Logic and Logical Philosophy
Volume 23 (2014), 15–45
DOI: 10.12775/LLP.2013.036

Daniele Genito
Giangiacomo Gerla

CONNECTING BILATTICE THEORY WITH MULTIVALUED LOGIC

Abstract. This is an exploratory paper whose aim is to investigate the potentialities of bilattice theory for an adequate definition of the deduction apparatus for multi-valued logic. We argue that bilattice theory enables us to obtain a nice extension of the graded approach to fuzzy logic. To give an example, a completeness theorem for a logic based on Boolean algebras is proved.

Keywords: bilattice; fuzzy logic; world-based semantics; approximate reasoning

1. Introduction and preliminaries

Formal fuzzy logic (or fuzzy logic in narrow sense) is a chapter of formal logic strictly related to the theory of fuzzy subsets and connected with the tradition of multi-valued logic (see [12, 14, 15, 16, 18, 19]). Bilattice theory was introduced by Ginsberg [8] in the framework of logic programming to treat both truth and information from an algebraic point of view (see also Fitting [6, 7]). Its principal task is to give successful tools to face the difficulties arising from the acceptance of the negation in the body of the rules in a program.

In this exploratory paper we argue for the potentialities of bilattice theory for fuzzy logic. The basic idea is that if $\boldsymbol{V}$ is the "valuation structure" used to evaluate the formulas in a multi-valued logic, then it is useful to extend $\boldsymbol{V}$ to a bilattice $\boldsymbol{B}$ as a tool to define an adequate inferential apparatus. The elements of $\boldsymbol{B}$ are interpreted as information

Received May 14, 2012. *Revised* February 23, 2013. *Published online* October 9, 2013

pieces on the elements in $\boldsymbol{V}$, i.e., on the truth values of the formulas. Perhaps this gives an answer to the important question denounced by D. Dubois in [4], i.e., the existing confusion between truth-values and information states (see also [9]).

The paper is mainly addressed to propositional level and, to give an example, we apply the proposed apparatus to a logic with a world-based semantics. In [2] one considers the first order level and the attention is focused on multi-valued logic programming and fuzzy control.

Notice that the first, very interesting, proposal to connect fuzzy logic with bilattice theory was done probably by E. Turunen, M. Öztürk, A. Tsoukiás in [22] in connection with paraconsistent logic. Our approach is different since we attempt to distinguish the role (related with information and inference) of the bilattice from the role (semantics in nature) of the valuation structure. Also in the literature on fuzzy logic an analogous of the notion of bilattice is named intuitionistic fuzzy logic (see for example [17]). Our approach is different since we refer to the formal definition of fuzzy logic in Pavelka's sense in which a deduction apparatus is defined by a fuzzy subset of logical axioms and by fuzzy inference rules.

A useful tool we use in this paper is the notion of a closure operator and the associated one of a closure system. This in accordance with the abstract approach to fuzzy logic proposed in [9, 12]. We recall that, given a complete lattice L, a *closure operator* in L is a map $H\colon L \longrightarrow L$ such that

$$H(x) \geq x; x \geq y \Rightarrow H(x) \geq H(y); \quad H(H(x)) = H(x).$$

Given $\mathcal{M} \subseteq L$, the map $H_{\mathcal{M}}$ defined by setting

$$H_{\mathcal{M}}(x) = \inf\{z \in \mathcal{M} : z \geq x\}$$

is a closure operator we call *closure operator generated by* $\mathcal{M}$. A *closure system* is a subset C of L closed with respect to the finite and infinite meets. Given a closure operator H, the set $C_H = \{x \in L : H(x) = x\}$ of fixed points of H is a closure system and the closure operator associated with C_H coincides with H, i.e. $H(x) = \inf\{z \in C_H : z \geq x\}$. Consequently, if H and K are closure operators, $C_H \subseteq C_K$ entails $H \geq K$, i.e. $H(x) \geq K(x)$ for every $x \in L$. In particular, if two closure operators have the same fixed points, then they coincide. Given $\mathcal{M} \subseteq L$, the class

$$\langle \mathcal{M} \rangle = \{\inf_{i \in I} m_i : (m_i)_{i \in I} \text{ is a family of elements of } \mathcal{M}\}$$

is a closure system, we call *the closure system generated by* $\mathcal{M}$. Equivalently $\langle\mathcal{M}\rangle$ is the intersection of all the closure systems containing $\mathcal{M}$.

2. Bilattice theory

We list some basic notions in bilattice theory.

Definition 2.1. A *bilattice* is a structure $\boldsymbol{B} = (B, \leq_{\mathrm{t}}, \leq_{\mathrm{k}}, \mathit{False}, \mathit{True}, \bot, \top)$ such that $(B, \leq_{\mathrm{t}}, \mathit{False}, \mathit{True})$ and $(B, \leq_{\mathrm{k}}, \bot, \top)$ are bounded lattices. If both the orders are complete, then we say that $\boldsymbol{B}$ is *complete*. We denote by $\wedge_{\mathrm{t}}$ and $\vee_{\mathrm{t}}$, $\wedge_{\mathrm{k}}$, and $\vee_{\mathrm{k}}$ the lattice operations in the lattices $(B, \leq_{\mathrm{t}}, \mathit{False}, \mathit{True})$ and $(B, \leq_{\mathrm{k}}, \bot, \top)$, respectively. $\boldsymbol{B}$ is *interlaced* if all these operations are order preserving with respect to $\leq_{\mathrm{t}}$ and $\leq_{\mathrm{k}}$. $\boldsymbol{B}$ is *distributive* if all 12 distributive laws connecting $\wedge_{\mathrm{t}}$, $\vee_{\mathrm{t}}$, $\wedge_{\mathrm{k}}$, and $\vee_{\mathrm{k}}$ are valid. $\boldsymbol{B}$ satisfies *the decomposition property* provided that, for every $x \in B$,

$$x = (x \wedge_{\mathrm{k}} \mathit{True}) \vee_{\mathrm{k}} (x \wedge_{\mathrm{k}} \mathit{False}).$$

The ordering $\leq_{\mathrm{t}}$ is with respect to the degree of truth, the ordering $\leq_{\mathrm{k}}$ is related to information or knowledge. In the paper we will distinguish a lattice notion related to $\leq_{\mathrm{k}}$ from the same notion related to $\leq_{\mathrm{t}}$ in an evident way. For example we write $\mathrm{Sup}_{\mathrm{k}}$ and $\mathrm{Sup}_{\mathrm{t}}$ to denote the least upper bound with respect to $\leq_{\mathrm{k}}$ and $\leq_{\mathrm{t}}$, respectively. It is easy to prove that if a bilattice is distributive, then it is also interlaced and that an interlaced bilattice satisfies the decomposition property.

Definition 2.2. Assume that in a bilattice $\boldsymbol{B}$ an involution $\sim\colon B \longrightarrow B$ is defined such that

1. $x \leq_{\mathrm{t}} y \Rightarrow \sim y \leq_{\mathrm{t}} \sim x$ (*t*-order reversing)
2. $x \leq_{\mathrm{k}} y \Rightarrow \sim x \leq_{\mathrm{k}} \sim y$ (*k*-order preserving)
3. $\sim\sim x = x$

Then we say that $(B, \leq_{\mathrm{t}}, \leq_{\mathrm{k}}, \sim, \mathit{False}, \mathit{True}, \bot, \top)$ is a *bilattice with negation.*

We emphasize that since $\sim$ is order-reversing with respect to $\leq_{\mathrm{t}}$ and order-preserving with respect to $\leq_{\mathrm{k}}$,

$$\sim(x \wedge_{\mathrm{t}} y) = \sim(x) \vee_{\mathrm{t}} \sim(y), \qquad \sim(x \vee_{\mathrm{t}} y) = \sim(x) \wedge_{\mathrm{t}} \sim(y),$$
$$\sim(x \wedge_{\mathrm{k}} y) = \sim(x) \wedge_{\mathrm{k}} \sim(y), \qquad \sim(x \vee_{\mathrm{k}} y) = \sim(x) \vee_{\mathrm{k}} \sim(y),$$

for every x, y in $\boldsymbol{B}$. It is also immediate that $\sim \mathit{False} = \mathit{True}$, $\sim \mathit{True} = \mathit{False}$, $\sim\bot = \bot$, $\sim\top = \top$.

There are several ways to define a bilattice by starting from a bounded lattice $\boldsymbol{L} = (L, \leq, 0, 1)$. One way is to consider the set of intervals of $\boldsymbol{L}$ (see for example A. Pilitowska [20]) and it is related in a natural way to multi-valued logic. Indeed, an interval is interpreted as a constraint on a possible truth value.

Theorem 2.3. *Let $I(L)$ be a set of closed intervals of a bounded lattice $\boldsymbol{L}$ (included the empty set) and define the structure $\boldsymbol{I}(\boldsymbol{L}) = (I(L), \leq_{\mathrm{t}}, \leq_{\mathrm{k}}, \{0\}, \{1\}, L, \varnothing)$ in such a way that*

- *$\leq_{\mathrm{k}}$ is the dual of the inclusion relation,*
- *for every $[a, b]$, $[c, d]$ in $I(L) - \{\varnothing\}$, $[a, b] \leq_{\mathrm{t}} [c, d]$ provided that $a \leq c$ and $b \leq d$,*
- *$\{0\} \leq_{\mathrm{t}} \varnothing \leq_{\mathrm{t}} \{1\}$ and $\varnothing$ is not t-comparable with any other interval.*

Then $\boldsymbol{I}(\boldsymbol{L})$ is a bilattice we call interval bilattice associated with $\boldsymbol{L}$.

Proposition 2.4. *The interval bilattice $\boldsymbol{I}(\boldsymbol{L})$ satisfies the decomposition property and it is interlaced if and only if $\boldsymbol{L}$ is the two elements Boolean algebra. If $\boldsymbol{L}$ is complete, then $\boldsymbol{I}(\boldsymbol{L})$ is complete. Let $\sim$ be an involution in L, i.e. an order reversing map such that $\sim(\sim x) = x$, and set*

$$\sim[a, b] = [\sim b, \sim a]; \quad \sim\varnothing = \varnothing.$$

Then we obtain a negation in $\boldsymbol{I}(\boldsymbol{L})$.

Proof. To prove the decomposition property, we observe that, for every interval $[a, b]$,

$$[a, b] = [a, 1] \cap [0, b] = ([a, b] \wedge_{\mathrm{k}} \{1\}) \vee_{\mathrm{k}} ([a, b] \wedge_{\mathrm{k}} \{0\})$$

and

$$\varnothing = \{1\} \vee_{\mathrm{k}} \{0\} = (\varnothing \wedge_{\mathrm{k}} \{1\}) \vee_{\mathrm{k}} (\varnothing \wedge_{\mathrm{k}} \{0\}).$$

Moreover, due to the behavior of $\varnothing$, in the case $L \neq \{0, 1\}$, $\boldsymbol{I}(\boldsymbol{L})$ is not interlaced. Indeed, if c is an element of L different from 0 and 1, then $[0, 0] \leq_{\mathrm{t}} [c, c]$, while $[0, 0] \vee_{\mathrm{k}} [c, 1] = \varnothing$ and $[c, c] \vee_{\mathrm{k}} [c, 1] = [c, c]$. On the other hand the relation $\varnothing \leq_{\mathrm{t}} [c, c]$ is false. The remaining part of the proposition is trivial. $\dashv$

Another very elegant way to obtain a bilattice is the following one (see Fitting [7]).

THEOREM 2.5. *Let $\boldsymbol{L} = (L, \leq, 0, 1)$ be a bounded lattice and denote by $\boldsymbol{B}(\boldsymbol{L})$ the structure $(L \times L, \leq_{\mathrm{t}}, \leq_{\mathrm{k}}, \sim, (0,1), (1,0), (0,0), (1,1))$, where $\sim$ is defined by setting $\sim(x, x') = (x', x)$, and the relations $\leq_{\mathrm{t}}$, $\leq_{\mathrm{k}}$ are defined by setting*

$$(x, x') \leq_{\mathrm{t}} (y, y') \iff x \leq y \text{ and } x' \geq y',$$
$$(x, x') \leq_{\mathrm{k}} (y, y') \iff x \leq y \text{ and } x' \leq y'.$$

Then $\boldsymbol{B}(\boldsymbol{L})$ is an interlaced bilattice with negation we call the square bilattice associated with $\boldsymbol{L}$.

PROPOSITION 2.6. *$\boldsymbol{B}(\boldsymbol{L})$ is interlaced and therefore it satisfies the decomposition property. If L is complete (distributive) then $\boldsymbol{B}(\boldsymbol{L})$ is complete (distributive, respectively).*

The idea is that if a claim α is valued by (x, x'), then x is a measure of the information in favor of α and x' a measure of the information against α.

The following proposition shows a basic connection between $\boldsymbol{I}(\boldsymbol{L})$ and $\boldsymbol{B}(\boldsymbol{L})$.

PROPOSITION 2.7. *Let $\boldsymbol{L}$ be a bounded lattice with an involution $\sim$ and let $I_0(L)$ be the set of nonempty intervals of $\boldsymbol{L}$. Then by setting $h([a, b]) = (a, \sim b)$ we obtain an embedding of the structure $\boldsymbol{I_0}(\boldsymbol{L}) = (I_0(L), \leq_{\mathrm{t}}, \leq_{\mathrm{k}}, \sim, \{0\}, \{1\}, L)$ into the structure $(L \times L, \leq_{\mathrm{t}}, \leq_{\mathrm{k}}, \sim, (0, 1), (1, 0), (0, 0))$.*

In accordance with the fact that in a multi-valued logics the conjunction is usually interpreted by a non idempotent operation, we give the following extension of the notion of bilattice. Under the name *commutative monoid with a zero* we mean a structure $(D, \cdot, 0, 1)$ such that $(D, \cdot, 1)$ is a commutative monoid such that $x \cdot 0 = 0$ for every $x \in D$.

DEFINITION 2.8. An *extended bilattice* (see also [3] and [22]) is a structure $\boldsymbol{B} = (B, \leq_{\mathrm{t}}, \leq_{\mathrm{k}}, \otimes_{\mathrm{t}}, \sim, \mathit{False}, \mathit{True}, \perp, \top)$ such that

- $(B, \leq_{\mathrm{t}}, \leq_{\mathrm{k}}, \sim, \mathit{False}, \mathit{True}, \perp, \top)$ is a bilattice with negation,
- $(B - \{\top\}, \otimes_{\mathrm{t}}, \mathit{False}, \mathit{True})$ is a commutative monoid with zero which is infinitely distributive with respect to the bilattice operations.

In an extended bilattice we have that,

$$\top \otimes_{\mathrm{t}} \perp = \mathit{False}.$$

Indeed, $\perp \leq_{\mathrm{k}}$ *False* and therefore $\perp \otimes_{\mathrm{t}} \top \leq_{\mathrm{k}}$ *False* $\otimes_{\mathrm{t}} \top =$ *False* and *False* $\leq_{\mathrm{k}} \top$ and therefore *False* $=$ *False* $\otimes_{\mathrm{t}} \perp \leq_{\mathrm{k}} \top \otimes_{\mathrm{t}} \perp$.

3. Bilattices as an information framework on truth values

In the literature on multi-valued logic one refers to algebraic structures whose elements are truth values and whose operations are devoted to interpret the logical connectives. To fix the ideas, in this paper we refer to the following well known class of structures.

Definition 3.1. A *valuation structure* is a complete commutative residuated lattice with an involution, i.e. a structure $\boldsymbol{V} = (V, \leq, \otimes, \sim, 0, 1)$ such that $(V, \leq, 0, 1)$ is a complete lattice with $0 \neq 1$, $\sim$ is an involution and

(i) $(V, \otimes, 0, 1)$ is a commutative monoid with a zero
(ii) $\otimes$ is *infinitely distributive*, i.e., for every $x \in V$ and $C \subseteq V$, $C \neq \varnothing$,

$$x \otimes \mathrm{Sup}(C) = \mathrm{Sup}(x \otimes C) \text{ and } x \otimes \mathrm{Inf}(C) = \mathrm{Inf}(x \otimes C).$$

We call *truth values* the elements of V, we denote by $\rightarrow$ the residuum of $\otimes$ and we define the operation $\oplus$ by setting

$$x \oplus y = \sim(\sim x \otimes \sim y).$$

Obviously, $\sim$ is an isomorphism from $(V, \leq, \otimes, \sim, 0, 1)$ to $(V, \geq, \oplus, \sim, 1, 0)$ and this means that an obvious duality principle holds true. This entails, for example, that $\oplus$ is commutative, associative, order-preserving and such that $x \oplus 0 = x$, $x \oplus 1 = 1$. Moreover, $\oplus$ is infinitely distributive.

Definition 3.2. A *k-extension* of a valuation structure $\boldsymbol{V}$ is an extended bilattice $\boldsymbol{B} = (B, \leq_{\mathrm{t}}, \leq_{\mathrm{k}}, \otimes_{\mathrm{t}}, \sim, \mathit{False}, \mathit{True}, \perp, \top)$ together with an embedding $i \colon V \longrightarrow B$ of $\boldsymbol{V}$ into the reduct $(B, \leq_{\mathrm{t}}, \otimes_{\mathrm{t}}, \sim, \mathit{False}, \mathit{True})$.

The intended interpretation is that the elements in $\boldsymbol{B}$ are pieces of information on the truth values in V and that, given $\lambda \in V, i(\lambda)$ is the whole information we can obtain on λ in the "information framework" $\boldsymbol{B}$. In accordance, it is reasonable to define a piece of information on λ as a "fragment" of the whole information $i(\lambda)$.

Definition 3.3. Given $\lambda \in V$ and $x \in B$, we say that x *is a correct piece of information on* λ and we write $\lambda \vDash x$ if $x \leq_{\mathrm{k}} i(\lambda)$. We say that x is *satisfiable* provided that there is $\lambda \in V$ such that $\lambda \vDash x$ and we indicate by $Sat(\boldsymbol{B})$ the set of satisfiable elements in $\boldsymbol{B}$.

We refer mainly to two classes of k-extensions, the first one is related with the interval bilattices.

Proposition 3.4. *Let $\boldsymbol{V}$ be a valuation structure and assume that $\boldsymbol{B}$ is the interval bilattice $\boldsymbol{I}(\boldsymbol{V})$. Then we obtain a k-extension of V by considering the operation $\otimes_{\mathrm{t}}$ defined by the equations*

$$[a,b] \otimes_{\mathrm{t}} [c,d] = [a \otimes c, b \otimes d]$$
$$\{1\} \otimes_{\mathrm{t}} \varnothing = \varnothing \otimes_{\mathrm{t}} \{1\} = \varnothing \otimes_{\mathrm{t}} \varnothing = \varnothing$$
$$[a,b] \otimes_{\mathrm{t}} \varnothing = \varnothing \otimes_{\mathrm{t}} [a,b] = \{0\}([a,b] \neq \{1\})$$

and by the map $i\colon V \longrightarrow I(V)$ defined by setting

$$i(\lambda) = [\lambda, \lambda] = \{\lambda\}$$

In such a k-extension $\varnothing$ is the unique element in $\boldsymbol{B}$ which is not satisfiable while, for every nonempty interval $[a,b]$,

$$\lambda \vDash [a,b] \Leftrightarrow \lambda \in [a,b].$$

Proposition 3.5. *The definition of $\otimes_{\mathrm{t}}$ is partially justified by the following facts*

- *it is minimal with respect to the condition*
  if $\lambda \vDash x$ and $\mu \vDash y$ then $\lambda \otimes \mu \vDash x \otimes_{\mathrm{t}} y$,
- *if $\otimes$ is the meet in $(V, \leq, 0, 1)$ then $\otimes_{\mathrm{t}}$ is the meet in $(I(L), \leq_{\mathrm{t}}, \{0\}, \{1\})$.*

A second class of k-extensions is related with the square bilattice $\boldsymbol{B}(\boldsymbol{V})$.

Definition 3.6. Given a valuation structure $\boldsymbol{V} = (V, \leq, \otimes, \sim, 0, 1)$, we call *square k-extension of* $\boldsymbol{V}$ the k-extension obtained by adding to the bilattice $\boldsymbol{B}(\boldsymbol{V})$ the operation $\otimes_{\mathrm{t}}$ defined by setting

$$(x, x') \otimes_{\mathrm{t}} (y, y') = (x \otimes y, x' \oplus y')$$

and by considering the embedding $i\colon V \longrightarrow B(V)$ defined by setting

$$i(\lambda) = (\lambda, \sim \lambda).$$

In a square k-extension we have that

$$\lambda \vDash^* (a,b) \Leftrightarrow a \leq \lambda \text{ and } b \leq \sim\lambda,$$

and this means that an element (a,b) of $\boldsymbol{B}(\boldsymbol{V})$ carries the information "the truth degree is at least a and the falsity degree is at least b". Notice that, differently from the interval k-extension, there are several elements in $\boldsymbol{B}(\boldsymbol{V})$ which are not satisfiable. Indeed,

$$Sat(\boldsymbol{B}(\boldsymbol{V})) = \{(\lambda,\mu) \in V \times V : \mu \leq \sim\lambda\}.$$

The following proposition gives a justification for the proposed definition of $\otimes_{\mathrm{t}}$.

Proposition 3.7. *The definitions of i and $\otimes_{\mathrm{t}}$ in a square extension are compatible with the embedding h of $\boldsymbol{I_0}(\boldsymbol{V})$ into $\boldsymbol{B}(\boldsymbol{V})$ given in Proposition 2.7 and they are the only possible definitions in the case $\otimes_{\mathrm{t}}$ is distributive with respect to $\vee_{\mathrm{k}}$.*

Proof. Indeed, consider a k-extension of $\boldsymbol{V}$ in $\boldsymbol{B}(\boldsymbol{V})$ and assume that it is in accordance with h. Then the map $i\colon \boldsymbol{V} \longrightarrow \boldsymbol{B}(\boldsymbol{V})$ have to satisfy the equation $i(\lambda) = h(i([\lambda,\lambda])) = (\lambda, \sim\lambda)$. Moreover, the operation $\otimes_{\mathrm{t}}$ in $\boldsymbol{B}(\boldsymbol{V})$ have to satisfy the condition

$$h^{-1}((x,x') \otimes_{\mathrm{t}} (y,y')) = h^{-1}((x,x')) \otimes_{\mathrm{t}} h^{-1}((y,y'))$$

and therefore

$$\begin{aligned}(x,x') \otimes_{\mathrm{t}} (y,y') &= h(h^{-1}((x,x')) \otimes_{\mathrm{t}} h^{-1}((y,y')))\\ &= h([x,\sim x'] \otimes_{\mathrm{t}} [y,\sim y'])\\ &= h([x \otimes y, \sim x' \otimes \sim y'])\\ &= (x \otimes y, \sim(\sim x' \otimes \sim y'))\\ &= (x \otimes y, x' \oplus y')\end{aligned}$$

for every x, y, x', y' such that $x \leq \sim x'$ and $y \leq \sim y'$. Assume that $\otimes_{\mathrm{t}}$ is distributive with respect to $\vee_{\mathrm{k}}$, and that $y \leq \sim y'$, then, for every x, y, x', y' in V,

$$\begin{aligned}(x,x') \otimes_{\mathrm{t}} (y,y') &= ((x,0) \vee_{\mathrm{k}} (0,x')) \otimes_{\mathrm{t}} (y,y')\\ &= ((x,0) \otimes_{\mathrm{t}} (y,y')) \vee_{\mathrm{k}} ((0,x') \otimes_{\mathrm{t}} (y,y'))\\ &= (x \otimes y, 0 \oplus y') \vee_{\mathrm{k}} (0 \otimes y, x' \oplus y')\\ &= (x \otimes y, x' \oplus y').\end{aligned}$$

$\dashv$

Notice that an analogous of the k-extensions by the square bilattices is proposed in [22] by E. Turunen, M. Öztürk and A. Tsoukiás. Indeed, in [22] one considers the case $\boldsymbol{V}$ is a MV-algebra and one extends $\boldsymbol{V}$ into a suitable *MV-algebra of evidence matrices*. This algebraic structure is defined in a set of matrices like

$$\begin{pmatrix} f & k \\ u & t \end{pmatrix}$$

where the values f, k, u, t correspond to *falsehood*, *contradictory*, *unknown* and *truth*, respectively. Namely, every pair (x, x') in the square bilattice $\boldsymbol{B}(\boldsymbol{V})$ is associated with the matrix

$$\begin{pmatrix} x' \wedge \sim x & x \otimes x' \\ \sim x \otimes \sim x' & x \wedge \sim x' \end{pmatrix}$$

and one considers the set of so obtained matrices.

We conclude this section by recalling some basic notions in fuzzy set theory. Given a nonempty set S and a valuation structure $\boldsymbol{V}$, we denote by $\boldsymbol{V}^S$ the direct power of $\boldsymbol{V}$ with index set S and we call *V-subsets* of S the elements of $\boldsymbol{V}^S$. Given $x \in S$, the value $s(x)$ is interpreted as the membership degree of x in s. The set-theoretical nomenclature is used in an obvious way. For example the order relation in V^S is denoted by $\subseteq$ and named *inclusion relation*. The meet and the join in V^S are denoted by $\cap$ and $\cup$ and named *intersection* and *union*, respectively. The *complement of* s is the V-subset $-s$ defined by setting $(-s)(x) = \sim s(x)$ for every $x \in S$. We extend these definitions by considering a bilattice $\boldsymbol{B}$ in place of $\boldsymbol{V}$. For example, a *B-subsets of* S is a map $s\colon S \longrightarrow B$, we denote by $\subseteq_{\mathrm{k}}$ the k-order relation in $\boldsymbol{B}^S$ and we denote by $\cap_{\mathrm{k}}$ and $\cup_{\mathrm{k}}$ and $-$ the k-meet, the k-join and the complement, respectively.

4. Bilattice-based logics

We will define a bilattice-based multi-valued logic by extending the definition for fuzzy logic proposed by Pavelka (graded approach). Recall that if F is the set of formulas of a given language, then Pavelka calls *semantics* any class $\boldsymbol{M} \subseteq V^F$ of V-subsets of F and *model* an element of $\boldsymbol{M}$. This means that a model is identified with the valuation in V of the formulas it determines. We extend this definition by identifying a model with the information we can have on this model in a k-extension $\boldsymbol{B}$ of $\boldsymbol{V}$, i.e. with a map from F to $\boldsymbol{B}$.

Definition 4.1. Let F be a nonempty set whose elements we call *formulas* and let $\boldsymbol{B}$ be a k-extension of a valuation structure $\boldsymbol{V}$. Then a *B-valuation* is a map $v\colon F \longrightarrow B$ from F to B. We say that v is *pointwise satisfiable* if $v(\alpha)$ is satisfiable for every formula α. A *B-semantics*, in brief a *semantics*, is a non-empty class $\mathcal{M} \subseteq B^F$ of B-valuations. We call *interpretation* or *model* any element m in $\mathcal{M}$ and we say that m is *real* if $m(\alpha) \in i(V)$ for every $\alpha \in F$. A semantics $\mathcal{M}$ is *canonical* if all its interpretations are real.

Obviously, we are mainly interested in the canonical semantics. Nevertheless in [2], in accordance with the fact that in logic programming there is no drastic distinction between semantics and syntax, one consider also the possibility of Herbrand models (i.e. B-subsets of facts) which are not real.

Definition 4.2. Given a B-subset of formulas v, we say that $m \in \mathcal{M}$ *is a model of* v, in brief $m \vDash v$, if $m(\alpha) \vDash v(\alpha)$ for every $\alpha \in F$.

Then m is a model of v provided that $m(\alpha) \geq_{\mathrm{k}} v(\alpha)$ for every $\alpha \in F$, i.e. provided that $m \supseteq_{\mathrm{k}} v$. We interpret v as the available information on (the truth-values of) an unknown "world" m. These definitions extend Pavelka's definitions since Pavelka's semantics coincides with the canonical B-semantics, in a sense. Indeed, if $\boldsymbol{M} \subseteq V^F$ is a semantics in Pavelka's sense, then we obtain a canonical B-semantics by setting $\mathcal{M} = \{i \circ m\colon m \in \boldsymbol{M}\} \subseteq B^F$. Conversely, if $\mathcal{M} \subseteq B^F$ is a canonical B-semantics, then we obtain a semantics in Pavelka's sense by setting $\boldsymbol{M} = \{i^{-1} \circ m : m \in \boldsymbol{M}\} \subseteq V^F$. In the following we identify a model $m \in \boldsymbol{M}$ with $i \circ m$ and therefore $\mathcal{M}$ with $\boldsymbol{M}$. For example, we say that $m \in \boldsymbol{M}$ is a model of v provided that $i \circ m$ is a model of v.

Definition 4.3. The *logical consequence operator associated with* $\mathcal{M}$ is the operator $L_c\colon B^F \longrightarrow B^F$ defined by setting,

$$L_c(v)(\alpha) = \mathrm{Inf}_{\mathrm{k}}\{m(\alpha) \in B : m \in \mathcal{M} \text{ and } m \vDash v\}$$

for every $v \in B^F$ and $\alpha \in F$. The *B-subset of tautologies* is defined by setting

$$\mathrm{Tau} = L_c(v_{\perp}) = \mathrm{Inf}_{\mathrm{k}}\{m(\alpha) \in B : m \in \mathcal{M}\}.$$

We interpret $L_c(v)(\alpha)$ as the information on α shared by all the possible models of v. In particular, *Tau*(α) represents the information on α shared by all the possible models.

PROPOSITION 4.4. *Given a B-subset of formulas v, we have that*

$$L_c(v) = \bigcap\nolimits_{\mathrm{k}}\{m \in \mathcal{M} : m \supseteq_{\mathrm{k}} v\}.$$

Consequently, $L_c\colon B^F \longrightarrow B^F$ is a closure operator, the closure operator generated by $\mathcal{M}$.

In the case $\boldsymbol{B}$ is an interval bilattice (a square bilattice), we denote by v^+ and v^- (by v_+, v_-) two V-subsets of formulas such that $v(x) = [v^+(x), v^-(x)]$, $(v(x) = (v_+(x), v_-(x))$ respectively) for every $x \in F$. Then, if $\boldsymbol{B} = \boldsymbol{I}(\boldsymbol{V}), \mathcal{M}$ is the canonical semantics associated with a Pavelka semantics $\boldsymbol{M}$, and $m = i \circ m'$ with $m' \in \boldsymbol{M}$, then we have that $m \vDash v$ if and only if $v^+ \subseteq m' \subseteq v^-$ and, given $\alpha \in F$, $L_c(v)(\alpha)$ is the least interval containing the set $\{m'(\alpha) \in V : m' \in \boldsymbol{M}, v^+ \subseteq m' \subseteq v^-\}$ of possible values assumed by α, i.e. $L_c(v)(\alpha) = [\mathrm{Inf}\{m'(\alpha) : m' \in \boldsymbol{M}, v^+ \subseteq m' \subseteq v^-\}, \mathrm{Sup}\{m'(\alpha) : m' \in \boldsymbol{M}, v^+ \subseteq m' \subseteq v^-\}]$.

In particular,

$$\mathrm{Tau}(\alpha) = [\mathrm{Inf}\{m'(\alpha) : m' \in \boldsymbol{M}\}, \mathrm{Sup}\{m'(\alpha) : m' \in \boldsymbol{M}\}].$$

In the case $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{V}), m \vDash v$ if and only if $v_+ \subseteq m' \subseteq -v_-$ and therefore

$$\begin{aligned} L_c(v)(\alpha) = (&\mathrm{Inf}\{m'(\alpha) : m' \in \boldsymbol{M}, v_+ \subseteq m' \subseteq -v_-\},\\ &\sim\mathrm{Sup}\{m'(\alpha) : m' \in \boldsymbol{M}, v_+ \subseteq m' \subseteq -v_-\}).\end{aligned}$$

In particular,

$$\mathrm{Tau}(\alpha) = (\mathrm{Inf}\{m'(\alpha) : m' \in \boldsymbol{M}\}, \sim\mathrm{Sup}\{m'(\alpha) : m' \in \boldsymbol{M}\}).$$

In the case condition $m'(\neg\alpha) = \sim m'(\alpha)$ is satisfied, we have

$$\begin{aligned} L_c(v)(\alpha) = (&\mathrm{Inf}\{m'(\alpha) : m' \in \boldsymbol{M}, v_+ \subseteq m' \subseteq -v_-\},\\ &\mathrm{Inf}\{m'(\neg\alpha) : m' \in \boldsymbol{M}, v_+ \subseteq m' \subseteq -v_-\})\end{aligned}$$

and therefore,

$$\mathrm{Tau}(\alpha) = (\mathrm{Inf}\{m'(\alpha) : m' \in \boldsymbol{M}\}, \mathrm{Inf}\{m'(\neg\alpha) : m' \in \boldsymbol{M}\}).$$

Once the operator L_c is defined, the crucial question is to propose an algorithm to "calculate" the values $L_c(v)(\alpha)$, i.e. to calculate the information we can obtain on the truth value of α given the available information v. To this aim, we define a notion of deduction apparatus in Hilbert style, i.e. by logical axioms and inference rules. This is obtained by modifying Pavelka's definitions so that we refer to the knowledge order in $\boldsymbol{B}$ and not to the order in $\boldsymbol{V}$. For example, we say that a map $f\colon B^n \longrightarrow B$ is *k-continuous* if it preserves the k-inductive limits, i.e.

$$f(x_1, \dots, \mathrm{Sup_k}C, \dots, x_n) = \mathrm{Sup_k}\{f(x_1, \dots, x, \dots, x_n) \in B : x \in C\}$$

for every k-directed subset C of B.

Definition 4.5. Let $\boldsymbol{B}$ be a complete bilattice, then an *n-ary B-inference rule* is a pair $r = (r_{\mathrm{syn}}, r_{\mathrm{sem}})$ where r_{syn} is a partial n-ary operation in F (i.e., an inference rule in the usual sense) and r_{sem} is a k-continuous n-ary operation in $\boldsymbol{B}$. A *B-deduction apparatus*, in brief a *deduction apparatus*, is a pair (IR, la) such that la is a B-subset of formulas, we call *B-subset of logical axioms*, and IR is a set of B-inference rules.

Notice that the continuity condition is required to obtain that the deduction operator (see Definition 4.8), is a closure operator. We represent an application of an n-ary B-inference rule as follows

$$\left\langle \frac{\alpha_1, \dots, \alpha_n}{r_{\mathrm{syn}}(\alpha_1, \dots, \alpha_n)} \;\middle|\; \frac{b_1, \dots, b_n}{r_{\mathrm{sem}}(b_1, \dots, b_n)} \right\rangle$$

The intended meaning is that if $b_1, \dots, b_n$ are correct pieces of information on $\alpha_1, \dots, \alpha_n$, then $r_{\mathrm{sem}}(b_1, \dots, b_n)$ is a correct piece of information on $r_{\mathrm{syn}}(\alpha_1, \dots, \alpha_n)$.

Definition 4.6. A *proof* π *of a formula* α is any sequence $\alpha_1, \dots, \alpha_m$ of formulas in F such that $\alpha_m = \alpha$, together with the related "*justifications*", i.e., for any formula α_i, we must specify whether

(i) α_i is assumed as a logical axiom; or
(ii) α_i is assumed as an hypothesis; or
(iii) α_i is obtained by a rule (in this case we have to indicate also the rule and the formulas $\alpha_{i(1)}, \dots, \alpha_{i(n)}$ in $\alpha_1, \dots, \alpha_{i-1}$ used to obtain α_i).

Differently from the classical logic, the justifications are necessary to calculate the information furnished by the proof. As in classical logic, for $i \leq m$, the initial segment $\alpha_1, \ldots, \alpha_i$ of a proof $\alpha_1, \ldots, \alpha_m$ is a proof of α_i we denote by $\pi(i)$.

Definition 4.7. Given a proof $\pi = \alpha_1, \ldots, \alpha_m$ of α and a B-set of formulas $v\colon F \longrightarrow B$, the *information on α furnished by π given v* is the element $I(\pi, v)$ in B defined by induction on the length of π in accordance with the following rules:

$I(\pi, v) = la(\alpha_m)$ if α_m is assumed as a logical axiom,
$I(\pi, v) = v(\alpha_m)$ if α_m is assumed as an hypothesis,
$I(\pi, v) = r_{\mathrm{sem}}(I(\pi_{i(1)}, v), \ldots, I(\pi_{i(n)}, v))$ if α_m is obtained from $\alpha_{i(1)}, \ldots, \alpha_{i(n)}$ by the rule $r = (r_{\mathrm{syn}}, r_{\mathrm{sem}})$
with $i(1) < m, \ldots, i(n) < m$.

Notice that we have only two proofs of α with length 1. The formula α with the justification "α is assumed as a logical axiom" and the formula α with the justification "α is assumed as an hypothesis". So, the first two lines in the definition of I give also the induction basis.

Each proof of a formula α gives a different piece of information on the truth value of α. So, we have to fuse all these pieces of information.

Definition 4.8. We call *deduction operator* the operator $D\colon B^F \longrightarrow B^F$ defined by setting, for every $v \in B^F$ and $\alpha \in F$,

$$D(v)(\alpha) = \mathrm{Sup}_{\mathrm{k}}\{I(\pi, v) \in B : \pi \text{ is a proof of } \alpha\}.$$

These definitions suggest adding further hypotheses to the notion of deduction apparatus. For example, in the case of a binary inference rule it is natural to assume that evaluation part is a binary k-continuous operation $\otimes_{\mathrm{k}}$ which is commutative and associative. Indeed, the order of the formulas assumed either as hypotheses or logical axioms in a proof is not relevant. Moreover to be coherent with the usual notion of inference rule, it is natural to assume that $\mathit{True} \otimes_{\mathrm{k}} \mathit{True} = \mathit{True}$. We call *valuation product* an operation satisfying these conditions. Also, it is not restrictive to assume that in the considered deduction apparatus there are two structural rules

$$\left\langle \frac{\alpha}{\alpha} \mid \frac{x}{r_b(x)} \right\rangle \quad \text{(weakening rules)} \qquad \left\langle \frac{\alpha, \alpha}{\alpha} \mid \frac{x, y}{x \vee y} \right\rangle \quad \text{(fusion rule)}$$

where $r_b(x) = b$ if $x \geq b$ and $r(x) = x$ otherwise. The weakening rule enables us to claim that if we can prove α at degree x, then we can prove α at degree b for every $b \leq x$. The fusion rule enables us to fuse two different proofs π_1 and π_2 of α into an unique proof π of α in such a way that $I(\pi, v) = I(\pi_1, v) \vee_{\mathrm{k}} I(\pi_2, v)$. In accordance, the information-set $\{I(\pi, v) \in B : \pi \text{ is a proof of } \alpha\}$ on α is up-ward directed and we can obtain $D(v)(\alpha)$ as a direct limit of this set,

$$D(v)(\alpha) = \lim_{\mathrm{k}}\{I(\pi, v) : \pi \text{ is a proof of } \alpha\}.$$

The following rule is related with the question of the inconsistency.

$$\left\langle \frac{\alpha}{\beta} \mid \frac{x}{c(x)} \right\rangle \qquad \text{(inconsistency rule)}$$

where the map c is defined by setting $c(x) = \perp$ if $x \in \mathit{Sat}$ and $c(x) = \top$ otherwise. The inconsistency rule claims that if the information on a formula is not consistent, then the information on all the formulas is completely inconsistent.

THEOREM 4.9. *The deduction operator D associated with a B-deduction apparatus is a closure operator in the lattice $(B^F, \subseteq_{\mathrm{k}})$.*

PROOF. To prove that $D(v) \supseteq_{\mathrm{k}} v$ it is sufficient to observe that, given a formula α, the formula α together with the justification "by hypothesis" is a proof π of α such that $I(\pi, v) = v(\alpha)$. We can prove that D is monotone by proving that $I(\pi, v)$ is monotone with respect to v for every proof π of a formula α. To this aim it is sufficient to observe that the semantic component of the inference rules is k-monotone and to proceed by induction on the length of π. To prove that D is idempotent we have to prove that $D(v)$ is a fixed point for D and therefore that, given a formula α,

$$\mathrm{Sup}_{\mathrm{k}}\{I(\pi, D(v)) \in B : \pi \text{ is a proof of } \alpha\} \leq_{\mathrm{k}} D(v)(\alpha).$$

Equivalently, we have to prove that, for every proof $\pi = \alpha_1, \ldots, \alpha_m$ of α,

$$I(\pi, D(v)) \leq_{\mathrm{k}} D(v)(\alpha). \qquad (*)$$

We will proceed by induction on the length m of π. Now, if α_m is assumed either as a logical axiom or as a hypothesis, then $(*)$ is evident. Otherwise, assume that α_m is obtained by an n-ary inference rule and therefore that

$$I(\pi, D(v)) = r_{\mathrm{sem}}(I(\pi_{i(1)}, D(v)), \ldots, I(\pi_{i(n)}, D(v))),$$

where $\pi_{i(1)}, \ldots, \pi_{i(n)}$ are the proofs of the formulas $\alpha_{i(1)}, \ldots, \alpha_{i(n)}$, $i(1) < m, \ldots, i(n) < m$. Then, taking in account the induction hypothesis, the definition of $D(v)$ and the k-continuity of r_{sem}: $I(\pi, D(v)) \leq_{\text{k}} r_{\text{sem}}(D(v)(\alpha_{i(1)}), \ldots, D(v)(\alpha_{i(n)})) = r_{\text{sem}}(\text{Sup}_{\text{k}}\{I(\underline{\pi}, v) \in B : \underline{\pi}$ is a proof of $\alpha_{i(1)}\}, \ldots, \text{Sup}_{\text{k}}\{I(\underline{\pi}, v) \in B : \underline{\pi}$ is a proof of $\alpha_{i(n)}\}) = \text{Sup}_{\text{k}}\{r_{\text{sem}}(I(\underline{\pi}_{i(1)}, v), \ldots, I(\underline{\pi}_{i(n)}, v)) \in B : \underline{\pi}_{i(1)}$ is a proof of $\alpha_{i(1)}$, $\ldots, \underline{\pi}_{i(n)}$ is a proof of $\alpha_{i(n)}\} \leq_{\text{k}} D(v)(\alpha)$. ⊣

We are now ready to give the main definition in this paper.

Definition 4.10. Let $\mathcal{M}$ be a semantics and (IR, la) be a deduction apparatus. Then (IR, la) is *sound* with respect to $\mathcal{M}$ if $L_c(v) \supseteq_{\text{k}} D(v)$ for every $v \in B^F$. (IR, la) is *complete* with respect to $\mathcal{M}$ if $D(v) \supseteq_{\text{k}} L_c(v)$ for every $v \in B^F$. In the case (IR, la) is both sound and complete, i.e. $D = L_c$, we say that $(\mathcal{M}, IR, la)$ is a *bilattice based fuzzy logic* and that *the completeness theorem* holds true.

We emphasize that such an approach to multi-valued logic extends the graded approach proposed by Pavelka. This in spite of the fact that in the definitions of Pavelka the reference to the knowledge order is not apparent. Indeed, as a matter of fact, such a reference is implicit since the information represented by a fuzzy subset $v\colon F \longrightarrow V$ of formulas is that m is a model of v provided that $m \supseteq v$. Then the information carried on by v is that, for every formula α, $v(\alpha)$ represents a lower-bound constraint like "the truth value of α is greater or equal to $v(\alpha)$". This means that in the graded approach one manages interval constraints on truth values and not truth values and we have not confuse the truth value λ with the constraint $[\lambda, 1]$. Then it is possible to identify every valuation $v\colon F \longrightarrow V$ with the valuation $\underline{v}\colon F \longrightarrow I(V)$ defined by setting $\underline{v}(\alpha) = [v(\alpha), 1]$ for every $\alpha \in F$ and $\underline{v}(\alpha) = [0, 1] = \perp$ otherwise. More in general, it is evident that we can reformulate every notion in Pavelka's approach in the bilattice-based framework proposed in this paper.

It is easy to prove that the deduction operator is "compact" and "computable" in the sense proposed in [13].

5. Soundness and completeness

If we fix a B-deduction apparatus, then there is no difficulty to find a semantics such that the completeness theorem holds true. Indeed, it is sufficient to put the semantics equal to the class of fixed points of the deduction operator. A more complicate task is to fix a semantics $\mathcal{M}$ and to find a deduction apparatus adequate to $\mathcal{M}$. The following proposition is an immediate consequence of closure operator theory.

PROPOSITION 5.1. *Let $\mathcal{M}$ be a semantics and (IR, la) a deduction apparatus. Then (IR, la) is sound if and only if each fixed points of L_c is a fixed point of D. (IR, la) is complete if and only if each fixed point of D is a fixed point of L_c. So, the completeness theorem holds true if and only if L_c and D have the same fixed points.*

Regarding the fixed points of L_c, we have the following proposition whose proof is trivial.

PROPOSITION 5.2. *A valuation v is a fixed point of L_c if and only if v is a k-intersection of elements in $\mathcal{M}$. Equivalently, the set of fixed points of L_c is the closure system generated by $\mathcal{M}$.*

We characterize the fixed points of D as the B-subsets of formulas closed under deductions.

DEFINITION 5.3. A B-set of formulas v is *closed* with respect to the n-ary inferential rule r provided that,

$$v(r_{\text{syn}}(\alpha_1, \ldots, \alpha_n)) \geq_{\text{k}} r_{\text{sem}}(v(\alpha_1), \ldots, v(\alpha_n))$$

for every $\alpha_1, \ldots, \alpha_n$. We say that v is a *deduction closed* theory (in brief a *dc-theory*) of (IR, la) if v is closed with respect to all the inferential rules in IR and it k-contains the B-subset of logical axioms. A *dc*-theory v is *consistent* if it is different from $v_{\top}$.

Trivially, every B-subset of formulas is closed with respect to the two structural rules.

THEOREM 5.4. *Let v be a valuation, then v is a fixed point of D if and only if v is a dc-theory.*

PROOF. Assume that v is a *dc*-theory. To prove that v is a fixed point for D, we prove, by induction on the length of the proofs, that for every formula α and for every proof π of α

$$I(\pi, v) \leq_{\text{k}} v(\alpha) \qquad (\star)$$

In the case $n = 1$ $(\star)$ is trivial. Consider the case $n \neq 1$ and, by induction hypothesis, that $(\star)$ is satisfied by all the proofs whose length is less than n. Then again in the case α is assumed as a logical axiom or a hypothesis $(\star)$ holds true. Otherwise, there is an inference rule $r = (r_{\text{syn}}, r_{\text{sem}})$ such that $\alpha = r_{\text{syn}}(\alpha_{i(1)}, \dots, \alpha_{i(m)})$ with $1 \leq i(1) < n, \dots, 1 \leq i(m) < n$ and $I(\pi, v) = r_{\text{sem}}(I(\pi(i(1)), v), \dots, I(\pi(i(m)), v))$. Then by the closure of v, by induction hypothesis and the monotony of r_{sem}, we have that

$$\begin{aligned} v(\alpha) &= v(r_{\text{syn}}(\pi_{i(1)}, \dots, \pi_{i(m)})) \\ &\geq_{\mathrm{k}} r_{\text{sem}}(v(\alpha_1), \dots, v(\alpha_n)) \\ &\geq_{\mathrm{k}} r_{\text{sem}}(I(\pi(i(1)), v), \dots, I(\pi(i(m)), v)) = I(\pi, v). \end{aligned}$$

Conversely, assume that v is a fixed point of D. Then $v(\alpha) = \mathrm{Sup}_{\mathrm{k}}\{I(\pi, v) \in B : \pi \text{ is a proof of } \alpha\}$ and therefore $v(\alpha) \geq_{\mathrm{k}} I(\pi, v)$ for every proof π of α. By assuming that π is the proof of length 1 consisting in assuming α as a logical axiom, we obtain $v(\alpha) \geq_{\mathrm{k}} I(\pi, v) = la(\alpha)$. Then v k-contains *la*. Let r be an n-ary inference rule, then to prove that v is closed with respect to r, given $\alpha_1, \dots, \alpha_n$ we consider the proof π obtained by assuming $\alpha_1, \dots, \alpha_n$ as hypotheses and by applying the rule r. Such a proof proves the formula $\alpha = r_{\text{syn}}(\alpha_1, \dots, \alpha_n)$ and therefore

$$v(r_{\text{syn}}(\alpha_1, \dots, \alpha_n)) = v(\alpha) \geq_{\mathrm{k}} I(\pi, v) = r_{\text{sem}}(v(\alpha_1), \dots, v(\alpha_n)).$$

Thus v is closed with respect to (IR, la). ⊣

Definition 5.5. Given a semantics $\mathcal{M}$, a system *la* of logical axioms is *sound* if $la \subseteq_{\mathrm{k}} m$ for every $m \in \mathcal{M}$, i.e. $la \subseteq_{\mathrm{k}} Tau$. An inference rule is *sound* if, for every $\alpha_1, \dots, \alpha_n$,

$$m(r_{\text{syn}}(\alpha_1, \dots, \alpha_n)) \geq_{\mathrm{k}} r_{\text{sem}}(m(\alpha_1), \dots, m(\alpha_n)).$$

The proof of the following proposition is evident.

Proposition 5.6. *A deduction apparatus* (IR, la) *is sound with respect to a semantics* $\mathcal{M}$ *if and only if the inference rules and the logical axioms are sound with respect to* $\mathcal{M}$.

The following simple lemmas give useful criterions to obtain the completeness.

Lemma 5.7 (Bitangency principle). *Let* $\boldsymbol{B}$ *be a bilattice satisfying the decomposition property and assume that the following conditions hold true for every consistent dc-theory* v *and every formula* α,

(i) *there is a model* m_α *of* v *such that*

$$m_\alpha(\alpha) \wedge_{\mathrm{k}} True = v(\alpha) \wedge_{\mathrm{k}} True\,,$$

(ii) *there is a model m^α of v such that*

$$m^\alpha(\alpha) \wedge_{\mathrm{k}} \mathit{False} = v(\alpha) \wedge_{\mathrm{k}} \mathit{False}\,.$$

Then the deduction apparatus is complete.

Proof. We will prove that every fixed point v of D is a fixed point of L_c. It is not restrictive to assume that $v \neq v_\top$. Now in account of the hypotheses, for every formula α,

$$\begin{aligned} L_c(v)(\alpha) \wedge_{\mathrm{k}} \mathit{True} &= (\mathrm{Inf}_{\mathrm{k}}\{m(\alpha) \in B : m \vDash v\}) \wedge_{\mathrm{k}} \mathit{True} \\ &\leq_{\mathrm{k}} m_\alpha(\alpha) \wedge_{\mathrm{k}} \mathit{True} = v(\alpha) \wedge_{\mathrm{k}} \mathit{True} \end{aligned}$$

and

$$\begin{aligned} L_c(v)(\alpha) \wedge_{\mathrm{k}} \mathit{False} &= (\mathrm{Inf}_{\mathrm{k}}\{m(\alpha) \in B : m \vDash v\}) \wedge_{\mathrm{k}} \mathit{False} \\ &\leq_{\mathrm{k}} m^\alpha(\alpha) \wedge_{\mathrm{k}} \mathit{False} = v(\alpha) \wedge_{\mathrm{k}} \mathit{False}. \end{aligned}$$

Consequently,

$$\begin{aligned} L_c(v)(\alpha) &= (L_c(v)(\alpha) \wedge_{\mathrm{k}} \mathit{True}) \vee_{\mathrm{k}} (L_c(v)(\alpha) \wedge_{\mathrm{k}} \mathit{False}) \\ &\leq_{\mathrm{k}} (v(\alpha) \wedge_{\mathrm{k}} \mathit{True})) \vee_{\mathrm{k}} (v(\alpha) \wedge_{\mathrm{k}} \mathit{False}) = v(\alpha). \end{aligned}$$

Since L_c is a closure operator, it is sufficient to claim that v is a fixed point of L_c. Thus, the deduction apparatus is complete. $\dashv$

Assume that $\mathcal{M}$ is the canonical semantics associated with a Pavelka's semantics $\boldsymbol{M}$. Then in the case $\boldsymbol{B} = \boldsymbol{I}(\boldsymbol{V})$ we have that $\mathcal{M}$ satisfies the bitangency principle if and only if for every consistent *dc*-theory v and every formula α, there are two models m_α and m^α in $\boldsymbol{M}$ of v such that $m_\alpha(\alpha) = v^+(\alpha)$ and $m^\alpha(\alpha) = v^-(\alpha)$, respectively. Likewise, in the case $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{V})$, $\mathcal{M}$ satisfies the bitangency principle provided that for every consistent *dc*-theory v and every formula α, there are two models m_α and m^α in $\boldsymbol{M}$ of v such that $m_\alpha(\alpha) = v_+(\alpha)$ and $m^\alpha(\alpha) = v_-(\alpha)$.

There is a large class of semantics satisfying the condition $m(\neg\alpha) = {\sim}\, m(\alpha)$ for every model m. It is evident that in this case it is convenient to consider the following two rules:

$$\left\langle \frac{\neg\alpha}{\alpha} \mid \frac{b}{\sim b} \right\rangle \quad (\neg\text{-elimination}) \qquad \left\langle \frac{\alpha}{\neg\alpha} \mid \frac{b}{\sim b} \right\rangle \quad (\neg\text{-introduction})$$

An inferential apparatus containing these rules is *balanced*. Again, we call *balanced* a B-subset of formulas v closed with respect to these rules, i.e. if $v(\neg\alpha) = \sim v(\alpha)$ for every $\alpha \in F$. A semantics $\mathcal{M}$ is *balanced* if all its models are balanced.

LEMMA 5.8 (Tangency Principle). *Consider a balanced semantics and a balanced deduction apparatus. Also, assume that for every consistent dc-theory v and $\alpha \in F$ there is a model m_α of v such that*

$$m_\alpha(\alpha) \wedge_{\mathrm{k}} \mathit{True} = v(\alpha) \wedge_{\mathrm{k}} \mathit{True}.$$

Then the deduction apparatus is complete.

PROOF. By hypothesis, given a formula α there is a model $m_{\neg\alpha}$ such that $m_{\neg\alpha}(\neg\alpha) \wedge_{\mathrm{k}} \mathit{True} = v(\neg\alpha) \wedge_{\mathrm{k}} \mathit{True}$. Consequently, if we set $m^\alpha = m_{\neg\alpha}$

$$\begin{aligned} M^\alpha(\alpha) \wedge_{\mathrm{k}} \mathit{False} &= \sim(\sim m^\alpha(\alpha) \wedge_{\mathrm{k}} \sim \mathit{False}) = \sim(m^\alpha(\neg\alpha) \wedge_{\mathrm{k}} \mathit{True}) \\ &= \sim(v(\neg\alpha) \wedge_{\mathrm{k}} \mathit{True}) = (\sim v(\neg\alpha)) \wedge_{\mathrm{k}} (\sim \mathit{True}) \\ &= v(\alpha) \wedge_{\mathrm{k}} \mathit{False} \end{aligned}$$

and condition (ii) of Lemma 5.7 is satisfied. ⊣

6. Boolean logic and Kripke bilattices

To test our formalisms, we consider a logic related with the Kripke bilattice logics proposed by Ginsberg in [8]. Namely, given a nonempty set W whose elements we call *worlds*, we consider as a valuation structure the Boolean algebra $\boldsymbol{V} = (P(W), \subseteq, \cap, -, \varnothing, W)$ where $P(W)$ is the class of all the subsets of W. Also, we consider the square k-extension $\boldsymbol{B_W} = \boldsymbol{B}(\boldsymbol{V})$ of $\boldsymbol{V}$. This means that $\boldsymbol{B_W}$ is defined in $P(W)\times P(W)$, by setting

$$\begin{gathered} (X, Y) \leq_{\mathrm{k}} (X', Y') \Leftrightarrow X \subseteq X' \text{ and } Y \subseteq Y'; \\ (X, Y) \leq_{\mathrm{t}} (X', Y') \Leftrightarrow X \subseteq X' \text{ and } Y \supseteq Y'; \\ \sim(X, Y) = (Y, X); \\ (X, Y) \otimes_{\mathrm{t}} (X', Y') = (X \cap X', Y \cup Y'); \\ \perp = (\varnothing, \varnothing); \\ \top = (W, W); \\ \mathit{False} = (\varnothing, W); \\ \mathit{True} = (W, \varnothing); \\ i(X) = (X, -X). \end{gathered}$$

We consider a first order language with two logical connectives $\neg$ and $\wedge$ corresponding to the operations $\sim$ and $\otimes_{\mathrm{t}}$, respectively, and we denote by F the set of formulas of such a language. The intended meaning is that we assign to a formula α, the value (X, Y) provided that

- X is the set of worlds in which the available information says that α is true
- Y is the set of worlds in which the available information says that α is false.

Moreover, for every $X \in P(W)$ and $(A, B) \in B_W$,

$$X \vDash^* (A, B) \text{ provided that } A \subseteq X \text{ and } B \subseteq -X$$

and

$$Sat(\boldsymbol{B_W}) = \{(A, B) \in B_W : A \cap B = \varnothing\}.$$

Definition 6.1. We call *Boolean semantics* the set $\boldsymbol{M}$ of truth-functional valuations in the Boolean algebra $(P(W), \cap, -, \varnothing, W)$, i.e., the set of maps $m' \colon F \longrightarrow P(W)$ such that for every $\alpha, \beta \in F$

$$m'(\alpha \wedge \beta) = m'(\alpha) \cap m'(\alpha); \quad m'(\neg\alpha) = -m'(\alpha).$$

The *Kripke bilattice semantics* is the canonical B-semantics $\mathcal{M} = \{i \circ m' \in B : m' \in \boldsymbol{M}\}$ associated with $\boldsymbol{M}$.

The intended meaning of a model $m \in \mathcal{M}$ is that, given $\alpha \in F$, the value $m(\alpha) = (X, -X)$ is defined by the set X of worlds in which α is true and the set $-X$ of worlds in which α is false.

The following properties are well known.

Proposition 6.2. *Let m' be an element in $\boldsymbol{M}$ and assume that α and α' are logically equivalent in classical propositional calculus, then $m(\alpha) = m(\alpha')$. Moreover, $m(\alpha) = W$ for every tautology α and $m(\alpha) = \varnothing$ for every contradiction α. Finally, $m(\alpha) \cap m(\alpha \to \beta) \subseteq m(\beta)$. Consequently, $L_c(v)$ is compatible with the logical equivalence and, for every $m \in \mathcal{M}$,*

- *$m(\alpha) = (W, \varnothing) = True$ for every tautology α*
- *$m(\alpha) = (\varnothing, W) = False$ for every contradiction α*
- $\mathrm{Tau}(\alpha) = (W, \varnothing)$ *if α is a tautology*
- $\mathrm{Tau}(\alpha) = (\varnothing, W)$ *if α is a contradiction*
- $\mathrm{Tau}(\alpha) = (\varnothing, \varnothing)$ *if α is neither a tautology or a contradiction.*

To individuate a suitable inferential apparatus for this semantics, at first we give a “symmetric” version of the usual deduction apparatus in classical propositional calculus. Indeed, we denote by $\alpha \to_{\mathrm{t}} \beta$ the formula $\neg(\alpha \wedge \neg\beta)$, i.e. the usual implication and by $\alpha \rightarrow_f \beta$ the formula $\beta \wedge \neg\alpha$ which is classically equivalent to $\neg(\neg\alpha \rightarrow_{\mathrm{t}} \neg\beta)$. In correspondence, we define two rules. The *positive Modus Ponens*, in brief MP^+, giving β from α and $\alpha \rightarrow_{\mathrm{t}} \beta$ which is sound in a positive sense (i.e. if α and $\alpha \rightarrow_{\mathrm{t}} \beta$ are true, then β is true). The *negative Modus Ponens*, in brief MP^- giving β from α and $\alpha \rightarrow_f \beta$ and which is sound in a negative sense (i.e. if α and $\alpha \rightarrow_f \beta$ are false, then β is false).

Definition 6.3. Let LA be a set of logical axioms of the classical propositional calculus and denote by $\neg LA$ the set $\{\neg\alpha \in F : \alpha \in LA\}$. Then we say that a set T of formulas is a *positive theory* or that T *is closed with respect to positive proofs* provided that T contains LA and it is closed with respect to MP^+. We say that T is a *negative theory* or that T *is closed with respect to negative proofs* provided that T contains $\neg LA$ and it is closed with respect to MP^-.

Passing to our bilattices-based logic, we call *graded positive Modus Ponens* (in brief GMP^+) the rule defined by setting

$$\left\langle \frac{\alpha, \alpha \rightarrow_{\mathrm{t}} \beta}{\beta} \mid \frac{(A_+, A_-), (I_+, I_-}{(A_+, A_-) \wedge_{\mathrm{k}} (I_+, I_-)} \right\rangle$$

We call *negative Modus Ponens* (in brief MP^-), the rule defined by setting

$$\left\langle \frac{\alpha, \alpha \rightarrow_f \beta}{\beta} \mid \frac{(A_+, A_-), (I_+, I_-}{(A_+, A_-) \wedge_{\mathrm{k}} (I_+, I_-)} \right\rangle$$

Definition 6.4. We call *Kripke deduction system*, in brief K-*system*, the deduction system in B_W whose B_W-set *la* of logical axioms is defined by setting

$$la(\alpha) = \begin{cases} (W, \varnothing) & \text{if } \alpha \in LA \\ (\varnothing, W) & \text{if } \alpha \in \neg LA \\ (\varnothing, \varnothing) & \text{otherwise,} \end{cases}$$

and whose rules are GMP^+, GMP^-, the $\neg$-*elimination* rule, the $\neg$-*introduction* rule, and the inconsistency rule.

It is intended that the K-system contains the fusion rule and the weakening rule. Notice that this set of rules is not reduced to a minimum. As an example, we can consider the *reduced K-system* obtained by skipping out the rule MP^- and by substituting MP^+ with the following *weakened* MP^+ rule, in brief wMP^+, working only on the positive information

$$\left\langle \frac{\alpha, \alpha \rightarrow_{\mathrm{t}} \beta}{\beta} \mid \frac{(A_+, A_-), (I_+, I_-)}{(A_+ \cap I_+, \varnothing)} \right\rangle$$

Proposition 6.5. *The reduced K-system is equivalent to the K-system.*

Proof. It is evident that if α is a tautology of classical propositional calculus, then in the reduced system we can prove α at degree $(W, \varnothing)$. Then, to show that wMP^+ entails GMP^+, we can consider the following proof:

α	(A_+, A_-)	hypothesis
$\alpha \rightarrow_{\mathrm{t}} \beta$	(I_+, I_-)	hypothesis
$\neg(\alpha \rightarrow_{\mathrm{t}} \beta)$	(I_-, I_+)	$\neg$-introduction rule
$\neg(\alpha \rightarrow_{\mathrm{t}} \beta) \rightarrow_{\mathrm{t}} \neg\beta$	$(W, \varnothing)$	tautology
$\neg\beta$	$(I_-, \varnothing)$	wMP^+
β	$(\varnothing, I_-)$	$\neg$-elimination
β	$(A_+ \cap I_+, \varnothing)$	wMP^+ from the hypotheses
β	$(A_+ \cap I_+, I_-)$	fusion rule
β	$(A_+ \cap I_+, A_- \cap I_-)$	weakening rule.

To obtain MP^- from MP^+, we consider the following proof

α	(A_+, A_-)	hypothesis
$\alpha \rightarrow_f \beta$	(I_+, I_-)	hypothesis
$\neg\alpha \rightarrow_{\mathrm{t}} \neg\beta$	(I_-, I_+)	$\neg$-elimination
$\neg\alpha$	(A_-, A_+)	$\neg$-introduction
$\neg\beta$	$(A_- \cap I_-, A_+ \cap I_+)$	MP^+
β	$(A_+ \cap I_+, A_- \cap I_-)$	$\neg$-elimination rule. $\dashv$

In spite of this proposition, in our opinion the whole K-system is interesting by its symmetry. The following proposition is evident.

Proposition 6.6. *Given a valuation v, the following equivalences hold true*

(a) $v \supseteq_{\mathrm{k}} la \Leftrightarrow v_+(\alpha) = W$ *and* $v_-(\neg\alpha) = W$ *for every* $\alpha \in LA$,
(b) *v is closed with respect to MP^+* $\Leftrightarrow v_+(\beta) \supseteq v_+(\alpha) \cap v_+(\alpha \rightarrow_{\mathrm{t}} \beta)$ *for every α and β,*
(c) *v is closed with respect to the $\neg$-introduction and the $\neg$-elimination rules* $\Leftrightarrow v_+(\alpha) = v_-(\neg\alpha)$ *and* $v_-(\alpha) = v_+(\neg\alpha)$ *for every α.*
(d) *v is closed with respect to the inconsistency rule* $\Leftrightarrow$ *either v is pointwise satisfiable or $v = v_\top$.*

7. A bilattice isomorphic with $\boldsymbol{B_W}$

In order to find a completeness theorem relating $\mathcal{M}$ with the proposed K-system, it is useful to introduce the following bilattice.

Definition 7.1. Let $P(F)$ be the Boolean algebra of all the subsets of F and denote by $\boldsymbol{B_F}$ the associated product bilattice $\boldsymbol{B}(P(F))$. Then we call *formulas based bilattice* the bilattice B_F^W obtained as the direct power of $\boldsymbol{B_F}$ with index set W. We call W-*valuations* the elements of such a bilattice.

Then a W-valuation $U\colon W \longrightarrow B_F$ is defined by a pair (U_+, U_-) of functions from W into $P(F)$ whose intended interpretation is that, for every world w,

- $U_+(w)$ is the set of formulas the available information suggests to be true in w
- $U_-(w)$ is the set of formulas the available information suggests to be false in w.

Theorem 7.2. *The bilattices B_W^F and B_F^W are isomorphic. Namely, define the map $H\colon B_W^F \longrightarrow B_F^W$ by setting, for every $v \in B_W^F$,*

$$H(v)(w) = (T^v(w), F^v(w)),$$

where

$$T^v(w) = \{\alpha \in F : w \in v_+(\alpha)\} \text{ and } F^v(w) = \{\alpha \in F : w \in v_-(\alpha)\}.$$

Then H is an isomorphism from B_W^F and B_F^W whose inverse is the function $K\colon B_F^W \longrightarrow B_W^F$ such that, for every $U \in B_F^W$ and $\alpha \in F$,

$$K(U)(\alpha) = (\{w \in W : \alpha \in U_+(w)\}, \{w \in W : \alpha \in U_-(w)\}).$$

Proof. It is immediate that H and K are both one-to-one and $H^{-1} = K$. Moreover,

$$\begin{aligned}
u \leq_{\mathrm{k}} v &\iff u(\alpha) \leq_{\mathrm{k}} v(\alpha), \text{ for every } \alpha \in F \\
&\iff u_+(\alpha) \subseteq v_+(\alpha) \text{ and } u_-(\alpha) \subseteq v_-(\alpha), \text{ for every } \alpha \in F \\
&\iff \{\alpha \in F : w \in u_+(\alpha)\} \subseteq \{\alpha \in F : w \in v_+(\alpha)\} \text{ and} \\
&\qquad \{\alpha \in F : w \in u_-(\alpha)\} \subseteq \{\alpha \in F : w \in v_-(\alpha)\}, \text{ for every } w \in W \\
&\iff H(u)(w) \leq_{\mathrm{k}} H(v)(w), \text{ for every } w \in W \\
&\iff H(u) \leq_{\mathrm{k}} H(v).
\end{aligned}$$

and

$$
\begin{aligned}
u \leq_{\mathrm{t}} v &\iff u(\alpha) \leq_{\mathrm{t}} v(\alpha), \text{ for every } \alpha \in F \\
&\iff u_+(\alpha) \subseteq v_+(\alpha) \text{ and } u_-(\alpha) \supseteq v_-(\alpha), \text{ for every } \alpha \in F \\
&\iff \{\alpha \in F : w \in u_+(\alpha)\} \subseteq \{\alpha \in F : w \in v_+(\alpha)\} \text{ and} \\
&\qquad \{\alpha \in F : w \in u_-(\alpha)\} \supseteq \{\alpha \in F : w \in v_-(\alpha)\}, \text{ for every } w \in W \\
&\iff H(u)(w) \leq_{\mathrm{t}} H(v)(w), \text{ for every } w \in W \\
&\iff H(u) \leq_{\mathrm{t}} H(v).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
H(\sim v)(w) &= H((v_-, v_+))(w) \\
&= (\{\alpha \in F : w \in v_-(\alpha)\}, \{\alpha \in F : w \in v_+(\alpha)\}) \\
&= \sim(\{\alpha \in F : w \in v_+(\alpha)\}, \{\alpha \in F : w \in v_-(\alpha)\}) \\
&= \sim H(v)(w). \qquad \dashv
\end{aligned}
$$

Observe that, $H(v_\top)$ is the map constantly equal to $(\varnothing, \varnothing)$, $H(v_\top)$ the map constantly equal to (F, F) and $H(la)$ the map constantly equal to $(LA, \neg LA)$.

Definition 7.3. Let $U = (U_+, U_-)$ be an element in B_F^W, then we say that

- U is *pointwise satisfiable* if, for every $w \in W$, $U_+(w) \cap U_-(w) = \varnothing$
- U is *closed with respect to MP^+* if, for every $w \in W$, $U_+(w)$ is closed with respect to MP^+
- U is *closed with respect to MP^-* if, for every $w \in W$, $U_-(w)$ is closed with respect to MP^-
- U is *balanced* if, for every $w \in W$,

$$\alpha \in U_+(w) \Leftrightarrow \neg\alpha \in U_-(w) \text{ and } \alpha \in U_-(w) \Leftrightarrow \neg\alpha \in U_+(w).$$

- U is *complete* if, for every $w \in W$, $U_+(w)$ is a complete theory and $U_-(w) = -U_+(w)$.

Proposition 7.4. *Given $v \in B_W^F$,*

(i) *$v \supseteq_{\mathrm{k}} la \iff T^v(w) \supseteq LA$ and $F^v(w) \supseteq \neg LA$.*
(ii) *v is closed with respect to $GMP^+ \iff T^v(w)$ is closed with respect to MP^+ for every $w \in W$.*
(iii) *v is closed with respect to $GMP^- \iff F^v(w)$ is closed with respect to MP^- for every $w \in W$.*

(iv) *v is balanced $\iff$ v is closed with respect to the $\neg$-elimination and $\neg$-introduction rules $\iff$ $H(v)$ is balanced.*

(v) *v is closed with respect to the inconsistency rule $\iff$ either $H(v)$ is pointwise satisfiable or $H(v)$ is constantly equal with (F, F).*

PROOF. Equivalences i), ii), iii) and v) are all trivial. To prove iv), assume that v is balanced and therefore that, for every $\alpha, v_+(\alpha) = v_-(\neg\alpha)$ and $v_-(\alpha) = v_+(\neg\alpha)$. Then

$$\alpha \in T^v(w) \iff w \in v_+(\alpha) \iff w \in v_-(\neg\alpha) \iff \neg\alpha \in F^v(w)$$

and

$$\alpha \in F^v(w) \iff w \in v_-(\alpha) \iff w \in v_+(\neg\alpha) \iff \neg\alpha \in T^v(w)$$

and this proves that $H(v)$ is balanced. Conversely, assume that $H(v)$ is balanced, then

$$w \in v_+(\alpha) \iff \alpha \in T^v(w) \iff \neg\alpha \in F^v(w) \iff w \in v_-(\neg\alpha)$$

and

$$w \in v_-(\alpha) \iff \alpha \in F^v(w) \iff \neg\alpha \in T^v(w) \iff w \in v_+(\neg\alpha)$$

and this proves that v is balanced. $\dashv$

Now, we are able to characterize the models of our logic as the families of complete positive theories of classical logic.

PROPOSITION 7.5. *Assume that $m \in \mathcal{M}$. Then $H(m)$ is a complete W-valuation. Conversely, if U is a complete W-valuation then $K(U) \in \mathcal{M}$.*

PROOF. Assume that $m \in \mathcal{M}$, then it is immediate that, for every $w \in W$, $F^m(w) = -T^m(w)$. Moreover, in accordance with Proposition 6.2, m is closed with respect to GMP^+ and $m \supseteq_k la$. This entails that $T^m(w)$ contains LA and it is closed with respect to MP^+ and therefore that $T^m(w)$ is a positive theory. To prove the completeness of $T^m(w)$, observe that,

$$\alpha \in T^m(w) \iff w \in m(\alpha) \iff w \notin m(\neg\alpha) \iff \neg\alpha \notin T^m(w).$$

Conversely, assume that U is a complete W-valuation, i.e. that for every $w \in W, U_+(w)$ is a complete theory and that $U_-(w) = -U_+(w)$. Define

$m'\colon F \longrightarrow P(W)$ by setting $m'(\alpha) = \{w \in W : \alpha \in U_+(w)\}$. Then m' is truth-functional. Indeed, since $U_+(w)$ is closed under deductions,

$$w \in m'(\gamma \wedge \beta) \iff \gamma \wedge \beta \in U_+(w) \iff \gamma \in U_+(w) \text{ and } \beta \in U_+(w)$$
$$\iff w \in m'(\gamma) \text{ and } w \in m'(\beta) \iff w \in m'(\gamma) \cap m'(\beta)$$

and this proves that $m'(\gamma \wedge \beta) = m'(\gamma) \cap m(\beta)$. Moreover, since $U_+(w)$ is complete,

$$w \in m'(\neg\gamma) \iff \neg\gamma \in U_+(w) \iff \gamma \notin U_+(w) \iff w \notin m'(\gamma)$$

and this proves that $m'(\neg\gamma) = -m'(\gamma)$.

On the other hand, since for every $w \in W$,

$$w \in -m'(\alpha) \iff \alpha \notin U_+(w) \iff \alpha \in U_-(w),$$

we have also

$$\begin{aligned} m(\alpha) &= (m'(\alpha), -m'(\alpha)) \\ &= (\{w \in W : \alpha \in U_+(w)\}, \{w \in W : \alpha \in U_-(w)\}) = K(U)(\alpha). \dashv \end{aligned}$$

Corollary 7.6. *Given a W-valuation v, a model m of v exists if and only if there is a family $(T_w)_{w\in W}$ of positive complete theories such that $T^v(w) \subseteq T_w \subseteq -F^v(w)$. This model is obtained by setting, for every formula α, $m(\alpha) = (\{w \in W : \alpha \in T_w\}, \{w \in W : \alpha \notin T_w\})$.*

Proof. Let m be a model of v, then, $H(m)$ is a complete W-valuation and, since $m \geq_{\mathrm{k}} v$, we have that $H(m) \geq_{\mathrm{k}} H(v)$. Then $(H_+(m)(w))_{w\in W}$ is a family of positive complete theories such that $T^v(w) \subseteq H_+(\underline{m})(w) \subseteq -F^v(w)$.

Conversely, let $(T_w)_{w\in W}$ be a family of positive complete theories such that $T^v(w) \subseteq T_w \subseteq -F^v(w)$. Then we can consider the W-valuation U obtained by setting $U_+(w) = T_w$ and $U_-(w) = -T_w$. By definition U is complete and therefore by setting $m'(\alpha) = \{w \in W : \alpha \in T_w\}$ we obtain an element m of $\mathcal{M}$ such that $m = K(U)$. Since by hypothesis $T^v(w) \subseteq T_w$ and $-T_w \supseteq F^v(w)$, we have $m = K(U) \geq_{\mathrm{k}} v$, i.e. $\underline{m} \models v$. $\dashv$

Proposition 7.7. *Given $v \in B^F_W$, $v \neq v_\top$, the following are equivalent:*

(i) *v is a dc-theory.*

(ii) *$T^v(w)$ is a consistent positive theory for every $w \in W$ and $H(v)$ is balanced.*

(iii) *$F^v(w)$ is a consistent negative theory for every $w \in W$ and $H(v)$ is balanced.*

Proof. The implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are evident. To prove that ii) $\Rightarrow$ i) we observe that $H(v)$ is pointwise satisfiable and therefore v is closed with respect to the inconsistency rule. Indeed, assume that there are $\alpha \in F$ and $w \in W$ such that $\alpha \in T^v(w) \cap F^v(w)$. Then, since $H(v)$ is balanced, $\neg\alpha \in T^v(w)$ and therefore $T^v(w)$ is inconsistent. This contradict the hypothesis $T^v(w) \neq F$. To prove the closure of v with respect to MP^- we prove that $F^v(w)$ is closed with respect to MP^-. Now if α and $\alpha \rightarrow_f \beta$ are in $F^v(w)$, then $\neg\alpha$ and $\neg(\beta \wedge \neg\alpha) \in T^v(w)$. Since $T^v(w)$ is a positive theory, this means that $\neg\alpha$ and $\neg\alpha \rightarrow_t \neg\beta \in T^v(w)$ and therefore $\neg\beta \in T^v(w)$. Thus, since $H(v)$ is balanced, we can conclude that $\beta \in F^v(w)$. In a similar way one proves that (iii) $\Rightarrow$ (i). $\dashv$

Taking in account the results in Section 5 we are ready to prove a completeness theorem.

Theorem 7.8. *The K-system is complete with respect to the truth-functional semantics $\mathcal{M}$. So, the Kripke truth-functional semantics and the Kripke deduction system define a bilattice-based fuzzy logic.*

Proof. Since both the elements in $\mathcal{M}$ and the fixed points of D are balanced, it is possible to apply Lemma 5.8. Let v be a consistent *dc*-theory, then, since v is a fixed point of D, for every $w \in W$, $T^v(w)$ is a consistent positive theory such that $T^v(w) \supseteq \neg F^v(w)$ and $F^v(w) \supseteq \neg T^v(w)$. Define $U_\alpha = (U_+^\alpha, U_-^\alpha) \in B_F^W$ by setting:

- $U_+^\alpha(w)$ equal to any complete positive theory extending $T^v(w)$ in the case $\alpha \in T^v(w)$.
- $U_+^\alpha(w)$ equal to any complete positive theory extending $T^v(w) \cup \{\neg\alpha\}$ in the case $\alpha \notin T^v(w)$.
- $U_-^\alpha(w) = -U_+^\alpha(w)$.

Then $U_+^\alpha(w)$ is a complete extension of $T^v(w)$ such that, trivially,

$$\alpha \in U_+^\alpha(w) \iff \alpha \in T^v(w).$$

Now, by Corollary 7.6, U_α is associated with an element $m_\alpha \in \mathcal{M}$ where $m_\alpha(\beta) = (\{w \in W : \beta \in U_+^\alpha(w)\}, \{w \in W : \beta \notin U_+^\alpha(w)\})$ for every formula β. To prove that m_α is a model of v, we observe that, by definition, $U_+^\alpha(w) \supseteq T^v(w)$. To prove that $U_+^\alpha(w) \subseteq -F^v(w)$ we observe that, for every $\beta \in U_+^\alpha(w)$, $\neg\beta \notin U_+^\alpha(w)$ and therefore $\neg\beta \notin T^v(w)$. So $\beta \notin F^v(w)$. This proves that m_α is a model of v. To prove that $m_\alpha(\alpha) = v_+(\alpha)$, we observe that

$$w \in m_\alpha(\alpha) \Leftrightarrow \alpha \in U_+^\alpha(w) \Leftrightarrow \alpha \in T^v(w) \Leftrightarrow w \in v_+(\alpha). \qquad \dashv$$

Observe that the fixed points of D satisfy some natural algebraic properties. Indeed, denote by $F/_\equiv$ the Lindebaum algebra of the propositional calculus and for every valuation v set

$$[T^v(w)] = \{[\alpha] \in F/_\equiv : \alpha \in T^v(w)\}$$

and

$$[F^v(w)] = \{[\alpha] \in F/_\equiv : \alpha \in F^v(w)\}.$$

Then if v is a fixed point of D, $[T_w^v]$ is a proper filter and $[F_w^v]$ the corresponding dual ideal in $F/_\equiv$. If $v = m \in \mathcal{M}$, then $[T_w^v]$ is maximal and $[F_w^v]$ is its complement.

8. Final remarks and future works

We conclude the paper by listing some peculiar features of the proposed approach to multi-valued logic.

1. As it is evident in considering square k-extensions, "Truth" and "False" have a symmetric role in the deduction apparatus while the usual deduction apparatus move from the truth of the axioms to the truth of the theorems.

2. It is possible to represent both inconsistency and contradiction.

3. Designed values are not required. This is important since, in our opinion, it is questionable to accept the "cut" implicit in this notion. Indeed, if we assume that multi-valued logic is a way to represent the vagueness phenomenon, then no cut is justified.

4. There is a notion of "*approximate reasoning*". Differently from the classical logic different proofs of the same formula can give different pieces of information on this formula (giving two proofs can be better than giving one proof of the same formula).

5. The set of tautologies is substituted with the B-subset of tautologies Tau. The value Tau(α) represents the of a-priori information on α arising from the structure of α. In accordance, every formula is a tautology (and a contradiction) at a given degree. For example, if we

consider the valuation structure $([0,1], \leq, \wedge, 1-x, 0, 1)$, then there is no formula assuming either constantly the value 1. So, no tautology exists in the classical sense. On the other hand, if we refer to the related interval k-extension, we have that $\mathrm{Tau}(p \vee \neg p) = [0.5, 1]$, an information on $p \vee \neg p$.

6. In all the main multi-valued logics the deduction operator is not compact and it is not computable. In particular, the set of tautologies is not recursively enumerable [21]. Now, we can extend to the B-subsets of formulas the definitions proposed in [13] and therefore to give very reasonable definitions of compactness and computability for bilattice based logic. With respect to these definitions the deduction operator is compact and computable. In particular, the B-subset of tautologies in axiomatizable logic is recursively enumerable.

There are also several difficulties and open questions. For example, the passage from the lower-constraints approach proposed by Pavelka to the bilattice-based approach proposed in this paper increases considerably the possibility of representing information in the inferential processes for multi-valued logic. Nevertheless, the question arises whether this is sufficient or we have to look for further extensions. In its very interesting comments, the referee of this paper observed that the proposed bilattices are not able to represent several kinds of information on the truth value of the formulas. For example, consider the interval bilattice associated with the 5-element MV-algebra $\boldsymbol{V}$ for 5-valued Łukasiewicz logic. Moreover, observe that the formula $(\neg\alpha) \to \alpha$ can only have the value 0 or 1/2 or 1, but neither 1/4 nor 3/4. So we have that $\mathrm{Tau}((\neg\alpha) \to \alpha) = [0, 1] = V$ (no information) while it should be natural to put $\mathrm{Tau}((\neg\alpha) \to \alpha)$ equal to the non-interval $\{0, 1/2, 1\}$. Then, a natural candidate to substitute the class of intervals should be the whole power set $P(V)$ together with the dual of the inclusion as a knowledge order (see [9]). Unfortunately, there are at least two difficulties in this choice. The first one is to define a truth-order in $P(V)$. The latter is that in the case V infinite the cardinality of $P(V)$ is an obstacle for the effectiveness and the representation of the deduction processes.

Regarding to future works, since this is an exploratory paper aiming to give a new basis to multi-valued logic, it is evident that there are countless open questions. The first one is to find suitable completeness theorems (in the sense given by Definition 4.10) for the "translations" of the main multi-valued logics in our framework. Moreover, should be interesting to apply the proposed formalisms for logics which are possibilistic or probabilistic in nature. For example, we can consider a logic whose semantics is defined by the class of *finitely additive probabilities* in F, i.e. that class $\boldsymbol{M}$ of the maps $p\colon F \longrightarrow [0,1]$ compatible with the logical equivalence and such that

i)	$p(\alpha) = 1$	for every tautology α,
ii)	$p(\alpha) = 0$	for every contradiction α
iii)	$p(\alpha \vee \beta) = p(\alpha) + p(\beta) - p(\alpha \wedge \beta)$	for every $\alpha, \beta \in F$.

The resulting logic is related with the notions of lower envelope and upper envelope (see [10, 11] and [12]). In a similar way it is possible to define a bilattice semantics related with the necessity and possibility measures (see [1, 5] and [12]).

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Daniele Genito, Giangiacomo Gerla
Dipartimento di Matematica ed Informatica
Università di Salerno
via Ponte don Melillo 84084 Fisciano (SA)
gerla@unisa.it