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LOGICAL PROBLEMS WITH NONMONOTONICITY

Abstract. A few years ago, believing that human thinking is nonmonotonic, I tried to reconstruct a nonmonotonic reasoning by application of two monotonic procedures. I called them "step forward" and "step backward" (see [4]). The first procedure is just a consequence operation responsible for an extension of the set of beliefs. The second one, defined on the base of the *logic of falsehood* reconstructed for the given *logic of truthfulness*, is responsible for a reduction of the set of beliefs. Both procedures taken together were successfully verified by using so-called *AGM* (see [5]), postulates for *expansion, contraction* and *revision* formulated by Alchourrón, Gärdenfors and Makinson (e.g. [1]). Reasoning composed of the mutual application of both procedures seemed to be quite natural for modeling our thinking. At that time, I supposed that it should be nonmonotonic but I was wrong. It turned out impossible to satisfy a definition of the nonmonotonic inference by reasoning composed both steps. To understand why this is impossible, I began to analyze how nonmonotonicity is obtainable in some well-known cases in the literature. I analyzed the problem from two points of view: (1) non-formal examples for nonmonotonicity and (2) formal constructions of nonmonotonic operations/relations. The result of those investigations was astonishing: none of the considered by me cases of nonmonotonicity belonging to point (1) and almost none belonging to (2) satisfies the definition of nonmonotonic inference. Arguments against the nonmonotonic character of well-known examples for nonmonotonicity of human thinking are more precisely presented in [6]. I present them below an abbreviated version of them.

Keywords: monotonic inference; nonmonotonic inference; Makinson's nonmonotonic constructions; enthymeme; background assumptions; admissible valuations; nonstructural superclassical logic

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Introduction

Let us recall that an inference is *monotonic* iff all conclusions inferred from some set Z of premises are still conclusions of every superset of Z:

an inference $\vdash$ *is monotonic*	*iff*	*for any* p, q, Z: *if* $Z \vdash p$, *then* $Z \cup \{q\} \vdash p$.

A relation of inference is *nonmonotonic* iff it is not monotonic, thus:

an inference $\vdash$ *is nonmonotonic*	*iff*	*for some* p, q, Z *such that* $q \notin Z$: $Z \vdash p$ *and* $Z \cup \{q\} \nvdash p$.

From the definition above, it directly follows that nonmonotonicity cannot be recognized in one step of reasoning. There must be at least two inferential steps. In the first one, some sentence p is inferred from the set Z of premises, while in the second step, the same p cannot be inferred from the Z enlarged by some additional sentence(s).

The standard (obvious) logical understanding of these formal definitions requires comment: an inference is nonmonotonic, if it satisfies two conditions; *con1.* The name "Z" appearing in the expressions "$Z \vdash p$" and "$Z \cup \{q\} \vdash p$" must be the name of one and the same set of premises in both cases; *con2.* The symbol "$\vdash$" appearing in the expressions "$Z \vdash p$" and "$Z \cup \{q\} \vdash p$" must be the name of one and the same inference in both cases, i.e. given by the same set of axioms and rules. If the *definiens* of the definition is of the form:

(a) *for some* p, q, Z_1, Z_2 *such that* $Z_1 \neq Z_2$ *and* $q \notin Z_2$ ($Z_1 \vdash p$ *and* $Z_2 \cup q \nvdash p$);

it means that *con1* is not satisfied. If the *definiens* of the definition has the form:

(b) *for some* p, q, Z *such that* $q \notin Z$ ($Z \vdash_1 p$ *and* $Z \cup \{q\} \nvdash_2 p$);

then *con2* is not satisfied. In both cases, the *definiens* does not deal with nonmonotonicity. It can be even worse when both characteristics come together. Then, the *definiens* is as follows:

(c) *for some* p, q, Z_1, Z_2 *such that* $Z_1 \neq Z_2$ *and* $q \notin Z_2$ ($Z_1 \vdash_1 p$ *and* $Z_2 \cup q \nvdash_2 p$).

Obviously, none of the conditions (a), (b), (c) coincides with the monotonic case. Condition (a) defines neither monotonic, nor nonmonotonic inference, while (b) some collection of inferences. Similarly, (c) does not define one inference only, but a set of various inferences. Only in the first form does the definition deal with one logic. In both other cases, there is not one logic but a set of them.

In this paper, a standard understanding of the definition of nonmonotonic inference can be accepted. It means that nonmonotonicity is given by the correct form of the *definiens*. None of the three above forms (a, b, c) will be treated as defining monotonicity.

From the point of view of the definition, there are two kinds of problems with nonmonotonicity: non-formal and formal. It seems that in both cases it is much more difficult to find a nonmonotonic construction (i.e. a construction really satisfying the *definiens* discussed above) than it seems to be.

1. Non-formal arguments[1]

In the literature, there are given, in various versions, several examples presented as illustrations of the nonmonotonicity of human thinking. Unfortunately, none of them confirms that humans thinks nonmonotonically. A precise analysis will show that every case of human thinking has nothing in common with nonmonotonicity. Let us begin with the most popular problem.

1.1. "Tweety the Ostrich" and "The Meeting in the Pub"

From the premise "Tweety is a bird" we infer the conclusion "Tweety can fly". However, if we gain the new piece of information "Tweety is an ostrich" we have to reject the conclusion. Let, p = "Tweety is a bird", q = "Tweety is an ostrich", z = "Tweety can fly". Then, it seems that the scheme of the reasoning is the following

1st step. $\{p\} \vdash z$
2nd step. $\{p, q\} \nvdash z$

Such an interpretation coincides with nonmonotonicity. However, it is not difficult to notice that the reasoning commits the logical *error of generality*. A feature or quality of one logical[2] species is assigned to other (or to all) species of the same logical genus. It is because both

[1] For an extended version of point 1 only, see, Łukowski [6].

[2] "Logical" means "non-biological".

(or all) species belong to the same genus. It is the only reason for this erroneous assignment.

No one can say that the example does not represent our thinking. It is typical of human reasoning. We call this error of generality THINKING BY STEREOTYPES (thinking by types) or STEREOTYPICAL THINKING.[3] To the extent that we do not know, not remember or not recognize other species of a given genus, we treat the known, remembered or recognized by us species as the only ones representing that genus. In this case, stereotypical thinking is a result of the LACK OF KNOWLEDGE. Sometimes it is a kind of a SLOPPY THINKING — another source of stereotype. However, in a non-pathological form, thinking by stereotypes is a result of so-called ECONOMICAL THINKING. We do not always need to think about all the species of some genus, especially when they are unimportant from the point of view of the current context of communication. Usually, we know very well that some conclusion does not follow from a given premise. However, our reasoning follows the shorter way, and we do not overtly express all conditions necessary for correctness of the reasoning. We know very well that Tweety, as a bird can fly, if it is not: ostrich, penguin, kiwi; if its wings are not broken; if it is not too young, etc. Thus, our thinking is represented by the scheme:

$$\{p\} \nvdash z \ \textit{and}\ (\{q_1\} \vdash z, \dots, \{q_s\} \vdash z)\ \textit{and}\ (\{r_1\} \nvdash z, \dots, \{r_t\} \nvdash z)$$

where $p =$ "Tweety is a bird", $z =$ "Tweety can fly", $q_1, \dots, q_s$ — sentences stating that Tweety belongs to the flying species and it can fly (e.g. $q_1 =$ "Tweety is an adult and healthy sparrow", $q_2 =$ "Tweety is an adult and healthy duck", $q_3 =$ "Tweety is an adult and healthy swan", ...)[4], $r_1, \dots, r_t$ — sentences stating that Tweety belongs to the not-flying species or it cannot fly because of some other reason (e.g. $r_1 =$ "Tweety is an ostrich", $r_2 =$ "Tweedy is a penguin", $r_3 =$ "Tweedy has a broken left wing", ...).

Usually, our thinking is different from our talking presenting the thinking. Moreover, every utterance is always in some sense imprecise. It is impossible to use absolutely precise sentences. The level of preci-

[3] Stereotypical kind of thinking is especially evident in the supposedly nonmonotonic example of the *pacifists-Kwarks* and the *nonpacifist-Republicans*, e.g. Ginsberg, [2, 12].

[4] Not any sparrow, not any duck, not any swan. In other case we would obtain many versions of "Tweety the Ostrich": "Tweety the Sparrow with Broken Left Wing", etc.

sion of an utterance depends on the situational context — it is a function the degree of precision necessary for successful understanding and the degree of precision necessary for communicating economically (too long an utterance would be unsuccessful).

"Tweety the Ostrich" can be used to explain another and similar way of thinking, which also has nothing to do with nonmonotonicity. It is reasoning from open and hidden premises. This way of thinking will be here illustrated by "The Meeting in the Pub".

John has an appointment with Mark on Saturday night in the pub "The Ten Bells", but under two conditions: that Mark's sore throat will pass, and that John's mother will feel better. Let us assume, that on Saturday morning Mark's throat as well as John's mother have recovered. So, at the proper time, John will call a taxi and go to the pub. Unfortunately, just before leaving home John is informed by Anna that Mark has had a car accident and has been taken to St James's Hospital. Now John will not go to the pub: the new information invalidates an earlier premises of John's reasoning — the premise expressing his appointment with Mark. The erroneous interpretation suggests that the reasoning is represented by the schema:

$$\begin{array}{rl} 1^{\text{st}} \text{ step.} & \{p_1, \ldots, p_n\} \vdash z \\ 2^{\text{nd}} \text{ step.} & \{p_1, \ldots, p_n, p_{n+1}\} \nvdash z \end{array}$$

The notation suggests that there is nonmonotonicity here. Of course, it is not true.

It is not difficult to notice that not only the car accident could have made the meeting impossible. Many other events could have invalidated the meeting: forgetfulness on the part of John or Mark, their falling ill, etc. Usually, we do not mention all of them. The existence of such cases is obvious to everybody and so nobody overtly expresses even some of them.

The real schema of the reasoning is given below:

$$\begin{array}{rl} 1^{\text{st}} \text{ step.} & \{p_1, \ldots, p_n\} \cup \{q_1, \ldots, q_s\} \vdash z \\ 2^{\text{nd}} \text{ step.} & \{p_1, \ldots, p_n\} \cup \{q_1, \ldots, q_{i-1}, \neg q_i, q_{i+1}, \ldots, q_s\} \nvdash z, \\ & \text{for any } i \in \{1, \ldots, s\} \end{array}$$

The sentence z does not follow from the set $\{p_1, \ldots, p_n\}$ only. It can be accepted only if two sets of premises are accepted as true: $\{p_1, \ldots, p_n\}$ and $\{q_1, \ldots, q_s\}$. The first set consists of *open, overtly spoken* premises. The second one contains all *hidden*, obvious, *covertly expressed* premises.

Sentences $q_1, \ldots, q_s$ are known as *enthymematic* premises. Usually, most of $q_1, \ldots, q_s$ are even not realized by us. For $i \in \{1, \ldots, s\}$, q_i is the negation of the sentence expressing the case which, if true, means that z cannot be inferred. Thus, for to infer z, all sentences from $\{p_1, \ldots, p_n\} \cup \{q_1, \ldots, q_s\}$ must be true.

Since hidden premises are usually not realized by people, they can be represented by one, usually, unspoken premise q = "it is not true that something has happened (or is the case) which invalidates z". Then the schema has the following form:

1^{st} step. $\{p_1, \ldots, p_n\} \cup \{q\} \vdash z$
2^{nd} step. $\{p_1, \ldots, p_n\} \cup \{\neg q\} \nvdash z$

This second interpretation enables to avoid the problem with a possibly infinite amount of such hidden premises. Both schemata represent *reasoning from open and hidden premises*, which obviously is not non-monotonic. Premises $p_1, \ldots, p_n$ are *open (explicit, direct)* and $q_1, \ldots, q_s$ (or just q) are *hidden (implicit, indirect)*. All premises have to be true for the inference of z.

To recapitulate: let us notice that both "Tweety the Ostrich" and "The Meeting in the Pub" can be understood as cases of stereotypical thinking as well as reasoning from open and hidden premises. However, "Tweety the Ostrich" seems to be closer to thinking by stereotypes, while "The Meeting in the Pub" to reasoning from open and hidden premises.

1.2. "Medical Diagnosis"

The name "Medical Diagnosis" represents the broad class of cases of PRECISION-INCREASING REASONING.

Possessing medical test results $p_1, \ldots, p_n$, a physician decides that the patient suffers from illness z_1. Later, after receiving new information p_{n+1} the physician changes his mind. Now, he believes that the patient suffers from z_2 ($z_2 \neq z_1$). Of course, the diagnosis z_2 does not have to be definitive. The structure of the reasoning supposedly has the schema:

1^{st} step. $\{p_1, \ldots, p_n\} \vdash z_1$
2^{nd} step. $\{p_1, \ldots, p_n, p_{n+1}\} \vdash z_2$ and $\{p_1, \ldots, p_n, p_{n+1}\} \nvdash z_1$
3^{rd} step. $\{p_1, \ldots, p_n, p_{n+1}, p_{n+2}\} \vdash z_3$ and
$\{p_1, \ldots, p_n, p_{n+1}, p_{n+2}\} \nvdash z_1$, and $\{p_1, \ldots, p_n, p_{n+1}, p_{n+2}\} \nvdash z_2$

Etc.

Of course, a real situation cannot be formalized in such a way. It obvious medical knowledge that one set of symptoms can indicate various illnesses: $\{p_1, \ldots, p_n\} \vdash \{z_1, \ldots, z_k\}$. That is why a physician has to choose one illness: the most probable in the given situation. Thus, to choose one element from the set $\{z_1, \ldots, z_k\}$, he uses some non-logical criterion based on his the knowledge and experience, on statistics, on biological and geographical conditions characteristic for the region, and so on. New information can change his opinion, and suggest another choice. It means that the correct form of reasoning is the following:

1st step. $\{p_1, \ldots, p_n\} \vdash z_1 \vee z_2 \vee z_3 \vee \cdots \vee z_k$[5]
2nd step. $\{p_1, \ldots, p_n, p_{n+1}\} \vdash z_2 \vee z_3 \vee \cdots \vee z_k$
3rd step. $\{p_1, \ldots, p_n, p_{n+1}, p_{n+2}\} \vdash z_3 \vee \cdots \vee z_k$

Of course, there is no place for nonmonotonicity. It seems that hereit is possible to recognize even monotonic reasoning.

The nonmonotonic interpretation of "Medical Diagnosis" makes difficult research into a very interesting mode of human thinking: We employ *precision-increasing reasoning* when we are looking for something. This fundamental type of everyday thinking is the essence of all diagnosis: medical and any other. We employ a *reasoning increasing preciseness* when we are looking for something. Then, step by step, we eliminate checked places, as possible locations of the lost thing. Nonmonotonic interpretation seems to be especially artificial and completely senseless in this case. Using this reasoning, we are limiting step by step the scope of possible solutions. Fewer solutions means more precision, and so, better knowledge. The problem is also closely connected with the rationality of decision taking.

1.3. "Car in Front of the House"

The last example to be considered here illustrates a popular way of reasoning, which can be called thinking by the most frequent cases, a relative of the well-known phenomenon of automatic thinking. In fact, from the logical point of view, it is a case of ordinary error.

[5] Every step of the reasoning precedes a non-logical choice made by the physician. Thus, after the 1st step, the physician has to decide which sentences should be chosen from $\{z_1, z_2, z_3, \ldots, z_k\}$. After the 2^{nd} step, the choice is from the smaller set $\{z_2, z_3, \ldots, z_k\}$.

Let us suppose that without having made an appointment I would like to visit John. I see his car in front of his house, and so I think that he is at home. The long time I spend ringing the doorbell makes it clear that he probably is not at home. Calling on my mobile phone, I finally know that he is not at home.

Following the nonmonotonic interpretation, one recognizes two steps of thinking. In the first one, I infer the sentence $t =$ "John is at home now" from: $s =$ "John's car is standing in front of his house" together with $s \to t$. In the second step, I find that t is not true. This second step is not typical reasoning but "immediate" thinking based on simple observation. The acceptance of nonmonotonicity leads here to the acceptance of a rough contradiction. Indeed, after the second step of reasoning, at the same time I have to accept (by the definition of nonmonotonic inference) all previous premises i.e. $s \to t$, s, together with a new conclusion $\neg t$. It means that I have to accept a contradiction: t and $\neg t$. This example is probably one of the most illogical illustrations of nonmonotonicity.

The proper interpretation of "Car in Front of the House" is simple and monotonic. In the first step I infer $t' =$ "John should be at home now" from: $s =$ "John's car is standing in front of his house" and $s \to t'$. In the second step I believe or even know that $\neg t =$ "John is not at home now". There is no contradiction here, because John should be at home but he is not. This final conclusion is natural and obvious — John may be in the grocery or at neighbor's flat.

Is there any sense in paying the high a price of inconsistency in defence of the nonmonotonic interpretation of this uncomplicated situation, especially when a monotonic solution is so simple and natural? What sense is there in defending a nonmonotonic understanding, if the defence has to be based on an ordinary, and even primitive error? The last example — "Car in Front of the House" — is representative of broad class of so-called "examples for nonmonotonicity" the nonmonotonic interpretation of which would be impossible without committing a simple error. Amongst all errors that 'establish' a nonmonotonic interpretation, the most popular is probably an error of ambiguity.

The conclusion of the above analysis is simple: all considered nonformal cases are not nonmonotonic. Moreover, all these cases can be interpreted in a monotonic way, and not necessarily in the paradigm of classical logic. For the monotonic interpretation, it suffices that Modus

Ponens is the rule of reasoning. So, in particular, our interpretation coincides with our everyday thinking, as well as with classical logic.

Now let us see if the formal constructions proposed by Makinson are really nonmonotonic.

2. Formal constructions by Makinson

The constructions of nonmonotonic operations proposed by D. Makinson (see [9]) are commonly accepted as representing various classes of nonmonotonic inferences. All of them are founded on some nonstructural extensions of classical logic. For every class, a desired extension is obtained, respectively, by (see [9, p. 18]):

1. some additional assumptions (i.e. systematically accepted sentences), called "additional background assumptions";
2. restriction of the set of admissible valuations;
3. some new rules of inference.

Every kind of extension leads to a logic stronger than classical logic but which is still consistent — a nonstructural superclassical logic — which is a monotonic base for the appropriate type of nonmonotonic inference.

2.1. Inference with some additional background assumptions

The role of the additional background assumptions is played by those sentences which are true for us at the moment because "they are obvious", "everybody knows that they are true", etc. Let us assume that K (called by Makinson a set of *expectations*) is a set of all these *enthymemes*. They are used in the inference but this fact is not emphasized. In such a sense, K consists of background assumptions. Of course, a background assumption is still an assumpton if it is actually used in an inference. Otherwise, it is an assumption in name only.

Let L be the set of all formulas of the language, $K \subseteq L$, and Cn the classical consequence operation ($\vdash$ is the classical consequence relation). Cn_K is a *consequence of the axial assumptions* K ($\vdash_K$ is a *relation of the axial assumptions* K), if for any $A \subseteq L, x \in L$:[6]

$$\begin{array}{rcl} x \in \mathrm{Cn}_K(A) & \text{iff} & x \in \mathrm{Cn}(K \cup A) \\ (A \vdash_K x & \text{iff} & (K \cup A) \vdash x). \end{array}$$

[6]Of course, $x \in \mathrm{Cn}_K(A) (A \vdash_K x)$ if and only if, for any classical valuation v, if $v(K \cup A) = 1$, then $v(x) = 1$.

Obviously, the logic defined above is superclassical: $\mathrm{Cn} \leqslant \mathrm{Cn}_K(\vdash \subseteq \vdash_K)$. Since K is not closed under substitution, $\mathrm{Cn}_K(\vdash_K)$ is nonstructural and so can be consistent. In the second step of the construction it is defined $C_K(\mid\!\sim_K)$ in the following way:

$$\begin{array}{rcl} C_K(A) & = & \cap\{\mathrm{Cn}(K' \cup A) : K' \subseteq K \text{ and} \\ & & K' \text{ is maximally consistent with } A\} \\ A\mid\!\sim_K x & \text{iff} & (K' \cup A) \vdash x, \text{ for any } K' \subseteq K, \\ & & \text{maximally consistent with } A. \end{array}$$

Every operation identical with some C_K (i.e. for some K) as well as every relation identical with some $\mid\!\sim_K$ is, called by Makinson, a *background assumptions consequence.*

Next, Makinson shows that a background assumptions consequence is not monotonic, and so, it is nonmonotonic. Let $K = \{p \to q, q \to r\}$. Since $K \cup \{p\}$ is consistent, there is the only one maximally consistent with $\{p\}$ subset of K. This is K. Thus, $r \in C_K(\{p\})$, because $r \in \mathrm{Cn}(K \cup \{p\}) \neq L$. However, $r \notin C_K(\{p, \neg q\})$. Indeed, $K \cup \{p, \neg q\}$ is inconsistent, and so $K' = \{q \to r\}$ is the only maximally subset consistent with $\{p, \neg q\}$ subset of $K(r \notin \mathrm{Cn}(\{q \to r, p, \neg q\}))$. It means that, $r \in C_K(\{p\})$ and $r \notin C_K(\{p, \neg q\})$, although $\{p\} \subseteq \{p, \neg q\}$.

At first view, it seems that a background assumptions consequence is really nonmonotonic: for some $K, A, B \subseteq L$, $\alpha \in L : \alpha \in C_K(A)$ and $\alpha \notin C_K(A \cup B)$. However, the *definiens* of the definition of nonmonotonic operation will be satisfied only if all the rules and all the assumptions used in the first step of reasoning (i.e. $C_K(A)$) are also used (accessible) in the second step (i.e. $C_K(A \cup B)$). It seems that all the rules available in the first step of reasoning are also available in the second step. Unfortunately, this does not work in the case of the assumptions: some of them used in the first step of reasoning are simply forbidden for the next step(s). It is even stated frankly by Makinson that the essence of the construction is that the set of available background assumptions is changing with respect to A, the set of "open" assumptions. For every step of reasoning, i.e. for every set A of "open" assumptions, the selection of additional "hidden" background assumptions is unique. It defines which elements from K can be used in the case of A. The aim of such a procedure is clear: to avoid a contradiction. For every step, all those assumptions from K which together with A lead to contradiction are forbidden. It such a case, it is clear that no-one can say that the *definiens* of the definition of nonmonotonic inference is satisfied. Since, a premise is a sentence which is used in the inference, the set of premises is here changing for every step of the reasoning: previously-used assumptions are, in the next step, removed from the set of all assumptions.

It is interesting why it seems so difficult to notice that a background assumptions consequence is not nonmonotonic. The answer is rather simple: the whole secret lies in the notation. There is a big difference between these two notations of one and the same notion: "$C_K(A)$" and "$\cap\{\mathrm{Cn}(K' \cup A) : K' \subseteq K$ and K' is maximally consistent with $A\}$". The expression

$$\text{“}C_K(A) \text{ and } C_K(A \cup B)\text{”}$$

suggests that in both cases (i.e. $C_K(A)$ and $C_K(A \cup B)$) the whole set K is used. Quite the opposite impression is created by the equivalent expression

$$\begin{aligned} \text{“}\cap\{\mathrm{Cn}(K' \cup A) :\ & K' \subseteq K \\ & \text{and } K' \text{ is maximally consistent with } A\} \text{ and} \\ \cap\{\mathrm{Cn}(K' \cup A \cup B) :\ & K' \subseteq K \\ & \text{and } K' \text{ is maximally consistent with } A \cup B\}\text{”.} \end{aligned}$$

In the case of the first notation, it is difficult to understand how it is possible that a sentence belongs to $C_K(A)$ but does not belong to $C_K(A \cup B)$. Still thinking in a monotonic way, one can easily observe that the same problem does not exist in the case of the second format: a sentence can belong to the first intersection and yet not belong to the second one. It is quite obvious that by rejecting some premises we must reject some conclusions. With the second format, it is difficult to forget that assumptions from the set K can be allowed but can also be forbidden, depending on the step of reasoning (i.e. systematically accepted open assumptions). A precise and adequate notation should employ "K_A", instead of "K", for every A. Then the correct form of *definiens* would be as follows:

$$\text{for some } A \subseteq L \text{ and } p, q \in L, q \in C_{K_A}(A) \text{ and } q \notin C_{K_{A\cup\{p\}}}(A \cup \{p\}).$$

Traditionally, such notation is not used in the logical literature, and whilst it is not the aim of the paper to suggest changing the tradition, such a change would be here well-motivated. The main aim of the remark above is to explain why some monotonic operations seem to be nonmonotonic.

2.2. Inference by restriction of the set of admissible valuations

The second construction is based on the limitation of V, the set of all Boolean valuations. Makinson admits that, in effect, we receive almost the same set of operations as in the previous construction. Let $W \subseteq V, A \subseteq L, x \in L$. Then,

$$x \in \mathrm{Cn}_W(A)(A \vdash_W x) \text{ iff for any } v \in W, \text{ if } v(A) = 1, \text{ then } v(x) = 1.$$ [7]

Every consequence operation (relation) identical with some $\mathrm{Cn}_W(\vdash_W)$ is called an *axial-valuation consequence*. Similarly to the previous case, this consequence is superclassical and nonstructural ($\mathrm{Cn} \leqslant \mathrm{Cn}_W, \vdash \subseteq \vdash_W$), but contrary to the axial assumptions consequence, it is not dense. In such a sense, the present construction delivers not the same but almost the same operations as the previous one. More precisely, for any $K \subseteq L : \mathrm{Cn}_K = \mathrm{Cn}_W$, if $W = \{v \in V : v(K) = 1\}$. It means, that every axial assumptions consequence is an axial valuations consequence, but not the opposite. Instead only every dense axial valuations consequence is an axial assumptions consequence. Of course, they are monotonic.

A definition of default-valuations consequence uses a so-called *preferential model*: i.e. the set W ordered by $<$, an irreflexive and transitive relation on W. Let $\langle W, <\rangle$ be a preferential model, then

$$A \mid\!\sim_< x \text{ iff } v(x) = 1, \text{for any } v \in W \text{ minimal among all valuations from } W \text{ satisfying } A.$$ [8]

Every relation (operation) identical with $\mid\!\sim_<$, for some preferential model $\langle W, <\rangle$ is called a *preferential consequence* or *default-valuations consequence*. There are some notions useful for more clearly defining $\mid\!\sim_<$. Let $|A|_W = \{v \in W : v(A) = 1\}$, and, $\min_<|A|_W$ be a set of all elements minimal in $|A|_W$. Then,

$$A \mid\!\sim_< x \quad \text{iff} \quad v(x) = 1, \text{ for any } v \in \min_<|A|_W.$$

Makinson shows, that preferential consequence is nonmonotonic. Let us assume that the language contains p, q, r — the only atomic sentences. Moreover, let $W = \{v_1, v_2\}$ such, that $v_1(p) = v_2(p) = 1$, $v_1(q) = 0$, $v_2(q) = 1$, $v_1(r) = 1$, $v_2(r) = 0$. Thus, $v_1 < v_2$, and so, $\{p\} \mid\!\sim_< r$. Indeed, $\{v_1\}$ is a set of all elements minimal among all valuations satisfying $\{p\}$,

[7] Of course, $v(A) = 1$ if and only if $v(x) = 1$, for all $x \in A$.

moreover, $v_1(r) = 1$. However, it is not true, that $\{p \wedge q\} \mid\!\sim_< r$. In fact, $\{v_2\}$ is a set of all elements minimal among all valuations satisfying $\{p \wedge q\}$, and $v_2(r) = 0$. Thus, for some $\langle W, <\rangle, A, B \subseteq L$ and $\alpha \in L$: $A \mid\!\sim_< \alpha$, and not $A \cup B \mid\!\sim_< \alpha$. It means that preferential consequence is nonmonotonic.

In contrast to the first construction, the second one not only seems to be nonmonotonic, but really is nonmonotonic. It is so, however, in the same sense as a logic without stable rules is also nonmonotonic. But is a logic without a defined set of rules still a logic?

The preferential model indicates which valuations are more important than others, but an ordering (by $<$) of elements from W does not complete a selection. The most important is that only some special valuations can be used in the inference: the minimal elements in $\langle W, <\rangle$, satisfying the set of assumptions. Since a set W as well as the order on W is arbitrary, one can only use rules of inference chosen by oneself. Then everything can be inferred from a given set of premises. Such strongly selected valuations establish additional non-logical criteria of inference. Those few valuations work like nonstructural rules expressing our beliefs about the world. For example, using appropriate valuations, one can prove that every bird can fly or every horse is black. In such a way, a choice of minimal elements in the preferential model, i.e. some non-logical criterion, defines the logic of inference.[9] In the case of the second construction, there is no well-defined logic but some (possibly small) set of valuations expressing beliefs about the world, in a way similar to the beliefs given by the set of systematically accepted assumptions. This is not a logic but special complex of beliefs about reality. Beliefs generated by selected valuations have the form of implication: "assumptions $\longrightarrow$ conclusion". Such a "selectively prepared" inference makes possible to prove what we want. Thus, it is an inference without stable rules. It is not strange that such an inference is not monotonic, since it is not a logical inference. That is why, from the formal point of view, the second

[9] This situation looks like a fulfillment of a dream of Marxists logicians. They were looking for a special logic, so called, *dialectical logic*, giving a possibility of proving of everything what they need at the moment. Marxists had never constructed it, but it seems that the mythic dialectical logic should be nonmonotonic: an inference from already accepted assumptions A, is no more valid even when all assumptions from A are still valid. Probably, the second construction defines the logic for proving of everything what is desired by an agent.

construction is nonmonotonic, but in a way typical for reasoning without stable rules.

Makinson's second construction should definitely be more deeply explored. It seems aptly to cover our everyday thinking. The nonmonotonicity of the second construction is an "accidental" feature. What is most important is that beliefs hidden in "implicational" assumptions expressed by strongly selected valuations define a current inference. Such an inference can be changed at every moment. It is a real benefit accruing to the second construction. There is probably no other formal construction closer to human thinking.

2.3. Inference with some additional new rules for converting sentences

Since every rule of sentence's conversion (henceforth for simplicity: rule) is a pair of sentences $\langle \alpha, x\rangle$, a set of such rules is a binary relation R defined on the Cartesian product L^2. For $X \subseteq L$ and $R \subseteq L^2$, an *image* $R(X) = \{y \in L : \langle x, y\rangle \in R$, for some $x \in X\}$. X is closed on R, if $R(X) \subseteq X$. Rules from R are called by Makinson "inferential tickets" which allow us to travel from any set of assumptions (see [9, p. 85]). Consequently, a set of assumptions is called by him "a potential set of assumptions", and conclusion, "a potential conclusion".

Let A be a set of potential assumptions, and x a potential conclusion. Then, x is a conclusion of A modulo R (formally, $A \vdash_R x$ or $x \in \mathrm{Cn}_R(A)$), if and only if x belongs to every superset of A, closed on Cn and R. Thus,

$$\mathrm{Cn}_R(A) = \cap\{X : A \subseteq X, \mathrm{Cn}(X) \subseteq X \text{ and } R(X) \subseteq X\}.$$

Every operation (relation) identical with $\mathrm{Cn}_R(\vdash_R)$, for some set of rules R, is called an *axial-rules consequence*. It is not surprising that the new logic is monotonic; moreover $\mathrm{Cn} \leqslant \mathrm{Cn}_R, \vdash \subseteq \vdash_R$. It is an interesting fact that the set of all axial-assumptions consequences is the intersection of two sets: the first, of all axial-valuations consequences; and the second, of all axial-rules consequences. From a practical point of view, it is better to define $\mathrm{Cn}_R(\vdash_R)$ using an ordered set of rules. It is possible because

$$\mathrm{Cn}_R(A) = \cup\{A_n : n < \omega\}, \text{where } A_0 = \mathrm{Cn}(A) \text{ and } A_{n+1} = \mathrm{Cn}(A_n \cup R(A_n)) \quad .$$

Thus, let us assume that $\langle R\rangle = \{\langle a_i, x_i\rangle : i < \omega\}$. Then, the operation $\mathrm{Cn}_{\langle R\rangle}(A)$ is defined inductively:

> $\mathrm{Cn}_{\langle R\rangle}(A) = \cup\{A_n : n < \omega\}$, with $A_0 = \mathrm{Cn}(A)$ and $A_{n+1} = \mathrm{Cn}(A_n \cup \{x\})$, where $\langle a, x\rangle$ is the first rule in $\langle R\rangle$ such that $a \in A_n$, and $x \notin A_n$. If there is no such rule, $A_{n+1} = A_n$.

Let us notice that the A_n from the definition of $\mathrm{Cn}_R(A)$ are not sets appearing appearing in the definition of $\mathrm{Cn}_{\langle R\rangle}(A)$. However, $\mathrm{Cn}_R(A) = \mathrm{Cn}_{\langle R\rangle}(A)$. It means that, an order of $\langle R\rangle$ can be any.

The second definition of the axial-rules consequence makes the definition of *ordered default-rules consequence* easy and shows the essence of the newconsequence given below (see [9, p. 93]). Before adding a new singleton to the set of assumptions it is first checked whether its addition does not lead to inconsistency; thus every singleton can be added or rejected. Thus,

> $C_{\langle R\rangle}(A) = \cup\{A_n : n < \omega\}$, with $A_0 = \mathrm{Cn}(A)$ and $A_{n+1} = \mathrm{Cn}(A_n \cup \{x\})$, where $\langle a, x\rangle$ is the first rule in $\langle R\rangle$ such that $a \in A_n$, $x \notin A_n$ and x is consistent with A_n. If there is no such rule, $A_{n+1} = A_n$.

A selection of rules being the essence of the definition of $C_{\langle R\rangle}$ means that the way of ordering of R becomes important. As in the cases of previous constructions, it seems, that the ordered default-rules consequence is nonmonotonic. For example, let us take $A = \{a\}$, $B = \{a, \neg x\}$, $R = \{\langle a, x\rangle\}$. Then, $C_{\langle R\rangle}(A) = \mathrm{Cn}(\{a, x\})$ and $C_{\langle R\rangle}(B) = \mathrm{Cn}(B) = \mathrm{Cn}(\{a, \neg x\})$.

However, this consequence distorts the *definiens* of the definition of nonmonotonic inference. First of all, the set of rules is arbitrary. Moreover, depending on the set of assumptions, some rules are allowed or not. An appropriate ordering of the set of rules makes possible the inference of the desired conclusions. Such a situation has nothing common with nonmonotonicity. Instead of one logic, there is a set of logics. Let $\langle R\rangle X$ be an ordered set of rules used in the case of the set of assumptions X. Then, sets of sentences inferred from A, B, C are, respectively, $C_{\langle R\rangle A}(A), C_{\langle R\rangle B}(B), C_{\langle R\rangle C}(C)$, and the *definiens* which should define a nonmonotonic operation discussed here has in fact the form:

> for some a, b, A such that $b \notin A$ ($a \in C_{\langle R\rangle A}(A)$
> and $a \notin C_{\langle R\rangle(A\cup\{b\})}(A \cup \{b\})$).

This clearly shows that the ordered default-rules consequence is not nonmonotonic.

3. Final conclusions

Makinson's proposal of three constructions corresponds to a remark formulated by Poole:

> "In nonmonotonic reasoning we want to reach conclusions that we may not reach if we had more information. There seem to be two ways to handle this: we could change logic to be defeasible; or we could allow some premises of the logical argument that may not be allowed when new information is received. Default logic is a formalization of the latter; it provides rules that add premises to logical arguments." [7, p. 189]

Although this clarification may be an accurate summary of all constructions, it does not coincide with the definition of nonmonotonic inference. Operations/relations given by the first or the third formal construction seem to be nonmonotonic, but they are not — the *definiens* of the definition of nonmonotonic inference is not satisfied. The second construction seems to be quite different. Applying some non-structural rules expressed by strongly selected valuations means that there is no logic here, and, in particular, no monotonic logic. In this case nonmonotonicity is in some sense "accidental" and has a secondary significance. More important is that the second construction expresses typical everyday thinking depending on depending on our desires. That is why, this formal structure has a real and high value.

Although, it is difficult to defend the nonmonotonicity of Makinson's constructions their logical as well as philosophical value is great. Not only the second but every construction is extremely important from the point of view of our human ways of thinking. Serious research into our everyday modes of thought cannot ignore Makinson's proposal. Every construction of the class of operations successfully covers some special kind of human reasoning. Fixed and stable sets of rules, traditionally defining logical systems, have almost nothing common with our thinking. Makinson's constructions propose a realistic approach to our everyday reasoning. No construction is nonmonotonic in the sense of Tarski's condition for consequence operation as human thinking is not nonmonotonic. Every class of consequences proposed by Makinson is worthy of research but not in the paradigm of nonmonotonicity. They are important for the explanation of various ways of our thinking, among which probably none is nonmonotonic.

Similar examples of nonmonotonicity of human thinking that exist in the logical literature are worthy of the new interpretation, which might be a starting point for serious and important investigationinto explaining various patterns of human thinking, making them better understood.[10]

Finally, let us notice that from the logical point of view it is clear that the popular and common application of the term "nonmonotonicity" is not correct — the strictly and precisely defined notion has a totally not strict, metaphoric understanding. The situation is probably even worse because usually by "nonmonotonic" every case of thinking in which there is a rejection of previously accepted conclusion is named. Thus, "nonmonotonicity" can only have a rhetorical sense only, not a logical one.

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