma(\beta) \neq \hat{\alpha}_k$ , for  $\hat{\alpha}_k$  is not a tautology of  $\mathfrak{M}$  while  $\sigma(\beta)$  is. So for  $i = 0$  we have (9) and the statement “if  $i \geq 1$  then (7) and (10)” is satisfied vacuously.

Let  $1 \leq i_0 \leq p$  and assume that for all  $i < i_0$  (9) holds and if  $i \geq 1$  then (10). By (9), for  $i = i_0 - 1$ , we have

$$\sigma(\beta_1^{i_0-1}) \in S_k \text{ and } \sigma(\beta_1^{i_0-1}) \neq \hat{\alpha}_k. \quad (12)$$

By (8) applied to  $i = i_0$

$$\begin{aligned} \beta_1^{i_0-1} &= \lambda_{i_0}(\beta_1^{i_0}, \dots, \beta_{n_{i_0}}^{i_0}), \text{ so} \\ \sigma(\beta_1^{i_0-1}) &= \lambda_{i_0}(\sigma(\beta_1^{i_0}), \dots, \sigma(\beta_{n_{i_0}}^{i_0})). \end{aligned}$$

By (12) and Definition 16

$$\begin{aligned} \sigma(\beta_1^{i_0}) \in S_k, \{\sigma(\beta_2^{i_0}), \dots, \sigma(\beta_{l_{i_0}}^{i_0})\} \subset \equiv V, \\ \lambda_{i_0}(x_1, \dots, x_{l_{i_0}}; x_{l_{i_0}+1}, \dots, x_{n_{i_0}}) \in \hat{\mathcal{L}}, \end{aligned} \quad (13)$$

so in particular (10) holds for  $i_0$ . To finish the proof it remains to show that  $\sigma(\beta_1^{i_0}) \neq \hat{\alpha}_k$ . Suppose otherwise. Equations (8) yield:

$$\beta = \lambda_1(\beta_1^1, \dots, \beta_{n_1}^1)$$

$$\begin{aligned}\beta_1^1 &= \lambda_2(\beta_1^2, \dots, \beta_{n_2}^2) \\ &\vdots \\ \beta_1^{i_0-1} &= \lambda_{i_0}(\beta_1^{i_0}, \dots, \beta_{n_{i_0}}^{i_0}).\end{aligned}$$

Therefore  $|\beta| \geq \sum_{i=1}^{i_0} (n_i - 1) + |\beta_1^{i_0}|$  and since  $|\beta| \leq k$ , we get

$$\begin{aligned}\sum_{i=1}^{i_0} (n_i - 1) + |\beta_1^{i_0}| &\leq k, \text{ so} \\ \sum_{i=1}^{i_0} (n_i - 1) &\leq k - 1.\end{aligned}\tag{14}$$

Applying the substitution  $\sigma$  to the equations expanding  $\beta$ , we get:

$$\begin{aligned}\sigma(\beta) &= \lambda_1(\sigma(\beta_1^1), \dots, \sigma(\beta_{n_1}^1)) \\ \sigma(\beta_1^1) &= \lambda_2(\sigma(\beta_1^2), \dots, \sigma(\beta_{n_2}^2)) \\ &\vdots \\ \sigma(\beta_1^{i_0-1}) &= \lambda_{i_0}(\sigma(\beta_1^{i_0}), \dots, \sigma(\beta_{n_{i_0}}^{i_0})).\end{aligned}$$

By assumption that (10) holds for all  $i < i_0$  and by (13), we know that for all  $i \leq i_0$ , all  $j = 2, \dots, l_i$ ,

$$\sigma(\beta_j^i) \text{ is equivalent to a variable.}\tag{15}$$

Let  $v$  be a new variable and let

$$\begin{aligned}\gamma_1 &= \lambda_{i_0}(v, \sigma(\beta_2^{i_0}), \dots, \sigma(\beta_{n_{i_0}}^{i_0})) \\ \gamma_2 &= \lambda_{i_0-1}(\gamma_1, \sigma(\beta_2^{i_0-1}), \dots, \sigma(\beta_{n_{i_0-1}}^{i_0-1})) \\ &\vdots \\ \gamma_{i_0} &= \lambda_1(\gamma_{i_0-1}, \sigma(\beta_2^1), \dots, \sigma(\beta_{n_1}^1)).\end{aligned}$$

Let  $\gamma := \gamma_{i_0}$ . Then by our assumption that  $\sigma(\beta_1^{i_0}) = \hat{\alpha}_k$ , we get  $\gamma[\alpha_k/v] = \gamma[\sigma(\beta_1^{i_0})/v] = \sigma(\beta)$ . Also, by construction and (15),  $\gamma \in \hat{\mathcal{L}}$ ,  $v = l(\gamma)$  and

$$\begin{aligned}\#(\mathcal{C}(\gamma)) &\leq \\ &\#(\{\sigma(\beta_2^1), \dots, \sigma(\beta_{l_1}^1), \sigma(\beta_2^2), \dots, \sigma(\beta_{l_2}^2), \dots, \sigma(\beta_2^{i_0}), \dots, \sigma(\beta_{l_{i_0}}^{i_0})\}).\end{aligned}$$

This is so because for each  $i = 1, \dots, i_0$  and for each  $j = 2, \dots, l_i$  there is one variable in  $\sigma(\beta_j^i)$  that is in  $\mathcal{C}(\gamma)$  and some of these variables may be repeating. So

$$\#(\mathcal{C}(\gamma)) \leq \sum_{i=1}^{i_0} (l_i - 1) \leq \sum_{i=1}^{i_0} (n_i - 1).$$

By (14)

$$\#(\mathcal{C}(\gamma)) \leq k - 1. \quad (16)$$

But

$$\gamma[\alpha_k/v] = \sigma(\beta) \in E(\mathfrak{N}),$$

so by Lemma 19

$$\#(\mathcal{C}(\gamma)) \geq k$$

contradicting (16). Hence  $\sigma(\beta_1^{i_0-1}) \neq \alpha_k$ , which finishes the proof.  $\square$

**COROLLARY 21.** *Let  $k \geq 1, \beta \in \text{Te}_F, |\beta| \leq k$ . Let  $\sigma : \text{Te}_F \rightarrow \text{Te}_F$  be a substitution such that  $\sigma(\beta) \in E(\mathfrak{N}) \cap S_k$ . Then*

1.  $\beta \in \hat{\mathcal{L}}$ ,
2.  $\sigma(l(\beta)) \in S_k$  and
3.  $\sigma(\mathcal{C}(\beta)) \subset_{\equiv} V$ .

**PROOF.** By conditions (7), (8) and (10) in Lemma 20,  $\beta \in \hat{\mathcal{L}}$  and  $l(\beta) = \beta_1^p$ . So by (9),  $\sigma(l(\beta)) = \sigma(\beta_1^p) \in S_k$ . By (10), for each  $i = 1, \dots, p$  and for each  $j = 2, \dots, l_i$  the substitution  $\sigma(\beta_j^i)$  of the term  $\beta_j^i$  is equivalent to a variable, so the term  $\beta_j^i$  is also equivalent to a variable. The set  $\mathcal{C}(\beta)$  consists of variables equivalent to all such terms  $\beta_j^i$ . Therefore again by (10),  $\sigma(\mathcal{C}(\beta)) \subset_{\equiv} V$ .  $\square$

## 5. Main result

For  $k \geq 1$  let  $\mathcal{R}_k$  be the set of valid rules of  $\mathfrak{N}$ , of the form  $\frac{X}{\beta}$ , such that  $|\beta| \leq k$ . Let  $\bar{0}$  denote the valuation that assigns 0 to every variable.

**LEMMA 22.** *Let  $k \geq 1$ . Then the set  $E(\mathfrak{N}) \setminus S_k$  is closed under  $\mathcal{R}_k$ .*

**PROOF.** Assume that  $\frac{X}{\beta} \in \mathcal{R}_k$ , so  $|\beta| \leq k$ . Let  $\sigma$  be a substitution and assume that  $\sigma(X) \in E(\mathfrak{N}) \setminus S_k$ . For the proof by contradiction assume that  $\alpha = \sigma(\beta) \in E(\mathfrak{N}) \cap S_k$ . By Corollary 21,  $\beta \in \hat{\mathcal{L}}$ ,  $\sigma(l(\beta)) \in S_k$  and  $\sigma(\mathcal{C}(\beta)) \subset_{\equiv} V$ . Let  $v = l(\beta)$ . Then  $v \notin \mathcal{C}(\beta)$  and  $\sigma(v) \in S_k$ . Let us

define a valuation  $\varphi$  as follows: for  $w \in V$  let

$$\varphi(w) = \begin{cases} \bar{0}(\sigma(w)) & \text{for } w \neq v \\ 1 & \text{for } w = v \text{ if } \bar{0}(\sigma(v)) = 2 \\ 2 & \text{for } w = v \text{ if } \bar{0}(\sigma(v)) = 1. \end{cases} \quad (17)$$

Since  $\sigma(\beta)$  is a tautology,  $\bar{0}(\sigma(\beta)) = 2$ . Since  $v \notin \mathcal{C}(\beta)$ , for every  $w \in \mathcal{C}(\beta)$  we have  $\varphi(w) = \bar{0}(\sigma(w))$  and since  $\sigma(\mathcal{C}(\beta)) \subseteq_{\equiv} V$ ,  $\varphi(w) = \bar{0}(\sigma(w)) = 0$ . By Corollary 14,  $\varphi(\beta) \neq \bar{0}(\sigma(\beta))$ , so  $\varphi(\beta) \neq 2$ . Since  $\frac{X}{\beta}$  is valid, there must be a term  $\gamma \in X$  such that  $\varphi(\gamma) \neq 2$ . But  $\sigma(\gamma)$  is a tautology of  $\mathfrak{N}$ , so  $\bar{0}(\sigma(\gamma)) = 2$ , hence  $\varphi(\gamma) \neq \bar{0}(\sigma(\gamma))$ . By Corollary 15,  $\gamma \in \hat{\mathcal{L}}$ ,  $v = I(\gamma)$  and  $\mathcal{C}(\gamma) \subseteq \varphi^{-1}(\{0\})$ . So  $\sigma(I(\gamma)) = \sigma(v) \in S_k$  and for every  $w \in \mathcal{C}(\gamma)$  we have  $\bar{0}(\sigma(w)) = \varphi(w) = 0$ , so  $\sigma(w)$  is equivalent to a variable. Hence  $\sigma(\mathcal{C}(\gamma)) \subseteq_{\equiv} V$ . By Proposition 17,  $\sigma(\gamma) \in S_k$ , which contradicts the assumption that  $\sigma(X) \subseteq E(\mathfrak{N}) \setminus S_k$ .  $\square$

**THEOREM 23.** *Let  $\mathfrak{M} = \langle M, \{\cdot\}, \{2\} \rangle$ , where  $\cdot$  is either  $\cdot_1$  or  $\cdot_2$ . Let  $\mathfrak{N} = \langle M, F, \{2\} \rangle$  be a matrix term-equivalent to  $\mathfrak{M}$ . Then  $\mathfrak{N}$  is not finitely axiomatizable.*

**PROOF.** Assume that there is a finite set  $\mathcal{R}$  of rules in  $\Lambda_F$  such that  $\mathcal{R}$  is an axiomatization of  $\mathfrak{N}$ . Then there exists  $k \geq 1$  such that  $\mathcal{R} \subseteq \mathcal{R}_k$ . Let  $\alpha$  be a term in  $E(\mathfrak{N}) \cap S_k$  with the shortest proof by means of  $\mathcal{R}$ . By Lemma 18 such  $\alpha$  exists. Obviously, by length considerations,  $\alpha$  is not an axiomatic rule in  $\mathcal{R}$ . So there is a substitution  $\sigma$  and a rule  $\frac{X}{\beta} \in \mathcal{R}$  that was used at the end of this proof with the substitution  $\sigma$ . Then  $\sigma(X) \subseteq E(\mathfrak{N}) \setminus S_k$  and  $\sigma(\beta) = \alpha \in E(\mathfrak{N}) \cap S_k$ . This contradicts Lemma 22.  $\square$

**Acknowledgments.** I would like to thank Professor Wolfgang Rautenberg for his generous comments improving the presentation of the Abstract and Introduction to this paper. Also, I appreciate very much the thorough reading of the paper by the referee whose insightful remarks allowed me to correct several mistakes.



## References

- [1] Blok, W., and D. Pigozzi, “Potoalgebraic logics”, *Studia Logica*, 45 (1986): 337–369.
- [2] Blok, W., and D. Pigozzi, *Algebraizable Logics*, Memoirs of the AMS, 396, Providence, 1989. DOI: [10.1090/memo/0396](https://doi.org/10.1090/memo/0396)
- [3] Blok, W., D. Pigozzi, “Algebraic semantics for universal Horn logic without equality”, in: *Universal Algebra and Quasigroup Theory*, A. Romanowska and J. D. H. Smith (eds.), Heldermann Verlag, Berlin, 1992.
- [4] Dziobiak, W., “A finite matrix whose set of tautologies is not finitely axiomatizable”, *Reports on Mathematical Logic*, 25 (1991): 113–117.
- [5] Herrmann, B., and W. Rautenberg, “Finite replacement and Finite Hilbert-style axiomatizability”, *Mathematical Logic Quarterly*, 38 (1992): 327–344.
- [6] Pałasińska, K., “Three-element nonfinitely axiomatizable matrices”, *Studia Logica*, 53 (1994): 361–372.
- [7] Pałasińska, K., “No matrix term-equivalent to Wroński’s 3-element matrix is finitely based”, *Studia Logica*, 77 (2004): 413–423.
- [8] Rautenberg, W., “Two-element matrices”, *Studia Logica*, 40 (1981): 315–353.
- [9] Wojtylak, P., “Strongly finite logics: Finite axiomatizability and the problem of supremum”, *Bulletin of the Section of Logic*, 8 (1979): 99–111.
- [10] Wojtylak, P., “An example of a finite though finitely non-axiomatizable matrix”, *Reports on Mathematical Logic*, 17 (1984): 39–46.
- [11] Wroński, A., “A Three-element matrix whose consequence operation is not finitely based”, *Bulletin of the Section of Logic*, 8 (1979): 68–71.

KATARZYNA PAŁASIŃSKA  
Cracow University of Technology  
Department of Mathematics  
ul. Warszawska 24  
Kraków, Poland  
[kpalasinska@gmail.com](mailto:kpalasinska@gmail.com)