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TOPOLOGIES AND FREE CONSTRUCTIONS*

Abstract. The standard presentation of topological spaces relies heavily on (naïve) set theory: a topology consists of a set of subsets of a set (of points). And many of the high-level tools of set theory are required to achieve just the basic results about topological spaces.

Concentrating on the mathematical structures, category theory offers the possibility to look synthetically at the structure of continuous transformations between topological spaces addressing specifically how the fundamental notions of point and open come about. As a byproduct of this, one may look at the different approaches to topology from an external perspective and compare them in a unified way.

Technically, the category of sober topological spaces can be seen as consisting of (co)algebraic structures in the exact completion of the elementary category of sets and relations. Moreover, the same abstract construction of taking the exact completion, when applied to the category of topological spaces and continuous functions produces an extension of it which is cartesian closed. In other words, there is one general mathematical construction that, when applied to a very elementary category, generates the category of topological spaces and continuous functions, and when applied to that category produces a very suitable category where to deal with all sorts functions spaces.

Yet, via such free constructions it is possible to give a new meaning to Marshall Stone's dictum: "always topologize" as the category of sets and relations is the most natural way to give structure to logic and the category of topological spaces and continuous functions is obtained from it by a good mix of free — *i.e.* syntactic — constructions.

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1. Introduction

Topological spaces offer an excellent abstraction for many aspects of reality. As a mathematical abstraction, they require a deep understanding of the theory of sets as a topological space is far from being an algebraic structure. Indeed, the notion of topology imposes to use sets of sets as a basic piece of information.

Taking a different point of view via categorical constructions, it is possible to recognize topological spaces as structures in a category obtained by a finitary, syntactic construction from the category of sets and relations. The difficult requirement moves in this case to handling mathematical structures — *i.e.* categories — which are alas far from the common intuition, worse still often that intuition stops at categories of functions. Yet, once the step of viewing categories as structures is taken, irrespective of their actually consisting of just functions, it is possible to look at topological spaces as algebraic structures in some free extensions of the category of sets and relations. Crucially here one must notice that a free extension corresponds to a very specific syntactic construction. By "finitary" above, we meant that the actual production of the extending categorical structure is performed only with finite constructions applied to the data of the given category.

The first section introduces the notion of category and suggests a few examples in order to provide some intuition of how to think of a category as itself a mathematical structure, not as a universe of mathematical structures and transformations between these. It is only in this way that we can explain how the overall scheme to produce a category of topological spaces can be seen as based on solutions to universal problems about categories. A reader expert in category theory can simply skim rapidly through that section just to check the notation introduced.

In the second section, we recall the essential notions of monoid and of comonoid (taking full advantage of the freedom of categories where arrows are fundamental notions and need not be specific set-theoretic functions) in a category and develop it in the case of the category of sets and relations. We also recall some basic facts about comonoids. In the third section, we describe the first two syntactic constructions on categories: the Freyd completion and the category of commutative comonoids. We recall the universal problem each solves, so that we can explain how topological spaces can be recognized as comonoids in that. In the fourth section we introduce another syntactic construction (the exact completion) and go through the same steps as in the previous section to get to the category of frames and homomorphism in the next, fifth section. In the final section, we connect the analysis completed thus far with another result using exact completions pointing out some open problems.

2. The category of relations

Categories allow to approach mathematical structures and theories of those within a unified framework. The structures and the transformations between them are viewed as whole in a graph together with an operation of *composition*[1] on pairs of consecutive arrows which is associative and with unit loops, u_C for each node C. A node in a graph which is a category is usually termed *object*.

In the following we shall review very quickly a few basic notions that we shall need in other sections, but we invite the reader to refer to the extensive literature available to read more about some of the concepts introduced in the following, see *e.g.* (Borceux 1995; Freyd and Scedrov 1991; Lawvere and Schanuel 1997; Lawvere and Rosebrugh 2003; Mac Lane 1998; Mac Lane and Moerdijk 1992; McLarty 1995; Taylor 1999).

Some concrete examples of categories are

***WWW*:** objects are the web hosts in the World Wide Web; an arrow from a host to another is a path connecting the first to the second; composition is concatenation of paths; the unit at a host is the empty path from that host

***Hop*:** objects are the web hosts in the World Wide Web (like the previous one); an arrow from a host to another is a number counting the hops needed to connect the first to the second; composition is given by addition; the unit at an object is the zero hop from that object

***Route*:** objects are the street intersections on a city map; an arrow from an intersection to another is a drivable route; composition is concatenation; the unit at an intersection is a stop

***Tube*:** objects are the stations of the London Tube; an arrow from a station to another is a sequence of line connections which allow to

[1] Like many names in the literature in category theory, it has a strong similarity with some set-theoretic name, but there is no implied connection between the two; in particular, always remember that a category need hardly be a category of sets and functions. We shall alwyas write it by simply juxtaposing the two arguments.

go from the first to the second; composition is given by linking two sequences of connections; units are the empty sequences.

Abstract mathematical examples are

Path(G)**:** given any (directed, multilabelled) graph G, the objects are the same as those of G; an arrow from an object to another is a path in the graph G from the first object to the second; composition is concatenation; units are empty paths

***Num&Fct*:** objects are the finite numerals; an arrow from an object to another is a set-theoretic function from the first numeral to the second; composition is functional composition (usually written backwards); the unit at a numeral is the identity function on that numeral

***Fin&Fct*:** objects (form a class and) are the finite sets; an arrow from an object to another is a set-theoretic function from the first set to the second; composition is functional composition; units are the identity functions

***Set&Fct*:** objects (form a class and) are the sets; an arrow from an object to another is a set-theoretic function from the first set to the second; composition is functional composition; units are the identity functions

Mod$_{\mathcal{A}}$***&Hm*:** given an algebraic theory $\mathcal{A}$, objects (form a class and) are the models of $\mathcal{A}$; an arrow from a model to another is a homomorphism from the first to the second; composition is functional composition (since a functional composition of homomorphisms is a homomorphism); units are identity functions

***Set&Rel*:** objects (form a class and) are the sets; an arrow from an object to another is a set-theoretic relation from the first set to the second; composition is relational composition[2]; units are diagonal relations — the same as (the graphs of) identity functions

***Top&Cnt*:** objects (form a class and) are the topological spaces; an arrow from a topological space to another is a continuous function from the first to the second; composition is functional composition (since a functional composition of continuous functions is continuous); units are identity functions.

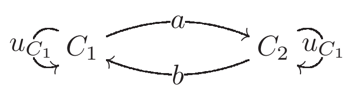
Clearly, ***WWW*** and ***Route*** are examples of categories of the form ***Path***(G). Comparing the various examples, the category ***Set&Fct*** con-

[2] Recall that, given relations $a \subseteq C \times D$ and $b \subseteq D \times E$, their relational composition is the relation $ab \subseteq C \times E$ of those pairs $\langle x, z\rangle \in C \times E$ such that $\exists_{y \in D}(\langle x, y\rangle \in a \wedge \langle y, z\rangle \in b)$.

tains the category $\boldsymbol{Fin\&Fct}$: there are fewer objects. Similarly the category $\boldsymbol{Set\&Rel}$ contains the category $\boldsymbol{Set\&Fct}$: the objects are the same, but there are definitely fewer arrows from a non-empty object to an(y)other object. There is an important difference between the two containments: the peculiarity in the first case is that the subcategory $\boldsymbol{Fin\&Fct}$ is completely determined from the data of the supercategory $\boldsymbol{Set\&Fct}$ and its objects, *i.e.* those sets which are finite. Such a subcategory is named *full*. Other examples of full subcategories are $\boldsymbol{Mod}_{\mathcal{M}}\boldsymbol{\&Hm}$ and its subcategory $\boldsymbol{Mod}_{\mathcal{G}}\boldsymbol{\&Hm}$ where $\mathcal{M}$ is the algebraic theory of monoids and $\mathcal{G}$ is the algebraic theory of groups, and $\boldsymbol{Mod}_{\mathcal{SL}_0}\boldsymbol{\&Hm}$ and its subcategory $\boldsymbol{Mod}_{\mathcal{B}}\boldsymbol{\&Hm}$ where $\mathcal{SL}_0$ is the algebraic theory of semilattices with a null element and $\mathcal{B}$ is the algebraic theory of Boolean algebras.

Categories and functors[3] are the best mathematical paradigm available to compare apparently different presentations of a mathematical concept, as it allows for a general notion of isomorphism between objects in a category and uses that to determine a general comparison test between categories which is weaker (and often apter) than the request that they be isomorphic. In the following we recall the notion of iso(morphism) in a category and of equivalence of categories.

Given a category $\boldsymbol{C}$, an arrow $C_1 \overset{a}{\longrightarrow} C_2$ in $\boldsymbol{C}$ is *iso* if it has a (necessarily unique) inverse, *i.e.* an arrow $C_2 \overset{b}{\longrightarrow} C_1$ such that in the diagram



compositions on all paths linking any two objects give the same result[4] where u_N denotes the unit loop on the object N.

An *equivalence of categories* captures formally how two presentations describe the same mathematical concept: it consists of a functor $F\colon \boldsymbol{C} \to \boldsymbol{D}$ such that, for every pair of objects C_1 and C_2 in $\boldsymbol{C}$ the function F restricted to the set of arrows from C_1 to C_2 is bijective onto the set of arrows from $F(C_1)$ to $F(C_2)$ and for every object D in $\boldsymbol{D}$, there is an object C in $\boldsymbol{C}$ and an iso from $F(C)$ to D.

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[3] It is the name of the obvious notion of "homomorphism from a category to another": a *functor* $F\colon \boldsymbol{C} \to \boldsymbol{D}$ is a function that maps each object of $\boldsymbol{C}$ to a object of $\boldsymbol{D}$ and each arrow a from object C_1 to object C_2 in $\boldsymbol{C}$ to an arrow $F(a)$ from object $F(C_1)$ to object $F(C_2)$ in $\boldsymbol{D}$ in such a way that composition and units are preserved.

[4] In the present case, the equalities are all those that can be derived from the two $u_{C_1} = ab$ and $u_{C_2} = ba$.

The inclusion functor from ***Num&Fct*** to ***Fin&Fct*** is an equivalence of categories since every finite set is in bijection with its cardinality.

We are interested in various other equivalences that will appear in the next sections.

The notion of iso between objects looks like a relaxed notion of equality and a prominent example of that is the structure that the cartesian product of sets induces on many of the examples we have listed.

In the category ***Num&Fct***, the product of two numerals is given by number multiplication and it is easy to define the product of two functions. The functor so obtained

$$\boldsymbol{Num\&Fct} \times \boldsymbol{Num\&Fct} \xrightarrow{\quad\times\quad} \boldsymbol{Num\&Fct}$$

enjoys properties reminiscent of those for a monoid. Similarly, in the category ***Set&Fct***, the cartesian product of two sets extends to a functor

$$\boldsymbol{Set\&Fct} \times \boldsymbol{Set\&Fct} \xrightarrow{\quad\times\quad} \boldsymbol{Set\&Fct}$$

only this time the monoid-like properties are "up to iso". And the same can be performed in the category ***Set&Rel***

$$\begin{array}{ccc}
\boldsymbol{Set\&Rel} \times \boldsymbol{Set\&Rel} & \xrightarrow{\quad\times\quad} & \boldsymbol{Set\&Rel} \\
\langle C_1, C_2 \rangle & \longmapsto & C_1 \times C_2 \\
\langle a_1, a_2 \rangle \downarrow\downarrow & \longmapsto & a_1 \otimes a_2 \downarrow \\
\langle D_1, D_2 \rangle & \longmapsto & D_1 \times D_2
\end{array}$$

where a little care must be put in the definition of the product of two relations as, given $a_1 \subseteq C_1 \times D_1$ and $a_2 \subseteq C_2 \times D_2$, one wants a relation $a_1 \otimes a_2 \subseteq (C_1 \times C_2) \times (D_1 \times D_2)$. So $a_1 \otimes a_2$ consists of those pairs of pairs $\langle\langle x, y\rangle, \langle u, v\rangle\rangle$ such that $\langle x, u\rangle \in a_1$ and $\langle y, v\rangle \in a_2$.

We shall consider in the remainder of the section how category theory allows to approach such notions as those of point or of open more freely by looking at some examples.

In the paradigmatic instance of the category ***Num&Fct***, the object $\{0\}$ encompasses the notion of constant as the constant functions are precisely those that are obtained by composition with a function into $\{0\}$, and also it represents the generic point as, considered any numeral

$\underline{n}$, the functions from $\{0\}$ to $\underline{n}$ are precisely the elements of $\underline{n}$. It is easy to see that the same analysis can be performed with a(ny) one-point topological space T in the category ***Top&Cnt***; on the other hand, in a category $\boldsymbol{Mod}_{\mathcal{A}}$***&Hm***, while the one-point algebra will certainly provide the notion of constant (homomorphism), this is not very useful. And the notion of generic point will be taken by the free algebra U on one generator — *e.g.* the additive monoid on the natural numbers in $\boldsymbol{Mod}_{\mathcal{M}}$***&Hm*** or the additive group on the integers in $\boldsymbol{Mod}_{\mathcal{G}}$***&Hm*** —, as all homomorphisms from U into another algebra A will be (in bijection with) the elements of the underlying set of A.

Similar to the last example, in the category ***Set&Rel*** the notion of constant is of very little interest (and provided by $\emptyset$) while a(ny) one-element set I represents the generic point. Note that the arrows from I to a set S in ***Set&Rel*** are (in bijection with) the subsets of S.

In all the examples considered, the generic point G allows to detect equality of parallel arrows, in other words, given two arrows $C_1 \underset{b}{\overset{a}{\rightrightarrows}} C_2$ such that all the composites

$$G \xrightarrow{p} C_1 \underset{b}{\overset{a}{\rightrightarrows}} C_2$$

as p varies among the arrows from G to C_1, are equal, one has that $a = b$.

We check that in ***Set&Rel*** any one-element set I enjoys that property. Indeed, let $C_1 \underset{b}{\overset{a}{\rightrightarrows}} C_2$ be two relations, *i.e.* $a, b \subseteq C_1 \times C_2$, and consider any $\langle x, z\rangle \in a$. Consider the relation $I \xrightarrow{p} C_1$ given by $p := I \times \{x\}$. Since $I \times \{z\} \subseteq pa = pb$, by definition of relational composition it must be that $\langle x, z\rangle \in b$. Hence every pair in a is also in b. The converse follows symmetrically.

3. Monoids and comonoids

There is an important characterization of the category ***Set&Fct*** within the category ***Set&Rel*** which is relevant for the sequel. It requires first to recognize an algebraic structure on each object of ***Set&Rel***, see (Carboni and Walters 1987). For the rest of the paper, we shall use I for a specific one-element set, say $\{0\}$.

Given a set C in ***Set&Rel***, write $\delta_C\colon C \to C \times C$ and $\varepsilon_C\colon C \to I$ for the relations

$$\delta_C := \{\langle x, \langle y, z\rangle\rangle \in C \times (C \times C) \mid x = y, x = z\}$$
$$\varepsilon_C := C \times I$$

Given a relation $a: C \to D$ the following holds:
(i) a is total if and only if the diagram

$$\begin{array}{ccc} C & & \\ {\scriptstyle a}\downarrow & \searrow{\scriptstyle \varepsilon_C} & \\ D & \xrightarrow{\varepsilon_D} & I \end{array}$$

commutes in ***Set&Rel***
(ii) a is single-valued if and only if the diagram

$$\begin{array}{ccc} C & \xrightarrow{\delta_C} & C \times C \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a \otimes a} \\ D & \xrightarrow{\delta_D} & D \times D \end{array}$$

commutes in ***Set&Rel***.[5]

In fact, all this sums up very well with the fact that each structure $(C, \delta_C, \varepsilon_C)$ is a commutative comonoid in ***Set&Rel***. We refer the reader to (Carboni and Walters 1987; Fox 1976) for the definition and the properties of commutative comonoid in a monoidal category and content ourselves with specifying what this amount to in the special case: a *commutative comonoid* in ***Set&Rel*** is a triple (C, m, e) where C is an object in ***Set&Rel***, and $m: C \to C \times C$ and $e: C \to I$ are arrows in ***Set&Rel*** such that the following diagrams commute

associativity:

$$\begin{array}{ccccc} C & \xrightarrow{m} & C \times C & & \\ {\scriptstyle m}\downarrow & & & \searrow{\scriptstyle m \otimes u_C} & \\ C \times C & \xrightarrow{u_C \otimes m} & C \times (C \times C) & \xrightarrow{a_{C,C,C}} & (C \times C) \times C \end{array}$$

neutrality:

$$\begin{array}{ccccc} C \times C & \xleftarrow{m} & C & \xrightarrow{m} & C \times C \\ {\scriptstyle u_C \otimes e}\downarrow & & \downarrow{\scriptstyle u_C} & & \downarrow{\scriptstyle e \otimes u_C} \\ C \times I & \xrightarrow{r_C} & C & \xleftarrow{l_C} & I \times C \end{array}$$

commutativity:

$$\begin{array}{ccc} C & \xrightarrow{m} & C \times C \\ & \searrow{\scriptstyle m} & \downarrow{\scriptstyle c_{C,C}} \\ & & C \times C \end{array}$$

[5] We sketch how to prove (ii): consider a pair $\langle x, \langle u, v \rangle \rangle$ in $C \times (D \times D)$. It is in the relation $\delta_C(a \otimes a)$ exactly when both $\langle x, u \rangle$ and $\langle x, v \rangle$ are in a. It is in the relation $a\delta_D$ exactly when both $\langle x, u \rangle$ and $\langle x, v \rangle$ are in a, and also $u = v$. Clearly $a\delta_D \subseteq \delta_C(a \otimes a)$, but the two coincide precisely when a is single-valued.

where the relations a, r, l and c are the obvious isos (in fact, bijections). A *homomorphism* between two comonoids (C, m, e) and (D, n, d) is an arrow $a: C \to D$ such that

$$\begin{array}{ccccc} & & C & \xrightarrow{\;m\;} & C \times C \\ & \swarrow^{e} & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a \otimes a} \\ I & \xleftarrow{\;d\;} & D & \xrightarrow{\;n\;} & D \times D \end{array}$$

We denote by CCom(***Set&Rel***, $\times$) the category of commutative comonoids and homonorphisms in ***Set&Rel***.

The unexperienced reader, though possibly stunned by the sheer complexity of the last arrows and diagrams, may feel encourage to press on by the following remark.

3.1 Remark. A relation $C \text{—}a\text{→} D$ is a function from C to D if and only if it is a homomorphism of comonoids from the comonoid $(C, \delta_C, \varepsilon_C)$ to the comonoid $(D, \delta_D, \varepsilon_D)$.

In other words, the functions $f: C \to D$ are exactly those relations from C to D which satisfy the two conditions (i) and (ii). So, the notion of function between sets can be recovered from that of relation by (co)algebraic means: the category ***Set&Fct*** of sets and functions can be seen as a category of (co)monoids on the category ***Set&Rel*** of sets and relations.

We shall see that topological spaces with continuous functions appear in a very similar fashion on a natural extension of the category ***Set&Rel***.

4. The Freyd completion

Recall that the category ***BP*** of basic pairs, as introduced in (Sambin and Gebellato 1
consists of

objects: arrows $r: X \to A$ in ***Set&Rel***. In this context, one calls the triple (X, A, r) a *basic pair*

an arrow $[a, a']: (X, A, r) \to (Y, B, s)$ is an equivalence class of pairs $\langle a, a' \rangle$ of ***Set&Rel*** such that

$$\begin{array}{ccc} X & \xrightarrow{\;a\;} & Y \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle s} \\ A & \xrightarrow{\;a'\;} & B \end{array}$$

commutes, with respect to the equivalence relation

$$\langle a, a' \rangle \sim \langle b, b' \rangle \Leftrightarrow as = bs [= ra' = rb']$$

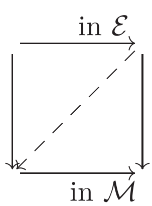
composition is given by pasting commutative squares

$$\begin{array}{ccccc} X & \xrightarrow{a} & Y & \xrightarrow{b} & Z \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} \\ A & \xrightarrow{a'} & B & \xrightarrow{b'} & C \end{array}$$

More explicitly, it is given on representative pairs as $(ab, a'b')$.

By its very definition, the category $\boldsymbol{BP}$ coincides with the free completion $\mathrm{Fr}(\boldsymbol{Set\&Rel})$ of the category $\boldsymbol{Set\&Rel}$ of sets and relations to a category with a proper factorization system. The general construction goes under the name of “Freyd completion of a category”, see (Freyd 1966; Grandis 2000). Recall from (Freyd and Kelly 1972) that a *factorization system* in a category $\boldsymbol{C}$ consists of a pair $(\mathcal{E}, \mathcal{M})$ of collections of maps of $\boldsymbol{C}$ such that

1. $\mathcal{E}$ and $\mathcal{M}$ are subcategories of $\boldsymbol{C}$, each containing all isomorphisms,
2. every map can be factored as the composition of a map in $\mathcal{E}$ followed by a map in $\mathcal{M}$,
3. each map in $\mathcal{E}$ is *orthogonal to* any map in $\mathcal{M}$, *i.e.* there is a unique diagonal fill-in in any commutative square of the form



It is *proper* if every map in $\mathcal{E}$ is epic, or equivalently, every map in $\mathcal{M}$ is monic.

Factorization systems generalize the structure of surjections and one-one functions on the category $\boldsymbol{Set\&Fct}$.

4.1 REMARK. The statement about the universal property satisfied by the Freyd completion of a category $\boldsymbol{C}$ is the following:
For a category $\boldsymbol{C}$, *the category* $\mathrm{Fr}(\boldsymbol{C})$ *has a proper factorization system and there is an embedding functor* $D\colon \boldsymbol{C} \to \mathrm{Fr}(\boldsymbol{C})$ *which is the universal*

solution among categories with proper factorization systems and a functor from $\boldsymbol{C}$*,* i.e. *for every category* $\boldsymbol{B}$ *with a proper factorization system and every functor* $G\colon \boldsymbol{C} \to \boldsymbol{B}$ *there is a functor* $G_s\colon \mathrm{Fr}(\boldsymbol{C}) \to \boldsymbol{B}$ *which maps the factorization system of* $\mathrm{Fr}(\boldsymbol{C})$ *into that of* $\boldsymbol{B}$ *and such that*

$$\begin{array}{ccc} \boldsymbol{C} & \xrightarrow{\;D\;} & \mathrm{Fr}(\boldsymbol{C}) \\ & {\scriptstyle G}\searrow & \downarrow {\scriptstyle G_s} \\ & & \boldsymbol{B}. \end{array}$$

The functor G_s *with that property is completely determined up to a unique natural iso.*

As already said, the construction of the category $\mathrm{Fr}(\boldsymbol{C})$ is that given in the particular instance of the category of basic pairs replacing ***Set&Rel*** with $\boldsymbol{C}$. Its syntactic nature is apparent by the finitary definition from the data of $\boldsymbol{C}$ and also by the universal property it satisfies. We shall see that all categorical constructions involved in the *de-pointification* of topological spaces have such a syntactic nature.

The product functor ***Set&Rel*** $\times$ ***Set&Rel*** $\xrightarrow{\quad\times\quad}$ ***Set&Rel*** extends directly to the category of basic pairs Fr(***Set&Rel***). Another similarity between ***Set&Rel*** and Fr(***Set&Rel***) is that each is equivalent to its opposite category.

In a sense, when we determined functions between sets as particular homomorphisms of comonoids we chose *one* direction in ***Set&Rel***. Similarly, one can do that in Fr(***Set&Rel***). As we have seen, a set enjoys a structure of commutative comonoid in the category ***Set&Rel***. And it is possible to do the same for a topological space (S, σ) in the category of basic pairs. First consider the relation $S \text{—}{\in}\text{→} \sigma$ and note that the diagram of relations

$$\begin{array}{ccccc} I & \xleftarrow{\;\varepsilon_S\;} & S & \xrightarrow{\;\delta_S\;} & S \times S \\ \downarrow {\scriptstyle u_!} & & \downarrow {\scriptstyle \in} & & \downarrow {\scriptstyle \in \otimes \in} \\ I & \xleftarrow{\;\{S\} \times I\;} & \sigma & \xrightarrow{\;\cap^{\mathrm{o}}\;} & \sigma \times \sigma \end{array}$$

commutes, providing the basic pair $(S, \sigma, \in)$ with a structure $C(S, \sigma)$ of commutative comonoid. Moreover, given any continuous function $f: (S, \sigma) \to (T, \tau)$, the diagram

$$\begin{array}{ccc} S & \xrightarrow{\quad f \quad} & T \\ \Big\downarrow{\scriptstyle \in} & & \Big\downarrow{\scriptstyle \in} \\ \sigma & \xrightarrow{(f^{-1})^{\mathrm{o}}} & \tau \end{array}$$

commutes so that $[f, (f^{-1})^{\mathrm{o}}]: (S, \sigma, \in) \to (T, \tau, \in)$ is an arrow of basic pairs which is clearly a homomorphism of comonoids.[6]

4.2 THEOREM. *The functor* $C: \boldsymbol{Top\&Cnt} \longrightarrow \mathrm{CCom}(\mathrm{Fr}(\boldsymbol{Set\&Rel}), \times)$, *which maps a topological space* (S, σ) *to the basic pair* $(S, \sigma, \in)$ *and a continuous function* $f: (S, \sigma) \to (T, \tau)$ *to the pair* $C(f) = [f, (f^{-1})^{\mathrm{o}}]$, *has a right adjoint. Moreover, the topological spaces on which the unit of the adjunction is a homeomorphism are precisely the sober spaces.*

The proof of the theorem can be found in (Bucalo and Rosolini 2006) and we refer the interested reader to that. There are two important points that we recall in the followings remarks

4.3 REMARK. The functor $C: \boldsymbol{Top\&Cnt} \longrightarrow \mathrm{CCom}(\mathrm{Fr}(\boldsymbol{Set\&Rel}), \times)$ is completely determined by a universal property of the category $\mathrm{CCom}(\mathrm{Fr}(\boldsymbol{Set\&Rel}), \times)$, which was proved by Thomas Fox in (Fox 1976) and whose statement is the following:

Given a symmetric monoidal category $(\boldsymbol{C}, \otimes, I, a, r, l, t)$, *the category* $\mathrm{CCom}(\boldsymbol{C}, \otimes)$ *of commutative comonoids and homomorphisms has finite (categorical) products and the forgetful functor* $U: \mathrm{CCom}(\boldsymbol{C}, \otimes) \to \boldsymbol{C}$ *is strict monoidal. Moreover, this is the universal solution among categories with finite products and a strict monoidal functor to* $\boldsymbol{C}$, i.e. *for every category* $\boldsymbol{A}$ *with finite products and every functor* $G: \boldsymbol{A} \to \boldsymbol{C}$ *which transforms the products in* $\boldsymbol{A}$ *into tensor products in* $\boldsymbol{C}$ *and takes the terminal object to the unit for the tensor*

$$G(A_1) \otimes G(A_2) \xrightarrow{\sim} G(A_1 \times A_2) \qquad\qquad I \xrightarrow{\sim} G(1)$$

[6] For a relation $C \xrightarrow{a} D$, we write $D \xrightarrow{a^{\mathrm{o}}} C$ for its opposite relation, *i.e.* $a^{\mathrm{o}} := \{\langle x, y\rangle \in D \times C \mid \langle y, x\rangle \in a\}$.

preserving all coherence isomorphisms, there is a functor $G_c\colon \boldsymbol{A} \to \mathrm{CCom}(\boldsymbol{C}, \otimes, I)$ *which preserves finite products and extends* G

$$\begin{array}{ccc} \boldsymbol{A} & & \\ {\scriptstyle G_c}\downarrow & \searrow^{G} & \\ \mathrm{CCom}(\boldsymbol{C}, \otimes, I) & \xrightarrow{\;U\;} & \boldsymbol{C}. \end{array}$$

The functor G_c *with that property is completely determined up to a unique natural iso.*

In the particular situation of topological spaces and comonoids of basic pairs, the category $\boldsymbol{C}$ is $\mathrm{Fr}(\boldsymbol{Set\&Rel})$, the tensor product $(X, A, r) \otimes (Y, B, s)$ is $(X \times Y, A \times B, r \otimes s)$, the category $\boldsymbol{A}$ is that of topological spaces and continuous functions $\boldsymbol{Top\&Cnt}$, and the functor $G\colon \boldsymbol{Top\&Cnt} \to \mathrm{Fr}(\boldsymbol{Set\&Rel})$ evaluates the basic pair $(S, \sigma, \in)$ of a topological space (S, σ). What the general theorem explains is that, in order to obtain the product preserving representation of topological spaces as comonoids of basic pairs, it is enough to check that G transforms a topological product into a product of basic pairs.

It is very useful to know that the underlying property for the representation functor in 4.2 is trivial to check, and that the reason for that resides in a general, syntactic construction on (monoidal) categories: that of taking the category of commutative comonoids and homomorphisms.

4.4 REMARK. The second observation about 4.2 is how the construction of the right adjoint is based on the notions of generic point, as discussed on p. 332, and on that of generic open.

Let $\mathbb{G}$ be the only comonoid structure on the basic pair $u_1\colon 1 \longrightarrow 1$, and let Σ be the comonoid

$$\begin{array}{ccccc} I & \xleftarrow{\quad t \quad} & \{0,1\} & \xrightarrow{\;\delta_{\{0,1\}}\;} & \{0,1\} \times \{0,1\} \\ \downarrow{\scriptstyle u_I} & & \downarrow{\scriptstyle \geq} & & \downarrow{\scriptstyle \geq \otimes \geq} \\ I & \xleftarrow{\{0\} \times I} & \{0,1\} & \xrightarrow{\quad \vee^{\mathrm{o}} \quad} & \{0,1\} \times \{0,1\} \end{array}$$

on the basic pair $\{0,1\} \xrightarrow{\geq} \{0,1\}$.

If a basic pair is of the form $(S, \sigma, \in)$ for some sober topological space (S, σ), then the points of S are in a one-one correspondence with the comonoid homomorphisms from $\mathbb{G}$ to the comonoid $C(S, \sigma)$.

Moreover, the open subsets of S are in a one-one correspondence with the comonoid homomorphisms from $C(S,\sigma)$ to Σ. And a point of S is in an open subset if and only if the composition of the corresponding homomorphisms is equivalent to the total homomorphism $[I\times\{0,1\}, I\times\{0,1\}]\colon \mathbb{G}\to\Sigma$.

The view of topological spaces as comonoids of basic pairs makes the approach to topology much more elementary as it avoids completely references to families of subsets. It also provides a standard approach to topological representations of preordered structures as such a piece of data prompts immediately the point of view of the preorder as a basic pair.

In that way, it provides standard tools with which to approach topology in a pointfree way. Indeed, one could take the underlying structure of comonoids of basic pairs in the perspective that, since all there is to know about a topological space is determined by its open sets, all one must do is to give a (relational) presentation for those, irrespective of what points one expects.

5. The exact completion

A category is *exact* (in the sense of Barr (Barr 1971)) if it has finite products, equalizers, quotients (*i.e.* coequalizers) of equivalence relations and ,moreover, these are effective and stable under pullbacks. Instances of exact categories are ***Fin&Fct***, ***Set&Fct***, $\boldsymbol{Mod_{\mathcal{A}}\&Hm}$ for $\mathcal{A}$ an algebraic theory.

The exact completion is another free construction on categories that is related to topology. The exact completion $\boldsymbol{Set\&Rel}_{\mathrm{ex}}$ of the category ***Set&Rel*** of sets and relations is (equivalent to) the category ***CmpLatt&SupPres*** of complete lattices and functions between those which preserve arbitrary sups.

The theorem about the universal property satisfied by the exact completion of a category $\boldsymbol{C}$ is similar to that for the Freyd completion, providing the universal solution for appropriate functors to exact categories.

Instances of exact completions are categories of the form $\boldsymbol{Mod_{\mathcal{A}}\&Hm}$, for $\mathcal{A}$ an algebraic theory; the category $\boldsymbol{Mod_{\mathcal{A}}\&Hm}$ is the exact completion of its full subcategory on the free algebras. Also the effective topos, the extension of the realizability interpretation of

(Kleene 1945) to full intuitionistic set theory, is an exact completion of the category of *assemblies*, see (van Oosten 2008), which offers a fitting interpretation of Martin-Löf intensional type theory.

Since in a exact category coequalizers and monos form a factorization system, by the universal property of the Freyd completion there is a canonical functor

$$Q\colon \mathrm{Fr}(\boldsymbol{Set\&Rel}) \to \boldsymbol{CmpLatt\&SupPres}$$

which maps a basic pair (X, A, r) to the complete sublattice $Q(X, A, r)$ of $\mathrm{P}(A)$ which is the image of $\mathrm{P}(X)$ via r. In fact, since every lattice is a homomorphic image of a powerset lattice and also a sublattice of a powerset lattice, it is easy to conclude that the canonical functor is an equivalence.

For the constructively minded reader, it would be relevant to note that that result is equivalent to the underlying logic of the universe of discourse being Boolean. And there are two other remarks:

- the functor does not transform the product functor on $\mathrm{Fr}(\boldsymbol{Set\&Rel})$ in the tensor product of complete lattices, see (Joyal and Tierney 1984; Bucalo and Rosolini 2006)[7].
- unlike $\boldsymbol{Set\&Rel}$ and $\mathrm{Fr}(\boldsymbol{Set\&Rel})$, there is no duality on the category $\boldsymbol{CmpLatt\&SupPres}$ which transforms the tensor product into itself.

In other words, when we look at the exact completion of $\boldsymbol{Set\&Rel}$, we have preemptively decided a direction of arrows, and the possibility to decide the direction of arrows for comonoid homomorphisms is already imposed.

On the other hand, given topological spaces (S, σ) and (T, τ), the functor $Q\colon \mathrm{Fr}(\boldsymbol{Set\&Rel}) \to \boldsymbol{CmpLatt\&SupPres}$ transforms the product $(\sigma \times \tau, S \times T, \ni \otimes \ni)$ of the basic pairs $(\sigma, S, \ni)$ and $(\tau, T, \ni)$ in the complete lattice $Q(\sigma, S, \ni) \otimes Q(\tau, T, \ni)$.

6. The opposite category of monoids

The closing remark in the previous section suggests that the representation of topological spaces should take the opposite direction in the

[7] The tensor product $L \otimes M$ of the complete lattices L and M is the set of Galois connections from L to M with the pointwise order.

category ***CmpLatt&SupPres***. Indeed, the category of locales and continuous functions is a full subcategory of the opposite of the category of commutative monoids and homomorphisms in the category ***CmpLatt&SupPres*** with respect to the tensor of complete lattices. A monoid in the category ***CmpLatt&SupPres*** is named a unital quantale and a fundamental reference for those is (Rosenthal 1990). The full subcategory of unital commutative quantales and homomorphisms whose opposite is the category of locales is the category of frames, see (Johnstone 1982; Joyal and Tierney 1984; Taylor 1999; Vickers 1989).

Since a sup-preserving function $M \otimes M \to M$ amounts to a binary operation $M \times M \to M$ which preserves sups in each argument separately, a monoid in ***CmpLatt&SupPres*** with respect to the tensor consists of a complete lattice with a binary operation of monoid on M which distributes over arbitrary sups.

6.1 Remark. Frames are those unital commutative quantales whose binary operation $\cdot\colon M \times M \to M$ is idempotent and whose unit is the top element of M. Indeed, since $\cdot$ preserves sups, it preserves the order; hence, for $a, b \in M$, $a \cdot b \leq a \cdot \top = a$. Moreover, for any $x \in M$ such that $x \leq a$ and $x \leq b$, one has that $x = x \cdot x \leq a \cdot b$.

Frames (aka locales) and (the opposite direction of) homomorphisms have become the most successful approach to topology in a pointless perspective, when one need not expect to have sufficient information to construct all the *points* of the intended space. It is not surprising that many constructions of frames have useful applications in constructive aspects of topology.

It is important to note that the constructions involved in the presentation of frames (or quantales, for that matter) are of a finitary, syntactic nature, based on the underlying data of the category of sets and relations. In a sense, in its various forms topology is a syntactic extension of the logical calculus of relations.

We conclude this section by reviewing briefly how the notions of generic point and of generic open which contribute to the construction of the right adjoint in 4.2 extends to the present situation by computing the correponding frames to the comonoids $\mathbb{G}$ and Σ: the first frame is $\mathrm{P}(I)$, the second is given by the frame of the only non-trivial topology ς on two points — *i.e.* with a single non-trivial open O.

Obvious care must be taken in computing points and opens of a given unital quantale M: the points are the homomorphisms from $p\colon M \to \mathrm{P}(I)$. Open subsets are determined by homomorphisms $\ell\colon \varsigma \to M$ as

$$U_\ell := \{p\colon M \to \mathrm{P}(I) \mid p(\ell(O)) = I\}.$$

7. Spaces of continuous functions

We conclude by mentioning some problems which involve the constructions we have presented in the previous sections.

It is well-known that the same construction of the exact completion, when applied to the category $\boldsymbol{Top\&Cnt}$, produces a very rich category which falls short of being a topos, see (Birkedal *et al.* 1998; Rosolini 2000; Carboni and Rosolini 2000). Beyond being an exact, full extension of the category of topological spaces and continuous functions, the category $\boldsymbol{Top\&Cnt}_{\mathrm{ex}}$ is locally cartesian closed, hence it is a setting where topological spaces can be treated as sets, and one can compute function spaces for any pair of topological spaces.

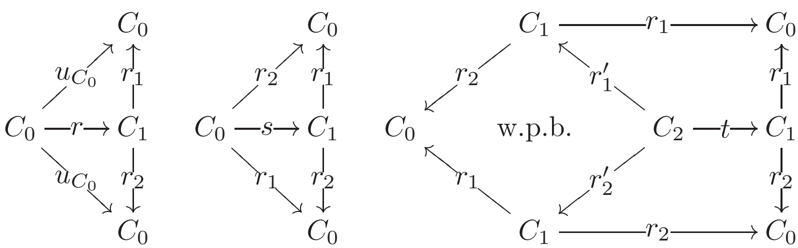
Again $\boldsymbol{Top\&Cnt}_{\mathrm{ex}}$ fits with the spirit of using syntactic constructions, but the syntactic construction, the exact completion, is applied to the very category of topological spaces and continuous functions. It would be interesting to consider if, for any of the syntactic construction considered in the previous sections such as $\mathrm{CCom}(\mathrm{Fr}(\boldsymbol{Set\&Rel}), \times)$, or the category of locales, the exact completion is locally cartesian closed.

Since we have to leave that as a conjecture, we only mention that, in case any one of those were cartesian closed, it would be a very useful setup for a constructive approach to algebraic constructions which require applications of the axiom of choice to produce actual topological spaces of functions.

A. The exact completion

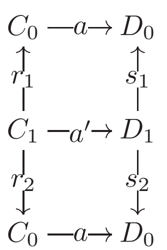
There are many references on exact completion and its various descriptions available in the literature, *e.g.* (Carboni 1995; Carboni and Magno 1982; Freyd and Scedrov 1991; Robinson and Rosolini 1990): we shall just recall that the elementary presentation of the exact completion $\boldsymbol{C}_{\mathrm{ex}}$ of a category $\boldsymbol{C}$ with weak finite limits consists of

objects: *equivalence spans* $C_1 \underset{r_2}{\overset{r_1}{\rightrightarrows}} C_0$ in $\boldsymbol{C}$, *i.e.* a pair of arrows from an object C_1 to an object C_0 such that there are arrows r, s and t making the following diagrams commute

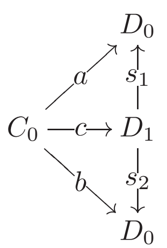


where w.p.b. indicates that the square is a weak pullback

an arrow from $C_1 \underset{r_2}{\overset{r_1}{\rightrightarrows}} C_0$ **to** $D_1 \underset{s_2}{\overset{s_1}{\rightrightarrows}} D_0$ is an equivalence class $[a]$ of arrows a of $\boldsymbol{C}$ such that there is an arrow a'



where $a \sim b$ if there is an arrow c



composition is defined using that of $\boldsymbol{C}$ on representatives.

It is a theorem that the data above define an exact category $\boldsymbol{C}_{\mathrm{ex}}$, and the free one such, see (Carboni and Magno 1982).

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