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Mirosława Kołowska-Gawiejnowicz

POWERSET RESIDUATED ALGEBRAS

Abstract. We present an algebraic approach to canonical embeddings of arbitrary residuated algebras into powerset residuated algebras. We propose some construction of powerset residuated algebras and prove a representation theorem for symmetric residuated algebras.

Keywords: residuated algebra; symmetric residuated algebra; powerset residuated algebra; canonical embedding

1. Introduction

Residuated algebras are models of some substructural logics. For example, residuated groupoids are models of Nonassociative Lambek Calculus [11] and other weak substructural logics [6]. Symmetric residuated groupoids are models of Lambek-Grishin Calculus [12], which is a symmetric extension of the Lambek Calculus [7]. There are also studied different extensions of this calculus by additional axioms such as Grishin axioms. Other logics of that kind are Multiplicative Linear Logics, corresponding to commutative involutive symmetric residuated semigroups, and their noncommutative and nonassociative variants such as InFL, InGL (see e.g. [1, 6]).

There are many natural constructions of multiple residuated groupoids, i.e. residuated groupoids with several residuation triples (see e.g. [4, 8]). Dual residuated groupoids, satisfying the residuation law with respect to dual ordering $\geq$, can be constructed by using an involutive negation, i.e. set complementation $\sim$ that defines the dual residuation triple:

$$X \oplus Y = \sim(\sim X \otimes \sim Y), X \obslash Y = \sim(\sim X \backslash \sim Y), X \oslash Y = \sim(\sim X / \sim Y).$$

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In the paper, some special construction of residuated algebras and dual residuated algebras is established. Recall that a residuated algebra is a generalization of a residuated groupoid in which binary operation $\otimes$ is replaced by an n-ary basic operation o and binary operations $\backslash$ and $/$ are replaced by n residual operations associated with o.

Our main result is the proof of a representation theorem for symmetric residuated algebras. As a special case of this theorem, we obtain a representation theorem for symmetric residuated groupoids. Analogous results have been obtained by the author for other classes of algebras such as (commutative) symmetric residuated semigroups, symmetric residuated unital groupoids and cyclic bilinear algebras [9].

The concept of residuation is closely related to Galois connections from Galois logics. Different kinds of generalized Galois logics called "gaggles" are studied by Bimbó and Dunn in [2]. The authors prove representation theorems for e.g. boolean, distributive, partial (multi-) gaggles. The paper on symmetric generalized Galois logics due to the same authors [3] contains topological representation results for a range of symmetric distributive lattice-ordered groupoids. Many-sorted gaggles and their canonical embeddings are discussed by Buszkowski in [5]. Algebras for Galois-style connections and their discrete duality are studied by Orłowska and Rewitzky in [13].

The paper is organized as follows. In Section 2 we set up notation and terminology. Section 3 provides a powerset construction of residuated algebras. The main result, a representation theorem for symmetric residuated algebras, is presented in Secion 4. In Section 5 we illustrate the construction of powerset residuated groupoids by an example and we establish a representation theorem for symmetric residuated groupoids.

2. Preliminaries

We begin this section with definitions of some notions.

Let $(A, \leq)$ be a poset. By a *residuation family on* $(A, \leq)$ we mean operations $o, {}^{i}o \colon A^n \to A$ such that the following equivalence holds:

$$o(a_1, \ldots, a_n) \leq b \ \text{ iff } \ a_i \leq {}^{i}o(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n)$$

for $i = 1, \ldots, n$.

A *dual residuation family on* $(A, \leq)$ is a residuation family on the dual poset $(A, \geq)$, it means: a family of operations o, o^{i}, for $i = 1, \ldots, n$, such that the operations $o, o^i\colon A^n \to A$ and the following equivalence holds:

$$b \leq o(a_1, \ldots, a_n) \text{ iff } o^i(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \leq a_i$$

for $i = 1, \ldots, n$.

We denote $[n] := \{1, 2, \ldots, n\}$.

A *residuated algebra* is a structure $(A, o, ({}^io)_{i\in[n]}, \leq)$ such that $(A, \leq)$ is a poset and n-ary operations o, ${}^io$ for $i = 1, \ldots, n$ are a residuation family.

A *dual residuated algebra* is a structure $(A, \omega, (\omega^i)_{i\in[n]}, \leq)$ such that $(A, \leq)$ is a poset and n-ary operations ω, ω^i for $i = 1, \ldots, n$ are a dual residuation family.

A *symmetric residuated algebra* is a structure $\boldsymbol{A} = (A, o, ({}^io)_{i\in[n]}, \omega, (\omega^i)_{i\in[n]}, \leq)$, if the $(o, ({}^io)_{i\in[n]}, \leq)$-reduct of $\boldsymbol{A}$ and the $(\omega, (\omega^i)_{i\in[n]}, \leq)$-reduct of $\boldsymbol{A}$ are a residuated algebra and a dual residuation algebra, respectively, for $i = 1, \ldots, n$. Some authors consider residuation algebras in a more general sense, see, e.g., Buszkowski [4] who admits several residuation families on the same poset.

3. A powerset construction

Let us denote by $\mathbf{A}$ a poset $(A, \leq)$. By $\mathbf{A}^{op}$ we denote the dual poset $(A, \geq)$. An *upset* is a set $X \subseteq A$ such that, if $x \in X$ and $x \leq y$, then $y \in X$ for all $x, y \in A$. A *downset* is a set $X \subseteq A$ such that, if $x \in X$ and $y \leq x$, then $y \in X$ for all $x, y \in A$. By the *principal upset* (*downset*) *generated by* $a \in A$ we mean the set of all $b \in A$ such that $a \leq b$ ($b \leq a$). We denote it $a^{\uparrow}$ ($a^{\downarrow}$).

We will denote:

$$\begin{aligned}
\mathcal{P}(A) &:= \{X : X \subseteq A\},\\
\mathcal{P}^{\uparrow}(\mathbf{A}) &:= \{X \subseteq A : X \text{ is an upset}\},\\
\mathcal{P}^{\downarrow}(\mathbf{A}) &:= \{X \subseteq A : X \text{ is a downset}\}.
\end{aligned}$$

Let $o\colon A^n \to A$. We define the operations $o\uparrow^{\mathcal{P}(A)}, o\downarrow^{\mathcal{P}(A)}$ from $\mathcal{P}(A)^n$ to $\mathcal{P}(A)$ as follows:

$$\begin{aligned}
&o\uparrow^{\mathcal{P}(A)}(X_1, ..., X_n) := \{z \in A : (\exists a_1 \in X_1, ..., a_n \in X_n)\ o(a_1, ..., a_n) \leq z\},\\
&o\downarrow^{\mathcal{P}(A)}(X_1, ..., X_n) := \{z \in A : (\exists a_1 \in X_1, ..., a_n \in X_n)\ z \leq o(a_1, \ldots, a_n)\}.
\end{aligned}$$

They possess residual operations ${}^{i}o\uparrow^{\mathcal{P}(A)}$, ${}^{i}o\downarrow^{\mathcal{P}(A)}$ defined in the following way:

$$\begin{aligned} {}^{i}o\uparrow^{\mathcal{P}(A)}(X_1,...,X_n) &:= \{a_i \in A : \forall j \neq i,\ a_j \in X_j \\ &\qquad \forall z(o(a_1,...,a_n) \leq z \Rightarrow z \in X_i)\}, \\ {}^{i}o\downarrow^{\mathcal{P}(A)}(X_1,...,X_n) &:= \{a_i \in A : \forall j \neq i,\ a_j \in X_j \\ &\qquad \forall z(z \leq o(a_1,...,a_n) \Rightarrow z \in X_i)\}. \end{aligned}$$

The above defined residuation operations for $i = 1, \ldots, n$ satisfy the general residuation laws:

$$o\uparrow^{\mathcal{P}(A)}(X_1,...,X_n) \subseteq Y \text{ iff } X_i \subseteq {}^{i}o\uparrow^{\mathcal{P}(A)}(X_1,...,X_{i-1},Y,X_{i+1},...,X_n),$$
$$o\downarrow^{\mathcal{P}(A)}(X_1,...,X_n) \subseteq Y \text{ iff } X_i \subseteq {}^{i}o\downarrow^{\mathcal{P}(A)}(X_1,...,X_{i-1},Y,X_{i+1},...,X_n)$$

for all $X_1, \ldots, X_n, Y \subseteq A$.

Observe that for any $X_1, \ldots, X_n \in \mathcal{P}(A)$:

$$o\uparrow^{\mathcal{P}(A)}(X_1, \ldots, X_n) \in \mathcal{P}^{\uparrow}(\mathbf{A}),$$
$$o\downarrow^{\mathcal{P}(A)}(X_1, \ldots, X_n) \in \mathcal{P}^{\downarrow}(\mathbf{A}).$$

If the operation o is isotone on i-th argument, then

$${}^{i}o\uparrow^{\mathcal{P}(A)}(X_1, \ldots, X_n) \in \mathcal{P}^{\uparrow}(\mathbf{A}),$$
$${}^{i}o\downarrow^{\mathcal{P}(A)}(X_1, \ldots, X_n) \in \mathcal{P}^{\downarrow}(\mathbf{A}).$$

In any residuated algebra with operations o and ${}^{i}o$, $i \in [n]$, o is isotone in the i-th argument and ${}^{i}o$ is isotone in the i-th argument and antitone in other arguments. In any dual residuation algebra the same holds for o, o^{i}. Consequently, $\mathcal{P}^{\uparrow}(\mathbf{A})$ is a subalgebra of the residuation algebra $(\mathcal{P}(A), o\uparrow^{\mathcal{P}(A)}, ({}^{i}o\uparrow^{\mathcal{P}(A)})_{i\in[n]}, \subseteq)$, and $\mathcal{P}^{\downarrow}(\mathbf{A})$ is a subalgebra of the algebra $(\mathcal{P}(A), o\downarrow^{\mathcal{P}(A)}, ({}^{i}o\downarrow^{\mathcal{P}(A)})_{i\in[n]}, \subseteq)$.

The operations $o\uparrow^{\mathcal{P}(A)}$, $({}^{i}o\uparrow^{\mathcal{P}(A)})_{i\in[n]}$ and $o\downarrow^{\mathcal{P}(A)}$, $({}^{i}o\downarrow^{\mathcal{P}(A)})_{i\in[n]}$ form residuation families.

For any $i = 1, \ldots, n$, we have defined the powerset structures $(\mathcal{P}(A), o\uparrow^{\mathcal{P}(A)}, ({}^{i}o\uparrow^{\mathcal{P}(A)})_{i\in[n]}, \subseteq)$ and $(\mathcal{P}(A), o\downarrow^{\mathcal{P}(A)}, ({}^{i}o\downarrow^{\mathcal{P}(A)})_{i\in[n]}, \subseteq)$. Both structures are residuated algebras. By $o\uparrow^{\mathcal{P}^{\uparrow}(A)}$ we denote the restriction of $o\uparrow^{\mathcal{P}(A)}$ to $\mathcal{P}^{\uparrow}(A)$, i.e., $o\uparrow^{\mathcal{P}^{\uparrow}(A)}(X_1, \ldots, X_n) = o\uparrow^{\mathcal{P}(A)}(X_1, \ldots, X_n)$, for $X_1, \ldots, X_n \in \mathcal{P}^{\uparrow}(A)$. The operation $o\downarrow^{\mathcal{P}^{\downarrow}(A)}$ is defined in a similar way.

The construction presented above can be used to construct a powerset algebra on the second level. Consider the subalgebra of $(\mathcal{P}(A), o\uparrow^{\mathcal{P}(A)}, ({}^i o\uparrow^{\mathcal{P}(A)})_{i\in[n]}, \subseteq)$ restricted to upsets on the first level. Next, we construct the second level algebra $(\mathcal{P}(\mathcal{P}^{\uparrow}(\mathbf{A})), O, ({}^i O)_{i\in[n]}, \subseteq)$, where $O = (o\uparrow^{\mathcal{P}^{\uparrow}(\mathbf{A})})\uparrow^{\mathcal{P}(\mathcal{P}^{\uparrow}(\mathbf{A}))}$. Thus, our construction is uniform, since the first level is related to the ground level in the same way as the second one to the first one.

4. Main result

In this section, we prove the main theorem of the paper.

Lemma 1. *If o, $({}^i o)_{i\in[n]}$ is a residuation family on $(A, \leq)$, then the mapping h such that $h(a) = \{X \in \mathcal{P}^{\uparrow}(\mathbf{A}) : a \in X\}$ is an embedding of $(A, o, ({}^i o)_{i\in[n]}, \leq)$ into $(\mathcal{P}(\mathcal{P}^{\uparrow}(\mathbf{A})), O, ({}^i O)_{i\in[n]}, \subseteq)$, where $O = (o\uparrow^{\mathcal{P}^{\uparrow}(\mathbf{A})})\uparrow^{\mathcal{P}(\mathcal{P}^{\uparrow}(\mathbf{A}))}$ and $O, ({}^i O)_{i\in[n]}$ is a residuation family on $(\mathcal{P}(\mathcal{P}^{\uparrow}(\mathbf{A})), \subseteq)$.*

Proof. First, we show that h preserves the order, i.e.

$$a \leq b \quad \text{iff} \quad h(a) \subseteq h(b), \quad \text{for all } a, b \in A.$$

($\Rightarrow$) Assume $a \leq b$. Let $X \in h(a)$. By the definition of h, $a \in X$. Since X is an upset, $a \in X$ and $a \leq b$ imply $b \in X$. Thus $X \in h(b)$.

($\Leftarrow$) Assume $h(a) \subseteq h(b)$. We have $a \in a^{\uparrow} \in h(a)$. Hence, $a^{\uparrow} \in h(b)$. By the definition of h, $b \in a^{\uparrow}$. It means that $a \leq b$.

We recall the definitions of operations $O, ({}^i O)_{i\in[n]}$:

$$\begin{aligned} O(\mathcal{X}_1, \ldots, \mathcal{X}_n) := \{&Z \in \mathcal{P}^{\uparrow}(\mathbf{A}) : \\ &(\exists X_1 \in \mathcal{X}_1, \ldots, X_n \in \mathcal{X}_n)\ o\uparrow^{\mathcal{P}^{\uparrow}(\mathbf{A})} (X_1, \ldots, X_n) \subseteq Z\},\end{aligned}$$

$$\begin{aligned} {}^i O(\mathcal{X}_1, \ldots, \mathcal{X}_n) := \{&X_i \in \mathcal{P}^{\uparrow}(\mathbf{A}) : \forall j \neq i\ X_j \in \mathcal{X}_j\ \forall Z \in \mathcal{P}^{\uparrow}(\mathbf{A}) \\ &(o\uparrow^{\mathcal{P}^{\uparrow}(\mathbf{A})} (X_1, \ldots, X_n) \subseteq Z \Rightarrow Z \in \mathcal{X}_i)\}.\end{aligned}$$

We show that h preserves all operations. First, we show that

$$h(o(a_1, \ldots, a_n)) = O(h(a_1), \ldots, h(a_n)).$$

($\subseteq$) Let $Z \in h(o(a_1, ..., a_n))$. We have then $o(a_1, ..., a_n) \in Z$. Since $Z \in \mathcal{P}^{\uparrow}(\mathbf{A})$, by the definition of operation $o\uparrow^{\mathcal{P}^{\uparrow}(\mathbf{A})}$, $o\uparrow^{\mathcal{P}^{\uparrow}(\mathbf{A})} (a_1{\uparrow}, ..., a_n{\uparrow}) \subseteq$

Z. We have $a_i{\uparrow} \in h(a_i)$ for $i = 1, \ldots, n$. Then, by the definition of operation O, we obtain $Z \in O(h(a_1), \ldots, h(a_n))$.

($\supseteq$) Let $Z \in O(h(a_1), \ldots, h(a_n))$. By the definition of operation O, there exist $X_i \in h(a_i)$ for $i = 1, \ldots, n$ such that $o\uparrow^{\mathcal{P}^{\uparrow}(\mathbf{A})} (X_1, \ldots, X_n) \subseteq Z$. By the definition of h, $a_i \in X_i$ for $i = 1, \ldots, n$. Hence by the definition of operation $o\uparrow^{\mathcal{P}^{\uparrow}(\mathbf{A})}$, $o(a_1, \ldots, a_n) \in o\uparrow^{\mathcal{P}^{\uparrow}(\mathbf{A})} (X_1, \ldots, X_n)$. Thus, $o(a_1, \ldots, a_n) \in Z$, and finally $Z \in h(o(a_1, \ldots, a_n))$.

Now, we show that $h({}^io(a_1, \ldots, a_n)) = {}^iO(h(a_1), \ldots, h(a_n))$.

($\subseteq$) Assume $X_i \in h({}^io(a_1, ..., a_n))$. We have then ${}^io(a_1, \ldots, a_n) \in X_i$. Take $X_j \in h(a_j)$ for all $j \neq i$ and $Z \in \mathcal{P}^{\uparrow}(\mathbf{A})$ such that ${}^io{\uparrow}^{\mathcal{P}(A)}(X_1, \ldots, X_n) \subseteq Z$. Since $a_j \in X_j$ for $j \neq i$ and $o(a_1, \ldots, a_{i-1}, {}^io(a_1, \ldots, a_n), a_{i+1}, \ldots, a_n) \leq a_i$, so $a_i \in {}^io\uparrow^{\mathcal{P}^{\uparrow}(\mathbf{A})}(X_1, \ldots, X_n)$. Hence $a_i \in Z$. Thus $Z \in h(a_i)$ and $X_i \in {}^iO(h(a_1), \ldots, h(a_n))$.

($\supseteq$) Assume $X_i \in {}^iO(h(a_1), \ldots, h(a_n))$. We have $a_j{\uparrow} \in h(a_j)$ for all $j \neq i$. By the definition of operation $(o\uparrow^{\mathcal{P}^{\uparrow}(\mathbf{A})})\uparrow^{\mathcal{P}(\mathcal{P}^{\uparrow}(\mathbf{A}))}$, for all $Z \in \mathcal{P}^{\uparrow}(A)$ the following implications hold: if $o\uparrow^{\mathcal{P}^{\uparrow}(\mathbf{A})} (a_1{\uparrow}, \ldots, a_{i-1}{\uparrow}, X_i, a_{i+1}{\uparrow}, \ldots, a_n{\uparrow}) \subseteq Z$, then $Z \in h(a_i)$. We have then $o\uparrow^{\mathcal{P}^{\uparrow}(\mathbf{A})} (a_1{\uparrow}, \ldots, a_{i-1}{\uparrow}, X_i, a_{i+1}{\uparrow}, \ldots, a_n{\uparrow}) \in h(a_i)$, and hence $a_i \in o\uparrow^{\mathcal{P}^{\uparrow}(\mathbf{A})} (a_1{\uparrow}, \ldots, a_{i-1}{\uparrow}, X_i, a_{i+1}{\uparrow}, \ldots, a_n{\uparrow})$. By the definition of operation $o\uparrow^{\mathcal{P}^{\uparrow}(\mathbf{A})}$, there exist $b_j \in a_j{\uparrow}$ and $x_i \in X_i$ for $j \neq i$ such that $o(b_1, \ldots, b_{i-1}, x_i, b_{i+1}, \ldots, b_n) \leq a_i$. Hence $x_i \leq {}^io(b_1, \ldots, b_{i-1}, a_i, b_{i+1}, \ldots, b_n)$ and ${}^io(b_1, \ldots, b_{i-1}, a_i, b_{i+1}, \ldots, b_n) \leq {}^io(a_1, \ldots, a_i, \ldots, a_n)$, so ${}^io(a_1, \ldots, a_i, \ldots, a_n) \in X_i$. Thus $X_i \in h({}^io(a_1, \ldots, a_n))$. $\square$

Let $(A, \leq_A)$, $(B, \leq_B)$ be posets, and let $f\colon A \to B$, $g\colon B \to A$ be antitone mappings such that $g \circ f = i_A$, $f \circ g = i_B$ (f and g are involutive negations). Let o, $({}^io)_{i\in[n]}$ be a residuation family on $(A, \leq)$. Then the mappings:

$$
\begin{aligned}
{}^Do(b_1, \ldots, b_n) &:= f(o(g(b_1), \ldots, g(b_n))),\\
{}^D({}^io)(b_1, \ldots, b_n) &:= f({}^io(g(b_1), \ldots, g(b_n))),
\end{aligned}
$$

form a dual residuation family on $(B, \leq)$. It is easily seen that the following fact holds.

Fact 1. *If f and g are involutive negations, then ${}^D({}^io) = ({}^Do)^i$.*

A unary operation $\sim$ is called a *single involutive negation* if it satisfies the following two conditions:

$$\sim\sim a = a \qquad \text{(Double Negation)}$$
$$a \le b \Rightarrow \sim b \le \sim a \qquad \text{(Transposition)}$$

For any operation $o\colon A^n \to A$, we define the operation o^D:

$$o^D(a_1,\ldots,a_n) = \sim o(\sim a_1,\ldots,\sim a_n)$$

for all $a_1, \ldots, a_n \in A$.

Fact 2. *If $\sim$ is a single involutive negation on $(A,\le)$ and o, $({}^io)_{i\in[n]}$ is a residuation family on $(A,\le)$, then o^D, $(({}^io)^D)_{i\in[n]}$ is a dual residuation family on $(A,\le)$; also $({}^io)^D = (o^D)^i$.*

Lemma 2. *If ω, $(\omega^i)_{i\in[n]}$ is a dual residuation family on $(A,\le)$, then the same mapping h, $h(a) = \{X \in \mathcal{P}^{\uparrow}(\mathbf{A}) : a \in X\}$, is an embedding of $(A,\omega,(\omega^i)_{i\in[n]},\le)$ into the powerset residuated algebra $(\mathcal{P}(\mathcal{P}^{\uparrow}(\mathbf{A})),\Omega,(\Omega^i)_{i\in[n]},\subseteq)$, where the operation Ω is defined as $(\omega\downarrow^{\mathcal{P}^{\downarrow}(\mathbf{A})D})\downarrow^{\mathcal{P}(\mathcal{P}^{\uparrow}(\mathbf{A}))D}$ and Ω, $(\Omega^i)_{i\in[n]}$ is a dual residuation family on $(\mathcal{P}(\mathcal{P}^{\uparrow}(\mathbf{A})),\subseteq)$.*

Proof. Let ω, $(\omega^i)_{i\in[n]}$ be a residuation family on $(A,\ge)$. By Lemma 1, g defined by $g(a) = \{X \in \mathcal{P}^{\uparrow}(\mathbf{A}^{op}) : a \in X\}$ is an embedding of $(A,\omega,(\omega^i)_{i\in[n]},\ge)$ into $(\mathcal{P}(\mathcal{P}^{\uparrow}(\mathbf{A}^{op})),\Pi,({}^i\Pi)_{i\in[n]},\subseteq)$, where $\Pi = (\omega\uparrow^{\mathcal{P}^{\uparrow}(\mathbf{A}^{op})})\uparrow^{\mathcal{P}(\mathcal{P}^{\uparrow}(\mathbf{A}^{op}))}$. Clearly, $\mathcal{P}^{\uparrow}(\mathbf{A}^{op}) = \mathcal{P}^{\downarrow}(\mathbf{A})$, $\omega\uparrow^{\mathcal{P}^{\uparrow}(\mathbf{A}^{op})} = \omega\downarrow^{\mathcal{P}^{\downarrow}(\mathbf{A})}$.

Consequently, $\Pi = (\omega\downarrow^{\mathcal{P}^{\downarrow}(\mathbf{A})})\uparrow^{\mathcal{P}(\mathcal{P}^{\downarrow}(\mathbf{A}))}$. We have then

$$g(\omega(a_1,\ldots,a_n)) = \Pi(g(a_1),\ldots,g(a_n)),$$
$$g(\omega^i(a_1,\ldots,a_n)) = {}^i\Pi(g(a_1),\ldots,g(a_n)).$$

Observe that for g defined above the following equivalence holds:

$$a \le b \quad \text{iff} \quad g(b) \subseteq g(a), \qquad \text{for all } a,b \in A.$$

For any set U and $X \subset U$, we denote $\sim_U X := U - X$. For $\mathcal{X} \subseteq \mathcal{P}(A)$ we define $\mathcal{X}^{\sim} := \{\sim_A X : X \in \mathcal{X}\}$. Following [5], for $\mathcal{X} \subseteq \mathcal{P}^{\uparrow}(A)$ we define $\mathcal{X}^* := \sim_{\mathcal{P}^{\downarrow}(A)}(\mathcal{X}^{\sim}) = (\sim_{\mathcal{P}^{\uparrow}(A)}\mathcal{X})^{\sim}$. Clearly, $\mathcal{X}^* \subseteq \mathcal{P}^{\downarrow}(A)$. For $\mathcal{X} \subseteq \mathcal{P}^{\downarrow}(A)$ we define ${}^*\mathcal{X} := \sim_{\mathcal{P}^{\uparrow}(A)}(\mathcal{X}^{\sim}) = (\sim_{\mathcal{P}^{\downarrow}(A)}\mathcal{X})^{\sim}$. Similarly, ${}^*\mathcal{X} \subseteq \mathcal{P}^{\uparrow}(A)$.

For all $X \in \mathcal{P}^{\downarrow}(A)$ the following equivalence holds: $X \in \mathcal{X}^*$ iff $\sim_A X \notin \mathcal{X}$, and for all $X \in \mathcal{P}^{\uparrow}(A)$ there holds: $X \in {}^*\mathcal{X}$ iff $\sim_A X \notin \mathcal{X}$. Also ${}^*(\mathcal{X}^*) = \mathcal{X}$, $({}^*\mathcal{Y})^* = \mathcal{Y}$, for $\mathcal{X} \subseteq \mathcal{P}^{\uparrow}(A)$, $\mathcal{Y} \subseteq \mathcal{P}^{\downarrow}(A)$.

We have $h(a) = {}^*g(a)$. Let us define ${}^D\Pi\colon \mathcal{P}(\mathcal{P}^{\uparrow}(A))^n \to \mathcal{P}(\mathcal{P}^{\uparrow}(A))$ in the following way:

$$^D\Pi(\mathcal{X}_1, \ldots, \mathcal{X}_n) := {}^*\Pi(\mathcal{X}_1{}^*, \ldots, \mathcal{X}_n{}^*).$$

We have $h(\omega(a_1, \ldots, a_n)) = {}^D\Pi(h(a_1), \ldots, h(a_n))$.

The following equalities hold:

$$\begin{aligned} h(\omega(a_1, \ldots, a_n)) &= {}^*g(\omega(a_1, \ldots, a_n)) = {}^*\Pi(g(a_1), \ldots, g(a_n)) \\ &= {}^*\Pi(h(a_1)^*, \ldots, h(a_n)^*) = {}^D\Pi(h(a_1), \ldots, h(a_n)). \end{aligned}$$

We show that ${}^D\Pi = \Omega$.

For $X \in \mathcal{P}^{\uparrow}(\mathbf{A})$ and $\mathcal{X}_1, \ldots, \mathcal{X}_n \subseteq \mathcal{P}^{\uparrow}(\mathbf{A})$ the following formulas are equivalent:

- $X \notin {}^D\Pi(\mathcal{X}_1, \ldots, \mathcal{X}_n)$,
- $X \notin {}^*\Pi(\mathcal{X}_1{}^*, \ldots, \mathcal{X}_n{}^*)$,
- $\sim_A X \in \Pi(\mathcal{X}_1{}^*, \ldots, \mathcal{X}_n{}^*)$,
- $(\exists X_1, \ldots, X_n \in \mathcal{P}^{\downarrow}(\mathbf{A}))(X_1 \in \mathcal{X}_1{}^* \wedge \cdots \wedge X_n \in \mathcal{X}_n{}^* \wedge$
  $\omega \downarrow^{\mathcal{P}^{\downarrow}(\mathbf{A})} (X_1, \ldots, X_n) \subset \sim_A X)$,
- $(\exists X_1, \ldots, X_n \in \mathcal{P}^{\downarrow}(\mathbf{A}))(\sim_A X_1 \notin \mathcal{X}_1 \wedge \cdots \wedge \sim_A X_n \notin \mathcal{X}_n \wedge$
  $X \subset \sim_A \omega \downarrow^{\mathcal{P}^{\downarrow}(\mathbf{A})} (X_1, \ldots, X_n))$,
- $(\exists Y_1, \ldots, Y_n \in \mathcal{P}^{\uparrow}(\mathbf{A}))(Y_1 \notin \mathcal{X}_1 \wedge \cdots \wedge Y_n \notin \mathcal{X}_n \wedge$
  $X \subset \sim_A \omega \downarrow^{\mathcal{P}^{\downarrow}(\mathbf{A})} (\sim_A Y_1, \ldots, \sim_A Y_n))$,
- $(\exists Y_1, \ldots, Y_n \in \mathcal{P}^{\uparrow}(\mathbf{A}))(Y_1 \in \sim_{\mathcal{P}^{\uparrow}(\mathbf{A})} \mathcal{X}_1 \wedge \cdots \wedge Y_n \in \sim_{\mathcal{P}^{\uparrow}(\mathbf{A})} \mathcal{X}_n \wedge$
  $X \subset \omega \downarrow^{\mathcal{P}^{\downarrow}(\mathbf{A})D} (Y_1, \ldots, Y_n))$,
- $X \in (\omega\downarrow^{\mathcal{P}^{\downarrow}(\mathbf{A})D})\downarrow^{\mathcal{P}(\mathcal{P}^{\uparrow}(\mathbf{A}))}(\sim \mathcal{X}_1, \ldots, \sim \mathcal{X}_n)$,
- $X \notin \sim_{\mathcal{P}(\mathcal{P}^{\uparrow}(\mathbf{A}))}(\omega\downarrow^{\mathcal{P}^{\downarrow}(\mathbf{A})D})\downarrow^{\mathcal{P}(\mathcal{P}^{\uparrow}(\mathbf{A}))}(\sim \mathcal{X}_1, \ldots, \sim \mathcal{X}_n)$,
- $X \notin (\omega\downarrow^{\mathcal{P}^{\downarrow}(\mathbf{A})D})\downarrow^{\mathcal{P}(\mathcal{P}^{\uparrow}(\mathbf{A}))D}(\mathcal{X}_1, \ldots, \mathcal{X}_n)$,
- $X \notin \Omega(\mathcal{X}_1, \ldots, \mathcal{X}_n)$.

Since Π is completely additive, it admits residuals ${}^i\Pi$ for $i = 1, \ldots, n$. The operation ${}^D\Pi$ admits residuals ${}^D({}^i\Pi) = ({}^D\Pi)^i = \Omega^i$ for $i = 1, \ldots, n$. Consequently, $\Omega, (\Omega^i)_{i\in[n]}$ is a dual residuation family on $\mathcal{P}(\mathcal{P}^{\uparrow}(\mathbf{A}))$.

It is easy to show that h preserves residuals: $h(\omega^i(a_1, \ldots, a_n)) = \Omega^i(h(a_1), \ldots, h(a_n))$. Indeed: $h(\omega^i(a_1, \ldots, a_n)) = {}^*g(\omega^i(a_1, \ldots, a_n)) = {}^*({}^i\Pi(g(a_1), \ldots, g(a_n))) = {}^*({}^i\Pi(h(a_1)^*, \ldots, h(a_n)^*)) = {}^{Di}\Pi(h(a_1), \ldots, h(a_n)) = \Omega^i(h(a_1), \ldots, h(a_n))$. □

The main result of the paper is the following theorem.

Theorem 1. *Every symmetric residuated algebra* $\mathbf{A} = (A, o, (o^i)_{i\in[n]}, \omega, (\omega^i)_{i\in[n]}, \leq)$ *is embeddable into the powerset residuated algebra* $(\mathcal{P}(\mathcal{P}^{\uparrow}(\mathbf{A})), O, ({}^iO)_{i\in[n]}, \Omega, (\Omega^i)_{i\in[n]}, \subseteq)$, *where* $O = (o{\uparrow}^{\mathcal{P}^{\uparrow}(\mathbf{A})}){\uparrow}^{\mathcal{P}(\mathcal{P}^{\uparrow}(\mathbf{A}))}$ *and* $\Omega = (\omega\downarrow^{\mathcal{P}^{\downarrow}(\mathbf{A})D})\downarrow^{\mathcal{P}(\mathcal{P}^{\uparrow}(\mathbf{A}))D}$.

Proof. It is an immediate consequence of Lemma 1 and Lemma 2. □

5. A representation theorem for symmetric residuated groupoids

In this section we illustrate the previous considerations for the case of one residuation triple. Direct proofs of results from this section are given in [9] and they are based on [10].

Let us recall some basic definitions.

A structure $(A, \otimes, \leq)$ is a *partially ordered groupoid* (*p.o. groupoid*), if $\leq$ is a partial order and $\otimes$ is monotone on both arguments, i.e., $a \leq b$ implies $a \otimes c \leq b \otimes c$ and $c \otimes a \leq c \otimes b$, for $a, b, c \in A$.

A *residuated groupoid* is a structure $(A, \otimes, \backslash, /, \leq)$ such that $(A, \leq)$ is a poset, $(A, \otimes)$ is a groupoid, and $\otimes, \backslash, /$ satisfy the residuation law: $a \leq c/b$ iff $a \otimes b \leq c$ iff $b \leq a\backslash c$, for all $a, b, c \in A$.

A *dual residuated groupoid* is a structure $(A, \oplus, \obslash, \oslash, \leq)$ such that $(A, \leq)$ is a poset, $(A, \oplus)$ is a groupoid, and $\oplus, \obslash, \oslash$ satisfy the dual residuation law: $c \oslash b \leq a$ iff $c \leq a \oplus b$ iff $a \obslash c \leq b$, for all $a, b, c \in A$.

A structure $\boldsymbol{A} = (A, \otimes, \backslash, /, \oplus, \obslash, \oslash, \leq)$ is called a *symmetric residuated groupoid* iff the $(\otimes, \backslash, /, \leq)$-reduct of $\boldsymbol{A}$ and the $(\oplus, \obslash, \oslash, \leq)$-reduct of $\boldsymbol{A}$ are a residuated groupoid and a dual residuated groupoid, respectively.

Starting from a p.o. groupoid $\boldsymbol{A} = (A, \otimes, \leq)$, one can define powerset algebra $(\mathcal{P}(A), \widehat{\otimes}, \widehat{\backslash}, \widehat{/}, \subseteq)$. For $X, Y, Z \subseteq A$, we define operations:

$$X \,\widehat{\otimes}\, Y := \{c \in A : \exists a \in X\ \exists b \in Y\ a \otimes b \leq c\},$$
$$X \,\widehat{\backslash}\, Z := \{b \in A : \forall a \in X\ \forall c \in M\ (a \otimes b \leq c \Rightarrow c \in Z)\},$$
$$Z \,\widehat{/}\, Y := \{a \in A : \forall b \in Y\ \forall c \in M\ (a \otimes b \leq c \Rightarrow c \in Z)\}.$$

The algebra $(\mathcal{P}(A), \widehat{\otimes}, \widehat{\backslash}, \widehat{/}, \subseteq)$ is a residuated groupoid.

Let us denote $\otimes$ by o. Then $\widehat{\otimes} = o\uparrow^{\mathcal{P}(A)}, \widehat{\backslash} = {}^2o\uparrow^{\mathcal{P}(A)}, \widehat{/} = {}^1o\uparrow^{\mathcal{P}(A)}$.

Starting from a p.o. groupoid $\boldsymbol{A} = (A, \oplus, \leq)$, we define a dual powerset algebra $(\mathcal{P}(A), \bar{\oplus}, \bar{\obslash}, \bar{\oslash}, \subseteq)$. For $X, Y, Z \subseteq A$, we define operations:

$$X \mathbin{\bar{\oplus}} Y := \{c \in A : \exists a \in X \ \exists b \in Y \ \ c \leq a \oplus b\},$$
$$X \mathbin{\bar{\obslash}} Z := \{b \in A : \forall a \in X \ \forall c \in A \ \ (c \leq a \oplus b \Rightarrow c \in Z)\},$$
$$Z \mathbin{\bar{\oslash}} Y := \{a \in A : \forall b \in Y \ \forall c \in A \ \ (c \leq a \oplus b \Rightarrow c \in Z)\}.$$

The algebra $(\mathcal{P}(A), \bar{\oplus}, \bar{\obslash}, \bar{\oslash}, \subseteq)$ is a residuated groupoid.

Let us denote $\oplus$ by ω. Then $\bar{\oplus} = \omega \downarrow^{\mathcal{P}(A)}, \bar{\obslash} = \omega^2 \downarrow^{\mathcal{P}(A)}, \bar{\oslash} = \omega^1 \downarrow^{\mathcal{P}(A)}$.

$(\mathcal{P}(A), \widehat{\otimes}, \widehat{\backslash}, \widehat{/}, \subseteq)$ and $(\mathcal{P}(A), \bar{\oplus}, \bar{\obslash}, \bar{\oslash}, \subseteq)$ can be expanded by the set complementation $\sim X = \{a \in A : \ a \notin X\}$ and we define dual operations on $(\mathcal{P}(A), \widehat{\otimes}, \widehat{\backslash}, \widehat{/}, \subseteq)$ as follows:

$$X \mathbin{\widehat{\oplus}} Y := \sim(\sim X \mathbin{\bar{\oplus}} \sim Y),$$
$$X \mathbin{\widehat{\obslash}} Z := \sim(\sim X \mathbin{\bar{\obslash}} \sim Z),$$
$$Z \mathbin{\widehat{\oslash}} Y := \sim(\sim Z \mathbin{\bar{\oslash}} \sim Y).$$

The algebra $\mathcal{P}(A), \widehat{\oplus}, \widehat{\obslash}, \widehat{\oslash}, \subseteq)$ is a dual residuated groupoid.

Under the notation from Section 4, operations $\widehat{\oplus}, \widehat{\obslash}, \widehat{\oslash}$ are as follows: $\widehat{\oplus} = \omega \downarrow^{\mathcal{P}(A)D}, \widehat{\obslash} = \omega^2 \downarrow^{\mathcal{P}(A)D}, \widehat{\oslash} = \omega^1 \downarrow^{\mathcal{P}(A)D}$.

The operations $\widehat{\oplus}, \widehat{\obslash}, \widehat{\oslash}$ can also be defined as follows:

$$X \mathbin{\widehat{\oplus}} Y := \{c \in A : \forall a, b \in A \ (c \leq a \oplus b \Rightarrow (a \in X \vee b \in Y))\},$$
$$X \mathbin{\widehat{\obslash}} Z := \{b \in A : \exists a \notin X \ \exists c \in Z \ \ c \leq a \oplus b\},$$
$$Z \mathbin{\widehat{\oslash}} Y := \{a \in A : \exists b \notin Y \ \exists c \in Z \ \ c \leq a \oplus b\}.$$

For any symmetric residuated groupoid $(A, \otimes, \backslash, /, \oplus, \obslash, \oslash, \leq)$, the algebra $(\mathcal{P}(A), \widehat{\otimes}, \widehat{\backslash}, \widehat{/}, \widehat{\oplus}, \widehat{\obslash}, \widehat{\oslash}, \subseteq)$ is a symmetric residuated groupoid.

The structure $(\mathcal{P}^{\uparrow}(\mathbf{A}), \widehat{\otimes}, \widehat{\backslash}, \widehat{/}, \widehat{\oplus}, \widehat{\obslash}, \widehat{\oslash}, \subseteq)$ is the algebra on the first level. We construct the higher-level algebra: $(\mathcal{P}(\mathcal{P}^{\uparrow}(\mathbf{A})), O, {}^2O, {}^1O, \Omega, \Omega^2, \Omega^1, \subseteq)$. Let us remind that $O = (o \uparrow^{\mathcal{P}^{\uparrow}(\mathbf{A})}) \uparrow^{\mathcal{P}(\mathcal{P}^{\uparrow}(\mathbf{A}))}$ and $\Omega = (\omega \downarrow^{\mathcal{P}^{\downarrow}(\mathbf{A})D}) \downarrow^{\mathcal{P}(\mathcal{P}^{\uparrow}(\mathbf{A}))D}$, so

$$\mathcal{X} \ O \ \mathcal{Y} = \{Z \in \mathcal{P}^{\uparrow}(\mathbf{A}) : \exists X \in \mathcal{X} \ \exists Y \in \mathcal{Y} \ \ X \widehat{\otimes} Y \subseteq Z\},$$
$$\mathcal{X} \ {}^2O \ \mathcal{Z} = \{Y \in \mathcal{P}^{\uparrow}(\mathbf{A}) : \forall X \in \mathcal{X} \ \forall Z \in \mathcal{P}^{\uparrow}(\mathbf{A}) \ (X \widehat{\otimes} Y \subseteq Z \Rightarrow Z \in \mathcal{Z})\},$$
$$\mathcal{Z} \ {}^1O \ \mathcal{Y} = \{X \in \mathcal{P}^{\uparrow}(\mathbf{A}) : \forall Y \in \mathcal{Y} \ \forall Z \in \mathcal{P}^{\uparrow}(\mathbf{A}) \ (X \widehat{\otimes} Y \subseteq Z \Rightarrow Z \in \mathcal{Z})\},$$
$$\mathcal{X} \ \Omega \ \mathcal{Y} = \{Z \in \mathcal{P}^{\uparrow}(\mathbf{A}) : \forall X \in \mathcal{P}^{\uparrow}(\mathbf{A}) \ \forall Y \in \mathcal{P}^{\uparrow}(\mathbf{A})(Z \subseteq X \widehat{\oplus} Y \Rightarrow X \in \mathcal{X} \vee Y \in \mathcal{Y}))\},$$

$$\mathcal{X}\ \Omega^2\ \mathcal{Z} = \{Y \in \mathcal{P}^{\uparrow}(\mathbf{A}) : \exists X \notin \mathcal{X}\ \exists Z \in \mathcal{Z}\ \ Z \subseteq X \widehat{\oplus} Y\},$$
$$\mathcal{Z}\ \Omega^1\ \mathcal{Y} = \{X \in \mathcal{P}^{\uparrow}(\mathbf{A}) : \exists Y \notin \mathcal{Y}\ \exists Z \in \mathcal{Z}\ \ Z \subseteq X \widehat{\oplus} Y\},$$

for all $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \subseteq \mathcal{P}^{\uparrow}(\mathbf{A})$.

The algebra $(\mathcal{P}(\mathcal{P}^{\uparrow}(\mathbf{A})), O, {}^2O, {}^1O, \Omega, \Omega^2, \Omega^1, \subseteq)$ is a symmetric residuated groupoid.

The following lemmas are special cases of lemmas 1 and 2, respectively.

Lemma 3. *If $\otimes, \backslash, /$ is a residuation family on $(A, \leq)$, then the mapping h such that $h(a) = \{X \in \mathcal{P}^{\uparrow}(\mathbf{A}) : a \in X\}$ is an embedding of $(A, \otimes, \backslash, /, \leq)$ into the powerset algebra $(\mathcal{P}(\mathcal{P}^{\uparrow}(\mathbf{A})), O, {}^1O, {}^2O, \subseteq)$.*

Lemma 4. *If $\oplus, \oslash, \oslash$ is a dual residuation family on $(A, \leq)$, then the same mapping h is an embedding of $(A, \oplus, \oslash, \oslash, \leq)$ into $(\mathcal{P}(\mathcal{P}^{\uparrow}(\mathbf{A})), \Omega, \Omega^1, \Omega^2, \subseteq)$.*

The following result is a special case of Theorem 1.

Theorem 2. *Every symmetric residuated groupoid $\mathbf{A} = (A, \otimes, \backslash, /, \oplus, \oslash, \oslash, \leq)$ is embeddable into the symmetric residuated groupoid $(\mathcal{P}(\mathcal{P}^{\uparrow}(\mathbf{A})), O, {}^1O, {}^2O, \Omega, \Omega^1, \Omega^2, \subseteq)$.*

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Mirosława Kołowska-Gawiejnowicz
Faculty of Mathematics and Computer Science
Adam Mickiewicz University
ul. Umultowska 87
61-614 Poznań, Poland
`mkolowsk@amu.edu.pl`